

ESSAYS IN MECHANISM DESIGN

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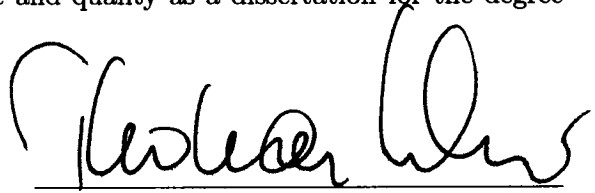
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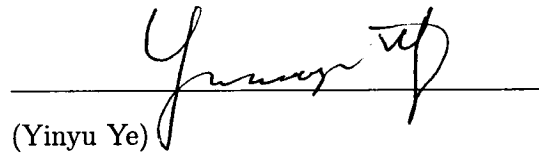
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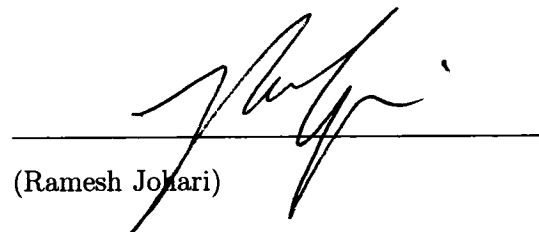
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Approved for the University Committee on Graduate Studies.



*To Sidhant*



# Abstract

This thesis is about two contributions to the theory of *mechanism design* and one application of this theory to the banking industry.

The first contribution tackles an important informational assumption made by most classical mechanism design models, the assumption that players know precisely the value of a resource that they wish to possess. We relax this assumption by modeling the fact that players learn the value of goods through possession. In this setting, we construct a dynamic allocation mechanism that achieves efficiency and other distributional objectives.

Our second contribution is the construction of a mechanism for allocating divisible goods (or multiple units of the same good) that leads to truthtelling and achieves efficiency subject to participation.

More precisely, chapter 2 explains our first contribution where we consider the problem of efficiently allocating a given resource or object repeatedly over time. The agents, who may temporarily receive access to the resource, learn more about its value through its use. When the agents' beliefs about their valuations at any given time are public information, this problem reduces to the classic multi-armed bandit problem, the solution to which is obtained by determining a Gittins index for every agent and allocating the resource to the agent with the highest index. In the setting we study, agents observe their valuations privately, and the efficient dynamic resource allocation problem under asymmetric information becomes a problem of truthfully



eliciting every agent's Gittins index. We introduce two "bounding mechanisms," under which agents announce types corresponding to Gittins indices either at least as high or at most as high as their true Gittins indices. Using an announcement-contingent affine combination of the bounding mechanisms it is possible to implement the efficient dynamic allocation policy.

Our second contribution is studied in chapter 3, where we provide a general mechanism for the allocation of a finite number of divisible resources to a finite number of heterogeneous agents. The mechanism allocates the resource in an ex-post efficient manner amongst participating agents, and is dominant-strategy incentive compatible. Vickrey-Clarke-Groves mechanisms for divisible goods, although efficient, might induce the seller to choose an inefficient allocation if the seller chooses to maximize revenues subsequent to bidders submitting their valuations. Our mechanism solves this problem.

The mechanism we propose is robust in the sense that equilibrium bid functions do not depend on any assumptions about other players. Transfers depend on posted price schedules and discounts, which are given based on the average bid function of the other firms evaluated at the fraction of the object not allocated to them.

In chapter 4 we invoke important results from auction theory to address the design of an auction in the credit card industry: the allocation of portfolios of delinquent debt to outside agencies for collections. We identify and discuss some of the important issues that arise in this auction, issues of portfolio selection, information dissemination, designing security-bids, bid-skewing, learning and moral hazard.

We start in chapter 1 with an overview of mechanism design and some of the theory's seminal results. We end with chapter 5 discussing directions of future research. The work in chapter 2 and chapter 3 is based on papers written with Thomas Weber.

# Acknowledgements

*Guru Govind dono khade, kake lagu paay  
Balihari Guru aapki Govind diyo milaay*

*–Sant Kabir*

*If my God and my Guru stood before me now, I'd bow down first to my Guru,  
For if it were not for him, my Lord, how could I have met you?*

Much of the work in this thesis has been joint work with Thomas Weber, to whom I shall always remain deeply indebted for a truly remarkable research experience. I am fortunate indeed to have spent the many hours staring at the whiteboard alongside him, hours when I have stood, sometimes in awe, sometimes in frustration and sometimes in joy and hoped that through all of this, some of Thomas' knowledge diffused into me. His tireless ability to generate ideas is a phenomenon to experience and learn from, and he shall continue to inspire me every day of my life.

It was my research stint with Prof. Yinyu Ye that first illustrated to me the exciting highs and lows of research and more importantly, allowed me to discover areas of research that irked my interest. Equally important in narrowing down my interests to game theoretic models was my experience at the Electric Power Research Institute working with Robert Entriken followed by work done at the Energy Modelling Forum.

Some of the professors whose courses left a very deep impression on my mind were Prof. Persi Diaconis, Dianne MacLagan, Paul Milgrom, Stephen Boyd and Roger Lee, though admittedly, every course at Stanford has been phenomenal.

I would like to thank American Express for funding my last year of research and providing an opportunity to work on the design of practical auctions. Amongst other teaching assistantships and research assistantships that have supported me through my career at Stanford, there's one that I must make a special mention of. Prof. Sheri Sheppard's funding me through the General Motors project in my first year as a master's student was what allowed me to think of a Ph.D and I owe a deep debt to her for that.

Many enlightening academic conversations with my friends from the Management Science and Engineering department is what makes the department so special to me. Amongst those, I especially cherish the many with Kahn and Gabriel though I'm surely missing the mention of many others.

My experience at Stanford outside of academics has been as enriching, if not more as my academics. Gaurav's friendship through the years, besides the friendship of all my other friends, has taken the meaning of friendship to an entirely other level, the eternity of which I shall always pray for. The corridors of the fourth floor have always been kept lively through the presence of Eymen and Jeffrey and working alone in the office has never felt lonely.

Five years is a long time to be away from one's family, but never for a day have I felt distant from my sister, my parents and my grandmother. I cannot feel fortunate enough to be the recipient of such unconditional love and guidance and it is standing on that foundation that this thesis has been possible.

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# Chapter 1

## Introduction

*It is only an auctioneer who can equally and impartially admire all schools of art.*

– OSCAR WILDE (1854 - 1900), THE CRITIC AS ARTIST, 1891

Mechanism design can be described as the process of devising a set of rules which govern the ways in which people can collectively arrive at desirable outcomes. In politics, the voting procedure that determines how voters' preferences translate to election winners is a mechanism. In economic models, the contracts between workers and employers to determine job functions and wages, or between buyers and sellers to determine prices of goods, or between partners in an organization to determine the allocation of organizational responsibilities are all mechanisms.

Mechanism design theory rigorously investigates the correspondence between normative goals and institutions designed to implement those goals. Given a normative goal or welfare criterion for a particular class of allocation problems, or domain of environments, mechanism design theory formally characterizes organizational mechanisms that will guarantee outcomes consistent with that goal, assuming the outcomes of any such mechanism arise from some specification of equilibrium behavior. Often also called *inverse game theory* mechanism design tries to design of rules to achieve

certain equilibria, whereas game theory predicts equilibrium behavior given a set of rules. The issues that intrigue a game theorist, however, are the same as those that intrigue a mechanism designer: how do results change if informational assumptions change? How do results depend on the distribution of preferences of the players or the number of players? How do results depend on the equilibrium concept governing behavior in the game? Some of the issues that arise in social choice theory and welfare economics are also at the heart of mechanism design theory: Are some first-best welfare criteria achievable? If not, what is the constrained second-best solution?

A framework for a general mechanism design problem is depicted in Figure 1, where the mechanism takes as inputs, the elements of the environment and implements desired outcomes. The environment comprises (i) participants and goods, (ii) the participants' possible *types*, that is, their preferences and beliefs. Our contributions address both of these aspects of the environment. The primary social choice outcome that we aim to achieve as a result of the mechanism is *efficiency*:<sup>1</sup> that is, to maximize the total social surplus. Our first contribution addresses the question of implementing efficient outcomes when people learn their preferences over time, and our second contribution addresses the same question when goods are divisible.

Section 1.1 motivates our first contribution through a simple example that illustrates why naive static mechanisms known to implement efficient outcomes in static settings could fail to do so in dynamic settings. The difficulty in implementing the efficient outcome using naive mechanisms motivates a more in-depth recapitulation of *implementation theory* in Section 1.2 where some of the cornerstone results characterizing the possibilities of implementing efficient outcomes under different equilibrium

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<sup>1</sup>Efficiency is one of the preeminent justifications given for auctioning public resources. For example, Vice-president Al Gore opened the 1994 Broadband PCS spectrum auction proclaiming, "Now we're using the auctions to put licenses in the hands of those who value them the most." In the Omnibus Budget Recognition Act of 1993, which authorized spectrum auctions, the U.S. Congress established the "efficient and intensive use of the electromagnetic spectrum" as a primary objective of the U.S. spectrum auctions.

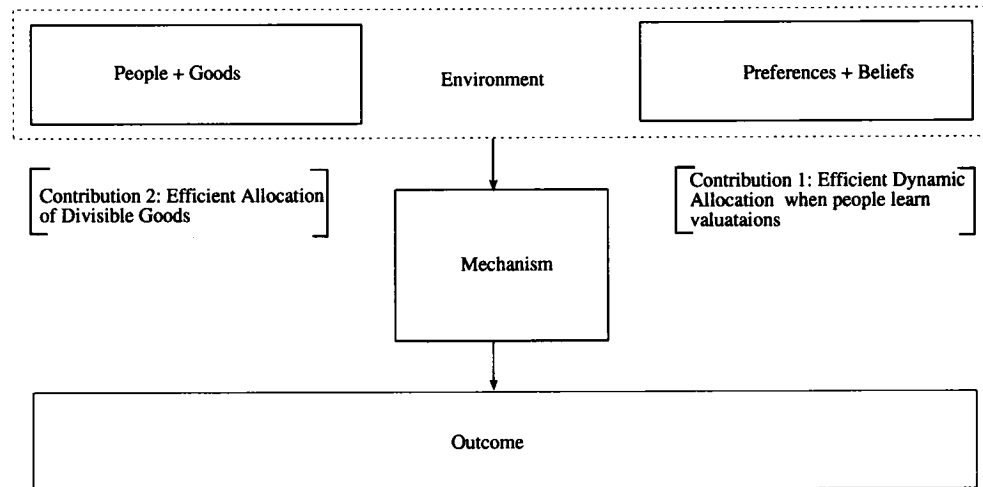


Figure 1.1: The contributions of this thesis as seen in a general mechanism design framework.

concepts are outlined. We then briefly discuss results related to dynamic mechanism design in Section 1.3.

1.4 deals with our second contribution where we summarize some of the results known for auctions of divisible goods and then illustrate a problem with classical mechanisms that are known to achieve efficient outcomes with divisible goods.

Besides theoretical contributions to the field of mechanism design, this thesis also describes the key features of a debt-lease auction currently being developed by a credit-card company: The *debt lease auction*. Section 1.5 motivates Chapter 4 which addresses the practical questions that are faced by a credit-card company in auctioning portfolios of delinquent debt to collections agencies. Since this chapter is mostly about developing auction mechanisms to maximize revenues for the credit card company, we summarize some of the key results that address revenue maximization from the auction theory literature in Section 1.6.

## 1.1 Motivation: Efficient Dynamic Allocation with Uncertain Valuations

Throughout the thesis we will deal with a setting where a single *principal* is faced with the problem of allocating a resource amongst  $N$  *agents*. In the classical static framework, when the agents have valuations about the resource that is privately known, and independent of the valuations of the other agents, the second-price auction is known to implement an efficient outcome. In the dynamic allocation problem that we consider in Chapter 2, a single resource is allocated repeatedly to one of the agents, and the agent allocated the resource receives a reward that is drawn from an unknown underlying reward distribution, and the sample realization helps the winning agent learn something about his underlying uncertain reward distribution.

**Example 1** Consider the dynamic example of two agents whose valuations are given in Figure 1.2, where the first agent has a deterministic underlying valuation sequence of 0.5 and the other agent's valuation is also a deterministic sequence of either 1 or 0, but which one it is, is not known to him. However, getting the resource in one time period would resolve the uncertainty. Assume that the agents discount future rewards by a discount factor  $\beta$ . If the second-price auction were to be run in this example where every agent announces his expected one-period reward, the agent with the highest announcement is awarded the resource and pays the second-highest announced expected one period reward. Assuming the agents announced their expected rewards truthfully, the allocation would still not be efficient as can be seen from the following argument. In period 0, the expected reward announcements of agent 1 and 2 are 0.5 and 0.25 respectively. This results in agent 1 receiving the resource and agent 2's uncertainty remaining yet unresolved. Since the informational state of the two players does not change from period 0 to period 1, the same allocation results in every

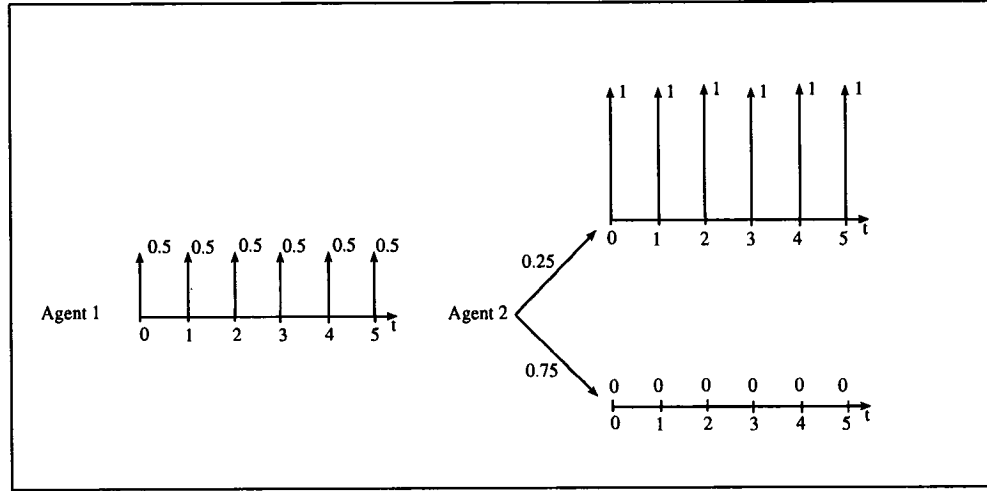


Figure 1.2: Reward sequences for the two agents in example 1

time period. This leads to a total surplus of  $0.5/\beta$ . Consider an alternate allocation strategy which yields higher payoffs for players that are patient enough: allocate the resource to agent 2. If he realizes that his reward sequence consists of *zeros* then switch allocations in future periods to agent 1, otherwise continue with agent 2. This yields expected payoffs of  $0.25 + 0.625\beta/(1 - \beta)$  which is higher for agents that are patient enough.

The second-price auction *implements* an efficient outcome in the static case. However, the value that agents associate with learning is an important factor in dynamic settings and is not accounted for by efficient static mechanisms. We present some of the important results from the theory of mechanism design that characterize the possibility of implementing certain social choice rules under different equilibrium concepts.



Complete Information	Incomplete Information
	

Figure 1.3: Implementability of social choice rules under different equilibrium concepts

## 1.2 Implementability Results

The two important categories of implementability results are based on whether agents have complete or incomplete information about the other players. The equilibrium concepts under which the requisite social choice rule is implemented also vary under the two settings also.

1. Complete information – Implementation via *dominant strategy* and *Nash equilibrium*
2. Incomplete information – Implementation via *dominant strategy* and *Bayesian Nash equilibrium*.

The problems we address in this thesis are all based in settings of incomplete information. We thus outline the important results about implementation in dominant strategies and Bayesian Nash equilibrium. However, we will briefly comment on Nash implementation as well.

Throughout this thesis, we will deal with *principal-agent* settings where agents have incomplete information. In such settings, if every agent has the appropriate incentives to announce outcome relevant information *truthfully*, then implementing efficient outcomes easy: once all agents truthfully reveal the outcome relevant information, the principal decides the optimal outcome and implements it. This makes *truthful implementation* of social choice functions particularly interesting to us.

### 1.2.1 Dominant-Strategy Implementation

This research, mainly following the seminal work of Gibbard (1973) and Satterthwaite (1975), has close connections with social choice theory. The most appealing notion of implementation is the one that makes the weakest assumptions about the agents' behavior: implementation in dominant strategy equilibrium. A dominant strategy equilibrium does not require the participating agents to have any beliefs about the other agents' preferences, let alone their behavior. Dominant strategy mechanisms also happen to be very *simple*.

For  $N$  players, let the profile of the agents' types be indexed by  $\theta = (\theta_1, \dots, \theta_N) \in \Theta_1 \times \dots \times \Theta_N \equiv \Theta$ . For a fixed player  $i$ , we let  $\theta_{-i}$  denote the type vector corresponding to all the other players. Let  $\mathcal{A}$  be the set of feasible outcomes. A choice rule is a correspondence  $f : \Theta \rightrightarrows \mathcal{A}$  that specifies a nonempty choice set  $f(\theta) \subseteq \mathcal{A}$  for every type vector  $\theta$ . We consider social choice functions that correspond to a particular allocation decisions  $x$ . When the type vector is  $\theta$ , let the  $N$  agents choose the vector of strategies  $(s_1(\theta), \dots, s_N(\theta))$ .

We restrict our attention to *direct revelation mechanisms* where the space of agents' strategies is restricted to the type space for every player, that is  $\hat{\theta}_i \equiv s_i(\theta) \in \Theta_i$ . Then  $\hat{\theta}_i$  is called the *announcement* of agent  $i$  and the principal implements the allocation  $x$  as a function of the announced types. Let  $u_i(x, \theta_i) = u_i(x(\hat{\theta}), \theta_i)$  denote player  $i$ 's utility function as a function of the allocation decision and the type vector.



**Definition 1** (DOMINANT-STRATEGY EQUILIBRIUM) *A strategy profile  $(\hat{\theta}_1, \dots, \hat{\theta}_N)$  is a dominant strategy equilibrium if, for all  $\theta_i \in \Theta_i$ ,*

$$u_i(x(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) \geq u_i(x(\tilde{\theta}_i, \hat{\theta}_{-i}), \theta_i) \quad (1.1)$$

*for all  $\tilde{\theta}_i \in \Theta_i$  and  $\hat{\theta}_{-i} \in \Theta_{-i}$*

A *truthful* direct revelation mechanism is one where the optimal announcement for every agent is his true type, that is  $\hat{\theta}_i = \theta_i$  for all  $i$ . The *Revelation Principle*<sup>2</sup> specifies that we can, without loss of generality, restrict attention to direct revelation mechanisms when looking for mechanisms to implement choice rules. Although the revelation principle allows for implementation under various equilibrium concepts, we present here a version for dominant strategy implementation.

**Theorem 1** (REVELATION PRINCIPLE) *If  $f$  is fully implementable in a dominant strategy equilibrium, then it must also be truthfully implementable in a dominant strategy equilibrium.*

To see why, replace any non-revelation mechanism  $g$  which fully implements  $f$  in dominant strategies by a revelation mechanism  $g^*$  which mimics it. Let the dominant strategies  $(\hat{\theta}_1, \dots, \hat{\theta}_N)$  correspond to the mechanism  $g$ , then announcing the truth  $g^*$  leads to the same outcome:  $g^*[\theta_1, \dots, \theta_N] \equiv g[\hat{\theta}_1, \dots, \hat{\theta}_N]$ . Clearly, for each agent  $i$ , announcing the truth  $\theta_i$  in  $g^*$  must be a dominant strategy, because  $\hat{\theta}_i$  is a dominant strategy in  $g$  – hence  $g^*$  truthfully implements  $f$  in dominant strategies.

### Quasi-linear Preferences

Many of the early results in implementation theory were negative results, proving the difficulty of implementing truthful mechanisms in dominant strategies when the

<sup>2</sup>Gibbard(1973), Green and Laffont (1977), Dasgupta, Hammond and Maskin (1978) and Myerson (1979)

domain of possible preferences that the agents are allowed to have is large. This included work by Gibbard (1973), Satterthwaite (1975), Zhou (1991) and focused the attention on restricting the domain of preferences. One such domain restriction is to quasi-linear preferences: each agent has a utility function  $u_i(x, \theta_i) = v_i(x, \theta_i) + \tau_i$ , where  $x$  is the allocation decision and  $\tau_i$  is a *transfer* payment which may be positive or negative. The set  $\mathcal{A}$  comprises the allocation decision  $x$  together with the vector of transfers  $(\tau_1, \dots, \tau_N)$ . Denote the set of feasible decisions by  $\mathcal{X}$ . Roberts (1979) proves the following powerful result for generically single valued choice rules

**Theorem 2** (Roberts, 1979) *Consider the domain of all quasi-linear preferences. Let the set  $\mathcal{X}$  of decisions be finite, but contain at least three outcomes. Take some generically single-valued choice rule  $f = (x, \tau_1, \dots, \tau_N)$  for which the range of  $x$  is the whole of  $\mathcal{X}$ . If (i)  $f$  is truthfully implementable in dominant strategies, then there exists some non-negative vector  $(k_1, \dots, k_N) \neq 0$  and some real-valued function  $u_0$  defined on  $\mathcal{X}$  such that*

$$x(\theta) = \arg \max_{x \in \mathcal{A}} \left\{ u_0(x) + \sum_{i=1}^N k_i u_i(x, \theta_i) \right\} \quad (1.2)$$

and (ii) if  $k_i > 0$ ,  $\tau_i(\theta) = \frac{1}{k_i} \left[ u_0(x(\theta)) + \sum_{i=1}^N k_i u_i(x(\theta), \theta_i) \right] + H_i(\theta)$  where  $H_i$  is some real-valued function independent of  $u_i(\cdot, \theta)$

In the special case where the  $k_i$ 's are equal and  $u_0 \equiv 0$ , the decision  $x(\theta)$  will be the Pareto efficient one,  $x^*(\theta)$  for each type realization  $\theta$ . Hence the transfers reduce to the transfers proposed by Vickrey (1961), Clarke (1971) and Groves (1973). Clarke (1971) exhibited a particularly interesting member of this class in which

$$H_i(\theta) = - \max_{x \in \mathcal{X}} \sum_{j \neq i} u_j(x, \theta) \quad (1.3)$$

the so-called “pivotal” mechanism because only agents who actually influence which decision in  $\mathcal{X}$  are taken get transfers. The last part of Roberts’ theorem in this special case implies that when the domain of quasi-linear preferences is sufficiently rich, only the Groves mechanisms can (truthfully) implement the Pareto efficient decision rule  $x^*$  in a dominant strategy equilibrium.

Another important result demonstrated by Green and Laffont (1979) for the Groves mechanisms applies to the choice rules specified by Roberts: if  $k_i > 0$  for all  $i$ , then the sum of transfers cannot always be zero. That is, mechanisms cannot be budget balanced. It is important to stress that balancedness is a prerequisite for Pareto efficiency of the choice rule  $f$ . This means that as in the case without quasi-linearity, Pareto efficient strategy-proof choice rules do not exist for the universal quasi-linear domain of Roberts’ theorem. (In chapter 2, the principal serves as a sink for all the transfers ensuring budget balance across all the players.)

This leads to a focus on choice rules that maximize the sum of the agents’ expected utilities prior to the realization of the types. Laffont and Maskin (1982) show that in the case when the decision space is binary, that is,  $x \in \{0, 1\}$  no mechanism can do better than the Groves mechanism in maximizing agents’ aggregate expected surplus. These results however depend on the separability of preferences and are not very robust and do not extend to the case of non quasi-linear preferences. However, the domain of quasi-linear payoffs is rich enough for it to be interesting in its own right.

**REMARK:** Nash Implementation: implementation via Nash equilibrium instead of dominant strategies does not expand the space of mechanisms implementable truthfully as it can be shown that a choice rule can be truthfully implemented in Nash equilibrium only if it can be truthfully implemented in dominant strategies (Dasgupta, Hammond and Maskin, 1979). However, in general, more can be implemented in Nash equilibria than in dominant-strategy equilibria. Maskin (first published 1977, revision 1998) showed that any Nash implementable choice rule must be monotonic

and conversely that for three or more agents, any monotonic choice rule satisfying no veto power is Nash implementable. Here, a social choice rule  $f$  is *monotonic* if, whenever  $a \in f(\theta)$  for some outcome  $a$  for some type  $\theta$  but  $a \notin f(\hat{\theta})$  for some other type  $\hat{\theta}$  then there must exist some agent  $i$  and some outcome  $y$  such that when the type vector realized is  $\theta$ ,  $i$  prefers  $a$  to  $y$ , but when the type vector realized is  $\hat{\theta}$ ,  $i$  strictly prefers  $y$  to  $a$ .  $f$  satisfies *no veto power* if, whenever some outcome  $a$  is top-ranked in  $A$  for at least  $N - 1$  players in some type realization  $\theta$ , then  $a \in f(\theta)$ . The monotonicity condition requirement is quite a serious restriction: it may not be possible to Nash implement choice rules that are concerned with distribution as well as efficiency.

### 1.2.2 Bayesian Nash Implementation

In chapter 2 we construct *bounding mechanisms* that are direct revelation mechanisms implementable in dominant strategies. However, they are not truthful. We then use these bounding mechanisms to construct a direct revelation mechanism that is truthfully implementable in a Bayesian Nash equilibrium.

We continue on the approach based on the Revelation Principle to study truthful implementation of allocation rules referred to as *weak implementation* (Dasgupta, Hammond and Maskin, 1979; Palfrey and Srivastava, 1993). An allocation rule is said to be truthfully implementable in a weak implementation sense if there is some equilibrium of some mechanism that leads to the desired allocation rule. *Full implementation*, on the other hand, must have the property that *all* equilibria lead to the desired allocation rule. Whereas in the previous section we used the word type to speak about preferences of the agents, here we let the type  $\theta$  represent not only the preferences, but also the information each player has about the preferences of the

other players and about their information. Let

$$U_i^x(\tilde{\theta}_i, \theta_i) = E_\theta \left[ u_i(x(\tilde{\theta}_i, \hat{\theta}_{-i}), \theta_i) | \theta_i \right] \quad (1.4)$$

denote agent  $i$ 's *interim* utility of an allocation decision  $x \in X$  resulting from announcement  $\tilde{\theta}_i$  where  $E_\theta$  denotes the expectation over the type vector  $\theta$ .

**Definition 2 (BAYESIAN NASH EQUILIBRIUM)** *Announcement strategies  $\hat{\theta}$  constitute a Bayesian Nash equilibrium if, for all  $i$  and for all  $\theta_i, \hat{\theta}_i$  is such that*

$$U_i^x(\hat{\theta}_i, \theta_i) \geq U_i^x(\tilde{\theta}_i, \theta_i) \quad (1.5)$$

for all  $\tilde{\theta}_i \in \Theta_i$

Postlewaite and Schmeidler (1986) showed that an incomplete information version of Maskin's monotonicity condition was necessary for implementation via Bayesian equilibrium. Palfrey and Srivastava (1989) prove that an extended version of the condition called *Bayesian Monotonicity*, is necessary and along with a self-selection condition, is sufficient for implementation.

**Bayesian Monotonicity:** In a direct mechanism, let  $\alpha$  denote a deception such that  $\alpha$  involves some of the players falsely reporting their types. This defines a new social choice function (or allocation rule), which we call  $\phi_\alpha$ . Sometimes, it is possible to add messages, augmenting the direct mechanism into an "indirect" mechanism that implements  $x$ . Given that the incentive compatibility condition holds, the implementation problem boils down to determining for which social choice functions, it is possible to augment the direct mechanism to eliminate unwanted Bayesian equilibria. This is the so called method of *selective elimination* that is used in most of the constructive proofs in implementation theory. There is a simple necessary condition called Bayesian monotonicity for this to be possible. A social choice correspondence  $f$

is Bayesian monotonic if, for every  $\phi \in f$  and for every joint deception  $\alpha : \Theta \rightarrow \Theta$  such that  $\phi_\alpha \notin f$ , there exists an agent  $i$ , a type  $\theta_i \in \Theta_i$  and an allocation rule  $y : \Theta \rightarrow \mathcal{A}$  such that  $U_i^{\phi_\alpha}(\hat{\theta}_i, \theta_i) < U_i^{y_\alpha}(\hat{\theta}_i, \theta_i)$  and for all  $\vartheta_i \in \Theta_i$ ,  $U_i^\phi(\hat{\vartheta}_i, \vartheta_i) \geq U_i^y(\hat{\vartheta}_i, \vartheta_i)$ .

Palfrey and Srivastava, (1989) show that under certain assumptions on the diffused nature of beliefs, a social choice function  $\phi$  (or allocation rule  $x$  corresponding to  $\phi$ ) is Bayesian Nash implementable if and only if it satisfies incentive compatibility and Bayesian monotonicity. In the context of efficiently allocating a resource repeatedly over time it is not easy to determine a priori whether or not the Bayesian monotonicity condition is satisfied. We take an alternate constructive approach using the multi-armed bandit problem and develop a family of mechanisms that yield incentive compatibility. In the next section, we motivate the importance of modeling repeated interaction amongst agents and summarize some of the important results from the theory of *dynamic mechanism design*.

## 1.3 Dynamic Mechanism Design

The underlying resource allocation problem in mechanism design is often recurrent and informational asymmetries are such that each agent's private information is updated as a consequence of both his interaction with other agents and his access to private information sources.

Many practical resource allocation problems display this recurrence. Consider, for instance, a central research facility and several research groups that wish to use the facility. These allocations are made repeatedly for relatively short periods of time, and the effectiveness of the research groups in using these facilities is privately known only by them. Contracts to operate public facilities such as airports and to use natural resources such as forests, are renewed periodically, and have the feature that the agencies that operate the facilities or use the resources learn most about the value

of the resource. Such problems also arise within a firm: the repeated allocation of an expert employee to one of several internal projects, where the employee's productivity with respect to the different tasks at hand is privately learned by the respective project managers, is another instance of the dynamic resource allocation problem discussed in this chapter. A recent phenomenon in the world of auctions is the auction of advertising space associated with search results displayed by online search companies such as Google and Yahoo.<sup>3</sup> In this auction, advertisers repeatedly bid for their displayed ranks and privately observe the internet traffic and transactions that result from the advertisement. A common feature of all of these problems is that the authority allocating the access rights to the critical resource in question does not actually observe the rewards obtained from its use. This information needs to be elicited from the agents competing for the resource's allocation in each round.

The design of multi-agent resource allocation mechanism should therefore take into account the following two key features: (i) repeated interaction; (ii) evolving private information.

Our objective is to design a mechanism that guarantees the long-run (infinite-horizon) efficient repeated allocation of an object when agents privately observe sample realizations of their valuations whenever they possess the object. In a static one-period model, a simple Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973), such as a second-price auction, leads to efficiency. In that setting it is assumed that each agent knows his own valuation precisely and only the static incentives of each agent need to be accounted for to implement efficient mechanisms. Compared to the static situation, the dynamic incentive-compatibility constraints are in some sense more stringent: while in a static mechanism it suffices to prevent deviations in which an agent pretends to be of another type, in a dynamic

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<sup>3</sup>Much of Google's \$6 billion in revenues comes today from selling advertising space and almost half of that arises out of "keyword auctions".

### 1.3 Dynamic Mechanism Design

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mechanism an agent can make his reporting strategy contingent on the information he has gleaned about his own and other's types from his past interactions.

The mechanism we construct uses classic findings in the theory of sequential single-person decision making as building blocks, notably the so-called *multi-armed bandit* problem. Thompson (1933, 1935) posed the first bandit problem. He considered two Bernoulli processes and took a Bayesian point of view. The objective was to maximize the expected number of successes in the first  $n$  trials. After Thompson, the bandit problem received little attention until it was studied by Robbins (1952, 1956), who also considered two arms and suggested a myopic selection strategy. These approaches essentially compared different strategies and looked for uniformly dominating strategies. Bradt et al. (1956) took a Bayesian approach which is the approach taken by much of the recent bandit literature.

The following table summarizes some of the important contributions to the literature of efficient allocation of resources.

	Private Information	Evolving Beliefs	Repeated Interaction	Infinite Horizon	Strategic Agents
Vickrey (1961)	✓	×	×	×	✓
Gittins, Jones (1974)	×	✓	✓	✓	×
Athey, Miller (2005)	✓	×	✓	✓	✓
Bergemann, Välimäki (1996)	×	✓	✓	✓	✓
Bapna, Weber (2006)	✓	✓	✓	✓	✓

Much of the extant literature focusses on the construction of mechanisms for situations in which agents interact only once or in which the agents' private information is publicly released after each round (Bergemann and Välimäki, 1996) or is perfectly recurrent in the sense that past private type realizations have no bearing upon future private type realizations (Athey and Miller, 2005), so that the issues of learning and experimentation do not arise. Most of the available dynamic mechanisms are either



for a single agent, or for a continuum of agents where only the aggregate of the other agents matters to individual agents (Courty and Li, 2000; Levin, 2003). In addition to the literature on mechanism design, there are also a few results on self-enforcing equilibria in dynamic multi-agent games, such as optimal collusion with private information in a repeated game (Athey and Bagwell, 2001; Athey et al., 2004); relational contracts (Rayo, 2002; Levin, 2003) and the generic payoff attainability in a folk-theorem sense (Miller, 2004). For other recent results on certain aspects of mechanism design, see Bergemann and Välimäki (2005) on the role of information, and Martimort (2005) on the design of multilateral static contracts.

Myerson and Satterthwaite (1983) demonstrate that there does not exist any ex-ante budget-balanced, interim incentive-compatible, and interim individually rational mechanism (where interim refers to the stage when agents observe their own types but not those of others) that achieves efficiency in a static bilateral trade setting with asymmetric information. McAfee and Reny (1992) further show that this fundamental impossibility result does not hold true when the agents' types are not independent, as deviations from full disclosure may be detected using correlation between types. Chung and Ely (2002) find that imposing ex-post incentive-compatibility and individual-rationality constraints restores the Myerson-Satterthwaite impossibility result. The proof of the Myerson-Satterthwaite theorem hints at the fact that efficiency may be restored if the individual-rationality constraints can be relaxed, i.e., if agents are willing to make payments in excess of the expected gains from trade. The fundamental impossibility of efficient mechanisms can also be circumvented by introducing an independent party, the *principal*, who balances the budget by claiming all payments from agents and allocating the object accordingly. In our setting a principal naturally exists in the form of a central authority that allocates the object in question and evaluates the messages sent by the agents, on the basis of which allocations are made.

Indeed, recent research has brought the tools of dynamic programming and mechanism design closer to the analysis of ongoing relationships with time-varying hidden information. Abreu et al. (1986) capture the incentives provided by future play in a continuation value function that maps today's outcomes into future payoffs. Fudenberg et al. (1994) prove a folk theorem for repeated games with hidden action or information. Although they focus on the case where agents cannot make monetary transfers, their framework can be extended to accommodate monetary transfers. The problem we address is closely related to the one that was considered by Athey and Miller's (2005) exploration of institutional frameworks which support efficiency in an infinite-horizon repeated game with hidden valuations. The key difference between our setting and theirs is that in our setting the agents' private information evolves as a result of learning whereas in Athey and Miller's setting private information is independent from period to period and is independent of reward observations. The notion of long-run efficiency, which is achieved by the desired multi-armed bandit solution outcome, can be interpreted as the implementation of an optimal 'exploration-versus-exploitation' strategy.

## 1.4 Motivation: Divisible Good Auctions

Many resources that are allocated in practice, are divisible goods. Mechanisms for the allocation of such resources typically constitute bidders submitting demand functions stating their willingness to pay for different fractions of the object being allocated. Ranging from flower auctions in Aalsmeer (Netherlands), to treasury auctions (Back and Zender, 1993) in the US and more recently, auctions of shares at the time of the initial public offering of companies, there are plenty of examples of *divisible good auctions* or *multi-unit auctions* currently used. Even in procurement auctions, goods are often divisible and there, instead of demand functions, agents submit supply

functions. In electricity markets for instance, generators bid on their willingness to accept for different amounts of electricity generated to a centralized agency which determines a market clearing price after receiving all the bids and allocates shares of the demand accordingly. As long as goods are homogeneous, one can think of running a single share auction to split the good (or goods) amongst the bidders. Like flowers in Holland or shares of Google, auctions of multiple homogenous discrete goods sold at the same time can be modeled as an auction of a single divisible object.

A common auction mechanism used for divisible goods is a uniform-price auction where bidders simultaneously submit demand functions and the auctioneer determines a clearing price and all bids exceeding the clearing price are deemed winning bids at the clearing price. Unfortunately, in the uniform price auction every equilibrium yields inefficient outcomes with positive probability (Ausubel and Cramton, 2002). The reason for inefficiency is that uniform pricing creates strong incentives for “demand reduction”, that is, a bidder will bid less than his value for a marginal unit, in order to depress the price that he pays for inframarginal units. Large bidders reduce demand for additional items and so sometimes lose to smaller bidders with lower values.

In a pure private-values setting, Vickrey’s (1961) celebrated multi-unit auction is an effective static design to achieve efficiency. Again, after bidders submit sealed bids comprising demand curves, the auctioneer determines the clearing price and all bids exceeding the clearing price are deemed winning bids. The price paid for each unit won, however, is neither the amount of the bid nor the clearing price, but the opportunity cost of assigning this unit to the winning bidder. If  $v_i(x_i)$  denotes agent  $i$ ’s value function, implying an inverse demand function  $x_i(p)$  as a function of the price  $p$ .  $1 - x_{-i}$  denotes the residual supply after subtracting the demands of all the other players,  $p^*$  denotes the market-clearing price if all bidders participate in the auction, and  $p_{-i}^*$  denotes the market-clearing price in the absence of bidder  $i$ . The Vickrey auction awards a quantity of  $x_i(p^*)$  to bidder  $i$  and requires a payment equal

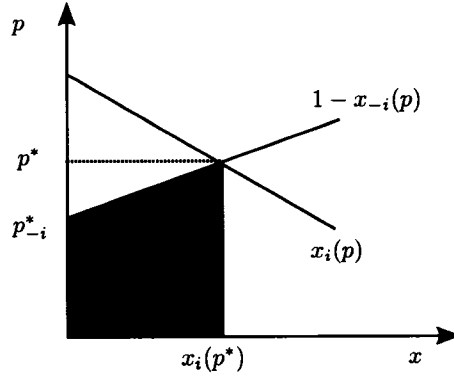


Figure 1.4: Vickrey payments for divisible resources.

to the area shaded in Figure 1.4

One of the problems of the VCG mechanism arises in the case of multi-object auctions, which can also be seen as a discrete version of divisible goods. The problem is one of the VCG payoffs not necessarily being in the core of the game and arises when players might value goods as complements. What this means, is that there might be suitable deviations for coalitions of players to deviate after the bids have been announced. These coalitions must include the seller, as without the sellers, the players get zero for sure. We are particularly interested in the singleton coalition comprised of just the seller. The problem can be illustrated through the following example.

**Example 2** Consider a Vickrey auction of one divisible resource to three bidders 1, 2 and 3. Bidder 1 is willing to pay \$1 for the whole resource and bidder 2 and 3 each seek just a half of the resource and are willing to pay up to \$1 for it. If a Vickrey auction is used and all the players play their dominant strategies, then bidders 2 and 3 are allocated a half each. The total price paid by the winning bidders, however, is zero. To see why, let us compute the price paid by bidder 2. The externality imposed

on the other players is the maximum value of the entire resource which is 1 minus the value of half of the resource, which is also 1. Similarly, player 3 also pays zero. The seller can perform strictly better by allocating the entire resource to player 2 (or allocating half to player 2 and disposing the other half) and the resulting Vickrey payment would be 1 which is individually rational for player 2 to pay, and will be paid. Thus if a seller commits to imposing VCG payments, but if the allocations cannot be monitored ex-post, a revenue maximizing allocation subject to using VCG payments need not be efficient.

In other work related to efficient auctions of multiple units of goods, Perry and Reny (2002) construct efficient mechanisms when types are interdependent. Wilson (1979) presents essentially the same idea as the Vickrey auction but in the form of rebates and shows that the revenues generated by the Vickrey mechanism are not necessarily higher than the uniform price auction. Ausubel (2004) constructs an ascending price auction which achieves the same outcome as a sealed bid Vickrey auction and Ausubel and Cramton (2004) show that for multiple units when perfect resale is allowed a Vickrey auction with reserve prices maximizes revenues. Grossman (1981) and Hart (1985) study supply function equilibria in the absence of uncertainty of demand and Klemperer and Meyer (1989) prove existence and uniqueness of equilibria in the presence of uncertainty. The supply function equilibrium aims to characterize the optimal bid function when the principal, after collecting the bid functions determines a uniform price to clear the market: matching the total supply at the price to the total demand.

## 1.5 Debt Collection Auctions

Whereas Chapters 2 and 3 are theoretical contributions, Chapter 4 is an application of auction theory to a practical setting.

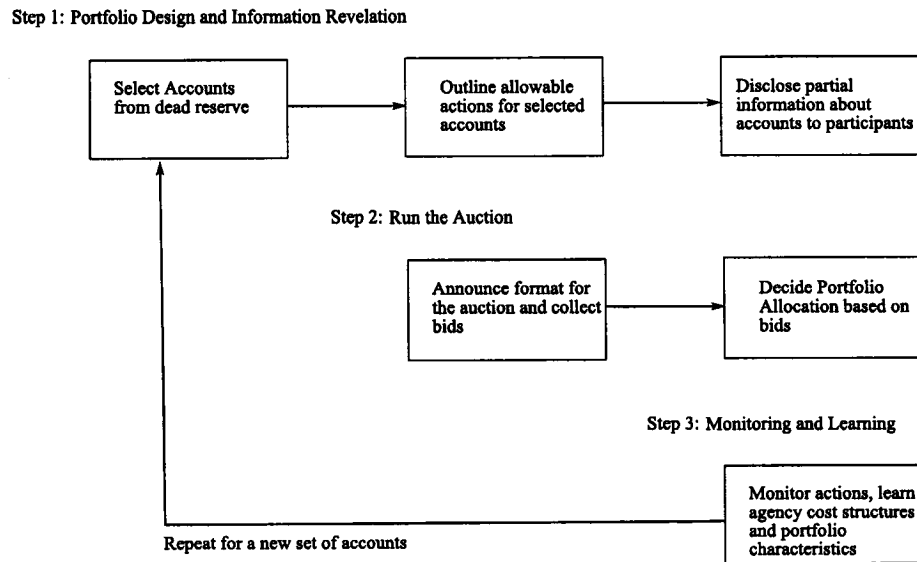


Figure 1.5: Main steps in a credit card company's decision process when designing a debt lease auction

Debt accounts that default are often transferred from the original creditor to collection agencies that specialize in retrieving debt. Credit-card companies have recently started conducting auctions to determine the allocation of accounts across different agencies. Some of the major decisions involved in designing these auctions are given in Figure 1.5. The primary objective of the credit-card company is to maximize its revenues, and in Chapter 4, we study the important characteristics of this auction that help identify ways in which revenue can be maximized and recent models that capture some of these characteristics. In the next section we present two key results from the classical theory of revenue maximization.

## 1.6 Revenue Maximization

So far, the primary objective of the mechanisms we have developed has been efficiency. In this section we touch upon two of the important results in addressing the question of maximizing principal revenues. Although the work we present does not directly

In practice, many designers have falsely high expectations about how changes in the rules can affect prices and payoffs. Many believe that auction procedures can alter auctioneer and bidder payoffs without changing allocations. We start by talking about the payoff equivalence theorem which shows that many different kinds of auctions lead to the same expected payoffs for the bidders and the revenue equivalence Theorem which shows a similar equivalence for seller's revenues. In the subsequent subsection, we assume quasi-linear payoffs, that is,  $u_i(x, \tau_i; \theta_i) = v_i(x, \theta_i) - \tau_i$ .

**Payoff Equivalence:** Assume that the valuations  $v_i(\cdot)$  are independently drawn from distributions  $F_i(\cdot)$  and are private, that is,  $v_i(x, \theta) = v_i(x, \theta_i)$  and suppose that  $s$  is a Bayes-Nash equilibrium with allocation and transfers  $(x, \tau)$ . Let  $V_i(\theta)$  denote the maximum expected payoff of player  $i$  of type  $\theta_i$  in the game. That is,

$$V_i(\theta_i) = \max_{\hat{s}_i} E [x(\hat{s}_i, s_{-i}(\theta_i)) \cdot v_i(\theta_i) - \tau_i(\hat{s}_i, s_{-i}(\theta_i))] \quad (1.6)$$

Then Myerson (1981) proves that the expected payoffs satisfy

$$V_i(\theta_i = t) = V_i(0) + \int_0^t E [x(\theta|\theta_i = s)] \cdot \frac{\partial v_i}{\partial s} ds \quad (1.7)$$

and expected payments must satisfy

$$E [\tau_i(\theta)|\theta_i = t] = -V_i(0) - \int_0^t E [x(\theta|\theta_i = s)] \cdot \frac{\partial v_i}{\partial s} ds + E [x(\theta)|\theta_i = t]v_i(t)$$

The (risk neutral) payoff equivalence theorem applies to bidder payoffs, but it

also has immediate implications for the seller's revenues. The original theorem of this sort is Myerson's revenue equivalence theorem and shows that as long as agents are risk neutral, have independent, identically drawn valuations, have common knowledge about the other bidders and the agent with the lowest type earns zero surplus, then any two auctions that result in identical winners, result in identical revenues for the principal.

**Revenue Maximizing Auctions:** Myerson (1981) posed the question of designing a sales mechanism that would maximize expected revenues of the seller, considering *any* possible mechanism in which the buyers would voluntarily participate (It is easy to raise lots of money, even if there's nothing to sell, if one can impose involuntary taxation). Myerson maintained the assumptions of risk neutrality and independent bidder types. A mechanism is voluntary if for every player  $i$  and type  $\theta_i$ , the maximal expected utility satisfies  $V_i(\theta_i) \geq 0$ . The expected revenue of the principal in following a mechanism  $\mu$  when agents follow a Bayes-Nash strategy  $s$ , is given by  $W_\mu(s) = E[\sum_{i=1}^N \tau_i(s_1(\theta_1), \dots, s_N(\theta_N))]$ . The following optimal auction theorem is a version of Myerson's (1981) theorem presented by Milgrom (2004)

**Theorem 3** *Consider a standard independent private values model with a single good for sale. For each  $i$ , define  $m_i(\theta_i) \equiv v_i(\theta_i) - (1 - \theta_i) \frac{dv_i}{d\theta_i}$ , and suppose that  $m_i$  is an increasing function. Further suppose that  $v_1(0) = \dots = v_N(0)$ . Then mechanism  $\mu$  is revenue maximizing if it satisfies  $V_i(0) = 0$  and has the following decision function:*

$$x_i(\theta) = \begin{cases} 1 & \text{if } m_i(\theta_i) > \max(0, \max_{j \neq i} m_j(\theta_j)) \\ 0 & \text{otherwise} \end{cases} \quad (1.8)$$

*Furthermore, at least one such mechanism exists.*





## Chapter 2

# Efficient Dynamic Allocation with Uncertain Valuations

*One good turn deserves another.*

– GAIUS PETRONIUS, SATYRICON (CIRCA 66 AD)

### 2.1 Introduction

In this chapter we construct an efficient Bayesian mechanism for a principal who repeatedly allocates an object to one of  $n$  agents who privately learn the object's value. An interesting feature of our analysis is that the efficient mechanism is obtained as an affine combination of two very simple “bounding mechanisms.” Each of these bounding mechanisms provides ‘one-sided incentives’ resulting *either* in all agents’ (weakly) over-reporting their types *or* their under-reporting their types, so that both are only imperfectly suited for the task. The particular affine combination that is realized may vary from period to period and from agent to agent, as it depends for any given agent on that agent’s message to the principal. The message in turn depends

on the agent's accumulated information about the object and about the other agents. Our method of combining bounding mechanisms to implement efficient allocations may also be used to approximately achieve other distributional goals conditional on maximizing welfare, such as the implementation of any particular type-dependent affine combination of the bounding mechanisms or (not unrelated) optimizing the principal's revenue, leading to a variety of attainable dynamic payoff profiles.

In our problem of designing a mechanism for efficient dynamic allocation of an object, we admit the possibility that the agents' private information evolves over time through learning and is not necessarily disclosed after each round. Indeed, in many real-world settings the value of obtaining access to a particular resource (or of being simply involved in associated trades or contracts) is not exactly known to agents and the valuation of future transactions depends on past experience, potentially evolving over time. One may, at a given time, think of an agent's "experience" in this context as the realization of a signal correlated with the payoff obtained from using the resource the next time. For modeling simplicity we assume that the private information that an agent obtains consists of an actual reward observation, derived from using the object. Each observation strengthens the agent's beliefs about the payoffs he would likely obtain the next time he is able to obtain the object.

Our infinite-horizon model naturally captures (through stochastic discounting) the "going concern" in a situation where the time horizon for the repeated interactions is not known, which realistically is the case in many, if not most, interesting dynamic resource allocation problems.

We make extensive use of the symmetric-information model of the dynamic allocation problem solved in the optimization literature in the 1970's under the name of the *multi-armed bandit problem*. In this problem each arm (i.e., each winning agent) of the bandit receives information about its private rewards and the gambler (i.e., the principal) observes nothing about the private rewards. In their seminal paper Gittins

and Jones (1974) showed that the optimal policy has a particularly simple structure allowing the formulation of the optimal policy as an “index” policy, under which the threshold or “index” of any arm is computed independently of all other arms and the optimal choice for the gambler at each time consists in pulling (i.e., selecting) the arm with the highest index. This chapter is organized as follows. In Section 2.2 we outline our model, describing the way in which agents learn their valuations and interact with other agents. We also introduce our key assumptions, mainly pertaining to stochastic dominance and Bayesian regularity (requiring e.g., conjugate prior and sampling distributions), and their implications for the first-best outcome, i.e., the solution to the multi-armed bandit problem. In Section 2.3 we introduce the two bounding mechanisms which are ‘upwards’ and ‘downwards’ incentive compatible, respectively. These two mechanisms are the building blocks for our efficient mechanism that is constructed in Section 2.4 using the necessary optimality conditions for the agents’ optimal revelation problem. An affine combination of the two bounding mechanisms yields a stationary and “locally” Bayesian incentive compatible mechanism.

In Section 2.5 we verify global incentive compatibility, which obtains provided that agents are sufficiently impatient. We extend our method to settings in which the desired efficient outcome is to be achieved, at least approximately, using a prespecified affine combination of the bounding mechanisms. The associated results can be used to address concerns about the payoff profile that the principal (or the agents) may have, including approximate revenue maximization. Section 2.7 provides a number of practical applications before we summarize our findings in Section 2.8.

## 2.2 The Model

We now introduce the basic setup of our model before formulating and interpreting the key assumptions under which our results obtain. Lastly, we specify the socially

efficient first-best outcome in the form of a Gittins-index allocation policy.

### 2.2.1 Basic Setup

We consider the problem of efficiently allocating an indivisible object to one of  $n$  agents repeatedly over time. The real-valued reward  $R_t^i \in \mathcal{R} \subseteq [R, \bar{R}]$  (with  $R < \bar{R}$ ) that agent  $i \in \mathcal{N} = \{1, \dots, n\}$  receives, conditional on winning the object at time  $t \in \{0, 1, 2, \dots\}$ , is an element of a uniformly bounded sequence  $R^i = \{R_t^i\}_{t=0}^\infty$  of i.i.d. random variables (a *sampling process*). The sampling process itself is drawn from a family  $\Sigma$  of distributions each with smooth density  $\sigma(\cdot|\alpha^i) \in \Sigma$  (the *sampling technology*), where  $\alpha^i \in \mathcal{A}$  is an index,  $\mathcal{A} = [\underline{a}, \bar{a}]$  (with  $\underline{a} < \bar{a}$ ) a compact interval,<sup>1</sup> and  $\Sigma = \bigcup_{\alpha^i \in \mathcal{A}} \{\sigma(\cdot|\alpha^i)\}$  denotes the family of distributions for simplicity as the collection of sampling-process densities. Since agents do not know the value of  $\alpha = (\alpha^1, \dots, \alpha^n)$ , each agent  $i \in \mathcal{N}$  forms a belief about  $\alpha$  which is updated in each period. Conditional on his reward observations up to time  $t$ , agent  $i$ 's beliefs about his own parameter  $\alpha^i$  are given by a probability density  $f^i(\cdot|\theta_t^i)$  over the set  $\mathcal{A}$ . To keep the discussion simple, we assume that  $f^i(\cdot|\theta_t^i)$  is continuous, is conjugate with respect to the sampling technology  $\sigma$ , and can be indexed by a real-valued parameter  $\theta_t^i \in \Theta$  which summarizes the agent's private information about his observed rewards. We refer to  $\theta_t^i$  as the *type* (or *state*) of agent  $i$  at time  $t$  and to the compact set  $\Theta = [\underline{\theta}, \bar{\theta}]$  (with  $\underline{\theta} < \bar{\theta}$ ) as the agents' (common) *type space*. Initially (at time  $t = 0$ ) each agent  $i$  observes his own type  $\theta_0^i$ , and his beliefs about any other agent  $j$ 's type are distributed with the continuous density  $\pi_0^j(\cdot|i)$ . After enjoying a reward  $R_t^i$  at time  $t$ ,

<sup>1</sup>It is possible to reinterpret our findings in a discrete setting, where  $\mathcal{A}$  is an ordered subset of a (finite-dimensional) Euclidean space. Similarly, by suitably interpreting formulas in the standard Lebesgue-integration framework, the support of rewards  $\mathcal{R}$  can be discrete. We choose a smooth structure for expositional purposes, since assumptions can be stated very clearly.

agent  $i$  updates his posterior to  $\theta_t^i$  using Bayes' rule, so that<sup>2</sup>

$$f^i(\alpha^i|\theta_{t+1}^i) = \frac{\sigma(R_t^i|\alpha^i)f^i(\alpha^i|\theta_t^i)}{\int_{\mathcal{A}} \sigma(R_t^i|\hat{\alpha}^i)f^i(\hat{\alpha}^i|\theta_t^i)d\hat{\alpha}^i} \equiv f^i(\alpha^i|\varphi^i(\theta_t^i, R_t^i)). \quad (2.1)$$

This implicitly defines this agent's state transition function  $\varphi^i : \Theta \times \mathbb{R} \rightarrow \Theta$ , which maps the agent's current state  $\theta_t^i$  at time  $t$  into his next state  $\theta_{t+1}^i$  at time  $t+1$  based on his current payoff observation  $R_t^i$ , i.e.,

$$\theta_{t+1}^i = \varphi^i(\theta_t^i, R_t^i). \quad (2.2)$$

The stationary reward density, conditional on agent  $i$ 's type, is given by

$$p^i(R_t^i|\theta_t^i) = \int_{\mathcal{A}} \sigma(R_t^i|\hat{\alpha}^i)f^i(\hat{\alpha}^i|\theta_t^i)d\hat{\alpha}^i \quad (2.3)$$

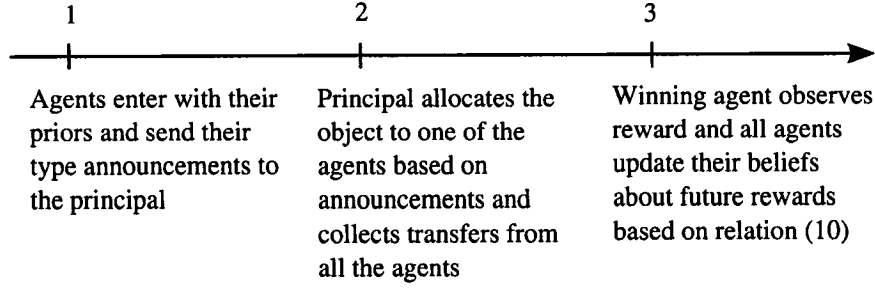
for all  $i \in \mathcal{N}$  and all  $t \in \mathbb{N}$ .

At any given time  $t$ , the specific sequence of events is as follows (cf. Figure 2.1). First, any agent  $i \in \mathcal{N}$  enters with his prior beliefs about the rewards in the event that he wins the object at time  $t$ . His prior beliefs are summarized by his type  $\theta_t^i$ . Based on his beliefs the agent makes an announcement  $\hat{\theta}_t^i = \vartheta_t^i(\theta_t^i)$ , where  $\vartheta_t^i$  denotes his announcement-strategy at time  $t$ . Second, the principal collects the vector  $\hat{\theta}_t = (\hat{\theta}_t^1, \dots, \hat{\theta}_t^n)$  of all agents' announcements and allocates the object to agent  $k = \delta_t(\hat{\theta}_t) \in \mathcal{N}$ . Any agent  $i \in \mathcal{N}$  then pays the transfer  $\tau_t^i(\hat{\theta}_t)$  to the principal. Third, agent  $k$  receives the private reward  $R_t^k$ . All agents subsequently update their beliefs using relation (2.10) below (a simple generalization of (2.2) for all participating agents). This three-step sequence is repeated at each time  $t \in \mathbb{N}$ .<sup>3</sup>

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<sup>2</sup>The prior density  $f^i(\cdot|\theta_t^i)$  and the posterior density  $f^i(\cdot|\theta_{t+1}^i)$  are of the same family of probability distributions, as a consequence of  $\sigma$  being conjugate to  $f^i$  by assumption. For examples of conjugate families of one-parameter distributions, see e.g., Raiffa and Schlaifer (1961).

<sup>3</sup>An assumption that is implicit in our model is that the principal is able to commit at time  $t = 0$  to the mechanism for all future times  $t > 0$ .

Figure 2.1: Timing within any single period  $t \in \mathbb{N}$ .

### 2.2.2 Key Assumptions

To obtain natural results, it is important that any agent  $i$ 's past reward observations, for which a sufficient statistic at time  $t$  is his type  $\theta_t^i$ , induce an ordering of his actions on  $\Theta$ . In other words, given a higher reward observation in the current period, the agent should *ceteris paribus* not expect his reward to decrease in the next period.

**Assumption A1 (Reward Log-Supermodularity).** *The conditional reward density  $p^i(R|\theta)$  is log-supermodular on  $\mathcal{R} \times \Theta$  for all  $i \in \mathcal{N}$ .*

Log-supermodularity of agent  $i$ 's conditional reward density  $p^i$  induces the monotone-likelihood ratio order, which – stronger than first-order stochastic dominance – guarantees that the conditional expectation  $E[R_t^i|\theta_t^i]$  is increasing in his type  $\theta_t^i$ .<sup>4</sup>

We also require that state transitions increase both in the size of the current reward observation and in the current state. Thus, the observation of a larger reward guarantees the transition (via relation (2.2)) to a posterior state that is not smaller than the posterior state reached otherwise. Furthermore, given two prior states  $\theta_t^i < \tilde{\theta}_t^i$  for an agent  $i$ , his corresponding states  $\theta_{t+1}^i$  and  $\tilde{\theta}_{t+1}^i$  in the next period for the same reward observation satisfy  $\theta_{t+1}^i \leq \tilde{\theta}_{t+1}^i$ .

<sup>4</sup>It actually guarantees that  $E[u^i(R_t^i)|\theta_t^i]$  is increasing in  $\theta_t^i$ , where  $u^i(\cdot)$  may be any nondecreasing utility function. In our setup we can think of the rewards directly in terms of utilities, without any loss in generality.

**Assumption A2 (State-Transition Monotonicity).** *The state-transition function  $\varphi^i(\theta, R)$  is nondecreasing on  $\mathcal{R} \times \Theta$  and all  $i \in \mathcal{N}$ .*

Using implicit differentiation, state-transition monotonicity is equivalent to

$$\left. \frac{\partial(\sigma/p^i)}{\partial(R, \theta)} \right|_{(R, \theta)} f_2^i(R|\varphi^i(\theta, R)) = \left( \frac{\partial}{\partial R} \left( \frac{\sigma(R|\theta)}{p^i(R|\theta)} \right), \frac{\partial}{\partial \theta} \left( \frac{\sigma(R|\theta)}{p^i(R|\theta)} \right) \right) f_2^i(R|\varphi^i(\theta, R)) \geq 0, \quad (2.4)$$

with

$$f_2^i(R|\theta) = \frac{\partial f^i(R|\theta)}{\partial \theta},$$

for all  $(R, \theta) \in \mathcal{R} \times \Theta$  and all  $i \in \mathcal{N}$ , provided that  $\mathcal{R}$  is a full support of the conditional reward distributions and that all parametrizations are smooth. Beyond implying monotonicity of the expected rewards and the state transitions, assumptions A1 and A2 are useful in establishing monotone comparative statics of the first-best dynamic allocation policy, detailed in Section 2.2.4.

### 2.2.3 The Mechanism Design Problem

A single decision maker (the “principal”) has at any time  $t \in \mathbb{N}$  the ability to allocate a critical resource (object) to one of  $n$  agents. To do so she uses a mechanism, which determines an allocation of the object and monetary transfers from the agents to the principal as a function of messages sent by the individual agents to the principal at the beginning of each time period. The general design of such a mechanism involves choosing an appropriate message space and an appropriate allocation rule  $\mu$ , which at each time  $t$  maps any vector of all agents’ messages into a vector of transfers  $\tau_t = (\tau_t^1, \dots, \tau_t^n) \in \mathbb{R}^n$  with payments from the different agents to the principal, and a deterministic allocation  $\delta_t \in \mathcal{N}$  which denotes the agent who obtains the object at time  $t$ . By the revelation principle (Gibbard, 1973; Myerson, 1979), it is possible without loss of generality to restrict attention to *direct revelation mechanisms*, which



(i) identify the message space with the type space and (ii) agents report their types truthfully. That is, at time  $t$  any agent  $i$ 's message  $\hat{\theta}_t^i$  lies in his type space  $\Theta$ . Furthermore, the agents' truthfulness means that

$$\hat{\theta}_t^i = \theta_t^i \quad (2.5)$$

for all  $i \in \mathcal{N}$  and all  $t \in \mathbb{N}$ . In designing such a mechanism, the principal may have several objectives. In this chapter we assume that his primary objective is to achieve an efficient allocation. Since by Lemma 1 any agent's Gittins index varies monotonically with his type, a truthful announcement allows the principal to simulate the agent's decision process and compute that agent's Gittins index. Hence, efficiency can automatically be achieved if the mechanism is Bayesian incentive compatible.<sup>5</sup>

**Welfare Objective: Efficient Allocation.** The principal wishes to maximize the discounted sum of rewards derived by each agent over all time periods. If we assume for simplicity (but without loss of generality) that all agents share a *common discount factor*  $\beta \in (0, 1)$ , the principal would wish to find an (intermediate-)Pareto-efficient allocation policy  $\delta^* = (\delta_0^*, \delta_1^*, \dots)$  which solves

$$\delta_t^*(\hat{\theta}_t) \in \arg \max_{\delta_t \in \mathcal{N}} E \left[ R_t^{\delta_t} + \sum_{s=t+1}^{\infty} \beta^{s-t} R_s^{\delta_s^*(\hat{\theta}_s)} \middle| \hat{\theta}_t \right] \quad (2.6)$$

for all  $t \in \mathbb{N}$  and all  $\hat{\theta}_t \in \Theta^n$ , where the expectation is taken over all future reward trajectories, given the allocation policy chosen by the principal and the agents' current message vector.

**Distributional Objectives.** It turns out that there may be several (or, in fact, a whole class of) mechanisms that are Bayesian incentive compatible. Of all Bayesian

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<sup>5</sup>We discuss individual rationality below, cf. relation (2.22) in Section 2.6.

incentive-compatible efficient mechanisms the principal may wish to select a revenue-maximizing mechanism, for instance. We discuss the question of how the principal may achieve this or other secondary objectives concerning the principal's distributional preferences in Section 2.6.

### 2.2.4 The Efficient Outcome

Let us now examine the first-best solution (efficient outcome) in the absence of informational asymmetries between the agents and the principal. If the principal could perfectly observe the rewards obtained by the agents, then an optimal solution to (2.6) can be obtained in the form of the celebrated “index policy.” Indeed, Gittins and Jones (1974) showed that at each time  $t$  it is possible to assign to any agent  $i$  an index  $\gamma_t^i(\theta_t^i)$  which depends only on that agent's current state  $\theta_t^i$  and satisfies

$$\gamma_t^i(\theta_t^i) = \sup\{z \in \mathbb{R} : G^i(\theta_t^i, z) > z\} = \inf\{z \in \mathbb{R} : G^i(\theta_t^i, z) = z\}, \quad (2.7)$$

where the value  $G^i(\theta_t^i, z)$  is for any “retirement reward”  $z \in \mathbb{R}$  determined implicitly as a solution of the Bellman equation<sup>6</sup>

$$G^i(\theta_t^i, z) = \max \left\{ z, E_{R_t^i} \left[ R_t^i + \beta G^i(\varphi^i(\theta_t^i, R_t^i), z) \mid \theta_t^i \right] \right\}. \quad (2.8)$$

Letting  $\pi$  denote any stopping-time policy and  $T(\pi)$  the corresponding (stochastic) stopping time, we can write agent  $i$ 's Gittins index at time  $t$  in the form of an annuity,

$$\gamma_t^i(\theta_t^i) = \sup \left\{ \hat{\gamma} : \sup_{\pi} E \left[ \sum_{s=0}^{T(\pi)} \beta^s (R_{t+s}^i - \hat{\gamma}) \mid \theta_t^i \right] \geq 0 \right\}. \quad (2.9)$$

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<sup>6</sup>The classic definition of the Gittins index derives from a lump-sum retirement reward  $z \in \mathbb{R}$  that makes a gambler indifferent between retiring now or continuing to play the bandit. In our context, for each agent the principal chooses between taking a retirement reward (effectively barring this agent from consideration) and continuing to allocate to this agent.

Using our assumptions in Section 2.2.2, the type monotonicity carries over to any agent's Gittins index, which is a measure of his future expected reward.

**Lemma 1** *Under assumptions A1 and A2 agent  $i$ 's Gittins index  $\gamma_t^i(\theta_t^i)$  is nondecreasing in  $\theta_t^i$  and stationary for all  $i \in \mathcal{N}$ .*

It is clear from relation (2.7) that agent  $i$ 's Gittins index in fact does not depend on the time instant  $t$ , since his reward distribution is also stationary. In what follows we therefore omit the time from the Gittins indices. The optimal policy based on the index function is simple: at any given time, compute the indices for all the agents and assign the object to the agent with the highest index.<sup>7</sup> In general the principal cannot observe the rewards earned by the agents, and thus never knows the state any of the agents is in. In the following sections we construct a Bayesian incentive compatible mechanism that leads to the agents' truthful revelation of their private information.

## 2.3 Bounding Mechanisms

We now construct the most important building blocks for our truthful mechanism: two mechanisms that, while both allocating the object to the agent with the highest (indirectly) announced Gittins index, use different transfers. In the first mechanism (A) the winning agent pays the Gittins index of the agent with the second-highest announced type, while in the second mechanism (B) the winner pays the expected reward of the agent with the second-highest announced type.

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<sup>7</sup>Tsitsiklis (1994) provides a simple and Weber (1992) an intuitive proof of optimality of the Gittins index policy.

**Mechanism A.** Consider the mechanism  $\mathcal{M}_A = (\delta_A, \tau_A)$  with

$$\delta_A(\hat{\theta}_t) \in \arg \max_{i \in \mathcal{N}} \gamma^i(\hat{\theta}_t^i)$$

and

$$\tau_A^i(\hat{\theta}_t) = \begin{cases} \max_{j \neq i} \gamma^j(\hat{\theta}_t^j), & \text{if } \delta_A(\hat{\theta}_t) = i, \\ 0, & \text{otherwise.} \end{cases}$$

The mechanism  $\mathcal{M}_A$  achieves “upward incentive compatibility” in the following sense.

**Proposition 1** *Under mechanism  $\mathcal{M}_A$  it is dominant-strategy incentive compatible for every agent  $i$  to announce a type whose corresponding Gittins index is at most as high as his true Gittins index, i.e.,  $\hat{\theta}_t^i$  is such that  $\gamma^i(\hat{\theta}_t^i) \leq \gamma^i(\theta_t^i)$  for all  $t$ .*

The intuition behind the proof is as follows: if at any given time an agent were to overbid and win, when truthfulness would have led him to losing, he ends up making a loss, similar to the weak dominance of truthful revelation in a static second-price auction. To prove in this context that the agent would make a loss, it is necessary to show that for every strategy involving an exaggerated type announcement in one period, there exists a better *truthful* strategy. Since the agents pay in terms of Gittins indices (which themselves correspond to the retirement rewards that would make agents indifferent between participating and dropping out), we can construct strategies that dominate over-announcing. In this manner we obtain one-sided truthfulness in weakly dominated strategies for any discount factor  $\beta \in (0, 1)$ .<sup>8</sup> Interestingly, since future transfers of an agent depend on the other agents’ beliefs about their own rewards, an agent might under mechanism  $\mathcal{M}_A$  have a strict incentive to allow other agents to win. The following example illustrates this effect: while agents under  $\mathcal{M}_A$  have no incentive to overstate their Gittins indices, some agent might strictly prefer

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<sup>8</sup> Mechanism  $\mathcal{M}_A$  can be shown to be fully truthful if the Gittins index process is a martingale. However, the Gittins index is not always a martingale. A characterization of the martingale properties of the Gittins index is, to the best of our knowledge, not available at present.

to understate his Gittins index in equilibrium. The main reason for understating one's Gittins index under  $\mathcal{M}_A$  is that it may be more beneficial for a patient agent to resolve another agent's valuation uncertainty (which might lead to lower future transfer payments) than to be forced to continuously pay high as a consequence of the other party's high announcements (driven by a lack of information paired with high expectations).

**Example 1** Consider two agents, 1 and 2. We assume that agent 1's reward sequence is deterministic and constant, i.e.,

$$f^1(\alpha^1|\theta_0^1) = \lambda_{\theta_0^1}(\alpha^1); \quad \text{and} \quad \sigma(R_t^1|\alpha^1) = \lambda_{\alpha^1}(R_t^1)$$

for all  $t \in \mathbb{N}$ , with  $\theta_0^1 \in (0, 1)$ , where  $\lambda_{\theta^k}(\xi) \equiv \lambda(\xi - \theta^k)$ . The *Dirac* distribution (density)  $\lambda(\cdot)$  is such that  $\lambda(\xi) = 0$  for all  $\xi \neq 0$  and  $\int_{-\varepsilon}^{\varepsilon} \lambda(\xi) d\xi = 1$  for any  $\varepsilon > 0$ . Agent 2's reward sequence is also constant, but the constant stream is unknown at time 0, so that

$$f^2(\alpha^2|\theta_0^2) = \theta_0^2 \lambda_1(\alpha^2) + (1 - \theta_0^2) \lambda_0(\alpha^2) \quad \text{and} \quad \sigma(R_t^2|\alpha^2) = \lambda_{\alpha^2}(R_t^2).$$

Given a common discount factor  $\beta \in (0, 1)$ , the agents' initial Gittins indices can be computed explicitly to  $\gamma^1(\theta_0^1) = \theta_0^1$  for agent 1 and

$$\gamma^2(\theta_0^2) = \sup \left\{ \hat{\gamma} : \frac{\theta_0^2(1 - \hat{\gamma})}{1 - \beta} - (1 - \theta_0^2)\hat{\gamma} \geq 0 \right\} = \frac{\theta_0^2}{1 - (1 - \theta_0^2)\beta}$$

for agent 2. Suppose further that at time  $t = 0$  the agents know each others' types, while they are not known to the principal. It is clear that no agent has any incentive to announce a type higher than his Gittins index, so that  $\vartheta_0^i(\theta_0^i) \leq \theta_0^i$  for  $i \in \{1, 2\}$  and mechanism  $\mathcal{M}_A$  is therefore upward incentive compatible. Consider now the case

when

$$\theta_0^2 < \left( \frac{1 - \beta}{1 - \theta_0^1 \beta} \right) \theta_0^1.$$

Since then agent 1's Gittins index is strictly larger than agent 2's (i.e.,  $\gamma^1(\theta_0^1) > \gamma^2(\theta_0^2)$ ), he has the option to win the object by announcing his type truthfully in every period, a strategy that yields an expected net present value of  $(\theta_0^1 - \gamma^2(\hat{\theta}_0^2))/(1 - \beta)$  depending on agent 2's announcement  $\hat{\theta}_0^2 = \vartheta_0^2(\theta_0^2)$ . On the other hand, given any nonzero announcement by the second agent, agent 1 can choose to loose the object by announcing  $\hat{\theta}_0^1 < \hat{\theta}_0^2$ , in which case with probability  $1 - \theta_0^2$  he can obtain the object by announcing  $\hat{\theta}_t^1 = \varepsilon$  for all  $t \geq 1$ , where  $\varepsilon > 0$  is an arbitrarily small constant. Agent 1's expected net present value of this alternative strategy is  $\theta_0^1(1 - \theta_0^2)\beta/(1 - \beta)$ . Since for agent 2, announcing his type truthfully is an undominated strategy, loosing the object initially is always a better strategy for agent 1, as long as

$$\beta > \beta_0 = \frac{1 - \sqrt{\theta_0^2/\theta_0^1}}{1 - \theta_0^2}.$$

Note that  $\beta_0 < 1$ , since  $\theta_0^1\theta_0^2 < 1$ . Thus, there is an equilibrium in which agent 1 would strictly prefer to pursue a nontruthful strategy.  $\square$

The problem of underbidding disappears if the principal makes the winner pay the second-highest agent's expected rewards. This is because losing when an agent can win, does not change the expected payments when he does win (by the martingale nature of expected rewards). However, since the Gittins index is never smaller than the expected per-period reward, agents might want to deviate by overbidding.

**Mechanism B.** If we define a mechanism  $\mathcal{M}_B = (\delta_B, \tau_B)$  with  $\delta_B = \delta_A$  and

$$\tau_B^i(\hat{\theta}_t) = \begin{cases} E[R_t^j | \hat{\theta}_t^j] & \text{if } i = \delta_B(\hat{\theta}_t), \text{ where } j \in \arg \max_{j \neq \delta_B(\hat{\theta}_t)} \gamma^j(\hat{\theta}_t^j), \\ 0, & \text{otherwise,} \end{cases}$$

then this mechanism achieves “downward incentive compatibility” in the following sense.

**Proposition 2** *Under mechanism  $\mathcal{M}_B$  it is dominant-strategy incentive compatible for every agent  $i$  to announce a type whose corresponding Gittins index is at least as high as his true Gittins index, i.e.,  $\hat{\theta}_t^i$  is such that  $\gamma^i(\hat{\theta}_t^i) \geq \gamma^i(\theta_t^i)$  for all  $t$ .*

The following example illustrates the fact that mechanism  $\mathcal{M}_B$  might not be upward incentive compatible.

**Example 2** Consider the same two agents as in Example 1. Conditional on an announcement profile  $(\hat{\theta}_0^1, \hat{\theta}_0^2)$  under mechanism  $\mathcal{M}_B$  agent  $i \in \{1, 2\}$ , in case he wins the object, needs to pay the other agent’s expected per-period reward  $E[R_0^j | \hat{\theta}_0^j] = \hat{\theta}_0^j$ , where  $j = 3 - i$ , as evaluated by that agent’s announcement. The interesting situation is such that agent 1’s type (identical to his Gittins index) lies between agent 2’s type and agent 2’s Gittins index, i.e.,

$$\theta_0^2 < \gamma^1(\theta_0^1) = \theta_1^0 < \frac{\theta_0^2}{1 - (1 - \theta_0^2)\beta} = \gamma^2(\theta_0^2).$$

Since by Proposition 2 no agent understates his true Gittins index, agent 2 announces at least his true type  $\theta_0^2$ . On the other hand, agent 2 has no incentive to announce a higher type than  $\theta_0^2$ , since the payment of  $\theta_0^1$  in case of winning would exceed his expected per-period payoff. Given that agent 2 bids truthfully, agent 1 has an incentive to announce a type  $\hat{\theta}_0^1 > \theta_0^1$ , for his actual payment of  $\theta_0^2$  under  $\mathcal{M}_B$  will be strictly less than his Gittins index.  $\square$

## 2.4 Local Incentive Compatibility

Given the two bounding mechanisms (satisfying upward and downward incentive compatibility respectively) it is possible, under some weak additional assumptions, to construct a mechanism which satisfies necessary conditions for Bayesian incentive compatibility (BIC) and can thus be used to obtain a *local* implementation of an efficient dynamic Bayesian allocation mechanism.

We assume that at the end of each period all agents' announcements are made public by the principal, just as they would be after a simultaneous-bid open-outcry auction. Under a (nonstationary) mechanism  $(\mu_t)_{t \in \mathbb{N}}$  each agent  $i$  can update his private information about his rewards at time  $t$  only when he wins the object in that round. Agent  $i$ 's state evolves, therefore, according to

$$\theta_{t+1}^i = \begin{cases} \varphi^i(\theta_t^i, R_t^i), & \text{if } \delta(\hat{\theta}_t^i, \hat{\theta}_t^{-i}) = i, \\ \theta_t^i, & \text{otherwise,} \end{cases} \quad (2.10)$$

where  $(\hat{\theta}_t^i, \hat{\theta}_t^{-i})$  is the vector of the agents' announcements. Let us now focus on the hypothetical situation in which all agents announce their types truthfully in each round. If agent  $j$  wins the auction at time  $t$ , then while everybody (including the principal) has perfect type information about any agent other than  $j$ , only agent  $j$  knows his state  $\theta_{t+1}^j = \varphi^j(\theta_t^j, R_t^j)$ , as he is the only one to observe his per-period reward  $R_t^j$  in that round. Hence, *any agent  $i$ 's beliefs* at time  $t + 1$  about *all other agents* can be summarized by the distribution (density) vector  $\pi_{t+1}^{-i} = (\pi_{t+1}^k)_{k \in \mathcal{N} \setminus \{i\}}$ ,

$$\pi_{t+1}^k(\xi; \theta_t^k) = \begin{cases} \lambda_{\theta_t^k}(\xi), & \text{if } k \neq j, \\ \int_{\mathcal{R}} p^j(R_t^j | \theta_t^j) dR_t^j |_{\varphi^j(\theta_t^j, R_t^j) = \xi}, & \text{otherwise,} \end{cases} \quad (2.11)$$

for all  $\xi \in \Theta$  and all  $t \geq 0$ . As pointed out earlier, before bidding the very first time,



at  $t = 0$ , each agent  $i$  may hold different beliefs  $\pi_0^k(\cdot|i)$  about any other agent  $k$ . The belief  $\pi_t^{-i}(\cdot)$  denotes the probability density that all agents other than agent  $i$  have types lower than  $\theta$ . We also note that the mechanism designer always shares the beliefs of the non-winning agents, which means that her beliefs are summarized by the full vector  $(\pi_t^1(\cdot; \theta_{t-1}^1), \dots, \pi_t^n(\cdot; \theta_{t-1}^n))$  for  $t > 0$  and  $(\pi_0^k(\cdot|l \neq k))_{k \in \mathcal{N}}$  for  $t = 0$ , where  $l \in \mathcal{N} \setminus \{k\}$  is arbitrary.

We now consider the construction of a locally incentive compatible mechanism. To ensure *global* incentive compatibility further restrictive assumptions are needed, which we discuss in the next section. Given our nonstationary mechanism  $(\mu_t)_{t \in \mathbb{N}} = (\delta_t, \tau_t)_{t \in \mathbb{N}}$  and an announcement strategy profile  $\vartheta = ((\vartheta_t^i(\cdot), \dots, \vartheta_t^n(\cdot)))_{t \in \mathbb{N}}$ , agent  $i$ 's expected utility at time  $t$  is

$$\begin{aligned} U_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}) &= E_{\theta_t^{-i}, R_t^i} \left[ \mathbf{1}_{\{\delta_t(\hat{\theta}_t^i, \vartheta_t^{-i}(\theta_t^{-i})) = i\}} \left( R_t^i - \tau_t^i(\hat{\theta}_t^i, \vartheta_t^{-i}(\theta_t^{-i})) \right) \right. \\ &\quad \left. + \beta [U_{t+1}^i(\vartheta_{t+1}^i(\theta_{t+1}^i), \theta_{t+1}^i; \pi_t^{-i})]_{|\theta_{t+1}^i = \varphi^i(\theta_t^i, R_t^i)} \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i} \right] \\ &\quad + \beta E_{\theta_t^{-i}} \left[ \mathbf{1}_{\{\delta_t(\hat{\theta}_t^i, \vartheta_t^{-i}(\theta_t^{-i})) \neq i\}} [U_{t+1}^i(\vartheta_{t+1}^i(\theta_{t+1}^i), \theta_{t+1}^i; \pi_t^{-i})]_{|\theta_{t+1}^i = \theta_t^i} \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i} \right] \end{aligned} \quad (2.12)$$

where his announcement at time  $t$  is  $\hat{\theta}_t^i = \vartheta_t^i(\theta_t^i)$ . Assuming now that the transfers of this mechanism  $\mathcal{M}(\eta) = (\delta_t(\cdot), (\tau_t^1(\cdot; \eta_t^1), \dots, \tau_t^n(\cdot; \eta_t^n)))_{t \in \mathbb{N}}$  are smoothly parameterized by  $\eta = (\eta_t^1, \dots, \eta_t^n)_{t \in \mathbb{N}}$  with  $\eta_t^i \in \mathbb{R}$  for all  $i$  and  $t$ , we can rewrite agent  $i$ 's expected payoff at  $t$  in the form

$$U_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta^i) = \bar{u}_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}) - \bar{\tau}_t^i(\hat{\theta}_t^i; \pi_t^{-i}, \eta_t^i, \eta_{t+1}^i, \dots), \quad (2.13)$$

where

$$\bar{u}_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}) = \sum_{s=t}^{\infty} \beta^{s-t} E_{\theta_s^{-i}, R_s^i} \left[ \mathbf{1}_{\{\delta_s(\vartheta_s^i(\theta_s^i), \vartheta_s^{-i}(\theta_s^{-i})) = i\}} R_s^i \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i} \right],$$

and

$$\begin{aligned} \bar{\tau}_t^i(\hat{\theta}_t^i; \pi_t^{-i}, \eta_t^i, \dots) = \\ \sum_{s=t}^{\infty} \beta^{s-t} E_{\theta_s^i, \theta_s^{-i}} \left[ \mathbf{1}_{\{\vartheta_s^i(\theta_s^i), \vartheta_s^{-i}(\theta_s^{-i})=i\}} \tau_s^i(\vartheta_s^i(\theta_s^i), \vartheta_s^{-i}(\theta_s^{-i}); \eta_s^i) \middle| \hat{\theta}_t^i; \pi_t^{-i}, \eta_t^i, \dots \right]. \end{aligned} \quad (2.14)$$

The key idea for obtaining Bayesian incentive compatibility is to let the principal make the parameter  $\eta_t^i$  at each time  $t$  dependent on agent  $i$ 's announcement  $\hat{\theta}_t^i$ . Naturally, the parameter  $\eta_t^i$  then also depends on the beliefs about the other agents' types, so that in fact  $\eta_t^i = \eta_t^i(\hat{\theta}_t^i; \pi_t^{-i})$ . If agent  $i$  finds it optimal to announce his type truthfully at time  $t$ , then the first-order necessary optimality condition,

$$\frac{\partial}{\partial \hat{\theta}_t^i} \bigg|_{\hat{\theta}_t^i = \theta_t^i} \left[ \bar{u}_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}) - \bar{\tau}_t^i(\hat{\theta}_t^i; \pi_t^{-i}, \eta_t^i, \eta_{t+1}^i, \dots) \right] = 0, \quad (2.15)$$

is satisfied, as long as his type  $\theta_t^i$  is an interior point of the type space  $\Theta$ . If, using the bounding mechanisms  $\mathcal{M}_A$  and  $\mathcal{M}_B$  identified in the last section, we set  $\delta_A = \delta_B \equiv \delta$  and

$$\tau_t^i(\hat{\theta}_t^i, \hat{\theta}_t^{-i}; \eta_t^i, \pi_t^{-i}) = \eta_t^i(\hat{\theta}_t^i; \pi_t^{-i}) \left( \tau_A^i(\hat{\theta}_t^i, \hat{\theta}_t^{-i}) - \tau_B^i(\hat{\theta}_t^i, \hat{\theta}_t^{-i}) \right) + \tau_B^i(\hat{\theta}_t^i, \hat{\theta}_t^{-i}), \quad (2.16)$$

then the system of equations (2.15) in  $\eta_t^i, \eta_{t+1}^i, \dots$  becomes *independent* of time, as long as the announcement strategy profile  $\vartheta$  is stationary with the possible exception of  $\vartheta_t^i$ . In other words, given a truthful announcement policy by all agents other than agent  $i$ , agent  $i$  will find it optimal to announce his type truthfully if deviating from this strategy is not optimal in any period.<sup>9</sup> We therefore obtain that condition (2.15) yields a stationary solution  $\eta_t^i \equiv \eta^i$ , and thus  $\bar{u}_t^i \equiv \bar{u}^i$  and  $\bar{\tau}_t^i \equiv \bar{\tau}^i$  for all times  $t \in \mathbb{N}$ ,

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<sup>9</sup>This corresponds to the well-known one-stage deviation principle for infinite-horizon games (Fudenberg and Tirole, 1991, p. 110). Its application is justified, since the game is 'continuous at infinity' as a result of the uniform boundedness of all agents' per-period payoffs.

which can be obtained by solving the ordinary differential equation (ODE)

$$\eta^i(\theta_t^i; \pi_t^{-i}) \left( \frac{\partial}{\partial \eta^i} \bigg|_{\substack{\hat{\theta}_t^i = \theta_t^i \\ \eta^i = \eta^i(\theta_t^i)}} \bar{\tau}^i(\hat{\theta}_t^i; \eta^i, \pi_t^{-i}) \right) = \frac{\partial}{\partial \hat{\theta}_t^i} \bigg|_{\hat{\theta}_t^i = \theta_t^i} \left( \bar{u}^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}) - \bar{\tau}^i(\hat{\theta}_t^i; \eta^i, \pi_t^{-i}) \right) \quad (2.17)$$

for all  $\theta_t^i \in \Theta$ . The details are provided in the proof of the following result.

**Proposition 3 (Local BIC)** *If the dynamic allocation mechanism  $\mathcal{M}(\eta)$  is Bayesian incentive compatible it solves (2.17), so that for any  $\theta_t^i \in \Theta$  it is*

$$\eta^i(\theta_t^i; \pi_t^{-i}) = \int_{\theta}^{\theta_t^i} \frac{\frac{\partial}{\partial \hat{\theta}_t^i} \big|_{\hat{\theta}_t^i = \xi} \left( \bar{u}^i(\hat{\theta}_t^i, \xi; \pi_t^{-i}) - \bar{\tau}_B^i(\hat{\theta}_t^i; \pi_t^{-i}) \right) d\xi}{\Delta^i(\theta_t^i - \xi; \pi_t^{-i}) \Delta^i(\xi; \pi_t^{-i})}, \quad (2.18)$$

where  $\Delta^i(\hat{\theta}_t^i; \pi_t^{-i}) = \bar{\tau}_A^i(\hat{\theta}_t^i; \pi_t^{-i}) - \bar{\tau}_B^i(\hat{\theta}_t^i; \pi_t^{-i})$ . The functions  $\bar{\tau}_A^i$  and  $\bar{\tau}_B^i$  are obtained from  $\tau_A^i$  and  $\tau_B^i$  in the same way as  $\bar{\tau}^i$  is obtained from  $\tau^i$ .

As a consequence of Proposition 3 we can interpret the parameter  $\eta^i$  as providing an affine combination of the bounding mechanisms  $\mathcal{M}_A$  and  $\mathcal{M}_B$ . In the special case when  $\eta^i \in [0, 1]$  one may interpret the resulting *convex* combination as a *randomization* between of two bounding mechanisms  $\mathcal{M}_A$  and  $\mathcal{M}_B$ .

## 2.5 Global Incentive Compatibility

To guarantee that the mechanism implied by the results in the last section is globally incentive compatible, one needs to ensure that, in addition to relation (2.17), the expected future gross reward  $\bar{U}^i$  from winning is supermodular. It turns out that global BIC can indeed be established in this manner, provided that agents are sufficiently impatient, i.e., their common discount factor  $\beta \in (0, 1)$  is small enough,

and their conditional reward densities are parameterized ‘strongly’ in a sense that is made precise below. As a consequence of the reward log-supermodularity guaranteed by Assumption A1, the expected per-period reward, conditional on winning, is monotonically increasing in an agent’s type. That is,

$$E_{\theta_t^{-i}, R_t^i} \left[ \mathbf{1}_{\{\hat{\theta}_t^i > \max_{j \neq i} \theta_t^j\}} R_t^i \middle| \theta_t^i, \pi_t^{-i} \right], \quad (2.19)$$

is increasing in  $\theta_t^i$ , given any possible announcement  $\hat{\theta}_t^i \in \Theta$  and beliefs  $\pi_t^{-i}$ . Let us denote agent  $i$ ’s optimal expected utility (or “value function”) by

$$\bar{U}^i(\theta_t^i; \pi_t^{-i}) = \max_{\hat{\theta}_t^i \in \Theta} U^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}). \quad (2.20)$$

The next lemma shows that under the reward monotonicity and state transition monotonicity assumptions we made in Section 2.2, the value function is monotonic in agent  $i$ ’s type.<sup>10</sup>

**Lemma 2** *Under assumptions A1 and A2 any agent  $i$ ’s value function  $\bar{U}^i(\theta_t^i; \pi_t^{-i})$  is nondecreasing on  $\Theta$ , given any beliefs  $\pi_t^{-i}$  about the other agents’ types.*

The monotonicity follows from the monotonicity of the expected per-period rewards conditional on the type (implied by Assumption A1), together with the fact that state transitions are monotonic in observations and prior states (implied by Assumption A2). To obtain a (strong) sufficient condition for global BIC, namely supermodularity of agent  $i$ ’s utility  $U^i(\hat{\theta}, \theta; \pi_t^{-i})$  in  $(\hat{\theta}, \theta)$ , for a certain range of  $\beta$ s, it is important that the agent’s value function is Lipschitz, at least in the case in which he has imperfect knowledge about at least one other agent.

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<sup>10</sup>The value-function monotonicity is not immediately implied by the reward monotonicity and Gittins-index monotonicity, as it includes the (expected) transfer payments in the current and all future periods.

**Lemma 3** *Let  $t \geq 1$  and  $i \in \mathcal{N}$ . Under assumptions A1 and A2 agent  $i$ 's value function  $\bar{U}^i(\cdot; \pi_t^{-i})$  after losing the object at  $t - 1$  is Lipschitz on  $\Theta$ , i.e., there exists a constant  $K > 0$  such that  $|\bar{U}^i(\theta; \pi_t^i) - \bar{U}^i(\hat{\theta}; \pi_t^{-i})| \leq K|\theta - \hat{\theta}|$  for all  $\theta, \hat{\theta} \in \Theta$ .*

The proof of the foregoing result is constructive in the sense that a Lipschitz constant  $K$  can be obtained explicitly as a function of model primitives. If the agent won the object in the last round, Lemma 3 may not hold true, since, given truthful announcement-strategies by the other agents, agent  $i$  knows their types precisely, rendering his payoff as a function of his announcement possibly discontinuous, i.e., not Lipschitz.<sup>11</sup> The two properties established by Lemma 2 and Lemma 3 help us establish the supermodularity of agent  $i$ 's value function for sufficiently impatient agents. In addition, the parametrization of the agent's conditional expected per-period reward  $E[R_t^i|\theta]$  needs to be *strong* in the sense that there exists a constant  $\rho > 0$ , such that

$$\theta, \tilde{\theta} \in \Theta \text{ and } \theta > \tilde{\theta} \Rightarrow E[R_t^i|\theta] - E[R_t^i|\tilde{\theta}] \geq \rho(\theta - \tilde{\theta}) \quad (2.21)$$

for any  $i \in \mathcal{N}$ .

**Proposition 4 (Global BIC)** *Suppose that relation (2.21) holds for any agent  $i \in \mathcal{N}$ . Under assumptions A1 and A2 there exists a  $\beta_0 > 0$  such that for all  $\beta \in (0, \beta_0)$  the mechanism  $\mathcal{M}(\eta)$  satisfies global BIC, i.e., agent  $i$  always (weakly) prefers telling the truth:*

$$U^i(\theta, \theta; \pi_t^{-i}) \geq U^i(\hat{\theta}, \theta; \pi_t^{-i})$$

for any  $\theta, \hat{\theta} \in \Theta$  and beliefs  $\pi_t^{-i}$ .

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<sup>11</sup>Suppose that agent  $j \neq i$  wins the object at time  $t - 1$ , then  $\pi_t^j$  is a continuous function. It is the boundedness of  $\pi_t^j$  that is critical for establishing the Lipschitz property of  $U^i$ .

As long as agents are sufficiently impatient, the last result guarantees global Bayesian incentive compatibility of the mechanism  $\mathcal{M}(\eta)$ , which thus efficiently allocates an indivisible object in each period  $t \in \mathbb{N}$ .

We construct an example to show that although very patient agents might choose to lie about their types, sufficiently impatient agents would choose to play truthfully.

**Example 3** Consider the case of two agents and a single object allocated repeatedly between the two agents. Agent 1 knows his valuation for the object,  $v_1$  to be \$1 with certainty. Agent 2, however, is not certain and thinks that there is a  $1/3$  chance that his valuation could be \$1. It could however, also turn out to be \$0 with probability  $2/3$ . He would find out which is the case only after being allocated the object once. Agent 1's Gittins index is  $\gamma^1 = 1$ . If at time period  $t = 1$  agent 2 observes a zero reward, his optimal stopping time would be  $\tau = 1$ , while if he observes a reward of 1, it would be  $\tau = \infty$ . The Gittins index for agent 2 at time 0,  $\gamma^j$ , can be computed using the equation

$$\gamma^j = \sup \left\{ \hat{\gamma} : \frac{1 - \gamma}{3(1 - \beta)} + \frac{2}{3}(-\hat{\gamma}) \geq 0 \right\}.$$

This gives us  $\gamma^j = 1/(3 - 2\beta)$  and an expected reward of  $1/3$ . We consider the mechanism  $\mathcal{M}(\eta)$  parameterized by an announcement dependent parameter  $\eta(\hat{\theta})$  and show that for very patient agents ( $\beta = 1 - \varepsilon$  for some  $\varepsilon > 0$  close to zero), truthtelling cannot necessarily be induced: for agent 1, to announce his Gittins index truthfully at time 0 would imply a win. If this were the optimal strategy, then it would imply truthtelling in every subsequent time period (since agent 1 learns no new information about the object's value) and would give agent 1 a total reward of

$$\frac{1}{1 - \beta} \left[ 1 - \frac{\eta(\hat{\theta})}{3 - 2\beta} - \frac{1 - \eta(\hat{\theta})}{3} \right].$$

If, however, agent 1 decided to announce a very low Gittins index at time 0 (say, zero), to ensure that agent 2 wins, and realized his true type, then if agent 2 turned out to have a zero valuation, agent 1 would pay zero from time 1 onwards. This strategy would earn agent 1 an expected payoff of  $\frac{2\beta}{3(1-\beta)}$ . Provided that  $\eta \leq 1$  (cf. the individual-rationality constraint (2.22) below), for agent 1 to announce his true type, it must be the case that

$$\min_{\eta \leq 1} \left\{ \frac{1}{1-\beta} \left[ 1 - \frac{\eta}{3-2\beta} - \frac{1-\eta}{3} \right] \right\} = \frac{3-4\beta}{3(1-\beta)(3-2\beta)} \geq 0,$$

which is satisfied, as long as  $\beta \leq \beta_0 = 3/4$ .  $\square$

**REMARK (PERFECT BAYESIAN EQUILIBRIUM):** the mechanism we have developed induces not only a Bayes-Nash equilibrium, but (in conjunction with an appropriate belief system) a perfect Bayesian equilibrium (PBE). This is true because the agents are making sequentially rational decisions at any point, and *on-the-equilibrium* path the Baye's updated agents' beliefs are consistent with the principal's announcements. Because of the separating nature of the equilibrium, (the announcement corresponding to every type is unique) there is no *off-the-equilibrium* deviation that can have consistent beliefs: the only set of consistent beliefs that any agent might have on seeing an announced type are to trust that to be the true type. Suppose an out-of-equilibrium announcement  $\hat{\theta}_t^i \notin \varphi^i(\theta_{t-1}^i, \mathcal{R})$  is made with ex-post beliefs of other agents, then the principal treats this announcement as if  $i$  had announced  $\tilde{\theta}_t^i = \min \varphi^i(\theta_{t-1}^i, \mathcal{R})$ . This is one set of beliefs that induces a PBE.

## 2.6 Implementation of Distributional Objectives

We have shown in Sections 2.4 and 2.5 that, under certain conditions, the mechanism designer can implement the first-best outcome with a Bayesian incentive-compatible

mechanism  $\mathcal{M}(\eta)$ , which is a type-contingent affine combination of the bounding mechanisms  $\mathcal{M}_A$  and  $\mathcal{M}_B$ . Based on the vector of truthful type announcements, implementing the efficient outcome determines in each round the agent who obtains the object (the one with the highest Gittins index). Note that therefore  $\delta_A = \delta_B = \delta$ , so that the function  $\eta = (\eta^1, \dots, \eta^n)$  solely affects the transfers between the agents and the principal. Furthermore, local BIC determines  $\eta$  only up to a constant, so that an adjustment of the revenue distribution between principal and agents may be possible via appropriate shifts, provided that the global BIC properties remain unaffected.

Let  $\bar{\eta} = (\bar{\eta}^1, \dots, \bar{\eta}^n) : \Theta^n \rightarrow \mathbb{R}_+^n$  be an absolutely continuous function and let  $\delta : \Theta^n \rightarrow \mathcal{N}$  with  $\delta(\theta) \in \arg \max_{i \in \mathcal{N}} \gamma^i(\theta^i)$  for any  $\theta = (\theta^1, \dots, \theta^n) \in \Theta^n$  be the welfare-maximizing first-best allocation function. In a complete information setting, provided that agents participate, the ex-ante payoff to the principal is

$$\bar{\Pi}(\bar{\eta}) = \int_{\Theta^n} V(\tau^1(\theta^1, \dots, \theta^n; \bar{\eta}^1(\theta^1)), \dots, \tau^n(\theta^1, \dots, \theta^n; \bar{\eta}^n(\theta^n))) dH(\theta^1, \dots, \theta^n),$$

where the smooth utility function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  captures the principal's discounting, risk, and fairness preferences when evaluating the transfers  $\tau^i(\cdot; \bar{\eta}^i) = \bar{\eta}^i(\tau_A^i(\cdot) - \tau_B^i(\cdot)) + \tau_B^i(\cdot)$  pointwise, and  $H : \Theta^n \rightarrow [0, 1]$  is a cumulative distribution function representing her prior beliefs about the agents' types (identical to the agents' prior beliefs in our common-knowledge setting). The precise form of the principal's payoff functional  $\bar{\Pi}$  is unimportant. The only feature necessary for our results is that  $\bar{\Pi}(\bar{\eta})$  is a continuous operator.<sup>12</sup>

Rather than solving the principal's payoff maximization problem subject to the agent's participation, we assume that a solution  $\bar{\eta}$  has already been found. For

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<sup>12</sup>We say that the operator  $\bar{\Pi}$  is continuous if for any  $\varepsilon > 0$  there exists a constant  $d = d(\varepsilon) > 0$  such that  $\|\bar{\eta} - \eta\|_\infty < d$  implies that  $|\bar{\Pi}(\bar{\eta}) - \bar{\Pi}(\eta)| < \varepsilon$ , where  $\|\bar{\eta} - \eta\|_\infty = \text{ess sup}_{\theta \in \Theta} \max_{i \in \mathcal{N}} \{|\bar{\eta}^i(\theta) - \eta^i(\theta)|\}$ .



instance, if the principal is strictly revenue-maximizing, in which case her utility becomes  $V = v(\sum_{i=1}^n \tau^i(\theta^1, \dots, \theta^n; \bar{\eta}^i(\theta^i)))$  (with  $v$  a smooth increasing function), then  $\bar{\eta}^i = 1$  for all  $i \in \mathcal{N}$  corresponding to a ‘dynamic second-price Gittins-index auction.’ The following result establishes that, whatever the (sufficiently smooth) solution  $\bar{\eta}$  might be under complete information, the principal can come arbitrarily close in terms of payoff with a local BIC mechanism  $\mathcal{M}(\hat{\eta})$ , if only the approximating  $\hat{\eta}$  is chosen such that  $\|\bar{\eta} - \hat{\eta}\|_\infty$  is small enough.

**Proposition 5** *For any  $\varepsilon > 0$  there exists a number  $N > 0$ , such that  $|\bar{\Pi}(\bar{\eta}) - \bar{\Pi}(\hat{\eta})| < \varepsilon$ , where  $\hat{\eta} = (\hat{\eta}^1, \dots, \hat{\eta}^n)$  is a piecewise absolutely continuous function and  $\hat{\eta}^i : \Theta \rightarrow \mathbb{R}$  solves the ODE (2.17) on the interval  $\mathcal{I}_k = [(\bar{\theta} - \underline{\theta})(k-1)/N, (\bar{\theta} - \underline{\theta})k/N)$  for all  $i \in \mathcal{N}$  and all  $k \in \{1, \dots, N\}$ .*

If for a given  $\varepsilon > 0$  the mechanism  $\mathcal{M}(\hat{\eta})$  is globally BIC, then the principal has met her revenue-distribution objective up to an epsilon while at the same time implementing the first-best outcome. It is important to note that in order to ensure agent  $i$ ’s participation, the principal, when optimizing her revenue objectives, needs to satisfy the constraint

$$\bar{\eta}^i \leq 1, \tag{2.22}$$

as long as agent  $i$ ’s presence yields to a net increase in the expected payoff.

## 2.7 Application: Probationary Contracts

In this section we provide a concrete example as an illustration for our methods. Although the Gittins index can be solved for numerically, analytical expressions are often difficult to obtain<sup>13</sup>. Therefore, we assume a simple stochastic reward structure

<sup>13</sup>Katehakis and Veinott (1987) have shown that the Gittins index for a bandit in state  $\theta$  may be interpreted as a particular component of the maximum-value function associated with the ‘restart in  $\theta$ ’ process, a simple Markov Decision Process to which standard solution methods for computing

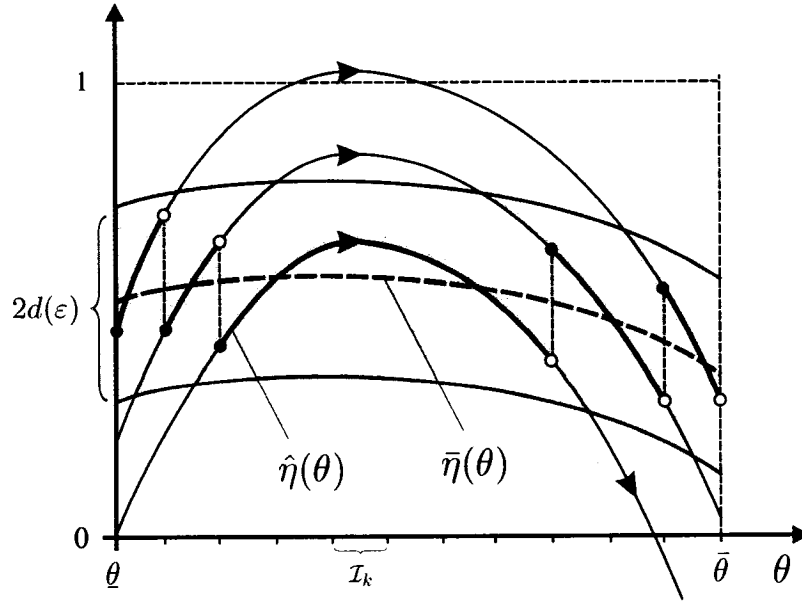


Figure 2.2: Approximate implementation.

where agents can get realizations from one of several deterministic reward sequences, but initially do not know which one. We note that versions of this example have appeared in the literature (cf. footnote 14), without concern for the issues raised by asymmetric information and mechanism design.

Many bilateral relationships in real life are *probationary* in the sense that two parties evaluate each other for a limited ‘courtship period’ before they decide about exercising the option of recontracting more permanently by entering into a long-term contract (Sadanand et al., 1989). Large manufacturing firms often use parallel sourcing (Richardson, 1993) before committing to a single supplier, for the multiple relationship-specific investments may be too costly compared to the expected benefit of keeping the suppliers in competition with each other. In real-estate transactions

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optimal policies (e.g., successive approximation) apply. In general, however, the state space may be extremely large, rendering computations difficult. Lovejoy (1991) provides an excellent survey of techniques used to compute Gittins indices, both exactly and approximately.

there is typically an ‘in-escrow’ period between the initial agreement and the closing of the deal, when either party performs its due diligence and may renege at a specified penalty cost. Similar due-diligence periods exist for acquisitions and mergers of corporations. Lastly, a couples’ personal engagement, perhaps fortunately, does not always result in a marriage. There is no shortage of examples.

We elaborate on a setting where a firm may keep candidates on probation for a while before deciding to extend an offer of regular employment.<sup>14</sup> Consider an employer faced with two candidates for employment. Each candidate  $i \in \{1, 2\}$  is either productive ( $\alpha^i \geq 0$ ) or unproductive ( $\alpha^i < 0$ ), but his suitability for the job can be evaluated only on the job. Given the job for a trial period  $t$ , candidate  $i$  has a probability  $\theta_t^i \in [0, 1] = \Theta$  of learning his productivity  $\alpha^i \in [-1, 1] = \mathcal{A}$ , in which case he would be able to produce a stream of stochastic rewards  $R_t^i$ , with an expected per-period payoff proportional to his productivity. More specifically, given  $\alpha^i$  candidate  $i$ ’s per-period rewards are distributed with probability density

$$\sigma(R_t^i | \alpha^i) = \begin{cases} \alpha^i \lambda_b(R_t^i) + (1 - \alpha^i) \lambda_0(R_t^i), & \text{if } \alpha^i \geq 0, \\ -\alpha^i \lambda_a(R_t^i) + (1 + \alpha^i) \lambda_0(R_t^i), & \text{otherwise,} \end{cases}$$

where  $R_t^i \in \{a, 0, b\} \subset \mathcal{R} = [a, b]$  with  $a < 0 < -a < b$ . At time  $t = 0$ , each

<sup>14</sup>Weitzman (1979) first used the multi-armed-bandit framework to analyze decisions in the labor market. In what he termed ‘Pandora’s Problem,’ an agent attempts to find an optimal job by searching across  $n$  boxes (the firms), where each inspection is costly and the value of the object in the box (the job with random rewards) can be discovered only after some stochastic waiting time. What later came to be known as the Gittins index is what Weitzman referred to as the reservation wage for a job, the agent’s optimal policy being to accept the job offering the highest reservation wage. Weitzman’s model, however, did not allow workers to learn: everything that needed to be learned was done as soon as the box was opened (job offer accepted). Adam (2001) extended Weitzman’s model to allow for the agents to learn more about the job as they perform it. In that setting learning is allowed to occur in a Bayesian, non-parametric or ad-hoc fashion. Gittins (1989) also draws on labor market examples to motivate the multi-armed-bandit problem. Similarly, Lippman and McCall (1981) and Morgan (1985) deal with the problem of optimal search strategies from a worker’s point of view: analyzing the decision of whether to accept a job offer or to search for a new one.

candidate  $i$  privately observes the parameter  $\theta_0^i \in \Theta$ . Conditionally on his information  $(\theta_t^i, \omega_t^i)$  at any time  $t \in \mathcal{N}$ , his beliefs about his productivity  $\alpha^i$  are distributed with the probability density

$$f^i(\alpha^i | \theta_t^i, \omega_t^i) = \omega_t^i \lambda(\alpha^i - \theta_t^i) + (1 - \omega_t^i) \lambda(\alpha^i + \theta_t^i),$$

where  $\omega_t^i \in \{0, 1/2, 1\} = \Omega$  is a second, possibly public, component of candidate  $i$ 's type. The second component is necessary to ensure that  $f^i$  is indeed conjugate to the sampling technology  $\sigma$ . Initially  $\omega_0^i = 1/2$  for all  $i \in \{1, 2\}$ . Given the simple stochastic structure of our model, the state-transition function for both candidates is

$$\varphi((\theta, \omega), R) = \begin{cases} (1, 0), & \text{if } R = a, \\ (\theta, \omega), & \text{if } R = 0, \\ (1, 1), & \text{if } R = b, \end{cases}$$

for all  $(\theta, \omega) \in \Theta \times \Omega$  and all  $R \in \mathcal{R}$ . Given the principal's mechanism  $\mathcal{M}(\eta)$ , candidate  $i$ 's posterior information  $(\theta_{t+1}^i, \omega_{t+1}^i)$  can be computed using (2.10),

$$(\theta_{t+1}^i, \omega_{t+1}^i) = \begin{cases} \varphi((\theta_t^i, \omega_t^i), R_t^i), & \text{if } \delta(\hat{\theta}_t^i, \hat{\theta}_t^{-i}) = i, \\ (\theta_t^i, \omega_t^i), & \text{otherwise,} \end{cases}$$

where  $\hat{\theta}_t^i \in \Theta$  corresponds to agent  $i$ 's announcement at time  $t$ . Note that candidate  $i$ 's productivity becomes apparent on the  $N^i$ -th day, where  $N^i$  is a geometrically distributed Bernoulli random variable with parameter  $\theta_0^i$ . At the end of any given time period  $t$ , the agent who wins the job contract in that period can have three possible realizations: he can (i) realize that his reward in every subsequent period will be  $a < 0$ , (ii) realize that his reward in every subsequent period will be  $b > 0$ , or (iii) not realize any new information about his productivity by observing a zero per-period reward. For simplicity, we assume that each candidate has a reservation utility of zero and would therefore quit in case (i). Let us now compute candidate  $i$ 's

Gittins index at any time  $t$ , when he has not yet realized his final reward stream (i.e., when  $\omega_t^i = 1/2$ ). The utility of that reward stream net of a retirement reward  $\gamma$  at the optimal stopping policy is  $u_0^i(\theta_0^i; \gamma)$ , whereas given an optimal stopping policy it is  $u_a^i(\gamma) = a - \gamma$  if he realizes reward stream  $a$  and  $u_b^i(\gamma) = (b - \gamma)/(1 - \beta)$  if he realizes reward stream  $b$ . Thus,

$$\begin{aligned} u_0^i(\theta_0^i; \gamma) &= \frac{\theta_0^i}{2}(a + b) - \gamma + \beta \left( \frac{\theta_0^i}{2} (u_a^i(\gamma) + u_b^i(\gamma)) + (1 - \theta_0^i) u_0^i(\theta_0^i; \gamma) \right) \\ &= (1 - (1 - \theta_0^i)\beta)^{-1} \left( \frac{\theta_0^i}{2} \left[ (1 + \beta)a + \left( 1 + \frac{\beta}{1 - \beta} \right) b \right] - \gamma \left[ 1 + \frac{\theta_0^i}{2} \left( \beta + \frac{\beta}{1 - \beta} \right) \right] \right) \end{aligned}$$

Setting the right-hand side of the last relation to zero thus yields candidate  $i$ 's Gittins index, provided he is still in the uncertain state, i.e.,  $\omega_t^i = 1/2$ . In sum, we thus obtain

$$\gamma^i(\theta, \omega) = \begin{cases} 0, & \text{if } \omega = 0, \\ \frac{((1 - \beta^2)a + b)(\theta/2)}{1 - \beta[1 - (2 - \beta)(\theta/2)]}, & \text{if } \omega = 1/2, \\ b, & \text{if } \omega = 1, \end{cases}$$

for all  $(\theta, \omega) \in \Theta \times \Omega$ . Let  $j = 3 - i$ . We have that

$$E[R_t^j | \theta_t^j, \omega_t^j] = \int_{\Theta} R_t^j p^j(R_t^j | \theta_t^j, \omega_t^j) dR_t^j = ((1 - \omega_t^j)a + \omega_t^j b) \theta_t^j,$$

since  $p^j(R | \theta, \omega) = ((1 - \omega)\lambda_a(R) + \omega\lambda_b(R))\theta + (1 - \theta)\lambda_0(R)$  for all  $(\theta, \omega) \in \Theta \times \Omega$  and all  $R \in \mathcal{R}$ . At time  $t$ , let us examine the case where the uncertainty for both agents is still unresolved, i.e., for any agent  $i$  it is  $\theta_t^i \in (0, 1)$  and  $\omega_t^i = \omega = 1/2$ . In that case, agent  $i$ 's utility can be written implicitly,

$$\begin{aligned} U^i(\hat{\theta}, (\theta, \omega); \pi_t^j, \eta^i) &= \int_0^{\hat{\theta}} \int_{\mathcal{R}} (R + \beta \bar{U}^i(\varphi^i((\theta, \omega), R); \lambda_{\theta_t^j}, \eta^i)) p^i(R | \theta) dR - \tau^i(\theta_t^j; \eta^i(\hat{\theta})) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\ &\quad + \int_{\hat{\theta}}^{\theta} \beta \bar{U}^i((\theta, \omega); \pi_{t+1}^j(\cdot; \theta_t^j), \eta^i) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j, \end{aligned} \quad (2.23)$$

in terms of agent  $i$ 's value function evaluated at the possible outcome states in the next period (i.e., at time  $t + 1$ ). To derive an explicit expression we first consider the

two terms on the right-hand side of (2.23) that are not directly related to current-period transfers, namely expected current-period rewards in case of winning and expected value in case of winning (in which case the other agent's type will be perfectly known at time  $t + 1$ ). Agent  $i$ 's expected current-period rewards in case of winning are given by

$$\int_0^{\theta} \left( \int_{\mathcal{R}} R p^i(R|\theta, \omega) dR \right) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j = ((1 - \omega)a + \omega b) \theta \int_0^{\theta} \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j, \quad (2.24)$$

while his expected value in case of winning evaluates to

$$\begin{aligned} & \int_{\mathcal{R}} \bar{U}^i(\varphi((\theta, \omega), R); \lambda_{\theta_t^j}, \eta^i) p^i(R|\theta, \omega) dR \\ &= \theta \left[ (1 - \omega) \bar{U}^i(\varphi((\theta, \omega), a); \lambda_{\theta_t^j}, \eta^i) + \omega \bar{U}^i(\varphi((\theta, \omega), b); \lambda_{\theta_t^j}, \eta^i) \right] \\ & \quad + (1 - \theta) \bar{U}^i(\varphi((\theta, \omega), 0); \lambda_{\theta_t^j}, \eta^i) \\ &= \theta \omega \bar{U}^i((1, 1); \lambda_{\theta_t^j}, \eta^i) + (1 - \theta) \bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) \\ &= \theta \omega \frac{b - \tau^i(\theta_t^j; \eta^i(1))}{1 - \beta} + (1 - \theta) \bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i). \end{aligned} \quad (2.25)$$

As pointed out earlier, in the case of winning at time  $t$  agent  $i$  knows agent  $j$ 's type  $\theta_{t+1}^j = \theta_t^j$  perfectly well at time  $t + 1$ . We thus examine separately the two possibilities of agent  $i$ 's type  $\theta$  either exceeding  $\theta_t^j$  or not (the case of equality being in expectation *ex ante* irrelevant, since it occurs with probability zero). For  $\theta > \theta_t^j$ , agent  $i$ 's value function in case of winning becomes

$$\bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) = \int_{\mathcal{R}} \left( R - \tau^i(\theta_t^j; \eta^i(\theta)) + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_t^j}, \eta^i) \right) p^i(R|\theta) dR$$

$$\begin{aligned}
&= \theta((1-\omega)a + \omega b) - \tau^i(\theta_t^j; \eta^i(\theta)) + \beta\theta\omega\bar{U}^i((1,1); \lambda_{\theta_t^j}, \eta^i) + (1-\theta)\beta\bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) \\
&= \frac{1}{1-(1-\theta)\beta} \left[ \theta \left[ (1-\omega)a + \omega b + \beta\omega\bar{U}^i((1,1); \lambda_{\theta_t^j}, \eta^i) \right] - \tau^i(\theta_t^j; \eta^i(\theta)) \right] \\
&= \frac{1}{1-(1-\theta)\beta} \left[ \theta(1-\omega)a + \frac{\theta\omega b}{1-\beta} - \frac{\beta\theta\omega\tau^i(\theta_t^j; \eta^i(1))}{1-\beta} - \tau^i(\theta_t^j; \eta^i(\theta)) \right]
\end{aligned} \tag{2.26}$$

$$(2.27)$$

In case of loosing, his expected value is

$$\begin{aligned}
&\int_{\hat{\theta}}^{\bar{\theta}} \bar{U}^i((\theta, \omega); \pi_{t+1}^j(\cdot; \theta_t^j), \eta^i) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\
&= \int_{\hat{\theta}}^{\bar{\theta}} \left[ (1-\omega^j)\theta_t^j \bar{U}^i((\theta, \omega); \lambda_a, \eta^i) + (1-\theta_t^j)\bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) \right] \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j
\end{aligned} \tag{2.28}$$

If, on the other hand,  $\theta < \theta_t^j$ , then he obtains

$$\begin{aligned}
\bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) &= \beta\theta_t^j\omega^j\bar{U}^i((\theta, \omega); \lambda_a, \eta^i) + (1-\theta_t^j)\bar{U}^i((\theta, \omega); \lambda_{\theta_t^j}, \eta^i) \\
&= \frac{\beta\theta_t^j\omega^j\bar{U}^i((\theta, \omega); \lambda_a, \eta^i)}{1-(1-\theta_t^j)\beta},
\end{aligned} \tag{2.29}$$

where

$$\bar{U}^i((\theta, \omega); \lambda_a, \eta^i) = \frac{\theta\omega b}{(1-\beta)(1-(1-\theta)\beta)}.$$

If we set

$$\bar{\gamma}(\hat{\theta}) \equiv \int_0^{\hat{\theta}} \gamma^j(\theta_t^j) \pi_t^j(\theta_t^j) d\theta_t^j = \frac{(1-\beta^2)a + b}{(2-\beta)^2\beta^2} \left( 2(1-\beta) \ln \frac{2(1-\beta)}{2(1-\beta) + (2-\beta)\beta\hat{\theta}} + (2-\beta)\beta\hat{\theta} \right)$$

and

$$\bar{r}_\omega(\hat{\theta}) \equiv \int_0^{\hat{\theta}} E[R_t^j | \theta_t^j] \pi_t^j(\theta_t^j) d\theta_t^j = \frac{((1-\omega)a + \omega b) \hat{\theta}^2}{2},$$

then by substituting equations (2.24)–(2.29) in equation (2.23) and subsequently integrating with respect to beliefs that are, for computational simplicity, uniformly

distributed (i.e.,  $\pi_t^j(\theta_t^j; \theta_{t-1}^j) \equiv 1$ ) we obtain agent  $i$ 's expected utility in the additively separable form (2.13), i.e.,

$$U^i(\hat{\theta}, (\theta, \omega); \pi_t^j, \eta^i) = \bar{u}^i(\hat{\theta}, (\theta, \omega); \pi_t^j) - \bar{\tau}^i(\hat{\theta}, (\theta, \omega); \pi_t^j, \eta^i),$$

where

$$\begin{aligned} \bar{u}^i(\hat{\theta}, (\theta, \omega); \pi_t^j) &= \hat{\theta}\theta((1-\omega)a + \omega b) + \beta\hat{\theta}\theta \left[ \frac{b\omega(1 + (1-\beta)(1-\theta))}{(1-\beta)(1 - (1-\theta)\beta)} + \frac{(1-\theta)(1-\omega)a}{1 - (1-\theta)\beta} \right] \\ &+ \beta \int_{\hat{\theta}}^{\theta} \left[ \frac{\theta\theta_t^j\omega b(1-\omega^j)}{(1-\beta)(1 - (1-\theta)\beta)} + \frac{\beta\theta\omega b\theta_t^j\omega^j(1-\theta_t^j)}{(1-\beta)(1 - (1-\theta)\beta)(1 - (1-\theta_t^j)\beta)} \right] d\theta_t^j, \end{aligned}$$

and

$$\begin{aligned} \bar{\tau}^i(\hat{\theta}, (\theta, \omega); \pi_t^j, \eta^i) &= \frac{(1 - (1 - \omega\theta)\beta)\bar{r}_\omega(\hat{\theta}) + ((1 - \beta)\eta^i(\theta) + (\beta\omega\theta)\eta^i(1))(\bar{\gamma}(\hat{\theta}) - \bar{r}_\omega(\hat{\theta}))}{(1 - \beta)(1 - (1 - \theta)\beta)}. \end{aligned} \quad (2.30)$$

We are now ready to explicitly construct the announcement-dependent affine combination in our efficient dynamic mechanism (using Proposition 3) and then verify global BIC (using Proposition 4), computing an explicit upper bound  $\beta_0$  for the discount rate. We then turn to the principal's approximate revenue maximization subject to efficiency (using Proposition 5).

**Local Incentive Compatibility (Computing  $\eta^i$ ).** Differentiating with respect to  $\hat{\theta}$ , the first-order conditions  $\left. \frac{\partial U^i(\hat{\theta}, \theta; \pi_t^{-i})}{\partial \hat{\theta}} \right|_{\hat{\theta}=\theta} = 0$  can be written as the linear ODE (2.17), i.e., explicitly in the form

$$\dot{\eta}^i(\theta) + g(\theta)\eta^i(\theta) = h(\theta) \quad (2.31)$$

for all  $\theta \in [0, 1]$ , where  $g(\theta) = (1 - (1 - \theta)\beta)^{-1}(\bar{\gamma}(\theta) - \bar{r}_\omega(\theta))^{-1}[\theta((1 - \omega)a + \omega b) - \bar{\gamma}(\theta)]$



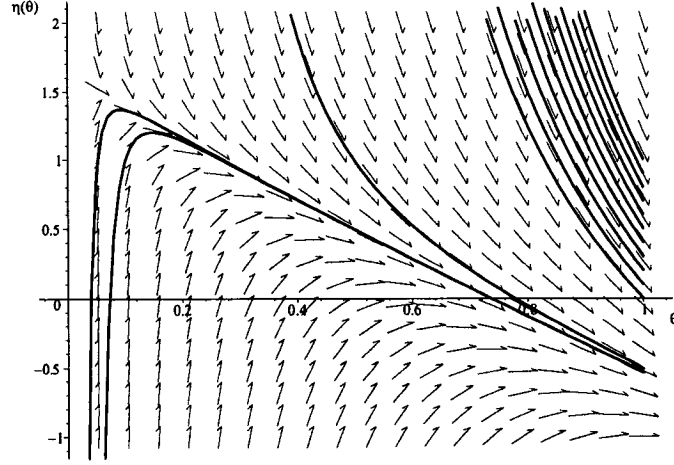


Figure 2.3: Vector field and trajectories generated by (2.31) for different initial values  $\eta^i(1) \in [0, 1]$ .

and

$$h(\theta) = \frac{1}{\bar{\gamma}(\theta) - \bar{r}_\omega(\theta)} \left\{ \frac{\beta\omega\theta [\eta^i(1)\bar{\gamma}(\theta) + (1 - \eta^i(1))r_\omega(\theta)]}{(1 - \beta)(1 - (1 - \theta)\beta)} - \frac{\theta b\omega(1 - \theta(1 - \beta))}{(1 - \beta)(1 - (1 - \theta)\beta)} \right. \\ \left. - \frac{\theta(1 - \omega)a}{1 - (1 - \theta)\beta} + \beta \left[ \frac{\theta^2\omega b(1 - \omega^j)}{(1 - \beta)(1 - (1 - \theta)\beta)} + \frac{\beta\theta^2\omega b\omega^j(1 - \theta)}{(1 - \beta)(1 - (1 - \theta)\beta)^2} \right] + \bar{r}_\omega(\theta) \right\}.$$

The right-hand side,  $h(\theta)$ , of the linear ODE (2.31) depends on the value of its solution at  $\theta = 1$ . Thus, by prescribing this value as ‘initial condition,’ we obtain a well-defined initial-value problem which can be solved using either formula (2.18) in Proposition 3 or direct application of the variation-of-constants formula (Petrovski, 1966, p. 20). A number of solution trajectories  $\eta^i(\theta)$ ,  $\theta \in [0, 1]$  of (2.31) corresponding to different initial values  $\eta^i(1)$  are depicted in Figure 2.3 for  $(a, b) = (-1, 2)$  and  $\omega_t^i = 1/2$ .

**Global Incentive Compatibility.** To verify that the reward parametrization is strong according to (2.21), which is by Proposition 4 sufficient for global incentive compatibility (provided the discount rate  $\beta$  is small enough), we note that  $E[R_t^i|\theta] -$

$E[R_t^i | \tilde{\theta}] = \rho(\theta - \tilde{\theta})$ , where  $\rho = ((1 - \omega)a + \omega b) > 0$ . Following the argument of the proof of Lemma 3, notice that

$$0 \leq \bar{U}^i \leq \frac{b}{1 - \beta} \equiv Q < \infty.$$

The first inequality in the last relation stems from our assumption that all agents are free to participate in the principal's mechanism. As a result, for any  $\hat{\theta}, \hat{\theta}, \theta \in \Theta$ :

$$|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \theta; \pi_t^j)| \leq \int_{\hat{\theta}}^{\hat{\theta}} ((1 - \beta)Q + 2\beta Q) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j.$$

To establish the Lipschitz property of  $U^i$ , notice first that

$$|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \theta; \pi_t^j)| \leq M_1 |\hat{\theta} - \hat{\theta}|,$$

where  $M_1 = b \max_{(\theta_t^j, \theta_{t-1}^j) \in \Theta^2} \{\pi_t^j(\theta_t^j; \theta_{t-1}^j)\}$ , suppressing the dependence on time  $t$ . On the other hand, for any  $\hat{\theta}, \theta, \tilde{\theta} \in \Theta$  it is

$$\begin{aligned} |U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j)| &\leq \int_{\hat{\theta}}^{\hat{\theta}} \left| \int_{\mathcal{R}} Q(p^i(R|\theta) - p^i(R|\tilde{\theta})) dR \right| \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\ &\leq \int_{\hat{\theta}}^{\hat{\theta}} |(\theta - \tilde{\theta})| \int_{\mathcal{R}} Q((1 - \omega)\lambda_a(R) + \omega\lambda_b(R) + \lambda_0(R)) dR \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\ &\leq \left( \frac{2(b - a)}{1 - \beta} \right) |\theta - \tilde{\theta}| \equiv M_2 |\theta - \tilde{\theta}|. \end{aligned}$$

We have therefore shown that  $|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j)| \leq M_1 |\hat{\theta} - \hat{\theta}| + M_2 |\theta - \tilde{\theta}|$ , so that

$$|\bar{U}^i(\theta; \pi_t^j) - \bar{U}^i(\tilde{\theta}; \pi_t^j)| \leq K |\theta - \tilde{\theta}|,$$

where  $K = 2 \max\{M_1, M_2\} = 4(b - a)/(1 - \beta)$ . Thus, as long as

$$\beta \leq \beta_0 = \frac{\rho}{K} = \frac{(a + b)(1 - \beta_0)}{8(b - a)} = \frac{a + b}{9b - 7a} < 1.$$

Proposition 4 guarantees global incentive compatibility.

**Approximate Revenue Maximization Subject to Efficiency.** Using the  $\eta^i$  computed above to construct an incentive-compatible affine combination of the bounding mechanisms, the firm can guarantee efficient experimentation and truthful announcements of an agent's private information about his productivity. For a profit-maximizing firm, however, the distributional constraints for the principal would imply choosing  $\bar{\eta}^i \leq 1$  such that

$$\bar{\eta}^i(\hat{\theta}; \omega_t) \in \arg \max_{\eta^i \leq 1} \int_{\Theta} \left( \sum_{i=1}^2 \bar{\tau}^i(\hat{\theta}, (\theta_t^i, \omega_t^i); \pi_t^j, \eta^i) \right) \pi_t^i(\theta_t^i) d\theta_t^i$$

for all  $\hat{\theta} \in \Theta$ . By Proposition 5 the optimal  $\bar{\eta}^i$  (provided sufficient regularity) can be approximated by a piecewise absolutely continuous function  $\hat{\eta}^i$  that satisfies the equation (2.31) a.e. on a partition of  $\Theta$ .

**REMARK.** A special case of this probationary contracts model discussed above is when the uncertainty about an agent's deterministic reward stream is resolved as soon as he obtains the resource once. Then the solution to equation (2.31) becomes the trivial solution  $\eta(\theta) \equiv 0$ . In this situation the mechanism  $\mathcal{M}_B$  with transfers  $\tau_B$  corresponding to the second-highest one-period expected rewards guarantees truthfulness.

## 2.8 Chapter Summary

The literature on solving dynamic and stochastic optimization (or dynamic and stochastic scheduling) problems has evolved tremendously over the last 50 years and a large number of decision problems have successfully been addressed using the models and algorithms developed therein. These, however, have only now begun to be applied to multi-agent systems, where the agents' objectives need not be aligned with a social planner's objectives. In this chapter, we construct an efficient dynamic mechanism where agents learn their valuations privately. We find two bounding mechanisms based on the multi-armed bandit problem which satisfy one-sided Bayesian incentive-compatibility conditions. A type-dependent affine combination of these two mechanisms leads to local Bayesian incentive compatibility." We show that if agents are sufficiently impatient, global Bayesian incentive compatibility of the mechanism can be guaranteed. In addition, the principal may be able to achieve separate distributional objectives (e.g., maximizing his own revenues or achieving some fairness goal), reallocating surplus between him and the agents. We show that local Bayesian incentive compatibility can almost everywhere be preserved while at the same time it is possible to approximate the principal's payoff corresponding to any (not necessarily Bayesian incentive compatible) affine combination of the bounding mechanisms arbitrarily closely.

## Appendix: Proofs

**Proof of Lemma 1.** By Assumption A1 the conditional expectation  $E[R_t^i | \theta_t^i]$  is nondecreasing in  $\theta_t^i$ , and by Assumption A2 the state-transition function  $\varphi^i(\theta_t^i, R_t^i)$  is nondecreasing in both  $R_t^i$  and  $\theta_t^i$ . From (2.8) we have that

$$G^i(\theta, z) = \max \{ z, E_{R_t^i} [R_t^i + \beta G^i(\varphi^i(\theta, R_t^i), z) | \theta] \}$$

for all  $\theta \in \Theta$ . If for a given  $z$  the function  $G^i(\cdot, z)$  is *not* nondecreasing, then there exists a nonempty compact interval  $[m, M] \subset \Theta$  (with  $m < M$ ) such that  $G^i(\cdot, z)$  is *strictly* decreasing on  $(m, M)$ . As a result,

$$G^i(\theta, z) > G(M, z) = \max \{z, E_{R_t^i} [R_t^i + \beta G^i(\varphi^i(\theta, R_t^i), z) | \theta = M]\} \geq z$$

for all  $\theta \in (m, M)$ . We first show that  $G^i(\cdot, z)$  is differentiable on  $(m, M)$ . Indeed, the last relation implies that

$$G^i(\theta, z) = E_{R_t^i} [R_t^i + \beta G^i(\varphi^i(\theta, R_t^i), z) | \theta]$$

for all  $\theta \in (m, M)$ . As a consequence of the smoothness of our distributions,  $G^i(\cdot, z)$  is differentiable on the interior of any compact interval where its slope is negative. On the compact set we can select an interval  $(m, M)$  which contains the smallest value of the derivative of  $G^i(\cdot, z)$ , which we denote by  $G_\theta^i(\cdot, z)$ . Assume that this lowest value  $G_\theta^i$  is attained at the point  $\check{\theta} \in (m, M)$ . Consider now a small neighborhood  $\mathcal{U}$  of the point  $\check{\theta}$  and a restricted compact support  $\mathcal{R} \subset \mathcal{R}$  for agent  $i$ 's rewards that ensures that state transitions from the state  $\check{\theta}$  end up in  $\mathcal{U}$ , i.e.,

$$\varphi^i(\check{\theta}, R) \in \mathcal{U}$$

for all  $R \in \mathcal{R}$ . Then there exists a neighborhood  $\mathcal{U}_1 \subset \mathcal{U}$  such that

$$\varphi^i(\theta, R) \in \mathcal{U}$$

for all  $\theta \in \mathcal{U}_1$  and all  $R \in \mathcal{R}$ . If we denote agent  $i$ 's expected reward conditional on his type  $\theta_t^i$  being equal to  $\theta \in \Theta$  by  $\bar{R}^i(\theta) = E[R_t^i | \theta_t^i = \theta]$ , then

$$G^i(\theta, z) = \bar{R}^i(\theta) + \int_{\mathcal{R}} G^i(\varphi^i(\theta, R), z) p^i(R | \theta) dR = \bar{R}^i(\theta) + G^i(\bar{\varphi}^i(\theta, z), z)$$

for all  $\theta \in (m, M)$ . The function  $\bar{\varphi}^i(\theta, z)$  is an average state-transition function, implicitly defined by the last equality. For all  $\theta \in \mathcal{U}_1$  the average state-transition function maps to states in  $\mathcal{U}$ , provided that  $\mathcal{U}_1$  is chosen small enough. Thus, by taking the derivative with respect to  $\theta$  we obtain (using Assumption A2) that

$$G_\theta^i(\theta, z) = \bar{R}_\theta^i(\theta) + \beta G_\theta^i(\bar{\varphi}^i(\theta, z), z) \bar{\varphi}_\theta^i(\theta, z) \geq \bar{R}_\theta^i(\theta) + \beta G_\theta^i(\bar{\varphi}^i(\check{\theta}, z), z) \bar{\varphi}_\theta^i(\theta, z),$$

where the subscript  $\theta$  denotes the corresponding derivatives with respect to  $\theta$ . Hence,

$$(1 - \beta \bar{\varphi}_\theta^i(\check{\theta}, z)) G_\theta^i(\check{\theta}, z) \geq \bar{R}_\theta^i(\check{\theta}) \geq 0,$$

while at the same time  $G_\theta^i(\check{\theta}, z) < 0$ , which for any given  $\beta \in (0, 1)$  can only hold if  $\bar{\varphi}_\theta^i(\check{\theta}, z) > 1$ . However, the latter is impossible, for under Bayesian updating, as a consequence of the averaging, the posterior cannot adjust faster than the prior varies, i.e., we must have that

$$\bar{\varphi}_\theta^i(\theta, z) \leq 1$$

for all  $\theta \in (m, M)$ . But this provides the desired contradiction and we have therefore shown that the function  $G^i(\cdot, z)$  is increasing for any given retirement reward  $z \in \mathcal{R}$ . Given the monotonicity of  $G^i(\cdot, z)$ , agent  $i$ 's Gittins index must be monotonically increasing. To see this, consider  $\hat{\theta}, \tilde{\theta} \in \Theta$  with  $\hat{\theta} < \tilde{\theta}$ . By definition, the Gittins index is the smallest retirement reward  $z$ , for which function  $G^i(\cdot, z) = z$ . Since by monotonicity  $G^i(\tilde{\theta}, z)$  lies above  $G^i(\hat{\theta}, z)$  for every  $z$ , the smallest  $z$  such that  $G^i(\tilde{\theta}, z) = z$  exceed

the corresponding value for  $G^i(\hat{\theta}, z)$ . The stationarity of the Gittins index obtains, since by relation (2.9) it does not depend on the time instant  $t$ . This concludes our proof. ■

**Proof of Proposition 1.** We focus on a setting with two agents, as our arguments generalize to  $N$  agents in a straightforward manner. Agent  $i \in \{1, 2\}$  is of type  $\theta_t^i$  and let  $j = 3 - i$ . We show that at time  $t$  this agent announces a distribution parameter  $\hat{\theta}_t^i$  which corresponds to a Gittins index not larger than his true Gittins index, i.e.,  $\gamma^i(\hat{\theta}_t^i) \leq \gamma^i(\theta_t^i)$ . Let  $\vartheta^i = (\vartheta_t^i(\cdot))_{t \in \mathbb{N}}$  denote agent  $i$ 's announcement strategy under which he overstates his Gittins index at time  $t$ . The interesting case occurs when his time- $t$  announcement  $\hat{\theta}_t^i = \vartheta_t^i(\theta_t^i)$  is such that

$$\gamma^i(\theta_t^i) < \gamma^j(\theta_t^j) < \gamma^i(\hat{\theta}_t^i), \quad (2.32)$$

i.e., when agent  $i$  wins the object, even though truthful revelation would have caused him to lose. For any stopping time  $\hat{T}^i \geq 0$  we have, for all  $\theta_t^j$  satisfying relation (2.32), by the definition (2.9) of the Gittins index that

$$E \left[ \sum_{s=0}^{\hat{T}^i} \beta^s (R_{t+s}^i - \gamma^j(\theta_t^j)) \middle| \theta_t^i \right] < E \left[ \sum_{s=0}^{\hat{T}^i} \beta^s (R_{t+s}^i - \gamma^i(\theta_t^i)) \middle| \theta_t^i \right] = 0, \quad (2.33)$$

where  $T^i$  would be the optimal stopping time if agent  $i$  had paid  $\gamma^i(\theta_t^i)$  from time  $t$  onwards.

Relation (2.33) expresses the fact that winning results at time  $t$  in an expected net *loss* for agent  $i$ , regardless of his stopping-time strategy. Given any stopping time  $\hat{T}^i$ , agent  $j$  (who loses in periods  $t$  through  $t + \hat{T}^i$ ) remains in the same state, so that  $\theta_{t+\hat{T}^i+1}^j = \theta_t^j$ . Hence, agent  $i$ 's losing the object at time  $t + \hat{T}^i + 1$  implies that

$$\gamma^i(\hat{\theta}_{t+\hat{T}^i+1}^i) < \gamma^j(\hat{\theta}_{t+\hat{T}^i+1}^j) = \gamma^j(\theta_t^j),$$

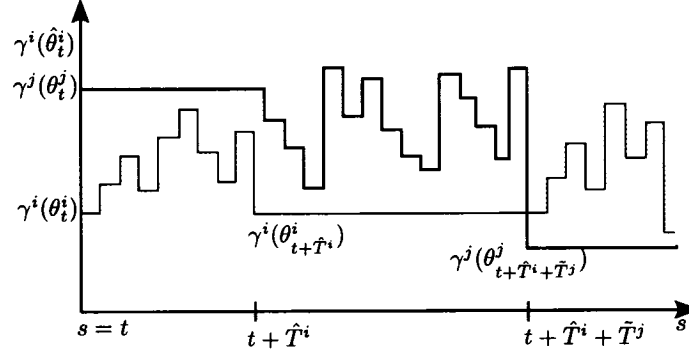


Figure 2.4: Evolution of the Gittins indices under the announcement strategy profile  $\vartheta(\cdot)$ .

where agent  $j$ 's announcement is  $\hat{\theta}_t^j = \vartheta_t^j(\theta_t^j)$  with  $\gamma^j(\hat{\theta}_t^j) \leq \gamma^j(\theta_t^j)$ . We now construct an alternate strategy  $\tilde{\vartheta}^i = (\tilde{\vartheta}_t^i(\cdot; \hat{T}^i))_{t \in \mathbb{N}}$  for agent  $i$ , which is strictly better for him than  $\vartheta^i$ . Without loss of generality, the two strategies coincide in the periods  $0, \dots, t-1$ . At time  $t$ , agent  $i$  announces  $\tilde{\theta}_t^i = \tilde{\vartheta}_t^i(\theta_t^i; \hat{T}^i)$ , corresponding to the future value of his Gittins index under the previous strategy (and thus  $\hat{T}^i$ ), such that

$$\gamma^i(\tilde{\theta}_t^i) = \gamma^i(\hat{\theta}_{t+\hat{T}^i+1}^i) < \gamma^j(\hat{\theta}_t^j).$$

This would result in agent  $j$ 's winning the object from time  $t$  until time  $t + \tilde{T}^j$  for some appropriate stopping time  $\tilde{T}^j$ . In that case agent  $j$ 's type at the beginning of period  $t + \tilde{T}^j + 1$  is the same as the state he would have finished in at time  $t + T^j + 1$  if agent  $i$  had announced  $\hat{\theta}_t^i$  at time  $t$ . At time  $t + \hat{T}^i + \tilde{T}^j + 2$ , announcing the true Gittins index  $\gamma^i(\theta_t^i)$  would result in agent  $i$  winning the object from time  $t + \hat{T}^i + \tilde{T}^j + 2$  onwards, while paying only

$$\gamma^j(\theta_{t+\tilde{T}^j+1}^j) \leq \gamma^i(\theta_{\hat{T}^i}^i) < \gamma^i(\theta_t^i) < \gamma^j(\hat{\theta}_t^j).$$



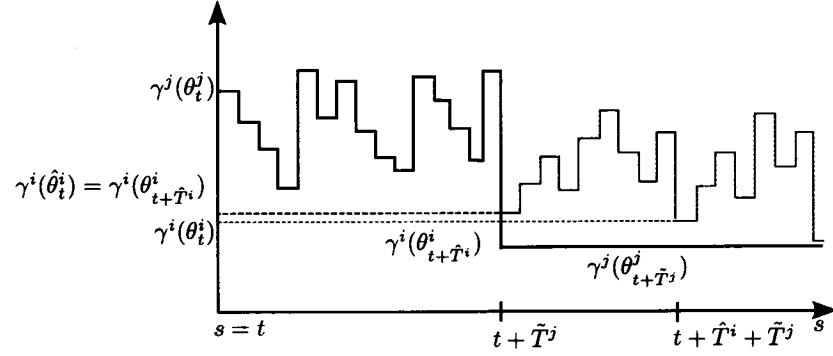


Figure 2.5: Evolution of the Gittins indices under the announcement strategy profile  $\tilde{\vartheta}(\cdot)$ .

For our argument we now fix the reward sample paths  $R^1$  and  $R^2$ . We assume that the reward of agent  $k \in \{1, 2\}$  after winning  $w$  times is  $r_w^k$ , where  $r_w^k = R_t^k$  if this event occurs at time  $t$ . In other words, we assume that independently of the particular time when agent  $k$  wins the object for the  $w$ -th time, the reward he obtains corresponds to the  $w$ -th element of the path  $\{r_1^k, r_2^k, \dots\}$ .<sup>15</sup> Assuming that agent  $i$  has won the item  $w(i)$  times before time  $t$ , we note (given the fixed sample paths) that at all outcomes  $r_{w(i)}^i, \dots, r_{w(i)+\tilde{T}^i+1}^i$ , for which agent  $i$  paid  $\gamma^j(\theta_t^j)$  under  $\vartheta^i$ , he pays  $\gamma^j < \gamma^j(\theta_t^j)$  under  $\tilde{\vartheta}^i$ , and the game from  $t + \tilde{T}^i + 1$  onwards is the same as that under the strategy  $\vartheta^i$ . It follows that strategy  $\tilde{\vartheta}^i$  is strictly better for agent  $i$  than strategy  $\vartheta^i$ , which concludes our proof. ■

**Proof of Proposition 2.** We consider the case of two agents,  $i \in \{1, 2\}$  and  $j = 3 - i$ . The arguments in the proof extend to more than two agents. Let  $\gamma^i(\theta_t^i) > \gamma^j(\theta_t^j)$ . We show that at any time  $t \in \mathbb{N}$  agent  $i$  announces a distribution parameter  $\hat{\theta}_t^i$  corresponding to a Gittins index at least as large as his true Gittins, i.e.,  $\gamma^i(\hat{\theta}_t^i) \geq$

<sup>15</sup>Since the rewards paths are i.i.d., fixing them in such a way is without loss of generality. This trick allows us to easily compare the consequences of changes to an agent's strategy.

$\gamma^i(\theta_t^i)$ . Let  $\vartheta^i = (\vartheta_t^i(\cdot))_{t \in \mathbb{N}}$  denote agent  $i$ 's announcement strategy. Suppose that at a given instant  $t$  the agents' optimal stopping times are  $T^i$  and  $T^j$ , conditionally on paying their respective Gittins indices in every period. From the definition (2.9) of the Gittins index we obtain that

$$E \left[ \sum_{s=0}^{T^i} \beta^s (R_{t+s}^j - \gamma^j(\theta_t^j)) \middle| \theta_t \right] \leq E \left[ \sum_{s=0}^{T^i} \beta^s (R_{t+s}^i - \gamma^i(\theta_t^i)) \middle| \theta_t \right] = 0.$$

The inequality results from the fact that  $T^j$  is the optimal stopping time for agent  $j$ . Indeed, for any other stopping time  $T^i$ , if the inequality were reversed, then  $\gamma^j(\theta_t^j)$  could be increased and a supporting policy resulting in  $T^i$  could be used. The foregoing relations imply that

$$E \left[ \sum_{s=0}^{T^i} \beta^s [(R_{t+s}^i - R_{t+s}^j) + (\gamma^j(\theta_t^j) - \gamma^i(\theta_t^i))] \middle| \theta_t \right] \geq 0.$$

If we let  $\bar{T}^i$  denote the first time for which  $\gamma^i(\theta_{\bar{T}^i}^i) < \gamma^j(\theta_{\bar{T}^i}^j)$ , then necessarily  $T^i < \bar{T}^i$ . This is because by hypothesis  $\gamma^i(\theta_t^i) > \gamma^j(\theta_t^j)$ , and  $T^i$  can be described as the first time after which agent  $i$ 's discounted net payoff crosses  $\gamma^i(\theta_t^i)$  from above. Agent  $i$ 's expected reward until time  $\bar{T}^i$  can be expressed recursively in the form

$$\begin{aligned} \hat{U}_{\bar{T}^i}^i(\vartheta_t^i(\theta_t^i), \theta_t^i; \mathbf{1}_{\theta_t^j}) &= E \left[ \sum_{s=0}^{T^i} \beta^s (R_{t+s}^i - R_{t+s}^j) \right] + E_{\theta_{T^i}^i} [\mathbf{1}_{\{\gamma^i(\theta_{T^i}^i) > \gamma^j(\theta_{T^i}^j)\}} U_{\bar{T}^i}^i(\vartheta_{T^i}^i(\theta_{T^i}^i), \theta_{T^i}^i; \lambda_{\theta_{T^i}^j})] \\ &\geq E \left[ \sum_{s=0}^{T^i} \beta^s (\gamma^i(\theta_t^i) - \gamma^j(\theta_t^j)) \right] + E_{\theta_{T^i}^i} [\lambda_{\{\gamma^i(\theta_{T^i}^i) > \gamma^j(\theta_{T^i}^j)\}} U_{\bar{T}^i}^i(\vartheta_{T^i}^i(\theta_{T^i}^i), \theta_{T^i}^i; \lambda_{\theta_{T^i}^j})]. \end{aligned}$$

Since  $\gamma^i(\theta_t^i) > \gamma^j(\theta_t^j)$ , the first term on the right-hand side is nonnegative, that is, until time  $T^i$  agent  $i$  obtains a nonnegative payoff (i.e., until he down-crosses  $\gamma^i(\theta_t^i)$  but is still above  $\gamma^j(\theta_t^j)$ ). The continuation payoff can again be written recursively as the sum of a nonnegative term and a continuation payoff. Since all the terms are

nonnegative, this results in

$$\hat{U}_{\bar{T}^i}^i(\vartheta_t^i(\theta_t^i), \theta_t^i; \lambda_{\theta_t^i}) \geq 0.$$

Thus, we see that the expected payoff up to time  $\bar{T}^i$  from winning an object by bidding honestly is non negative for agent  $i$ . Consider any alternate strategy  $\hat{\vartheta}_t^i(\cdot)$  where an agent deviates by announcing a type lower than his true type bidding his true value. The interesting case would be when he loses as a result, when announcing the true type would have resulted in a win. That is,

$$\gamma^i(\hat{\theta}_t^i) < \gamma^j(\theta_t^j) < \gamma^i(\theta_t^i).$$

This deviation would result in agent  $j$  winning the object for  $T^j$  periods. Then the expected payment that agent  $i$  makes when he wins the object at time  $t + T^j + 1$  is the same as the expected payment that he would make had he won at time  $t$ . This is because for any  $\theta_t^j$ ,

$$E \left[ E \left[ R_{t+T^j}^j \mid \theta_{t+T^j}^j \right] \mid \theta_t^j \right] = E \left[ R_{t+T^j}^j \mid \theta_t^j \right].$$

Thus, the expected payment that agent  $i$  would make if he won at time  $t + T^j + 1$  would be

$$E_{\theta_t^j} \left[ E \left[ E \left[ R_{t+T^j}^j \mid \theta_{t+T^j}^j \right] \mid \theta_t^j \right] \right] = E_{\theta_t^j} \left[ E \left[ R_{t+T^j}^j \mid \theta_t^j \right] \right].$$

Therefore, under strategy  $\hat{\vartheta}_t^i(\cdot)$ , the expected payment at time  $t + T^j$  is the same as the expected payment at time  $t$  under strategy  $\vartheta_t^i(\cdot)$ . As in the proof of Proposition 1, we fix the reward sample paths  $R^1$  and  $R^2$ . We assume that the reward of agent  $k \in \{1, 2\}$  after winning  $w$  times is  $r_w^k$ , where  $r_w^k = R_t^k$  if this event occurs at time  $t$ . In other words, we assume that independently of the particular time when agent  $k$

wins the object for the  $w$ -th time, the reward he obtains corresponds to the  $w$ -th element of the path  $\{r_1^k, r_2^k, \dots\}$ . Thus, the rewards and expected payments under either strategy,  $\vartheta_t^i(\cdot)$  or  $\tilde{\vartheta}_t^i(\cdot)$  are the same conditional on agent  $i$  winning. However, whereas under  $\vartheta_t^i(\cdot)$  agent  $i$  wins at time  $t$ , under  $\tilde{\vartheta}_t^i(\cdot)$  agent  $i$  wins at time  $t + T^j + 1$  and, as long as  $T^j > 0$  (which is the case if agent  $i$  loses at time  $t$ ), the nonnegative reward  $U_{T^j}$  is discounted under  $\tilde{\vartheta}_t^i(\cdot)$  by  $\beta^{T^j+1}$ , and the subsequent game remains identical, rendering the strategy  $\tilde{\vartheta}_t^i$  strictly worse than  $\vartheta_t^i(\cdot)$  for agent  $i$ , which completes our proof. ■

**Proof of Proposition 3.** The proof has been largely provided in the text. Here we give more detail regarding the stationarity of the solution to the system of first-order conditions (2.15). Indeed, given  $\vartheta_s^i(\theta_s^i) = \theta_s^i$  for all  $s > t$ ,  $\hat{\theta}_t^i = \vartheta_t^i(\theta_t^i)$ , and  $\vartheta_s^{-i}(\theta_s^{-i}) = \theta_s^{-i}$  for all  $s \geq t$ , agent  $i$ 's expected utility can be written non-recursively in the form

$$\begin{aligned} U_t^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta) &= E_{\theta_t^{-i}, R_t^i} \left[ \mathbf{1}_{\{\delta(\hat{\theta}_t^i, \theta_t^{-i})=i\}} \left( R_t^i - \tau_t^i(\hat{\theta}_t^i, \theta_t^{-i}; \eta_t^i) \right) \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta_t^i \right] \\ &\quad + \sum_{s=t+1}^{\infty} \beta^{s-t} E_{\theta_s^i, \theta_s^{-i}, R_s^i} \left[ \mathbf{1}_{\{\delta(\theta_s^i, \theta_s^{-i})=i\}} \left( R_s^i - \tau_s^i(\theta_s^i, \theta_s^{-i}; \eta_s^i) \right) \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta_s^i \right] \\ &= E_{\theta_t^{-i}, R_t^i} \left[ \mathbf{1}_{\{\delta(\hat{\theta}_t^i, \theta_t^{-i})=i\}} \left( R_t^i - \eta_t^i(\hat{\theta}_t^i) \tau_A^i(\hat{\theta}_t^i, \theta_t^{-i}) - (1 - \eta_t^i(\hat{\theta}_t^i)) \tau_B^i(\hat{\theta}_t^i, \theta_t^{-i}) \right) \middle| \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta_t^i \right] \\ &\quad + \sum_{s=t+1}^{\infty} \beta^{s-t} \int_{\Theta^n} \left[ \mathbf{1}_{\{\delta(\theta_s^i, \theta_s^{-i})=i\}} \left( E[R_s^i | \theta_s^i] - \eta_s^i(\theta_s^i) \tau_A^i(\theta_s^i, \theta_s^{-i}) \right. \right. \\ &\quad \left. \left. - (1 - \eta_s^i(\theta_s^i)) \tau_B^i(\theta_s^i, \theta_s^{-i}) \right) \right] dP(\theta_s^i, \theta_s^{-i} | \hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}), \end{aligned}$$

where  $P$  denotes the probability that at time  $s$  agent  $i$  hits state  $\theta_s^i$ , and the other agents hit state  $\theta_s^{-i}$ , given that agent  $i$  announces  $\hat{\theta}_t^i$  at time  $t$  and makes truthful future announcements. Since the per-period rewards  $R_t^i$  are i.i.d. by assumption, agent  $i$ 's expected utility does not explicitly depend on time, provided that the coefficients  $\eta_t^i$  are stationary. Furthermore, when differentiating the right-hand side of the last relation with respect to agent  $i$ 's announcement  $\hat{\theta}_t^i$  only the time derivative  $\dot{\eta}_t^i$  appears. More specifically, we obtain agent  $i$ 's necessary optimality conditions for

truth-telling in the form

$$\left[ \frac{\partial}{\partial \hat{\theta}_t^i} + \eta_t^i(\theta_t^i) \frac{\partial}{\partial \eta_t^i} \right]_{|\hat{\theta}_t^i = \theta_t^i, \eta_t^i = \eta_t^i(\hat{\theta}_t^i), \eta_{t+1}^i = \eta_{t+1}^i(\theta_{t+1}^i), \dots} U^i(\hat{\theta}_t^i, \theta_t^i; \pi_t^{-i}, \eta_t^i, \eta_{t+1}^i, \dots) = 0,$$

which allows for a stationary solution  $\eta_t^i \equiv \eta^i$ . The latter can be obtained by solving the linear ODE (2.17) after substituting the appropriate expressions introduced in the additive relation (2.13). The explicit solution (2.18) of this linear ODE with variable coefficients can be obtained by the standard variation-of-constants formula (Petrovski, 1966, p. 20), which concludes our proof. ■

**Proof of Lemma 2.** For simplicity we consider a situation with only two agents,  $i \in \{1, 2\}$  and  $j = 3 - i$ . The argument is completely analogous for the general case of  $n$  agents. Let us denote the Dirac density centered around a given type  $\theta \in \Theta$  by  $\lambda_\theta$ , i.e.,  $\lambda_\theta(\xi) = \lambda(\xi - \theta)$  for all  $\xi \in \Theta$ . Consider agent  $i$ . Since the other agent  $j$  follows by assumption the truthful strategy  $\vartheta_t^j(\theta_t^j) = \theta_t^j$ , we can write agent  $i$ 's expected optimal utility (at his optimal announcement) in the form of a Bellman equation,

$$\begin{aligned} \bar{U}^i(\theta; \pi_t^j) = \max_{\hat{\theta} \in \Theta} & \left\{ \int_{\hat{\theta}}^{\hat{\theta}} \left[ \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_t^j}) \right) p^i(R|\theta) dR - \tau^i(\theta_t^j; \eta^i(\hat{\theta})) \right] \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \right. \\ & \left. + \int_{\hat{\theta}}^{\hat{\theta}} \beta \bar{U}^i(\theta; \pi_{t+1}^j(\cdot; \theta_t^j)) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \right\}, \end{aligned} \quad (2.34)$$

where

$$\tau^i(\theta_t^j; \eta^i(\hat{\theta})) = \eta^i(\hat{\theta}) \gamma^i(\theta_t^j) + (1 - \eta^i(\hat{\theta})) E[R_t^j | \theta_t^j]$$

represents the transfer from agent  $i$  to the principal, which is an affine combination (contingent on agent  $i$ 's announcement  $\hat{\theta} \in \Theta$ ) of the corresponding transfers under the bounding mechanisms  $\mathcal{M}_A$  and  $\mathcal{M}_B$  (cf. relation (2.16)). As usual, we denote agent  $i$ 's optimal announcement  $\hat{\theta}$  at time  $t$  by  $\hat{\theta}_t^i$ . In case agent  $i$  won the object at time  $t - 1$ , it is  $\pi_t^j = \lambda_{\theta_{t-1}^j}$  and the above expression for  $\bar{U}^i$  specializes to

$$\bar{U}^i(\theta; \lambda_{\theta_{t-1}^j}) = \max \left\{ \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_{t-1}^j}) \right) p^i(R|\theta) dR - \tau^i(\theta_{t-1}^j; \eta^i(\hat{\theta}_t^i)), \beta \bar{U}^i(\theta; \pi_{t+1}^j) \right\},$$

where  $\pi_{t+1}^j = \pi_{t+1}^j(\cdot; \theta_{t-1}^j)$  represents agent  $i$ 's updated beliefs about agent  $j$  at time  $t+1$  if agent  $j$  wins the object at time  $t$ . If, given any (possibly diffuse) prior  $\pi_t^j$  over agent  $j$ 's types,  $\bar{U}^i(\cdot; \pi_t^j)$  is strictly decreasing on a nonempty open subset of  $\Theta$ , then there necessarily exists a type  $\theta_t^j \in \text{int } \Theta$  such that  $\bar{U}^i(\cdot; \lambda_{\theta_t^j})$  is also strictly decreasing on a nonempty open subset  $(m, M)$  of  $\Theta$  (with  $m < M$ ). In other words, if with some imprecise knowledge about the other agent's type agent  $i$ 's expected utility is strictly decreasing, then *a fortiori* it is also strictly decreasing from *some* particular realizations of the other agent's type. To obtain a contradiction to the claim that  $\bar{U}^i(\cdot; \pi_t^j)$  is strictly decreasing somewhere on  $\Theta$ , it is therefore sufficient to examine the case when agent  $i$  has won the object at time  $t-1$  and he consequently knows agent  $j$ 's type (for the principal, by assumption, makes all type announcements public after every round).

After winning a round, agent  $i$  will, as a function of his type announcement  $\hat{\theta}$ , know exactly whether he wins the object or not. If his optimal announcement  $\hat{\theta}_t^i$  is such that he wins the object at time  $t$ , then

$$\bar{U}^i(\theta; \lambda_{\theta_{t-1}^j}) = \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_{t-1}^j}) \right) p^i(R|\theta) dR - \tau^i(\theta_{t-1}^j; \eta^i(\hat{\theta}_t^i)).$$

As in the proof of Lemma 1, we assume that the interval  $(m, M)$  contains the point where the slope of  $\bar{U}^i(\cdot; \lambda_{\theta_t^j})$  is the most negative. In addition, we choose the type  $\theta_t^j$  in the compact set  $\Theta$  such that  $K^i < 0$  represents the smallest slope of  $\bar{U}^i$ . Given that type (which we can omit in the notation), it is

$$\bar{U}^i(\theta) = \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R)) \right) p^i(R|\theta) dR - \kappa,$$

where  $\kappa = \tau^i(\theta_{t-1}^j; \eta^i(\hat{\theta}_t^i))$  is an appropriate constant. As a consequence of assumptions A1 and A2 we can show, using a technique exactly as in the proof of Lemma 1

(used there to establish the monotonicity of  $G^i$ ), that  $\bar{U}^i(\theta)$  cannot be decreasing. On the other hand, if agent  $i$ 's optimal announcement  $\hat{\theta}_t^i$  is such that he does not win the object at time  $t$ , then agent  $i$ 's per-period reward is zero and  $\bar{U}^i$  can be expressed in the form of (2.34) with  $\pi_t^j$  replaced by  $\pi_{t+1}^j$ . By recursion, we see that the monotonicity of the stationary  $\bar{U}^i$  can be influenced only by the per-period rewards in winning rounds, the utility for which is nondecreasing in type as a consequence of Assumption A1. Hence, the function  $\bar{U}^i(\cdot; \pi_t^j)$  is increasing, which concludes our proof. ■

**Proof of Lemma 3.** For simplicity we consider a situation with two agents,  $i \in \{1, 2\}$  and  $j = 3 - i$ . The generalization of our arguments to the general case of  $n > 2$  agents is straightforward. The proof proceeds in three steps.

*Step 1: the function  $p^i(R|\theta)$  is Lipschitz in  $\theta$  for any  $R \in \mathcal{R}$ .* From (2.3) we know that the stationary reward density, conditional on agent  $i$ 's type, is given by

$$p^i(R|\theta) = \int_{\mathcal{A}} \sigma(R|\alpha) f^i(\alpha|\theta) d\alpha$$

for all  $(R, \theta) \in \mathcal{R} \times \Theta$ . Hence, for any  $R \in \mathcal{R}$  and any  $\theta, \tilde{\theta} \in \Theta$  we have that

$$\left| p^i(R|\theta) - p^i(R|\tilde{\theta}) \right| \leq \int_{\mathcal{A}} \sigma(R|\alpha) \left| f^i(\alpha|\theta) - f^i(\alpha|\tilde{\theta}) \right| d\alpha \leq L^i |\theta - \tilde{\theta}|,$$

where (with  $f_2^i(\alpha|\theta) = \partial f^i(\alpha|\theta) / \partial \theta$ )

$$L^i = \max_{R \in \mathcal{R}} \left\{ \int_{\mathcal{A}} \sigma(R|\alpha) \left( \sup_{\xi \in \Theta} f_2^i(\alpha|\xi) \right) d\alpha \right\}.$$

*Step 2: the function  $U^i(\hat{\theta}, \theta; \pi_t^{-i})$  is Lipschitz in  $(\hat{\theta}, \theta)$  for any given non-atomistic continuous beliefs  $\pi_t^j$ .* Let agent  $i$ 's beliefs  $\pi_t^j$  about the other agent  $j$  at time  $t$  be fixed. Agent  $i$ 's expected utility (cf. relation (2.34) in the proof of Lemma 2) for

any  $\hat{\theta}, \theta \in \Theta$  is then

$$\begin{aligned}
 U^i(\hat{\theta}, \theta; \pi_t^j) &= \int_{\hat{\theta}}^{\bar{\theta}} \beta \bar{U}^i(\theta; \pi_{t+1}^j(\cdot; \theta_t^j)) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j, \\
 &+ \int_{\theta}^{\hat{\theta}} \left[ \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_t^j}) \right) p^i(R|\theta) dR - \tau^i(\theta_t^j; \eta^i(\hat{\theta})) \right] \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j
 \end{aligned} \tag{2.35}$$

where  $\bar{U}^i$  corresponds to agent  $i$ 's resulting value function (after maximization with respect to his announcement). The latter value function is, as a result of the uniform boundedness of the reward process  $\{R_t^i\}_{t \in \mathbb{N}}$ , bounded by a constant,  $Q$ , so that

$$0 \leq \bar{U}^i \leq \frac{\max\{|R|, |\bar{R}|\}}{1 - \beta} \equiv Q < \infty.$$

The first inequality in the last relation stems from our basic presumption that all agents are free to participate in the principal's mechanism. As a result, for any  $\hat{\theta}, \hat{\theta}, \theta \in \Theta$ :

$$|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \theta; \pi_t^j)| \leq \int_{\hat{\theta}}^{\bar{\theta}} ((1 - \beta)Q + 2\beta Q) \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j.$$

Since  $\pi_t^j$  does not contain atoms and is continuous by hypothesis, its maximum value on  $\Theta$  exists. Hence,

$$|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \theta; \pi_t^j)| \leq M_1 |\hat{\theta} - \hat{\theta}|,$$

where  $M_1 = \max\{|R|, |\bar{R}|\} \max_{(\theta_t^j, \theta_{t-1}^j) \in \Theta^2} \{\pi_t^j(\theta_t^j; \theta_{t-1}^j)\}$ , suppressing the dependence



on time  $t$ . On the other hand, for any  $\hat{\theta}, \theta, \tilde{\theta} \in \Theta$  it is

$$\begin{aligned} |U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j)| &\leq \int_{\theta}^{\hat{\theta}} \left[ \int_{\mathcal{R}} Q \left| p^i(R|\theta) - p^i(R|\tilde{\theta}) \right| dR \right] \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\ &\leq \int_{\theta}^{\hat{\theta}} \left[ \int_{\mathcal{R}} Q L^i |\theta - \tilde{\theta}| dR \right] \pi_t^j(\theta_t^j; \theta_{t-1}^j) d\theta_t^j \\ &\leq \left( \frac{(\tilde{\theta} - \theta)(\bar{R} - R) M_1 L^i}{1 - \beta} \right) |\theta - \tilde{\theta}| \equiv M_2 |\theta - \tilde{\theta}|, \end{aligned}$$

where we have used the result of Step 1. We have therefore shown that

$$|U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j)| \leq M_1 |\hat{\theta} - \tilde{\theta}| + M_2 |\theta - \tilde{\theta}|$$

for all  $\hat{\theta}, \tilde{\theta}, \theta, \tilde{\theta} \in \Theta$ , as long as  $\pi_t^j$  is bounded (i.e., in particular does not contain any atoms).

*Step 3: the function  $\bar{U}^i(\theta; \pi_t^j)$  is Lipschitz in  $\theta$  for any given non-atomistic continuous beliefs  $\pi_t^j$ .* By the definition (2.20) of agent  $i$ 's value function, it is

$$\bar{U}^i(\theta; \pi_t^j) = \max_{\hat{\theta} \in \Theta} U^i(\hat{\theta}, \theta; \pi_t^j)$$

for any  $\theta \in \Theta$ . Let  $\varepsilon > 0$  and  $\theta, \tilde{\theta} \in \Theta$  be fixed and consider any  $\hat{\theta}, \tilde{\theta} \in \Theta$  such that

$$\bar{U}^i(\theta; \pi_t^j) \leq U^i(\hat{\theta}, \theta; \pi_t^j) + \varepsilon \quad \text{and} \quad |\hat{\theta} - \tilde{\theta}| \leq \varepsilon \leq c|\theta - \tilde{\theta}| + \varepsilon,$$

for an appropriate constant  $c > 0$ . Hence, since  $U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j) \leq \bar{U}^i(\tilde{\theta}; \pi_t^j)$ , we have with the result proved in the previous step that

$$\begin{aligned} |\bar{U}^i(\theta; \pi_t^j) - \bar{U}^i(\tilde{\theta}; \pi_t^j)| &\leq |U^i(\hat{\theta}, \theta; \pi_t^j) - U^i(\hat{\theta}, \tilde{\theta}; \pi_t^j)| + \varepsilon, \\ &\leq M_1 |\hat{\theta} - \tilde{\theta}| + M_2 |\theta - \tilde{\theta}| + \varepsilon \\ &\leq K |\theta - \tilde{\theta}| + (K + 1)\varepsilon, \end{aligned}$$

where  $K = \max\{M_1, M_2\}(c + 1)$ . The claim then follows by letting  $\varepsilon \rightarrow 0+$ . ■

**Proof of Proposition 4.** For simplicity we consider a situation with only two agents,  $i \in \{1, 2\}$  and  $j = 3 - i$ . If  $n > 2$ , the arguments presented below go through without any significant changes. In case agent  $i$  wins the object at time  $t$ , he knows agent  $j$ 's type  $\theta_t^j = \theta_{t-1}^j$  from the principal's announcement in the last round. For any given  $\theta_t^j \in \Theta$  we can define agent  $i$ 's averaged state update  $\bar{\varphi}^i(\theta_t^i, \theta_t^j)$  implicitly through the relation

$$\int_{\mathcal{R}} \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_t^j}) p^i(R|\theta) dR = \bar{U}^i(\bar{\varphi}^i(\theta, \theta_{t-1}^j); \lambda_{\theta_t^j}),$$

where (as in the proof of Lemma 2)

$$\begin{aligned} & \bar{U}^i(\theta; \lambda_{\theta_t^j}) \\ &= \max \left\{ \int_{\mathcal{R}} \left( R + \beta \bar{U}^i(\varphi^i(\theta, R); \lambda_{\theta_t^j}) \right) p^i(R|\theta) dR - \tau^i(\theta_t^j; \eta^i(\hat{\theta}_t^i)), \beta \bar{U}^i(\theta; \pi_{t+1}^j) \right\}. \end{aligned} \quad (2.36)$$

By reward log-supermodularity (Assumption A1) and state-transition monotonicity (Assumption A2) we obtain that agent  $i$ 's averaged state update  $\bar{\varphi}^i(\cdot; \theta_t^j)$  is nondecreasing, independent of the other agent's type  $\theta_t^j$ . We now show that, as long as the agents' discount rate  $\beta \in (0, 1)$  is sufficiently small, agent  $i$ 's utility  $U^i(\hat{\theta}, \theta; \pi_t^j)$  is supermodular in  $(\hat{\theta}, \theta)$ , which implies (together with local BIC) global Bayesian incentive compatibility. By differentiating  $U^i(\hat{\theta}, \theta; \pi_t^j)$  in (2.35) with respect to agent  $i$ 's announcement  $\hat{\theta}$  we obtain that

$$U_1^i(\hat{\theta}, \theta; \pi_t^j) = \left( E[R_t^i|\theta] + \beta \left( \bar{U}^i(\bar{\varphi}^i(\theta, \hat{\theta}); \lambda_{\hat{\theta}}) - \bar{U}^i(\theta; \pi_{t+1}^j(\cdot; \hat{\theta})) - \tau^i(\hat{\theta}; \eta^i(\hat{\theta})) \right) \right) \pi_t^j(\hat{\theta}; \theta_{t-1}^j),$$

where  $U_1^i(\hat{\theta}, \theta; \pi_t^j) = \partial U^i(\hat{\theta}, \theta; \pi_t^j) / \partial \hat{\theta}$ . Hence, for any  $\hat{\theta}, \theta, \tilde{\theta} \in \Theta$  with  $\theta > \tilde{\theta}$  it is

$$U_1^i(\hat{\theta}, \theta; \pi_t^j) - U_1^i(\hat{\theta}, \tilde{\theta}; \pi_t^j) \geq 0$$

if and only if

$$E[R_t^i | \theta] - E[R_t^i | \tilde{\theta}] + \beta \left( \left[ \bar{U}^i(\bar{\varphi}^i(\theta, \hat{\theta}); \lambda_{\hat{\theta}}) - \bar{U}^i(\bar{\varphi}^i(\tilde{\theta}, \hat{\theta}); \lambda_{\hat{\theta}}) \right] - \left[ \bar{U}^i(\theta; \pi_{t+1}^j) - \bar{U}^i(\tilde{\theta}; \pi_{t+1}^j) \right] \right) \geq 0,$$

where  $\pi_{t+1}^j = \pi_{t+1}^j(\cdot; \theta_{t-1}^j)$  represents agent  $i$ 's updated beliefs about agent  $j$  at time  $t + 1$  if agent  $j$  wins the object at time  $t$  (a continuous, nonnegative, and bounded function). By monotonicity of agent  $i$ 's value function  $\bar{U}^i(\cdot; \lambda_{\hat{\theta}})$  (established in Lemma 2) and the monotonicity of the averaged state update  $\bar{\varphi}^i(\cdot, \hat{\theta})$  established above, it is  $\bar{U}^i(\theta; \pi_{t+1}^j) - \bar{U}^i(\tilde{\theta}; \pi_{t+1}^j) \geq 0$ . In addition, we have that by hypothesis the reward type-parametrization is 'strong' in the sense that  $E[R_t^i | \theta] - E[R_t^i | \tilde{\theta}] \geq \rho(\theta - \tilde{\theta})$  for some  $\rho > 0$ . Lastly, by Lemma 3, it is  $\bar{U}^i(\theta; \pi_{t+1}^j) - \bar{U}^i(\tilde{\theta}; \pi_{t+1}^j) \leq K(\theta - \tilde{\theta})$  for some constant  $K > 0$ , so that supermodularity of  $U^i$  obtains if

$$(\rho - \beta K)(\theta - \tilde{\theta}) \geq 0,$$

or, in other words, if

$$\beta \leq \frac{\rho}{K} \equiv \beta_0 \in (0, 1),$$

which completes our proof. ■

## Chapter 3

# Efficient Allocation of Divisible Goods

### 3.1 Introduction

An important deficiency in the Vickrey Clarke Groves (VCG) mechanism when dealing with divisible goods or multiple units of a homogeneous good, is that although it is efficient, the payments may not lie in the core of the game and might induce the seller to choose an inefficient allocation if she chooses to maximize revenues subsequent to bidders submitting their valuations. Our mechanism alleviates this problem by making the seller's revenue maximization problem choose the efficient allocation. We provide a general mechanism for the allocation of a finite number of divisible resources or multiple units of a homogenous good, to a finite number of heterogeneous agents. The mechanism is ex-post efficient (subject to participation) and dominant-strategy incentive compatible. Agents can have multidimensional pieces of private information that may be correlated. The mechanism is robust in the sense that equilibrium bid functions do not depend on any assumptions about other agents other

than smoothness of their otherwise arbitrary payoff functions.

The way we construct our mechanism, once the principal commits to using the suggested payment scheme, the best that she can do in terms of maximizing revenues after the agents announce their types is to allocate the resource efficiently.

Unlike the VCG mechanism, transfers depend on posted price schedules and discounts, which are given based on the average bid function of the other agents evaluated at the fraction of the resource not allocated to them. Our mechanism being a posted price mechanism, does suffer from the drawback that if the initial price posted is too high, then a fraction of the type space might prefer not to participate in the mechanism. The allocation made subsequently is efficient, but only amongst the participating agents.

The posted price is computed by the principal by maximizing her expected rewards knowing that agent participation is contingent on the posted price and expected rebates. We must note that our mechanism need not satisfy budget balance. Since it is well known (Green and Laffont, 1977) that there cannot exist allocation mechanisms (in the presence of asymmetric information) that at the same time are ex-post efficient, dominant-strategy incentive compatible, and balance the budget, our mechanism is in general not budget-balancing. We discuss the impact of this on the principals equilibrium revenues.

Since in our mechanism, the bid functions are a simple linear transformation of the value function of the agents, through arguments similar to the revelation principle, one can establish that truthtelling is optimal in dominant strategies. This makes our mechanism robust to the type space that the other agents' types are drawn from and the beliefs of the agents about each other. To elaborate, the mechanism is robust in the sense that equilibrium-bidding behavior does not depend heavily on common-knowledge assumptions and implementation is therefore be in dominant strategies and lead to truthful bidding behavior. Because of the simple equilibrium bid structure, it

is robust in the sense that it is easy to implement and its complexity does not increase substantially by adding more real-world features.

In what follows we first introduce the primitives of our model before we introduce our general mechanism and examine its equilibrium properties. We then turn to the question of budget balance and examine the principal's revenue generation possibilities using our mechanism. A number of simple examples are discussed to provide intuition for our results. We then extend the model to the case of a divisible asset auction, where the payoffs of the asset depend on effort exerted by the agents. This makes our problem a two stage problem where we run an auction in the first stage and face a moral hazard problem in the second. The efficient auction must take into account the consequences of the second stage. We conclude with a discussion of the contributions and limitations of the results, and directions for further research.

## 3.2 The Model

We consider  $N$  heterogeneous agents that have nonnegative valuations for fractional allocations of  $K$  available goods provided by a principal. The fractional allocation  $x_i^k$  of good  $k$  to agent  $i \in \{1, \dots, N\}$  is nonnegative (with  $N \geq 2$ ) and each good is fully assigned, i.e.,

$$\sum_{i=1}^N x_i^k = 1 \quad (3.1)$$

for  $k \in \{1, \dots, K\}$ . Agent  $i$ 's utility for his allocation  $x_i = (x_i^1, \dots, x_i^K)$  is denoted by  $v_i(x_i; \theta_i)$  where  $\theta_i \in \Theta_i$  represents agent  $i$ 's private information, or *type* as an element of its type space  $\Theta_i$  which is a bounded measurable subset of some finite dimensional Euclidean space. The function  $v_i : [0, 1]^K \times \Theta_i \rightarrow \mathbb{R}$  is assumed to be smooth and increasing in its first argument. We assume that the agents' type vector  $\theta = (\theta_1, \dots, \theta_N)$  is distributed with the joint cumulative distribution function

$F : \Theta \rightarrow [0, 1]$ , where  $\Theta = \Theta_1 \times \dots \times \Theta_N$  is the joint type space. Before the beginning of the game each agent  $i$  observes its type  $\theta_i$  privately; his beliefs about the other agents' types  $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_N)$  are thus distributed according to the conditional distribution function  $F(\theta_{-i}|\theta_i)$ , which is obtained via Bayesian updating. The sequence of events is as follows:

Stage 1: The principal announces and commits to her mechanism, which includes a fixed posted  $K$ -dimensional price vector  $p = (p^1, \dots, p^K)$ , the components of which correspond to the different available resources.

Stage 2: Each agent  $i$ , conditional on having observed his type  $\theta_i$ , then submits to the principal a real-valued bid function  $b_i(\cdot; \theta_i)$ , defined on  $[0, 1]^K$ .

Stage 3: After all bid functions have been collected the principal determines the agents' allocation matrix  $X = [x_i^k]_{i,k=1}^{N,K}$  as well as their transfer payments.

A peculiar feature of our mechanism is that the agents' bids actually act as discounts that will be subtracted from the payment implied by the posted price schedule. Each agent  $i$  makes a monetary transfer of  $t_i$  to the principal which depends on  $X$  and all other agents' bid functions. The precise dependence of the fractional allocations and monetary transfers on the agents' bid functions and the principal's payoffs is specified by her allocation mechanism.

### 3.3 The Mechanism

The principal's mechanism  $\mathcal{M}$  maps the agents' bid-function vector  $b = (b_1, \dots, b_N)$  to an allocation matrix  $X$  and a corresponding vector of monetary transfers  $t = (t_1, \dots, t_N)$ , such that

$$t_i(b, X) = p \cdot x_i - \frac{1}{N-1} \sum_{j \neq i} b_j(\bar{x}_{-j}; \theta_j) \quad (3.2)$$

### 3.3 The Mechanism

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where  $\bar{x}_{-i} = x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_N$  is the vector containing the respective sums of fractional good-allocations to agents other than agent  $i$ . Note that an average over the other agent's bid defines the discount that agent  $i$  obtains off the total payment  $p \cdot x_i = \sum_{k=1}^K p^k x_i^k$  which is implied by the posted price vector  $p$ . Each agent  $j$ 's bid function is evaluated not only at  $i$ 's allocation, but at the sum of all allocations other than  $j$ 's (which includes  $i$ 's allocation). The allocation matrix solves the principal's profit maximization problem,

$$X \in \arg \max_{\hat{X} = [\hat{x}_i^k] \in \Delta} \left\{ \sum_{i=1}^N t_i(b, \hat{X}) \right\} \quad (3.3)$$

where  $\Delta = \left\{ [x_i^k]_{i,k=1}^{N,K} : x_i^k \in [0, 1] \text{ and } \sum_{j=1}^N x_j^k = 1 \ \forall i, k \right\}$  describes the simplex of admissible allocation matrices where  $\mathbf{1}$  is a vector of ones. The principal, by maximizing her profits, trades off discounts versus revenues, subject to the constraints on the availability of resources. The first-order necessary optimality conditions for the principal's problem are

$$-\frac{1}{N-1} \sum_{i \neq l} \sum_{j \neq i} b_j'(\bar{x}_{-j}; \theta_j) = \lambda - \mu_l \quad (3.4)$$

for all  $l \in \{1, \dots, N\}$  where  $b_i'(\cdot; \theta_i)$  is the  $K$ -dimensional gradient of agent  $i$ 's bid function.  $\mu_l, \lambda \in \mathbb{R}^K$  are the  $K$ -dimensional Lagrange multipliers corresponding to the feasibility constraint  $x_l \geq 0$  and  $\bar{x} = 1$  respectively, where  $\bar{x} = x_1 + \dots + x_N$  is the sum of all allocation vectors. In addition, we have the complementary slackness conditions

$$\mu_l^k x_l^k = \lambda^k (1 - \bar{x}^k) = 0 \quad (3.5)$$



for all  $k \in \{1, \dots, K\}$ . The system of  $(2N + 1)K$  equations (3.4) and (3.5) implies an optimal allocation matrix  $X$ . Note that since  $\Delta$  is compact, by the Weierstrass theorem such an optimal solution exists as long as all bid functions are continuous. We show below (Proposition 1) that the equilibrium bid functions inherit all the smoothness properties of the agents' payoff functions, so that the existence of a solution to the intermediary's allocation problem is always guaranteed.

In order to determine their optimal bid functions, agents maximize their payoffs, given the principal's mechanism  $\mathcal{M}$  which, via the system (3.4) and (3.5), determines the allocation matrix  $X$ . Thus, in equilibrium, each agent  $i$ 's type  $\theta_i$  is mapped to an equilibrium bid function  $b_i(\cdot; \theta_i)$ . Given this mapping, it is therefore the realization of the type vector  $\theta$ , which determines the precise allocation matrix

$$X = G(\theta; b) = \begin{bmatrix} g_1(\theta; b) \\ \vdots \\ g_N(\theta; b) \end{bmatrix} \quad (3.6)$$

where  $G$  is a matrix-valued mapping from  $\Theta$  to  $\Delta$ . The vector-valued functions  $g_i$  describe agent  $i$ 's allocation  $x_i = g_i(\theta)$  for any given  $\theta \in \Theta$ . With this in mind, we are now able to formulate agent  $i$ 's bidding problem as follows:

$$\sup_{b_i \in L_\infty([0,1]^K, \mathbb{R}_+)} \left\{ \int_{\Theta_{-i}} [v_i(g_i(\theta_i, \theta_{-i}; b_i, b_{-i}); \theta_i) - p \cdot g_i(\theta_i, \theta_{-i}; b_i, b_{-i}) + \frac{1}{N-1} \sum_{j \neq i} b_j(\bar{g}_i(\theta_i, \theta_{-i}; b_i, b_{-i}); \theta_j) dF(\theta_{-i} | \theta_i)] \right\} \quad (3.7)$$

where, consistent with earlier notation,  $\bar{g}_{-j} = g_1 + \dots + g_{j-1} + g_{j+1} + \dots + g_N$ . This is an unconstrained variational problem, a solution to which can be found simply through pointwise maximization of the integrand with respect to  $g_i$  for any given  $\theta \in \Theta$ , which in turn determines agent  $i$ 's equilibrium allocation  $x_i$  and thus its

optimal bid  $b_i(x_i; \theta_i)$ .

**Proposition 1** *Given the principal's mechanism  $\mathcal{M}$ , the agents' bidding equilibrium is in dominant strategies. For any  $i \in \{1, \dots, N\}$  agent  $i$ 's equilibrium bid function is given by*

$$b_i(x_i; \theta_i) = v_i(\mathbf{1}; \theta_i) - v_i(\mathbf{1} - x_i; \theta_i) \geq 0 \quad (3.8)$$

for all  $x_i \in [0, 1]^K$  and all  $\theta_i \in \Theta_i$

Each agent  $i$ 's equilibrium bid function in Proposition 1 only depends on agent  $i$ 's gross payoff function  $v_i(\cdot; \theta_i)$  which is naturally part of agent  $i$ 's private information. In particular, it is independent of the distributional assumptions about the other agents's types. The intuition for this result is as follows. Provided truthful bidding, the principal, who collects the bid functions from all the agents, can make a proper allocation decision which reflects the full realization of the type vector  $\theta \in \Theta$ . Thus, anticipating this, each agent  $i$  can now use the principal's optimality conditions as a way of inferring system-wide marginal cost corresponding to the Lagrange multipliers associated with common constraints (such as  $x_1 + \dots + x_N = \mathbf{1}$ ) from its own gross payoff function. Note that for this to work, it is crucial that agent  $i$ 's payoffs are independent (at least before the principal optimizes) of its allocation and bidding function. In addition, the mechanism  $\mathcal{M}$  provides nice aggregation properties; for any feasible allocation matrix  $X \in \Delta$ , it needs to be true that  $b_j(\bar{x}_{-j}; \theta_j) = b_j(\mathbf{1} - x_j; \theta_j)$ , which, if the bidding function in Proposition 1 is substituted, provides incentives for the principal to maximize the sum of all agents' payoffs and this implement an ex-post efficient allocation of the goods. Let  $W(X, \theta) = \sum_{i=1}^N v_i(x_i; \theta_i)$ . Then, the following result gives a more precise meaning to this intuitive argument

**Proposition 2** *For any  $\theta \in \Theta$ , the principal's equilibrium payoff is given by  $W(X, \theta) - (W(\mathbf{1}, \theta) - \bar{p})$ , where  $\bar{p} = p^1 + \dots + p^K$  and  $\mathbf{1}$  is an  $N \times K$  matrix of ones.*

Since  $W(\underline{1}; \theta)$  is not attainable by any feasible allocation (unless all firms but one have gross payoffs that are identically equal to zero), it becomes clear that the principal is unable to extract all the surplus from the agent. In fact, as becomes clear later, by increasing the posted price, participation in the principal's mechanism will drop (eventually to zero for  $p$  large enough). Hence there exists an interior optimal posted price that maximizes the principal's expected revenue. Clearly this revenue maximization would need to depend on the principal's own beliefs about the distribution of the agents' types.

**Proposition 3** *Given any type vector  $\theta \in \Theta$ , under mechanism  $\mathcal{M}$  any positive equilibrium allocation matrix  $X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$  is determined by  $v_i'(x_i; \theta_i) = v_1'(x_1; \theta_1)$  for all  $i \in \{2, \dots, N\}$  and  $x_1 + \dots + x_N = \underline{1}$ .*

The reason for restricting the scope of Proposition 3 to equilibrium allocation matrices with positive entries, i.e., where no agent receives zero payoffs, is that then there is only a single equality constraint in the entire equilibrium problem, the Lagrange multiplier of which can be eliminated by setting the respective agent's marginal payoffs at the optimal allocation equal. This implies efficiency.

**Corollary 1** *Under mechanism  $\mathcal{M}$  any positive equilibrium allocation matrix  $X$  is efficient.*

A sufficient condition for interiority (i.e., positivity) of the equilibrium allocation matrix is that the marginal change of the agents' gross payoff functions is sufficiently close to zero allocations. If the slopes of all agents' payoffs remain finite within the set of possible allocations  $[0, 1]^K$ , then this is also sufficient. The following corollary of Proposition 3 uses a slightly stronger requirement as a sufficient condition for a somewhat stronger result.

**Corollary 2** *Let  $\theta \in \Theta$  and assume that the agents' payoff functions  $v_1(\cdot; \theta_1), \dots, v_N(\cdot; \theta_N)$  are concave in the fractional allocations they receive. If every entry of  $v_i'(0; \theta_i) > 0$  is large enough (e.g., ensured by the Inada-type conditions  $\frac{\partial v_i}{\partial x_i^k} \Big|_{(0, \theta_i)} = \infty$ ,  $k \in \{1, \dots, K\}$ ) for all  $i \in \{1, \dots, N\}$  then there exists a unique positive equilibrium allocation matrix  $X = G(\theta)$ .*

### 3.4 Revenue Maximization

From the results in the last section we conclude that mechanism  $\mathcal{M}$  is dominant-strategy incentive compatible and ex-post efficient. Clearly, this implies that it cannot at the same time be budget balanced and individually rational. Thus, the principal faces a choice: (i) to sufficiently subsidize the mechanism, ensuring that the agents always participate (“social objective”), i.e., set the posted price vector  $p$  to zero; or (ii) optimize the posted price vector  $p$  so as to maximize revenues (“capital objective”). Before discussing these two contrary objectives, we first need to understand exactly how the agents' participation decision can be characterized. The mechanism  $\mathcal{M}$  is ex-post individually rational if and only if

$$v_i(g_i(\theta); \theta_i) - p_i \cdot g_i(\theta) + \frac{1}{N-1} [(W(\mathbf{1}; \theta) - W(G(\theta); \theta)) - (v_i(\mathbf{1}; \theta_i) - v_i(g_i(\theta); \theta_i))] \geq 0 \quad (3.9)$$

for all  $\theta \in \Theta$  and all  $i \in \{1, \dots, N\}$ . This condition follows directly from ensuring nonnegativity of the maximand in agent  $i$ 's optimal bidding problem.

**Social Objective:** by charging a zero posted price ( $p = 0$ ), the principal can ensure full participation: any agent obtains at least zero and is therefore at least indifferent between participating or not. However, since the principal initially commits to the mechanism, its discounts to some agents may exceed their payments, so that the principal ends up out of the money. Depending on the principal's beliefs about

the distribution of the agents' types, this implies a lowest possible price that the principal would be able to charge in order for the mechanism  $\mathcal{M}$  to be ex-ante willing to run it. This would exclude some types from participating, but the mechanism still implements an efficient outcome.

**Capital Objective:** the principal's choice of the price vector  $p$  determines the types of agents that would find it ex-post individually rational to participate in the auction. The principal anticipates this and chooses a  $\bar{p}$  to maximize its expected total transfer conditional on the fact that the set of participating types ("participation set") is potentially a strict subset of the type space. More specifically, for any  $p \in \mathbb{R}^K$ , the participation set is given by

$$\Theta_0(p) = \{\theta \in \Theta : v_i(g_i(\theta; p); \theta_i) - t_i(b, G(\theta; p); \theta) \geq 0, i \in \{1, \dots, N\}\} \quad (3.10)$$

Then the price that the principal announces is given by

$$\bar{p} \in \arg \max_{\hat{p} \in \mathbb{R}^K} \int_{\Theta_0(p)} [W(X; \theta, \hat{p}) - (W(\mathbf{1}; \theta, \hat{p}) - \hat{p})] dF(\theta) \quad (3.11)$$

This price always exists since the principal's objective function is continuous.

**REMARK (RELATION TO ROBERTS' THEOREM):** Roberts' theorem, discussed in Chapter 1 gives a general characterization of truthfully implementable social choice functions. Given that the mechanism we construct induces truthful revelation, it is important to understand our mechanism in its context. For a finite set of alternatives  $A$ , the Roberts' theorem says that, if the domain of preferences is unrestricted, then the only truthfully implementable outcomes are ones that maximize the weighted social surplus. The domain of preferences is said to be unrestricted if it is equal to  $\mathbb{R}^{|A|}$  for all players.

As Lavi, Mu'alem and Nisan (2003) show, when the domain of preferences are not unrestricted (with multi-unit or divisible good auctions being two examples)

then outcomes besides weighted social surplus maximizers can be implemented. For example, in the case of a single dimensional value function (scalar value for the entire good) a mechanism that allocates the good to the agent with the highest value of  $(v_i)^i$  and charges  $(v_j)^{j/i}$  where  $j$  is the agent with the second highest value of  $(v_j)^j$  is also truthful.

For multi-unit auctions with a finite number of resources (as is the case in our auction), the domain of preferences is restricted enough for there to be mechanisms other than weighted surplus maximizers that implement outcomes truthfully. Just as the revenue-maximizing reserve-price second-price auction is not really a Roberts' mechanism even though the second-price is, ours cannot be characterized as a Roberts' mechanism even though the VCG mechanism for divisible goods is.

## 3.5 Examples

**Example 1** We now discuss a simple example explicitly to illustrate the mechanism introduced and discussed in the previous sections. This shows that the mechanism performs well, even when boundary allocations (where some agent obtains nothing of the good) are efficient. Consider two agents  $i, j \in \{1, 2\}$  with  $i \neq j$  and respective valuations of  $v_i(x_i; \theta_i) = \theta_i(x_i)^\alpha$  and  $v_j(x_j; \theta_j) = \theta_j(x_j)^\alpha$  for some  $\theta_i, \theta_j \in [0, 1]$  and some constant  $\alpha \geq 0$ . The principal solves

$$\max_{X \in \Delta} \{t_i(b, X) + t_j(b, X)\} = \max_{X \in \Delta} \{p - b_j(x_i; \theta_j) - b_i(x_j; \theta_i)\} \quad (3.12)$$

where  $X = \begin{bmatrix} x_i \\ x_j \end{bmatrix}$ . Firm  $i$ 's problem is

$$\max_{g_i \in [0,1]} \left\{ \theta_i (g_i)^\alpha - t_i \left( b, \begin{bmatrix} g_i \\ 1 - g_j \end{bmatrix} \right) \right\} = \max_{g_i \in [0,1]} \{ \theta_i (g_i)^\alpha - p g_i + b_j (g_i; \theta_j) \} \quad (3.13)$$

and using Proposition 1, agent  $i$ 's equilibrium bid-function becomes

$$b_i(x_i; \theta_i) = v_i(\mathbf{1}, \theta_i) - v_i(\mathbf{1} - x_i; \theta_i) = \theta_i(1 - (1 - x_i)^\alpha) \geq 0 \quad (3.14)$$

for all  $x_i \in [0, 1]$  and all  $\theta_i \in [0, 1]$ . Substituting the bid functions, together with the relation  $x_j = 1 - x_i$  in (3.12) the principal's problem can be rewritten in the form

$$\max_{x_i \in [0,1]} \{ p - b_j(x_i; \theta_j) - b_i(\mathbf{1} - x_i; \theta_i) \} = \max_{x_i \in [0,1]} \{ p - \theta_j(1 - (1 - x_i)^\alpha) - \theta_i(1 - x_i^\alpha) \} \quad (3.15)$$

For the rest of the analysis it is useful to distinguish two cases, depending on whether the agents' payoffs are convex (i.e.,  $\alpha \geq 1$ ) with nondecreasing returns, or concave (i.e.,  $\alpha \in (0, 1)$ ) with decreasing returns. The reason is that in the former situation, boundary allocations are efficient, while in the latter it is socially optimal to implement an interior solution where all agent obtain positive shares of the good.

**Case 1:  $\alpha \geq 1$ ; Convex Valuations (*Nondecreasing Returns*)**

The maximand in (3.15) is convex in  $x_i$  so that the maximizer lies at the boundary of  $[0, 1]$ ,

$$x_i = g_i(\theta) = \begin{cases} 1, & \theta_i \geq \theta_j, \\ 0, & \theta_i < \theta_j. \end{cases} \quad (3.16)$$

Note that this is indeed an efficient allocation since

$$g_i(\theta) \in \arg \max_{x_i \in [0,1]} \{ v_i(x_i; \theta_i) + v_j(\mathbf{1} - x_i; \theta_j) \} \quad (3.17)$$

### 3.5 Examples

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as can easily be verified. To verify ex-post individual rationality for the agents, we have that

$$v_i(g_i(\theta); \theta_i) - pg_i(\theta) + \frac{v_j(1; \theta_j) - v_j(1 - g_i(\theta)); \theta_j}{N - 1} = \begin{cases} \theta_i + \theta_j - p & \theta_i \geq \theta_j \\ 0 & \theta_i \leq \theta_j \end{cases} \quad (3.18)$$

for  $i \in \{1, 2\}$ , or, equivalently, if  $p \leq \theta_i + \theta_j$ ,

**Case 2:**  $\alpha \in (0, 1)$ ; Concave Valuations (*Decreasing Returns*)

Problem (3.15) has a unique interior solution,

$$x_i = g_i(\theta) = \frac{\theta_j^{1/(1-\alpha)}}{\theta_i^{1/(1-\alpha)} + \theta_j^{1/(1-\alpha)}} \quad (3.19)$$

The mechanism  $\mathcal{M}$  is by (3.9) ex-post individually rational if and only if

$$v_i(g_i(\theta); \theta_i) - pg_i(\theta) + \frac{v_j(1; \theta_j) - v_j(1 - g_i(\theta); \theta_j)}{N - 1} = \theta_j - p \frac{\theta_j^{1/(1-\alpha)}}{\theta_i^{1/(1-\alpha)} + \theta_j^{1/(1-\alpha)}} \quad (3.20)$$

for all  $i \in \{1, 2\}$ , or equivalently, if  $p \leq \frac{\theta_i^{1/(1-\alpha)} + \theta_j^{1/(1-\alpha)}}{(\max\{\theta_i, \theta_j\})^{\alpha/(1-\alpha)}}$ . Let us now briefly discuss the principal's possibility to influence the surplus distribution by modifying the posted price  $p$ . For simplicity, we restrict attention to case 1 (convex valuations) and assume that the principal's beliefs correspond to a uniform probability distribution over the type space  $\Theta = [0, 1] \times [0, 1]$ . In addition, for the mechanism to be well specified, it is necessary that both players participate (i.e.,  $N = 2$ ). In that situation, the principal's expected payoffs as a function of  $p$  are given by

$$(1 - p)^2 E_\theta [p - \max(\theta_i, \theta_j) | \theta_i, \theta_j \geq p] = \frac{(1 - p)^2 (5p - 2)}{3} \quad (3.21)$$

which attains its unique maximum at the intermediary's optimal posted price  $p = 3/5$ . Note that the smallest posted price at which the principal is still willing to run the



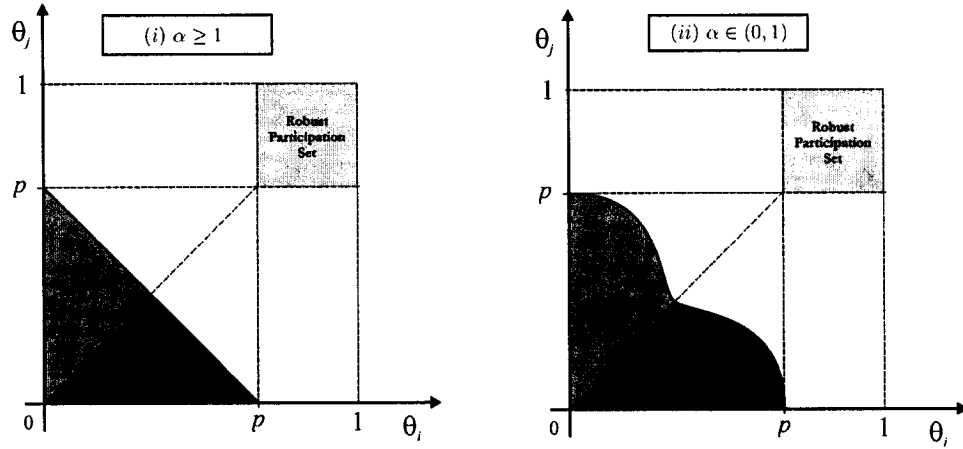


Figure 3.1: Firm participation in Example 1.

mechanism  $\mathcal{M}$  is  $2/5$ .

**Example 2** To illustrate the spirit in which our mechanism would potentially induce the principal into implementing the efficient auction, we present the following simple example based on Example 2 in discrete value functions. Although the theory developed is for the continuous case, the example illustrates the ideas behind how it might work for the continuous case, especially since the discrete case can be approximated using continuous functions. As in Example 2 we consider the auction of 1 divisible resource and let there be three bidders 1, 2 and 3 where the type  $\theta_i$  for each player belongs to the set  $\Theta = \{\theta^L, \theta^H\}$  and can be either with a probability of  $\frac{1}{2}$ . A bidder of type  $\theta_L$  values the entire resource at \$1 whereas a bidder of type  $\theta_H$  values half of the resource at \$ 1 and any additional amount is of no use. If the principal decided to use the Vickrey mechanism, then the optimal allocations,  $x_i$ 's and transfer payments  $\tau_i$ 's for the different type realizations are as follows:

### 3.5 Examples

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$\theta_1$	$\theta_2$	$\theta_3$	$x_1$	$x_2$	$x_3$	$\tau_1(x)$	$\tau_2(x)$	$\tau_3(x)$
$\theta^L$	$\theta^L$	$\theta^L$	1	0	0	1	0	0
$\theta^L$	$\theta^L$	$\theta^H$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
$\theta^L$	$\theta^H$	$\theta^H$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\theta^H$	$\theta^H$	$\theta^H$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0

where we have skipped the repetition of permutations and shown one of the possible efficient allocations when multiple are possible. This however does not change the total expected payments, hence we shall refrain from listing all possibilities. Taking expectations over the possible types, we get the expected revenues of the principal to be  $\frac{3}{4}$ . We now tabulate the allocations  $x_i$  and rebates  $b_i(x)$  that would be given under our mechanism if both players participated.

$\theta_1$	$\theta_2$	$\theta_3$	$x_1$	$x_2$	$x_3$	$b_1(x)$	$b_2(x)$	$b_3(x)$
$\theta^L$	$\theta^L$	$\theta^L$	1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\theta^L$	$\theta^L$	$\theta^H$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\theta^L$	$\theta^H$	$\theta^H$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\theta^H$	$\theta^H$	$\theta^H$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0

We can compute the largest posted price  $p$  that would make it possible for both players to participate in the game in expectation of the possible types of the other players, and the resulting allocations and rebates. This turns out to be  $\frac{36}{11}$  for  $\theta^H$  and 6 for player  $\theta^L$  which implies a maximum posted price of  $\frac{36}{11}$ . (If the price is raised higher then in the absence of player  $\theta^H$ , the expected payoffs of  $\theta^L$  also become negative). This yields an expected revenue of  $\frac{51}{44} > \frac{3}{4}$  for the principal implying that our mechanism not only ensures efficient allocation but also increases revenues for the principal in this example.

**REMARK (COMPUTING OPTIMAL ALLOCATION)** Note that as long as the value functions  $v_i(x_i, \theta_i)$  are additively separable across the  $K$  resources and for any resource

$k$ , piecewise linear on intervals  $[x_i^{k,j}, x_i^{k,j+1}]$ ,  $j = 1, \dots, J_i$  such that

$$v_i(x_i, \theta_i) = \sum_{k=1}^K v_i^k(x_i^k, \theta_i) \quad (3.22)$$

where

$$v_i^k(x_i^k, \theta_i) = a_i^{k,j}(\theta_i)x_i^k + b_i^{k,j}(\theta_i) \quad x_i^k \in [x_i^{k,j}, x_i^{k,j+1}]$$

such that the continuity condition  $a_i^{k,j}(\theta_i)x_i^{k,j+1} + b_i^{k,j}(\theta_i) = a_i^{k,j+1}(\theta_i)x_i^{k,j+1} + b_i^{k,j+1}(\theta_i)$  holds, although the function is non-smooth. The resulting allocation problem is a linear program

$$\begin{aligned} \max_x \quad & \sum_{i=1}^N v_i(x_i, \theta_i) \\ \text{s.t.} \quad & \sum_{i=1}^N x_i^k = 1, x_i^k \geq 0 \quad k = 1, \dots, K \end{aligned}$$

and can be solved in polynomial time using standard interior point algorithms (for a comprehensive treatise on interior point algorithms, see Ye (1997)).

### 3.6 Allocation with Moral Hazard

In this section, we introduce the fact that the rewards realized by the agents depend not only on the allocation  $x$ , but the effort exerted by the agent as well. Related literature on mechanism design with moral hazard includes McAfee and Mcmillan (1987) and Laffont and Tirole (1987). Let  $e_i = (e_i^1, \dots, e_i^K) \in E \subset \mathbb{R}^K$  denote the vector of effort exerted by agent  $i$  when his fractional allocation of the  $K$  goods is denoted by  $x_i = (x_i^1, \dots, x_i^K)$ . The output function  $y_i = y(x_i, e_i)$  is assumed to be the same for all agents, that is, given an allocation and an effort level, the output that each agent produces is the same. We assume that although the effort exerted by the agents is neither observable nor verifiable, the output can be both observed and

verified. The costs incurred by each agent,  $c(x_i, e_i; \theta_i)$ , however, depend on the type  $\theta_i$  of the agent. Both  $y : [0, 1]^K \times E \rightarrow \mathbb{R}^K$  and  $c : [0, 1]^K \times E \times \Theta \rightarrow \mathbb{R}$  are assumed to be smooth in  $x_i$  and  $(x_i, e_i)$  respectively. The assumptions on the distribution of the type space carry over from the basic setup of the problem described in Section 3.2. The sequence of events is as follows:

Stage 1: First the principal announces and commits to her mechanism  $\mathcal{M}^e$  which determines a fixed posted price vector  $p = (p^1, \dots, p^K)$  for the output of the agents.

Stage 2: The agents observe their private types  $\theta_i$  and submit real-valued bid functions  $b_i(y_i; \theta_i)$  defined on  $\mathbb{R}^K$  which is used to determine the agents' allocation matrix

Stage 3: After all bid functions have been collected the principal determines the agents' allocation matrix  $X = [x_i^k]_{i,k=1}^{N,K}$  as well as their transfer payments.

Stage 4: Based on the allocations, the agents choose their appropriate effort levels and generate output  $Y = [y_i^k]_{i,k=1}^{N,K}$

An extension of the idea of Section 3.3, helps us construct the principal's mechanism  $\mathcal{M}^e$  which maps the agents' bid-function vector  $b = (b_1, \dots, b_N)$  to an allocation matrix  $X$  and a corresponding vector of monetary transfers  $t = (t_1, \dots, t_N)$ . The transfers are such that

$$t_i(b, X) = p \cdot y(x_i, \bar{e}(x_i; \theta_i)) - \frac{1}{N-1} \sum_{j \neq i} b_j(y(\bar{x}_{-j}, \bar{e}(\bar{x}_{-j}; \theta_j)); \theta_j) \quad (3.23)$$

where  $\bar{x}_{-i} = x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_N$  is the vector containing the respective sums of fractional-good allocations to agents other than agent  $i$ . and  $\bar{e}(\bar{x}_{-j}; \theta_j) \in E$  is the optimal effort level exerted by agent  $j$  if he is allocated  $\bar{x}_{-j}$ . The transfers of

agent  $i$  are governed by the outputs other agents would have generated had they been allocated fractions not originally allocated to them. Knowing the optimal effort function  $\bar{e}(\cdot; \theta_j)$  is necessary to compute the transfers of the agents as given by equation (3.23). In equilibrium, the principal can compute  $\bar{e}(\cdot; \theta_j)$  for every agent  $j$  accurately based on his announced bid function  $b_j(\cdot)$  by the revelation principle. The following proposition determines the equilibrium bid functions for the agents in a dominant strategy equilibrium.

**Proposition 4** *Given the principal's mechanism  $\mathcal{M}^e$ , the agents' bidding equilibrium is in dominant strategies. For any  $i \in \{1, \dots, N\}$  agent  $i$ 's equilibrium bid function is given by*

$$b_i'(1-x_i; \theta_i) = \frac{y_1(x_i, \bar{e}(x_i; \theta_i)) - c_1(x_i, \bar{e}(x_i; \theta_i); \theta_i)}{y_1(1-\bar{x}_i; \bar{e}(1-x_i; \theta_i)) + y_2(1-\bar{x}_i; \bar{e}(1-x_i; \theta_i))\bar{e}_1(1-x_i; \theta_i)} \quad (3.24)$$

for all  $x_i \in [0, 1]^K$  and all  $\theta_i \in \Theta_i$

### 3.7 Chapter Summary

In this chapter we have constructed a mechanism for the allocation of divisible goods that implements the efficient outcome subject to participation constraints in dominant strategies and has the nice property that the principal's revenue maximizing allocation after the agents have announced their valuations is aligned in a way that it yields the efficient outcome. We extend the mechanism to a case where the realized rewards are dependent on costly effort exerted by the agent allocated the resource and has an inherent moral hazard phase following the auction.

## 3.8 Appendix: Proofs

**Proof of Proposition 1.** If we set  $B_i(X; \theta_{-i}) \equiv -\frac{1}{N-1} \sum_{j \neq i} b_j'(\bar{x}_{-j}; \theta_j)$ , then the first-order necessary optimality conditions (3.4) can be equivalently written in the form <sup>1</sup>

$$\sum_{i \neq l} B_i(X; \theta_{-i}) = \lambda - \mu_l \quad (3.25)$$

for all  $l \in \{1, \dots, N\}$  with  $j \neq l$  it is

$$\begin{aligned} \sum_{i \neq j} B_i(X; \theta_{-i}) &= \sum_{i \neq l} B_i(X; \theta_{-i}) + B_l(X; \theta_{-l}) - B_j(X; \theta_{-j}) \\ &= (\lambda - \mu_l) + B_l(X; \theta_{-l}) - B_j(X; \theta_{-j}) = \lambda - \mu_j \end{aligned} \quad (3.26)$$

which, together with (3.25), implies that  $B_l(X; \theta_{-l}) - \mu_l = B_j(X; \theta_{-j}) - \mu_j$  for all  $l, j \in \{1, \dots, N\}$ . Hence,

$$\begin{aligned} & -\frac{1}{N-1} \sum_{i \neq j} b_i'(\bar{x}_{-i}; \theta_i) - \mu_j \\ &= -\frac{1}{N-1} \left[ \sum_{i \neq l} b_i'(\bar{x}_{-i}; \theta_i) - [b_l'(\bar{x}_{-l}; \theta_l) - b_j'(\bar{x}_{-j}; \theta_j)] \right] - \mu_l \end{aligned} \quad (3.27)$$

which gives us that

$$\frac{b_l'(\bar{x}_{-l}; \theta_l)}{N-1} + \mu_l = \frac{b_j'(\bar{x}_{-j}; \theta_j)}{N-1} + \mu_j \quad (3.28)$$

for all  $l, j \in \{1, \dots, N\}$ . Let us now turn our attention to the agents' optimal bidding problem. Maximizing the integrand of agent  $i$ 's objective function pointwise with

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<sup>1</sup>For simplicity we have dropped the fixed price  $p$  from the principal's optimality conditions (to include it is sufficient to replace  $\lambda$  by  $\bar{\lambda} = \lambda - p$ ). Indeed, since all fractional allocations sum to one, the principal obtains  $p \cdot (x_1 + \dots + x_N) = \sum_{k=1}^K p^k$  for any given price vector  $p = (p^1, \dots, p^K)$ , independent of the allocation matrix  $X$ .

respect to  $x_i = g_i \in [0, 1]^K$  for all  $\theta \in [0, 1]^K$  yields the optimality conditions

$$v_i'(x_i; \theta_i) - B_i(X; \theta_{-i}) = p + \hat{\lambda}_i - \hat{\mu}_i \quad (3.29)$$

where  $\hat{\mu}_i, \hat{\lambda}_i \in \mathbb{R}_+^K$  are the Lagrange multipliers corresponding to the feasibility constraints  $0 \leq x_i \leq 1$ . Clearly, for any  $k \in \{1, \dots, K\}$  only one of the inequality constraints (either  $x_i^k \geq 0$  or  $x_i^k \leq 1$ ) can be binding, so that  $\hat{\lambda}_i^k \hat{\mu}_i^k = 0$ . Relation (3.28) implies that

$$-B_i(X; \theta_{-i}) = b_i'(x_{-i}^-; \theta_{-i}) + \sum_{j \neq i} (\mu_i - \mu_j) = b_i'(\bar{x}_{-i}; \theta_i) + N\mu_i - M \quad (3.30)$$

where  $M = \sum_{j=1}^N \mu_j$  is the aggregate shadow value associated with the lower allocation bound. Substituting (3.30) in (3.29) and realizing that  $\bar{x}_{-i} = 1 - x_i$  gives

$$v_i'(x_i; \theta_i) + b_i'(1 - x_i; \theta_{-i}) = p + \hat{\lambda}_i - \hat{\mu}_i - M + N\mu_i \quad (3.31)$$

Constructing  $\mu_i \geq 0$  such that

$$\mu_i = \frac{v_i'(x_i; \theta_i) + B_i(X; \theta_{-i}) + M}{N} \quad (3.32)$$

and substituting (3.29) into (3.31) we find that

$$v_i'(x_i; \theta_i) = b_i'(1 - x_i; \theta_i) \quad (3.33)$$

Depending on the realization  $\theta_{-i} \in \Theta_{-i}$  any such  $x_i \in [0, 1]^K$  may be considered optimal by the principal, so that the last relation should be considered an identity. Using the integration variable  $\zeta_i = 1 - x_i$  via integration of the (conservative) gradient field  $v_i'(1 - \zeta_i; \theta_i)$  along any path in  $[0, 1]^K$  from  $\mathbf{0}$  to  $x_i$  we thus find (3.8) which

completes the proof. ■

**Proof of Proposition 2.** Using the equilibrium bid schedule specified in Proposition 1, we find that the principal's payoff is

$$\sum_{i=1}^N t_i(b, X) = \bar{p} - \frac{1}{N-1} \sum_{i=1}^N \sum_{j \neq i} b_j(1 - x_j) \quad (3.34)$$

$$\begin{aligned} &= \bar{p} - \frac{1}{N-1} \sum_{i=1}^N \sum_{j \neq i} (v_j(\mathbf{1}; \theta_j) - v_j(x_j; \theta_j)) \\ &= \bar{p} - (W(\mathbf{1}; \theta) - W(X; \theta)) \end{aligned} \quad (3.35)$$

**Proof of Proposition 3.** If all entries in the equilibrium allocation matrix  $X$  are positive, then all Lagrange multipliers  $\mu_i$  and  $\hat{\mu}_i, \hat{\lambda}_i$  vanish for  $i \in \{1, \dots, N\}$ . Hence we conclude from (3.28) that  $b'_i(1 - x_i; \theta_i) = b'_1(1 - x_1; \theta_1)$  for all  $i \in \{1, \dots, N\}$ . Substituting the equilibrium bid schedules specified in Proposition 1, together with the definition of the set  $\Delta$  of feasible allocation matrices, yields the result. ■

**Proof of Corollary 1.** An efficient allocation matrix  $X$  maximizes the sum of the agents' payoffs,  $W(X, \theta) = \sum_{i=1}^N v_i(x_i; \theta_i)$ , subject to the feasibility constraint  $x_1 + \dots + x_N = 1$ . The first-order necessary optimality conditions are equivalent to the conditions specified in Proposition 3, which concludes the proof. ■

**Proof of Corollary 2.** Any agent for which  $v'_i(0, \theta_i) > 0$  is large would under mechanism  $\mathcal{M}$ , at the margin generate very large discount savings for the principal, which can be seen by inspecting the expression of the principal's equilibrium payoff (3.34) in the proof of Proposition 2. Since  $v'_j(x_j; \theta_j)$  is finite for any  $x_j$  with positive entries and any  $(j, \theta_j) \in \{1, \dots, N\} \times \Theta_j$ , it must be the case that  $x_i$  has positive entries in equilibrium for any resource  $k \in \{1, \dots, K\}$ . The uniqueness follows directly from the concavity of all agents' payoff functions. ■



**Proof of Corollary 4.** Stage 4 - Agents' Optimal Effort: We backwards induct by first computing the optimal effort that agent  $i$  would exert given his allocation  $x_i$ . Fixing  $X$ , agent  $i$ 's effort optimization problem can be formulated as

$$\max_{e \in E} \{y(x_i, e) - c(x_i, e; \theta_i)\} \quad (3.36)$$

Let the optimal effort be denoted by  $\bar{e}(x_i; \theta_i)$ .

Stage 3 - Principal's Optimal Allocation: Anticipating the agents' optimal effort functions, the principal chooses an allocation matrix  $X$  that solves

$$X \in \arg \max_{\hat{X} = [\hat{x}_i^k] \in \Delta} \left\{ \sum_{i=1}^N t_i(b, \hat{X}) \right\} \quad (3.37)$$

The first-order necessary optimality conditions which arise from (3.37) are

$$\sum_{i \neq l} B_i(X; \theta_i) = \lambda - \mu_l \quad (3.38)$$

where

$$\begin{aligned} B_i(X; \theta_i) &\equiv -\frac{1}{N-1} \sum_{j \neq i} b_j'(y(\bar{x}_{-j}; \bar{e}(x_{-j}; \theta_j)); \theta_j) \cdot [y_1(\bar{x}_{-j}; \bar{e}(x_{-j}; \theta_j)) + y_2(\bar{x}_{-j}; \bar{e}(x_{-j}; \theta_j)) \bar{e}_1(x_{-j}; \theta_j)] \end{aligned} \quad (3.39)$$

Stage 2 - Agents' Optimal Bid Function: Just as in Section 3.3, in equilibrium, each agents' type  $\theta_i$  is mapped to an equilibrium bid function  $b_i(\cdot; \theta_i)$  and the allocation matrix is given by

$$X = G(\theta; b) = \begin{bmatrix} g_1(\theta; b) \\ \vdots \\ g_N(\theta; b) \end{bmatrix} \quad (3.40)$$

where the vector valued functions  $g_i(\theta)$  describe agent  $i$ 's allocation  $x_i = g_i(\theta)$ . The agents' bidding problem can then be written as

$$\begin{aligned} \max_{b_i \in L_\infty([0,1]^K, \mathbb{R}_+)} \int_{\Theta_{-i}} & [y(g_i(\theta_i, \theta_{-i}; b_i, b_{-i}), \bar{e}(g_i(\theta_i, \theta_{-i}; b_i, b_{-i}); \theta_i)) \\ & - c(g_i(\theta_i, \theta_{-i}; b_i, b_{-i}), \bar{e}(g_i(\theta_i, \theta_{-i}; b_i, b_{-i}); \theta_i); \theta_i) - t_i(b, G(\theta, b_i, b_{-i}))] dF(\theta_{-i}|\theta_i) \end{aligned} \quad (3.41)$$

This is an unconstrained variational problem and by pointwise maximizing, with respect to  $g_i$  for any given  $\theta$ , which in turn determines agent  $i$ 's equilibrium allocation  $x_i$  and this it's optimal bid  $b_i(y(x_i, \bar{e}(x_i; \theta_i); \theta_i)$

$$(1-p)y_1(x_i, \bar{e}(x_i; \theta_i)) - c_1(x_i, \bar{e}(x_i; \theta_i); \theta_i) - B_i(X, \theta_{-i}) = \hat{\lambda}_i - \hat{\mu}_i \quad (3.42)$$

where  $\hat{\mu}_i$  and  $\hat{\lambda}_i \in \mathbb{R}^K$  are the Lagrange multipliers corresponding to the feasibility conditions  $0 \leq x_i \leq 1$ . From relation (3.38),

$$\begin{aligned} -B_i(X, \theta_{-i}) &= b_i'(y(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i)); \theta_i) \cdot [y_1(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i)) + y_2(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i))\bar{e}_1(x_{-i}; \theta_i)] \\ &\quad + N\mu_i - M \end{aligned} \quad (3.43)$$

where  $M = \sum_{j=1}^N \mu_j$ . Substituting (3.43) in equation (3.42), we get

$$\begin{aligned} & b_i'(y(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i)); \theta_i) \cdot [y_1(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i)) + y_2(\bar{x}_{-i}; \bar{e}(x_{-i}; \theta_i))\bar{e}_1(x_{-i}; \theta_i)] \\ & + (1-p)y_1(x_i, \bar{e}(x_i; \theta_i)) - c_1(x_i, \bar{e}(x_i; \theta_i); \theta_i) + N\mu_i - M = \hat{\lambda}_i - \hat{\mu}_i \end{aligned} \quad (3.44)$$

Constructing  $\mu_i \geq 0$  such that

$$\mu_i = \frac{y_1(x_i, \bar{e}(x_i; \theta_i)) - c_1(x_i, \bar{e}(x_i; \theta_i); \theta_i) + B_i(X; \theta_{-i}) + M}{N} \quad (3.45)$$

and substituting (3.42) into (3.44) we find that

$$b_i'(1 - x_i; \theta_i) = \frac{y_1(x_i, \bar{e}(x_i; \theta_i)) - c_1(x_i, \bar{e}(x_i; \theta_i); \theta_i)}{y_1(1 - \bar{x}_i; \bar{e}(1 - x_i; \theta_i)) + y_2(1 - \bar{x}_i; \bar{e}(1 - x_i; \theta_i))\bar{e}_1(1 - x_i; \theta_i)} \quad (3.46)$$

■

## Chapter 4

# Auctioning Debt Collection Contracts

*Thought and theory must precede all salutary action; yet action is nobler in itself than either thought or theory*

– WILLIAM WORDSWORTH (1770–1850)

### 4.1 Introduction

Creditors have recently begun using auctions to allocate delinquent debt accounts to collection agencies for retrieving outstanding debt. We refer to these as *auctions of debt collection contracts*. The practical details of the collections process make the design of these auctions extremely interesting, and it is important to see how current results from the theory of auctions might help design these effectively. Delinquent debt is handled in different ways by different creditors. We have been motivated by debt auctions used in the United States credit-card industry and begin the chapter by some figures illustrating the scale of delinquent debt market in the credit-card

industry.

**Credit Card Borrowing:** the total outstanding consumer credit in the United States as of March 2006 was 2.15 trillion dollars of which commercial banks and finance companies accounted for more than 1 trillion in loans. Almost 800 billion of this debt is accounted for by credit-card companies <sup>1</sup>. Research by the Federal Reserve indicates that household debt is at a record high relative to disposable income. Some analysts are concerned that this unprecedented level of debt might pose a risk to the financial health of American households. A high level of indebtedness among households could lead to increased household delinquencies and bankruptcies, which could threaten to worsen the condition of lenders if loan losses are greater than anticipated. Bankruptcy filings eclipsed the two million mark for the first time in the United States with 2,078,415 filings were reported in calendar year 2005<sup>2</sup>. This represents a record 30 percent increase over the total filings for the same period in 2004.

## 4.2 Debt Collection

Through the rest of this chapter, we shall refer to the credit-card company as the creditor. Once a borrowing account defaults, the creditor has several options that can be followed to retrieve or *collect* the outstanding debt. The collections process proceeds in stages where each stage is characterized by the type of actions the creditor takes to retrieve the debt. The actions taken in different stages vary from trying to locate the account holder, sending letters reminding of payments, threatening legal

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<sup>1</sup>Source: U.S. Public Interest Research Group and Federal Reserve Statistical Release. Though merchant credit has been as old as civilization, the present-day credit card industry in the United States originated in the nineteenth century. In the early 1800s, merchant and financial intermediaries provided credit for agricultural and durable goods, and by the early 1900s, major U.S. hotels and departmental stores issued paper identification cards to their most valued customers. (source: FDIC)

<sup>2</sup>Source: American Bankruptcy Institute

action or even taking legal action. The efficacy of the collections process rests highly on the interaction that takes place between the individual collector and the account holder making the job of a collector specialized enough for many creditors to rely on debt collection firms (agencies) to do the collections. The industry of collecting delinquent debt has grown significantly in the past few years with some estimates stating that there are almost 6,500 firms in North America the industry today.<sup>3</sup>

The allocation of delinquent accounts to collection firms takes place in batches, with portfolios of accounts assigned to a firm for a fixed period of time. Some creditors have now begun running auctions to determine the allocations of the accounts to different agencies and the corresponding payments made by them with the hope of maximizing the resulting revenues. In this chapter, we recognize that in designing any such auction, the game is usually bigger than what mathematical models might capture and we try to underscore the important aspects that need to be addressed when designing these auctions. Our main objective is to highlight all the important factors determining revenues and how they impact the auction design process. We explore ways in which some of the more recent theory in the field of auctions can be applied to design these auctions effectively.

The key decision variables of any auction are the manner in which it allocates the resource in question and the way in which payments are determined. The important factors that influence the final revenues of the creditor are shown in Figure 4.1. Although the relationships between the different factors will become clearer as we proceed, the diagram can be read as follows: The final revenues of the creditor are determined by the collections generated by the agency and the payment structure. The collections depend on the effort exerted by the agency and the intrinsic attributes of the allocated portfolio. Agencies would choose effort levels depending on their cost structures, their beliefs about the portfolio attributes and the incentives generated by

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<sup>3</sup>Source: [www.collectionindustry.com](http://www.collectionindustry.com), a Kaulkin Media publication.

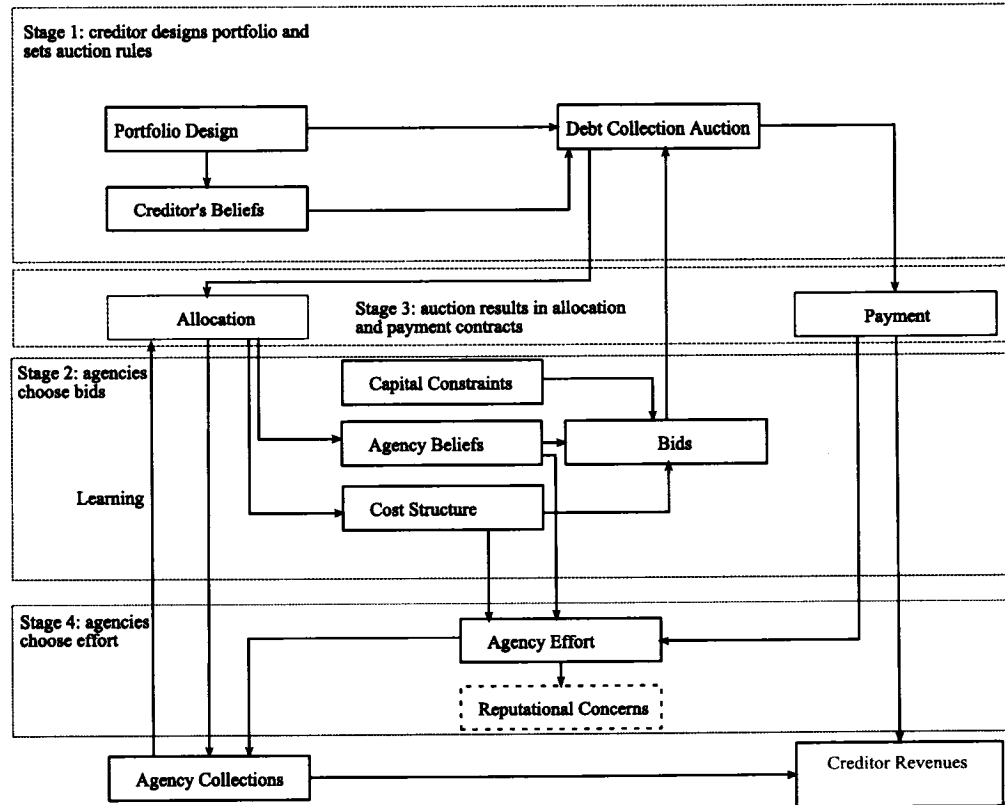


Figure 4.1: Influence diagram highlighting important factors affecting revenues in the debt collection auction problem.

the payment scheme. The allocation and the payment are determined by the design of the auction by taking the bids of the agencies as inputs. The bid, in turn, is chosen by an agency based on its cost structure, capital constraints and its beliefs about the portfolio attributes as well as beliefs about the potential bids of other agencies. The agency's beliefs and cost structure depend on the portfolio allocated. Before the auction begins, the creditor's beliefs play an important role in designing the portfolio that is auctioned. These beliefs are continuously updated based on the past performance of agencies and reputations arising as a result of effort exerted.

## 4.3 Account Allocation

### 4.3.1 Account Allocation as an Optimal Decision Problem

We first consider the allocation problem as an optimal decision problem rather than an auction. Here the payment structure can be assumed to be uniform across all the agencies, making the optimal allocation dependent on the creditor's beliefs about the attributes of the accounts and the agencies.

1. **Beliefs about Account Attributes:** delinquent accounts are typically characterized by the amount of outstanding debt and the creditor's assessment of the likelihood of the debt being paid back. The allocation of accounts to different agencies is based on these two fundamental attributes. The outstanding debt is an easily observable attribute, however, much of the art in collections surrounds a sound understanding of the likelihood of the debtor paying the outstanding balance. One might break down the likelihood of paying back according to the (i) willingness to pay, (ii) ability to pay. To understand the difference, one can think of fraudulent accounts where the willingness would be low, as opposed to debtors with genuine financial hardships where the ability would be low at



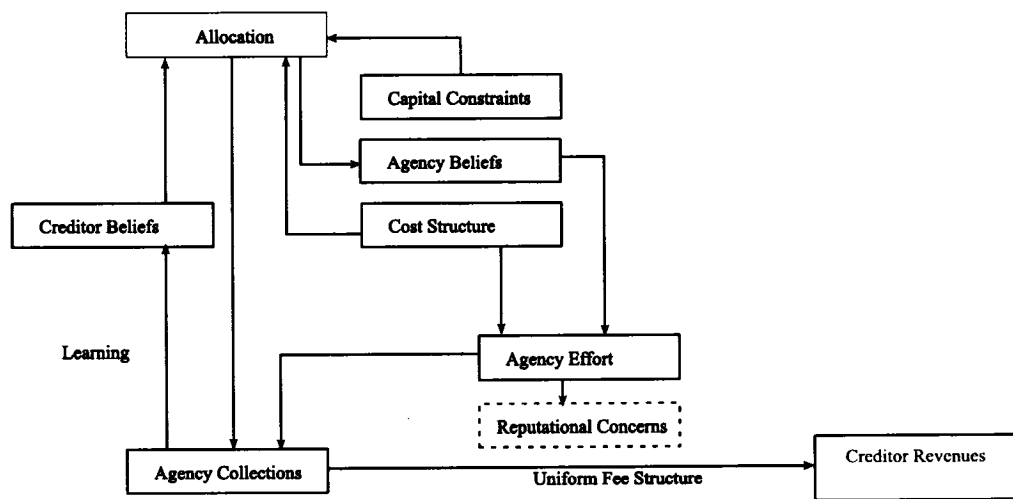


Figure 4.2: Influence diagram highlighting important factors influencing revenues in the *debt collection allocation* problem.

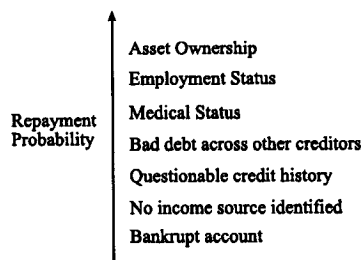


Figure 4.3: An example where different account attributes might indicate varying levels of repayment probabilities.

times despite good intentions. Through past historical records, external sources of credit history, discussions with the debtor and other informative actions, the creditor wishes to understand both the willingness and the ability of the debtor in paying the outstanding money back. The ownership of assets is a strong indicator of the ability to pay by indicates a lower willingness, however medical hardships might be interpreted as a low ability to pay although the willingness might be high. Other attributes include indicators of employment, transactions with other creditors, bankruptcy information, legal status, traceability and credit history.

2. **Beliefs about Agency Attributes:** The optimal allocation can be seen as a matching problem where the different types of accounts that constitute a portfolio need to be matched to different agencies. Based on agency specialization and past performance (*learning*) the creditor can compute the optimal match. This problem will be subject to capital constraints and incentive compatibility constraints for the agencies based on the creditor's estimates of their cost structures. Additional constraints corresponding to minimum experimental allocations to assist learning, can also be incorporated.

In reality, the collections process is a multi-stage process where any delinquent

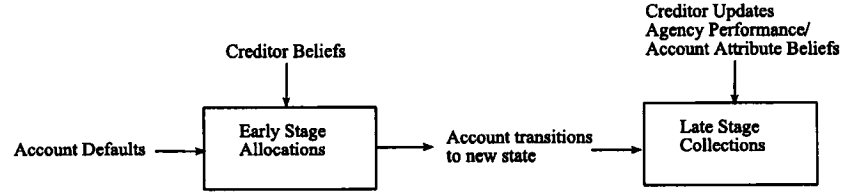


Figure 4.4: Multi-stage optimal allocation.

account travels from one agency to another as the time it remains delinquent increases, and more account relevant information is obtained in the process of collections. For example, if it is learnt that an account holder has filed for bankruptcy, then the account is routed to an agency that specializes in handling bankrupt accounts where understanding the complete financial status of the account holder becomes important. On the other hand if tracing the account holder becomes difficult then investigative agencies must be used. The optimal allocation problem then becomes a dynamic optimization problem where one can model the information gleaned about the account as its state at any given time to optimally allocate in the next. The problem effectively becomes a dynamic resource allocation problem which is provably NP-hard but fairly good approximations to the solution can be arrived at using fast algorithms (Farias and Van Roy, 2005)

### 4.3.2 Account Allocation through an Auction

We now consider the allocation problem in the context of a debt collection auction. As Figure 4.1 describes, the auction mechanism takes as inputs the bids placed by

the agencies and based on the beliefs of the creditor (about both account and agency attributes) and the *portfolio design*, determines the allocation of accounts. The way the beliefs about performance and account attributes influence the allocation remain the same as in the case with allocation as an optimal decision. Here we discuss how the portfolio design impacts the allocation. By portfolio design, we mean the segmentation of accounts into packages by the creditor before running the auction. The accounts can be considered as individual goods auctioned in packages and the way the creditor defines these packages impacts the outcome of the auction significantly.

If all the agencies had the same fee structure and if each account could be separately bid upon, the resulting allocation problem would be the same as the optimal allocation problem studied in the previous section. Bidding on each account separately is infeasible in most settings as there are millions of accounts typically and using a uniform price auction is not necessarily revenue maximizing for a set of non-homogenous goods (Wilson, 1979). Thus it is important to consider the effects of segmenting the set of accounts into different bundles.

A highly *segmented* portfolio with each segment having accounts with very similar attributes, and thus similar probabilities of paying back would lead to a different allocation profile compared to a *homogeneous* bundling where a random set of accounts to form a bundle with a mixed set of account attributes. The important factors in deciding whether to choose a segmented or a homogeneous portfolio are:

1. Participation: Whereas the bigger collection agencies are typically diversified and employ collectors who specialize in collecting debt from many types of accounts, the smaller agencies are more specialized and can only handle certain segments. A homogeneous portfolio might attract the larger agencies, however, small risk averse agencies might refrain from participating in auctions of high variance portfolios. This might be alleviated to a certain extent through

segmented portfolios although segments in which not many small agencies specialize might have few participants. The above consideration might lead to an elimination round where many types of accounts are dropped from consideration for auction, and then homogeneous or segmented portfolios are created from the remaining accounts.

2. Future benefits from learning: The argument given in favor of auctioning a diverse portfolio is that it leads to much greater learning about the different accounts than do segmented portfolios. Although diverse portfolios would result in greater information related to the performance and cost associated with many segments of accounts at the same time, they yield little information on bidding behavior if those segments were auctioned separately. It is also not clear whether they would yield higher or lower revenues.
3. Complementarities: The theory of combinatorial auctions tries to address questions that arise when one deals with multiple goods being auctioned at once. If the portfolio is completely homogenized (through randomization) then the auction becomes an auction of a divisible good (as studied in Chapter 3). If the portfolio is not randomized, then it becomes interesting to see whether different packages become substitutes or complements for agencies. This is important, because although revenue maximizing auctions have been characterized completely in the combinatorial auction literature (Cramton, Shoham and Steinberg, 2006, Likhodedov and Sandholm, 2005) the revenue generated by any combinatorial auction rests on whether the packages are complements or substitutes.

If an agency specializes in a particular segment of accounts (say bankrupt accounts) then its costs in collecting from other segments might outweigh the benefits making any other segment worthless and hence substitutes amongst

themselves. On the other hand an agency with relatively less experience might want to hedge by acquiring accounts from different segments making them complements.

4. Computational complexity: The computational complexity associated with expressing preferences across a segmented portfolio, and combinations of various segments is far greater than that associated with preferences for fractions of a homogenized portfolio and might sometimes be important enough to choose one option over the other.

## 4.4 Factors Influencing Bids

### 4.4.1 Capital Constraints and Cost Structure

Under strict budget constraints it is known that the first price auction performs better than the Vickrey auction (Milgrom, 2004) in maximizing revenues. The budget constraints implied by capital constraints faced by agencies suggests the use of the first-price auction. The primary costs that are incurred in the collections process are the salaries paid to the collectors. The other costs include paying external information sources for account attribute information, telephone and letter costs. The training that needs to be imparted to the employees implies significant capacity building costs. Thus, when bidding for a new portfolio, agencies give significant importance to capacity building costs implying a cost structure that exhibits both complementarities and substitute characteristics as shown in Figure 4.5.

### 4.4.2 Common Values

The underlying valuation of a debt portfolio is closely related across agencies. Some agencies might have significant experience in working on a particular segment of

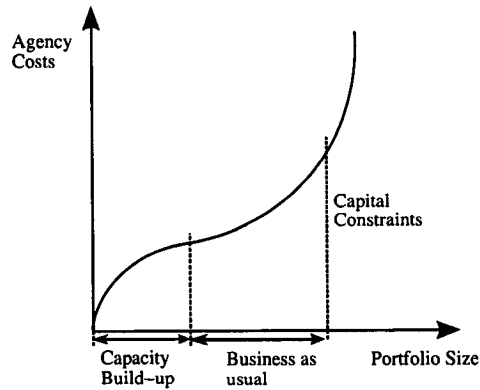


Figure 4.5: Typical cost structure curves for collection agencies.

accounts and might have better estimates of the potential collections from a given set of accounts, and might be able to retrieve more debt than newer entrants in that segment. However, the underlying characteristics of the account holders induce strong common value characteristics across agencies. The beliefs about the other agencies' cost structures also figures prominently in deciding one's own bids. The cost structures do vary in the initial capacity building phase, but in the business-as-usual segment they are again similar across agencies. We know from the *linkage principle* (Milgrom and Weber, 1982) that when bidder signals are strongly affiliated<sup>4</sup>, then Vickrey auctions yield greater revenues than first-price auctions. Thus given the strong common values nature of the problem at hand, one might consider using a second-price auction. We will see later that the structure of the bids matters significantly more than the auction format and the contrasting suggestions arising out of budget constraints and common values do not leave us with a clear winner.

<sup>4</sup>Two informative signals drawn from a distribution characterized by density  $f$  are said to be affiliated if  $\ln(f)$  is supermodular

##### 4.4.3 Belief Revelation

The amount of information that the creditor chooses to reveal about the intrinsic and relationship-specific attributes of accounts being auctioned plays an important role in shaping the beliefs the agents have about the portfolio valuation. The amount of information revealed to the agencies has a threefold impact on the outcome of the portfolio allocation.

The bids of the agencies changes as their own signals of portfolio quality and their beliefs about the signals that other agents receive change. From Milgrom and Weber (1982), we know that in most auctions, the auctioneer can increase expected revenues by disclosing greater information. More formally, if the bidders valuations are a function of a group of signals that are all affiliated, then it is in the interest of the creditor to make the information public. The information revealed being a direct measure of the possible collections if the creditor were to try and retrieve debt on it's own, it also implies a certain reserve price the the agencies must beat. This fact in isolation would imply the bids to increase with more information being released. The information can be revealed primarily in two ways

##### 1. Direct Revelation Methods:

- Creating segmented portfolios: choosing a segment of accounts for the portfolio reveals some information about the constituent accounts. Even within segmented portfolios, the degree of information could be varied by selecting the attributes that describe the segments.
- Allow agencies to sample accounts: the agencies are given a random sample of accounts selected from the portfolio to assess the quality of the remainder of the portfolio.

##### 2. Indirect Revelation Methods : In equilibrium, the various choices made by



the creditor - portfolio design, action restrictions, direct information revelation, choice of auction format, all send signals to the agencies about the quality of the accounts being auctioned. The problem of mechanism design with an informed principal was first studied by Myerson (1983) where the notion of inscrutable mechanisms is developed which do not disclose the information possessed by the principal.

Besides impacting bidding behavior, greater information means *lower entry costs* associated with participating in the auction. This happens in terms of both the risk premium associated with a more uncertain portfolio and the effort exerted in running tests on sample accounts. Also, the *effort level* chosen subsequently by the agencies changes with the prior information that the agencies start with when working on the accounts and hence impacts total collections.

## 4.5 Bid Structure

The design of an auction for debt collection contract requires specifying the structure of the bid functions and the payment function that maps agencies' allocations to their payments. The most important feature of debt collection contracts is that they yield uncertain rewards to the agency. In the next subsection, we describe revenue maximizing auctions where the space of bids have been expanded from standard *cash* bids (where lump-sum payments are made upfront) to outcome contingent bids, also known as *security-bids*. Bids on debt portfolios are usually multidimensional due to the different constituent elements giving bidders additional flexibility in dealing with the risk associated with bidding on portfolios with uncertain rewards. Athey and Levin (2001) study Timber auctions and model how this additional flexibility results in agents *skewing* bids across different constituents of timber portfolios. This arises due to informational asymmetries between bidders and auctioneers with respect to

the underlying composition of the portfolio. We translate the analysis to the setting of debt collection portfolios.

### 4.5.1 Realization-Contingent Bids

Bids on debt collection contracts can be viewed as securities that express an agent's willingness to make payments that are contingent on the asset's realized value. This of course might include bidding formats which disallow payments being contingent on realized outcomes (akin to a classic debt contract where the auctioneer would be paid a fixed amount upfront), however, we allow the possibility of a wide range of outcome contingent bid structures. Recent work by DeMarzo, Kremer and Skrzypacz (2005) (henceforth referred to as DKS) becomes particularly relevant to this setting and we draw on some of the key insights presented in that paper to compare different bidding securities. We shall also use insights from the paper when speaking about auction formats (including informal auctions) for different bid structures.

#### Security-Bid Classes

The different classes of securities include

1. Equity Bid: The agencies bid on the fraction of the realized payoff they are willing to pay to the creditor.
2. Debt Bid: The bids describe a fixed amount that the credit card company is paid.
3. Convertible Debt Bid: The creditor is promised a fixed amount or a fraction of the realized debt, whichever is larger. This is equivalent to a debt plus royalty rate contract.

4. Levered Equity bid: The creditor receives a fraction of the realized payoff after a fixed amount is paid to the agencies
5. Call Option Bid: The creditor receives a call option on the realized rewards and agents bid on the strike price
6. Debt Plus Equity Bid: The creditor is paid a fixed amount upfront and promised an equity in the realized rewards.

One might conjecture that the results from standard auction theory carry over to security bid auctions by simply replacing each security with its cash value. Unlike cash bids, however, the value of a security bid depends upon bidders' private information. This difference can have important consequences.

### Steepness

DKS define a notion of steepness of a class of security bids as follows. A security  $S_1$  crosses from below security  $S_2$  if it crosses once from below. A class of securities  $\mathbf{S}_1$  is steeper than  $\mathbf{S}_2$  if for any  $S_1 \in \mathbf{S}_1$  and  $S_2 \in \mathbf{S}_2$ ,  $S_1$  crosses  $S_2$  from below. Comparing the standard classes of securities, we can see from Figure 4.6 that the call option bid (levered equity) is steeper than the equity bid which in turn is steeper than the debt bid. In-fact levered equity is steeper than a debt plus equity security and would therefore yield higher revenues.

Keeping the format of the auction fixed, as long as securities within a particular class can be ordered based on realized rewards, under assumptions of symmetric independent private valuations and risk neutrality<sup>5</sup> steeper securities yield higher revenues. To see the intuition behind the relation between steepness and auction

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<sup>5</sup> We call a class of securities ordered if a 'lower' ranked security's payment at a lower realized reward is always less than that of a 'higher' ranked security at a higher realization

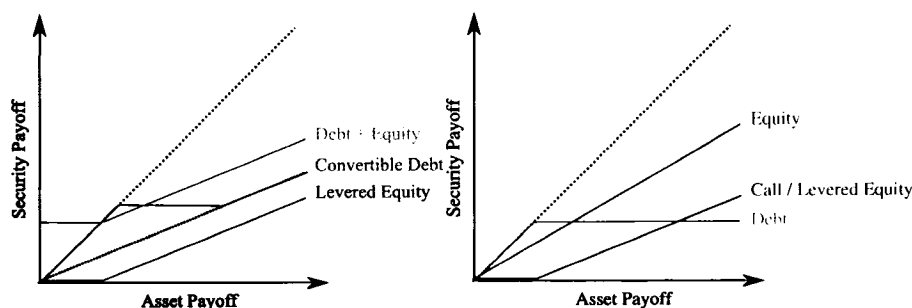


Figure 4.6: Security payoffs for levered equity, convertible debt and a debt plus equity security and the simple debt and equity securities.

revenues, consider the second-price auction. The bidders bid their reservation value and hence the security design impacts revenues only through the sensitivity of the security to the parameters that dictate the private information held by the players. So to compare two securities, comparison of the slopes of two securities where the expected payments are the same yields that a steeper security, having a higher slope at a point where it crosses from below another security, gives rise to higher revenues.

### Comparing Auction Formats

For *cash* auctions, where the winner pays a fixed cash amount, the revenue equivalence theorem (Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981) says that under certain assumptions such as risk neutrality, independent symmetric valuations, and lowest bidder earning no surplus, the expected revenues from the four auction formats are the same. However, revenue equivalence across auction formats does not always hold when security-bids are used. A first price auction with call options yields the highest expected revenues among all general symmetric mechanisms. The first price auction with standard debt yields the lowest expected revenue among all general symmetric mechanisms. It is also true that the design of the security is more important than the specific auction format: the revenue consequences of shifting from debt to

call options in a first price auction exceeds the consequences of any change in auction mechanism. The importance of leveraged securities (which are inherently steeper) in increasing revenues is emphasized by the observation that a security that pays the bidders cash for equity might raise more revenues than a call option. To implement this, if there are initial investments made by the agencies if they are allocated the portfolio, the creditor can reimburse the initial investment and claim almost all the revenues generated by the agency, guaranteeing individual rationality, and ensuring almost complete surplus extraction.

### 4.5.2 Package Equity Bids

In this section, we restrict attention to the package equity bids. That is, agencies pay fractions of collections as fees, but instead of scalar bids for the entire portfolio, bidders can bid a vector of equity bids for different constituents of the portfolio. A main motivation for restricting attention to equity auctions is to reduce the risk borne by the winner. If agencies face substantial risk in the rewards generated by the portfolio, they will require a risk premium and bid less aggressively. The analysis in this section is based on the timber auctions studied in Athey and Levin (2001).

Let there be  $N$  bidding agencies. We let  $Q$  denote the true value of the total retrievable debt and  $Q_e$  denote the creditor's estimate of it. The estimate is publicly announced at the start of the auction. For simplicity, we restrict attention to two kinds of constituent packages, "easy" accounts and "hard" accounts where the two differ in the costs of retrieval and yield per unit rewards  $v_1$  and  $v_2$  respectively. The true portfolio constitution can be described as  $(\rho_1, \rho_2)$ ,  $\rho_2 = 1 - \rho_1$ : the proportions of the two kinds of accounts. These proportions are initially unknown. The creditor, based on the information it possesses about the accounts, announces an estimate of  $\rho_1$  denoted by  $e_1$  (and  $e_2 = 1 - e_1$ ). Each bidder can run a set of tests on a sample subset of the portfolio and obtain information related to the accounts. Let

$s_1^j$  be agency  $j$ 's estimate of  $\rho_1$  after running the tests. We assume that  $Q$  becomes known after the agencies run their tests. We assume that the bidder's estimates are affiliated with the truth, are reasonably noisy and are correlated so that they are either all greater than or all less than the creditor's estimate. A bid is a price vector  $(b_1, b_2)$  that generates a total bid of  $B = Q_e \sum_i b_i e_i$ , and if it wins the auction, an agency pays  $P = Q \sum_i b_i \rho_i$ . In a sealed bid reserve price auction where each bidder submits a bid vector  $(b_1, b_2)$  and the creditor sets a reserve price  $r_1$  and  $r_2$  for the "easy" and "hard" types of accounts respectively, the bid decision can be divided into two stages: selecting a total bid  $B$ , and allocating that bid over the two types of accounts. Denoting  $\Delta b = b_1 - b_2$ , the ex post profits conditional on winning can be written as

$$\pi(\Delta b, B, \rho_1) = Q[(\Delta b - \Delta v)(e_1 - \rho_1) + (V - B)/Q_e] \quad (4.1)$$

where  $\Delta v = v_1 - v_2$  and  $V = Q_e \sum_v i e_i$  and fixing the total bid  $B$  the bid allocation problem can be written as

$$\max_{\Delta b} E_{\rho_1}[\pi(\Delta b, B, \rho_1) | e_1, s_1^j, \forall k \neq j, B^k < B] \quad (4.2)$$

$$\text{subject to } b_1 \geq r_1, b_2 \geq r_2 \quad (4.3)$$

It is useful to think of bid allocation as a portfolio problem where the bidder is "investing"  $\Delta b - \Delta v$  in a risky asset with return  $e_1 - \rho_1$ . By bidding a constant profit margin on each species (setting  $\Delta b = \Delta v$ ), a bidder eliminates risk from errors in estimates by the creditor. Athey and Levin (2001) show that the bidders skew their bids onto the types of accounts which they think have been overestimated. This lowers the expected payment for a given bid, but it means taking on additional risk. A risk-neutral bidder disregards the risk and skews to the maximum extent allowed by the reserve prices. When a bidder is risk-neutral, expected profits are linear. So long as the posterior mean of  $\rho_1$  is not equal to  $e_1$ , it will be optimal to place an extreme

skew. A bidder's allocation is monotone in his estimate: the higher his estimates of one type, the more he allocates his bid to the other type. Bidders whose estimates differ greatly from the creditor want to skew most aggressively. Since these bidders will also be the most optimistic about the gap between calculated bids and expected payments, it is natural that they should bid more as well. Bidders whose estimates differ greatly from the credit card company's estimates are most optimistic about their ability to drive a wedge between bid and payment. Interestingly, in equilibrium, bidders must skew to win the auction. If a bidder does not skew, a second bidder can earn greater profits by slightly raising the bid and skewing optimally. Following this logic, competition leads each bidder to choose  $B/Q_e$  greater than  $V/Q_e$ , the certain per unit value of winning when  $\Delta b = \Delta v$ .

## 4.6 Factors Influencing Effort

Once an agency is allocated accounts, the actions that it takes to retrieve the debt are determined by the payment that it has committed to, its beliefs about the account attributes and its cost structure. Beaudry (1994) studies a principal-agent moral hazard setting with an informed principal and shows that a principal who values effort highly will choose to induce effort by paying a high base wage and low bonuses. Chade and Silvers (2002) show that (i) a principal with a more informative technology ends up earning less profits than a principal with a less informative one does; (ii) compared to the complete information case, the actions implemented by the privately informed principal can be distorted; (iii) the agent can end up being better off when the principal has private information. However, the result holds in a contractual take-it-or-leave-it setting and not necessarily in an auctions setting.

In the context of auctions, an important insight from the security-bid literature is that if there are initial investments made by the agencies, the creditor can reimburse

the initial investment and claim almost all the revenues generated by the agency, guaranteeing individual rationality, and ensuring almost complete surplus extraction. DeMarzo, Kremer and Skrzypacz do point out though, that this might not be possible in the presence of moral hazard: if the winner's investment is not fully contractible, the winner might underinvest. They go on to show that if the investment is not contractible, reimbursement would not occur in equilibrium, or not be in the sellers best interest. It can also be shown that just as in cash auctions, sellers revenues can be increased by using reservation prices. The moral hazard setting for auctions of divisible goods has also been studied in Chapter 3.

In general, the creditor is limited in its ability to monitor the actions, otherwise it would be able to enforce contracts based not on the outcome but on the effort. The fact that the effort is usually not perfectly observable nor verifiable might make outcome contingent contractual provisions such as recall of the allocated portfolio or penalties in future auctions important. To a certain extent, the long-run nature of the creditor-agency relationship induces reputational concerns for the agency and alleviates some of the moral hazard concerns raised above. Through auditing procedures that many creditors put in place to monitor the actions of the agencies, deviating from expected effort levels is typically hard for agencies.

## 4.7 Learning and Repeated Interaction

The outcome of the collections problems give the creditor and the agency an opportunity to learn more about the type of accounts in the portfolio and help make better guesses about rewards expected from future portfolios. Since the cost associated with retrieving the delinquent debt is also not known upfront, the agency uses this to learn better its own cost structures to help future bids. Decisions made during the portfolio allocation stage. The creditor also learns about the agencies' beliefs through its



bids and makes future allocations based on this information.

Auction Feature	Recommendation
Format	First-Price
Bid Structure	Call Options
Commitment to format	Maximal
Information Revelation	Maximal
Repetition	Multiple Rounds

The performance of an agency can only be evaluated when it is assigned a portfolio. As pointed out in Chapter 2, in the case of repeated auctions, this has important consequences for the long term efficient and revenue maximizing auctions. The primary reason behind this is a *exploitation vs. experimentation* tradeoff. To elaborate, once certain agencies have been allocated the portfolio in some rounds of the auction, they have gathered more information about their own cost structures as well as reward distribution associated with the particular class of accounts. The credit card company wishes to use this information in making future allocations, but would also like to experiment with newer agencies so that in the long run, opportunities of high reward rates and low cost structures are not lost. To recall, this can be modeled as a *multi-armed bandit problem* under asymmetric information and the mechanism that achieves efficiency is a convex combinations of two one-sided incentive compatible constituent mechanisms which are similar to the second price auction, but instead of using the one period expected rewards as the metric to determine the winner, they use the Gittins index. Some of the assumptions implied in the model of chapter 2 that allow us to use the Gittins index for optimal allocations might not necessarily be applicable in the setting of the debt lease auction. In particular, switching costs between agencies, non stationarity of portfolio returns and common values are important deviations which might make the model restrictive. The costs that an agency incurs in acquiring and working on a new portfolio of accounts is largely determined

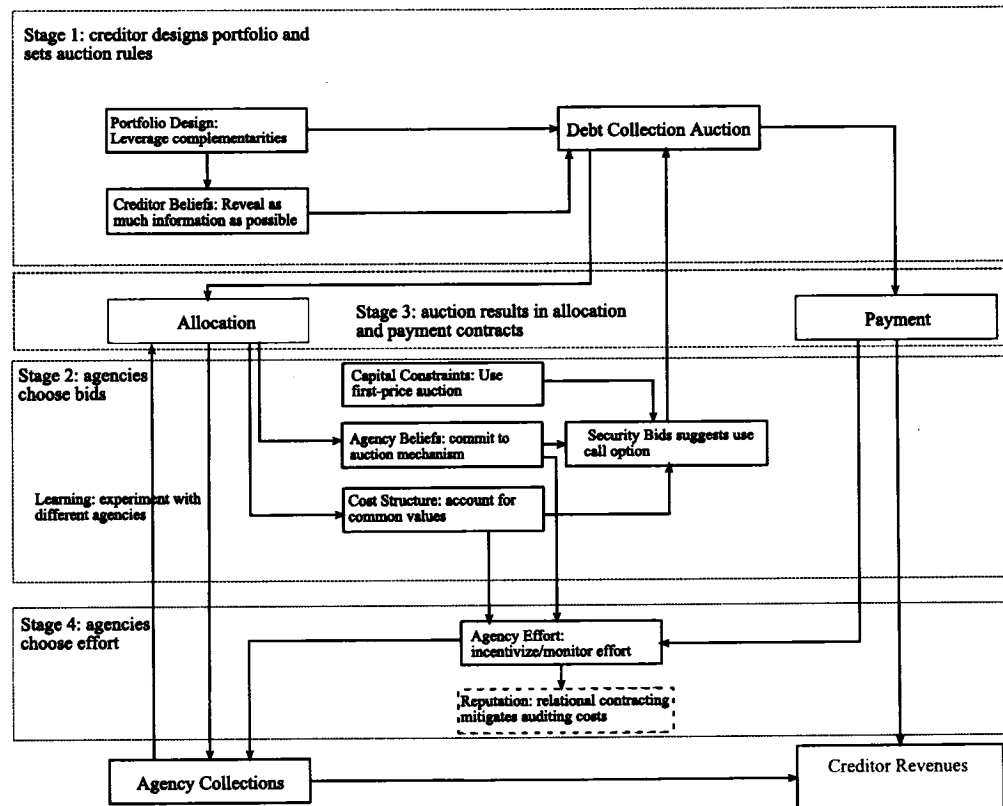


Figure 4.7: Some of the important insights drawn from the auction design literature in designing debt collection auctions.

by its capacity in terms of the number of collectors it hires and capital constraints that it faces. These costs and other overheads become clear only as the agency spends more time working on the new accounts and realizes the skill level of collectors and time required in collecting. Besides the overall size of the portfolio, the composition also impacts the costs in more ways than determining how many collectors of a certain skill are needed. It impacts it through making the portfolio more or less risky for the agency.

A highly segmented portfolio might lower agency costs through specialization but add substantial risk. This determines whether different components of the portfolio are substitutes or complements and thus largely impacts the outcome of the auction. Since the true composition is only revealed over time, the complementarities also become clear only as time progresses, and make learning all the more essential.

## 4.8 Chapter Summary

In this chapter we have studied some of the important practical aspects of a debt lease auction recently conducted by some credit card agencies. We explain the manner in which different factors influence the revenues of the creditor and draw on important recent results from the theory of auctions. Some of the important insights corresponding to each of the important factors influencing revenues are summarized in Figure 4.7.

## Chapter 5

# Discussion and Future Research

The *leitmotif* of this thesis has been to design mechanisms that can achieve efficiency in allocating resources. The first contribution is in designing a mechanism that achieves efficiency when players learn about their own valuations over a period of time through the use of the resource. The second contribution constructs a mechanism that aligns the profit maximizing incentives of resource allocators with efficiency in the case when goods are divisible. We also describe the important considerations in designing an auction for allocation of debt collection contracts. In this section discuss some of the possible directions for future research in these areas.

### 5.1 Dynamic Mechanism Design

The derivation for first mechanism began through the observation that in the absence of information asymmetries, the underlying problem reduces to the well known multi-armed bandit problem and hence the key issue in the asymmetric information case is to truthfully elicit the agents' Gittins index. Whereas two different mechanisms designed on the lines of a second-price auction failed to yield the requisite truthfulness, they guaranteed one-sided truthfulness. These then became the building blocks for a

mechanism that yielded incentive compatibility.

The multi-armed bandit problem has been studied extensively and has thus become a useful framework for to address questions involving dynamic decision making in the presence of uncertainty and learning. Albeit often computationally intense (cf. footnote 13), the multi-armed bandit framework has been applied in many diverse settings, ranging from machine scheduling in manufacturing (Gittins, 1989; Walrand, 1988; Van Oyen et al., 1992), job search in labor markets (Mortensen, 1986), search problems in oil exploration, target tracking (Gittins, 1989; DeGroot, 1970), resource allocation problems in communication networks (Ishikida, 1992), and industrial research under budget constraints (Gittins, 1989) to clinical trials (Katehakis and Derman, 1986).

In this thesis, we have used but the base vanilla flavor of the bandit problem where a fixed number of bandits earn independent rewards, time is in discrete periods, at most one bandit is experimented with at any time, only the bandit that is operated is allowed to change its state, there is no cost associated with switching between bandits, there are no budgets (or minimum reserves), and rewards evolve in a Markovian fashion. Relaxing these constraints has yielded a large variety of results on a rich class of bandit problems. As an example, the problem where bandits may change states even when they are not being experimented with, known as the ‘restless bandit problem,’ was first studied by Whittle (1988). This last framework is popularly chosen when information about one bandit’s reward rates influences the beliefs about others. This is the case when reward rates are correlated. When the number of agents is not fixed, but follows a birth-and-death process which depends on the bandit selected to operate, the problem is known as the ‘branching bandit problem’ and was first introduced by Meilijson and Weiss (1977). When rewards are non-Markovian, the problem can no longer be modeled as a classical Markov Decision Process. Mandelbaum (1986)

has studied the case of rewards modeled as multi-parameter processes such as diffusion processes. These problems are formulated as control problems over partially ordered sets (owing to the multi-dimensional nature of the parameters). Switching costs have been studied by Oyen et al. (1992), Banks and Sundaram (1994), Asawa and Teneketzis (1996), and others. There are other extensions also known as ‘arm-acquiring bandits,’ ‘superprocesses’ and ‘correlated bandits,’ which have been studied in the literature.

As we have seen through the mechanisms derived in this thesis, indexability is a crucial property of dynamic and stochastic optimization problems to design mechanisms which lead to optimal allocation of resources in multi-agent asymmetric information games. By an indexable bandit problem, we mean a bandit problem whose optimal solution is given by an index policy, i.e., at any stage the optimal policy is given by a comparison of indices computed for each bandit. A decomposable index policy is where a bandit’s index does not depend on any of the other bandits. As we have seen, the vanilla flavored problem can be solved using a decomposable Gittins index policy. A lot of the work done in trying to solve the various extensions of the bandit problem has tried to identify whether the optimal scheduling policy is an index policy. Not all of the the above extensions can be characterized by an index policy, let alone a decomposable one. Bertsimas and Niño Mora (1996) use a polyhedral approach to derive a general theory to characterize which bandit problems are indeed indexable. Extending the approach outlined in this thesis to other indexable problems would be a valuable next step.

## 5.2 Multi-Unit Auctions

This thesis has proposed a new mechanism for multi-unit auctions or auctions of divisible goods which aligns the revenue maximizing objectives of the seller with

efficiency to make it optimal for him to choose the efficient allocation after submission of bids. This was one of the problems that plagued the VCG mechanism when bidders might have complementarities in their preferences. The mechanism induces truthful revelation in dominant strategies subject to participation.

Despite the attractiveness of the dominant-strategy property of the VCG mechanism, it also has several possible weaknesses. As we had seen in example2, the auction might result in zero revenues, the seller's revenues might actually be non-monotonic in the set of bidders and the amounts bid, it is vulnerable to collusion by a coalition of losing bidders and it is vulnerable to shill bidding, or multiple bidding identities by a single bidder. In economic environments where every bidder has substitute preferences, the above listed weaknesses will never appear. However, if there is a single bidder whose preferences violate the substitutes condition, then with an appropriate choice of values for the remaining bidders, all of the above weaknesses will be presents. We have addressed the third issue here (where the coalition is the singleton seller). Whether our proposed mechanism addresses any of the other issues or not is still an open question and needs to be answered.

### 5.3 Debt Collection Contracts

This thesis captures some of the important factors that influence the revenues that a creditor expects when she auctions a debt collection contract to collection agencies. This fresh new practical problem has many interesting features that allow us to apply recent theory to provide insights that help design a revenue maximizing auction. There are some additional features such as the possibility of renegotiation and the fact that there are multiple auctioneers that also compete for the capacities of the agencies that are also very real and make the problem all the more interesting. Independently seen, many of the factors allow for precise modeling through existing models, however

### *5.3 Debt Collection Contracts*

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the optimal auction that accounts for all the practical details together eludes us and is an open field for new research.





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