Characteristics method for fluid/rigid systems
Application to self-propelled motions

J.-F. Scheid

Institut Elie Cartan de Nancy

Université de Lorraine, INRIA

Joint work with T. Takahashi, M. Tucsnak, (Nancy),
J. San Martín (University of Chile, Santiago),
L. Smaranda (University of Pitești, Romania)

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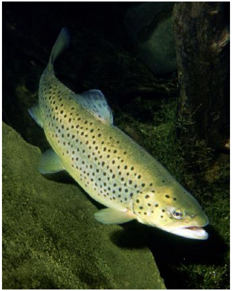
Introduction

Fluid-Structure interaction problems considered :

- Fluid/solid systems: rigid bodies immersed in a viscous fluid
- Fluid/deformable body

  Self-propulsion of a deformable body = motion by shape changes (bacteria, amoeba, ciliate, eel, fish, ...)

White blood cell vs bacterium
(David Rogers, Vanderbilt University, 1950)



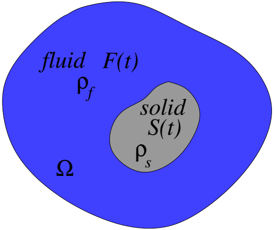
Outline

A. Rigid body system
   1. Monolithic finite elements scheme with a characteristics method
   2. Convergence result for the case $\rho_f = \rho_s$
   3. Modified characteristics method for the case $\rho_f \neq \rho_s$
   4. Link with the fictitious domain method

B. Self-propelled motion of deformable body
   1. Deformations
   2. The full system
   3. Numerical simulations for the fish-like swimming

A. The rigid body system

Fluid/rigid interaction. Rigid body $S(t)$ immersed into a fluid filling a bounded domain $F(t) = \Omega \setminus \overline{S}(t)$.



- ⋆ Incompressible viscous fluid : Navier-Stokes
- ⋆ Motion of the rigid body : Newton's laws
- ⋆ Continuity of the velocity fluid/rigid

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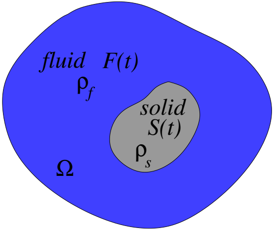

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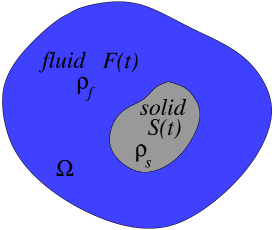

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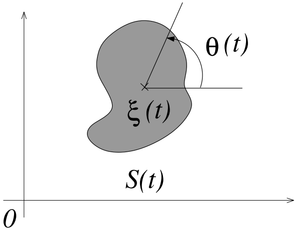
- The position of the body is unknown and results from the interaction with the fluid → free boundary problem.
- Constant densities ρ_f (fluid) and ρ_s (solid).

Fluid/rigid system

a- Rigid displacement (2D)

The **unknown** position of the rigid body $S(t)$ into the fluid is defined by :

- the position of the mass center $\boldsymbol{\xi}(t) \in \mathbb{R}^2$.
- an orientation angle $\theta(t)$.



Eulerian velocity field for the solid: for all $\mathbf{x} \in S(t), t \geq 0$

$$\mathbf{u}_S(\mathbf{x}, t) = \boldsymbol{\xi}'(t) + \theta'(t)(\mathbf{x} - \boldsymbol{\xi})^{\perp}$$

where $\mathbf{x}^{\perp} = (-x_2, x_1)^{\top}$.

b- The full system

$\Omega \subset \mathbb{R}^2$ is a bounded domain occupied by the fluid $F(t)$ together with the rigid body $S(t)$.

The **unknowns**:

- velocity $\mathbf{u}$ and pressure p into the fluid.
- position $\boldsymbol{\xi}$ of the mass center and orientation angle θ of the rigid body.

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The fluid:

- The Cauchy stress tensor

$$\boldsymbol{\sigma}(\mathbf{u}, p) = 2\nu D(\mathbf{u}) - pI_d$$

- $\nu > 0$ viscosity
- $D(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)/2$ symmetric strain tensor.

The full fluid/rigid system

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) - \nu \Delta \mathbf{u} + \nabla p = \rho_f \mathbf{f} \quad \text{in } F(t)$$

$$\text{div } \mathbf{u} = 0 \quad \text{in } F(t)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega$$

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$$m\boldsymbol{\xi}''(t) = -\int_{\partial S(t)} \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma + \int_{S(t)} \rho_s \mathbf{f}(\mathbf{x}, t) \, d\mathbf{x},$$

$$I\theta''(t) = -\int_{\partial S(t)} (\mathbf{x} - \boldsymbol{\xi})^{\perp} \cdot \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma + \int_{S(t)} \rho_s (\mathbf{x} - \boldsymbol{\xi})^{\perp} \cdot \mathbf{f}(\mathbf{x}, t) \, d\mathbf{x}$$

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$$\mathbf{u} = \boldsymbol{\xi}' + \theta' (\mathbf{x} - \boldsymbol{\xi})^{\perp} \text{ on } \partial S(t)$$

mass m and moment of inertia I of the rigid body.

Some numerical methods for fluid-rigid systems

– *ALE / moving mesh methods + fixed point.*
  Donea & Giuliani & Halleux 1982, Hughes & Liu & Zimmermann 1981, ..., Formaggia & Nobile 1999, Gastaldi 2001,2004, Legendre& Takahashi 2008

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– *Monolithic scheme (global weak formulation)*
• Distributed Lagrange multipliers/ fictitious domain, fixed mesh (eulerian method) : Glowinski & Pan & Hesla & Joseph & Périaux, 2000.
• San Martín & JFS & Takahashi & Tucsnak, 2005, San Martín & JFS & Loredana, 2011.
• Penalization methods and Level-set: Angot 1999, Bost & Cottet& Maitre 2010

1 - Monolithic finite elements scheme with characteristics

a- A global weak formulation

- Space of functions (velocity) which are *rigids* into the solid $S(t)$:

$$\mathcal{K}(t) = \{\mathbf{u} \in H_0^1(\Omega) \mid D(\mathbf{u}) = 0 \text{ in } S(t)\}$$

Remark: for all $\mathbf{v} \in \mathcal{K}(t)$, there exist $l_{\mathbf{v}} \in \mathbb{R}^2$ and $\omega_{\mathbf{v}} \in \mathbb{R}$ s.t.

$$\mathbf{v}(\mathbf{x}) = l_{\mathbf{v}} + \omega_{\mathbf{v}}(\mathbf{x} - \boldsymbol{\xi})^{\perp} \quad \forall \mathbf{x} \in S(t).$$

- The space for pressure :

$$\mathcal{M}(t) = \left\{ p \in L^2(\Omega) \mid \int_{\Omega} p \, d\mathbf{x} = 0, \; p = 0 \text{ in } S(t) \right\}$$

- Bilinear forms

$$a(\mathbf{u},\mathbf{v}) = 2\nu \int_{\Omega} D(\mathbf{u}) : D(\mathbf{v})\, d\mathbf{x} \quad \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega),$$

$$b(\mathbf{u},q) = -\int_{\Omega} (\mathrm{div}\,\mathbf{u}) q\, d\mathbf{x} \quad \forall \mathbf{u} \in H_0^1(\Omega),\ q \in L^2(\Omega)$$

- Material derivative of **v** associated to the velocity **u**

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{v}$$

Let $(\mathbf{u}, p, \boldsymbol{\xi}, \theta)$ be a strong solution to the fluid-rigid system.
$\mathbf{u}$ and p are extended to the whole domain Ω :

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\xi}'(t) + \theta'(t)(\mathbf{x} - \boldsymbol{\xi}(t))^{\perp} \quad \text{if } \mathbf{x} \in S(t),$$

$$p(\mathbf{x}, t) = 0 \quad \text{if } \mathbf{x} \in S(t).$$

Global mixed formulation.

For all $t \in (0, T)$,

$$\begin{array}{rcll} \displaystyle\int_\Omega \rho \frac{d\mathbf{u}}{dt} \cdot \mathbf{v}\, d\mathbf{x} + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & \displaystyle\int_\Omega \rho \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} & \forall \mathbf{v} \in \mathcal{K}(t) \\ b(\mathbf{u}, q) & = & 0 & \forall q \in \mathcal{M}(t) \end{array}$$

$$\text{density} \quad \rho(\cdot, t) = \begin{cases} \rho_f & \text{in } F(t) \\ \rho_s & \text{in } S(t) \end{cases}$$

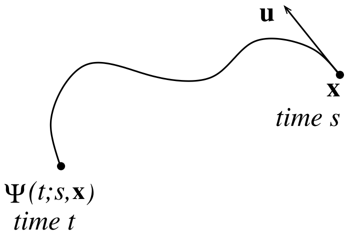
Material derivative of **u** and the characteristic function.

$$\frac{d\mathbf{u}}{dt}(\mathbf{x}, t_0) = \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)(\mathbf{x}, t_0) = \frac{d}{dt}[\mathbf{u}(\psi(t; t_0, x), t)]_{|t=t_0}$$

The characteristic function $\psi : (\mathbb{R}^+)^2 \times \Omega \to \Omega$ is the solution of

$$\begin{aligned} \frac{d}{dt}\psi(t; s, \mathbf{x}) &= \mathbf{u}(\psi(t; s, \mathbf{x}), t) \\ \psi(s; s, \mathbf{x}) &= \mathbf{x} \end{aligned}$$

$\psi(t; s, \mathbf{x})$ is the position of a particle at time t which is in $\mathbf{x}$ at time s.



b- Space and time discretisation - monolithic scheme

Triangulation $\mathcal{T}_h$ of the complete domain Ω (fluid+solid) $\rightarrow$ fixed mesh

Approximations with $\mathcal{T}_h$: velocity $\mathbf{u}_h^k \simeq \mathbf{u}(\cdot, t^k)$

pressure $p_h^k \simeq p(\cdot, t^k)$

mass center $\boldsymbol{\xi}_h^k \simeq \boldsymbol{\xi}(t^k)$

orientation angle $\theta_h^k \simeq \theta(t^k)$

Finite elements space for the velocity $\mathcal{K}_h^k \simeq \mathcal{K}(t^k)$

Finite elements space for the pressure $\mathcal{M}_h^k \simeq \mathcal{M}(t^k)$

Suppose the approximate solution $(\mathbf{u}_h^k, p_h^k, \boldsymbol{\xi}_h^k, \theta_h^k)$ is known at time $t = t^k$.
We aim to compute the solution at time $t = t^{k+1}$.

1. *Computation of the new position of the mass center and the new orientation angle of the solid.*

$$\boldsymbol{\xi}_h^{k+1} = \boldsymbol{\xi}_h^k + \Delta t\, \mathbf{u}_h^k(\boldsymbol{\xi}_h^k),$$

$$\theta_h^{k+1} = \theta_h^k + \frac{\Delta t}{I} \int_{S_h^k} \rho_s(\mathbf{u}_h^k(\mathbf{x}) - \mathbf{u}_h^k(\boldsymbol{\xi}_h^k)) \cdot (\mathbf{x} - \boldsymbol{\xi}_h^k)^{\perp}\, d\mathbf{x} \quad \text{with } S_h^k = S(\boldsymbol{\xi}_h^k, \theta_h^k)$$

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2. Monolithic scheme: *Global computation of the velocity* $\mathbf{u}_h^{k+1} \in \mathcal{K}_h^{k+1}$ *and the pressure* $p_h^{k+1} \in \mathcal{M}_h^{k+1}$.

   $\rightarrow$ approximations of the material derivative and the characteristic function

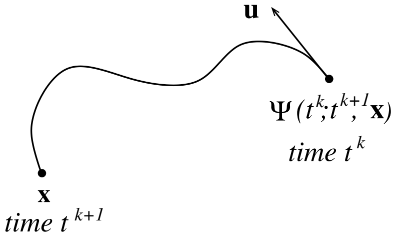
Time approximation of the characteristic function :

$$\frac{d\mathbf{u}}{dt}(\mathbf{x}, t^{k+1}) = \frac{d}{dt}\left[\mathbf{u}(\psi(t; t^{k+1}, \mathbf{x}), t)\right]_{|t=t^{k+1}}$$

$$\simeq \frac{\mathbf{u}(\mathbf{x}, t^{k+1}) - \mathbf{u}(\psi(t^k, t^{k+1}, \mathbf{x}), t^k)}{\Delta t}$$

$$\text{where} \begin{cases} \dfrac{d}{dt}\psi(t; t^{k+1}, \mathbf{x}) = \mathbf{u}(\psi(t; t^{k+1}, \mathbf{x}), t), \quad t \in [t^k, t^{k+1}] \\ \psi(t^{k+1}; t^{k+1}, \mathbf{x}) = \mathbf{x}. \end{cases}$$

$\psi(t^k; t^{k+1}, \mathbf{x})$ is the (exact) position of a particle at time t^k which is in $\mathbf{x}$ at time t^{k+1}.



2 - The homogeneous case of a constant density function ($\rho_f = \rho_s$)

[SMSTT05] J. San Martín, J.-F. Scheid, T. Takahashi and M. Tucsnak, Convergence of the Lagrange-Galerkin method for the equations modelling the motion of a fluid-rigid system, SIAM J. Numer. Anal., Vol. 43 (2005).

We assume that $\rho_f = \rho_s = 1$.

Find $\mathbf{u}_h^{k+1} \in \mathcal{K}_h^{k+1}$, $p_h^{k+1} \in \mathcal{M}_h^{k+1}$ s.t.

$$\begin{aligned}
\left(\frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t}, \phi \right) + a(\mathbf{u}_h^{k+1}, \phi) + b(\phi, p_h^{k+1}) &= (\mathbf{f}_h^{k+1}, \phi) \quad \forall \phi \in \mathcal{K}_h^{k+1}, \\
b(\mathbf{u}_h^{k+1}, q) &= 0 \quad \forall q \in \mathcal{M}_h^{k+1},
\end{aligned}$$

where $\overline{\mathbf{X}}_h^k$ is the approximate characteristic function at time t^k.

$\overline{\mathbf{X}}_h^k$ is the approximate position at time t^k of a particle which is in $\mathbf{x}$ at time t^{k+1}.

$$\overline{\mathbf{X}}_h^k = \tilde{\psi}(t^k; t^{k+1}, \cdot) \simeq \psi(t^k, t^{k+1}, \cdot)$$

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\tilde{\psi}(t; t^{k+1}, \mathbf{x}) & = & P\mathbf{u}_h^k(\tilde{\psi}(t; t^{k+1}, \mathbf{x})), \\ \tilde{\psi}(t^{k+1}; t^{k+1}, \mathbf{x}) & = & \mathbf{x}. \end{array} \right.$$

P is the orthogonal projector on the finite elements space

$$\mathcal{R}_h = \{\mathbf{rot}\,\varphi_h,\ \varphi_h \in E_h,\ \varphi_h = 0 \text{ on } \partial\Omega\}.$$

Interest of P : $\operatorname{div}(P\mathbf{u}_h^k) = 0$ in Ω (fluid+solid) and since $P\mathbf{u}_h^k = 0$ on $\partial\Omega$, we have $\overline{\mathbf{X}}_h^k(\Omega) = \Omega$.

The finite elements spaces $\mathcal{K}_h^k$ and $\mathcal{M}_h^k$.

Finite elements for Stokes: $\mathbb{P}_1$-bubble/$\mathbb{P}_1$, $\mathbb{P}_2/\mathbb{P}_1$, $\mathbb{P}_1$-iso-$\mathbb{P}_2/\mathbb{P}_1$, $\cdots$

- E_h : space of approximations for the pressure in the (classical) Stokes problem

  $E_h =$ space of $\mathbb{P}_1$ functions.

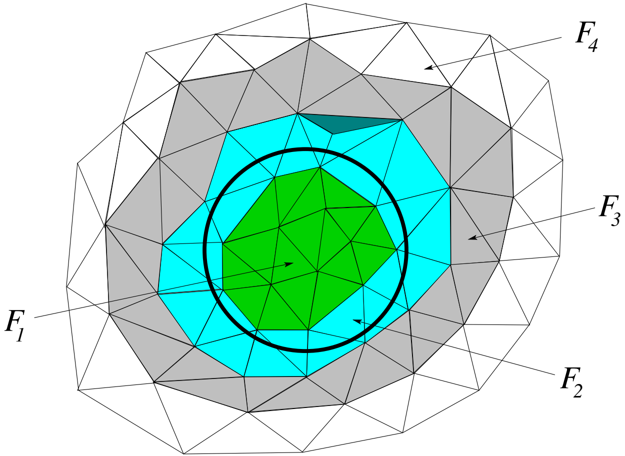
- W_h : space of approximations for the velocity in the (classical) Stokes problem

  $W_h =$ space of $\mathbb{P}_1$-bubble functions.

Conforming approximations:

$$\begin{aligned}\mathcal{M}_h^k &= E_h \cap \mathcal{M}(t^k) \quad \text{where } \mathcal{M}(t^k) = \mathcal{M}(\xi_h^k, \theta_h^k)\\ \mathcal{K}_h^k &= W_h \cap \mathcal{K}(t^k) \quad \text{where } \mathcal{K}(t^k) = \mathcal{K}(\xi_h^k, \theta_h^k)\end{aligned}$$

Let A_h^k be the union of all the triangles which intersect the solid body $S(\xi_h^k, \theta_h^k)$ (green and blue triangles)



Every function of $\mathcal{M}_h^k$ is null in A_h^k

Every function of $\mathcal{K}_h^k$ is rigid in A_h^k.

Convergence result ([San Martín-S. -Takahashi-Tucsnak, 2005])

Theorem . Let $C_0 > 0$ be a fixed constant. We suppose that Ω is polygonal convex and that the solution $(\mathbf{u}, p, \boldsymbol{\xi}, \theta)$ of the fluid-rigid system is regular. Assume that S is a *ball* and that $\text{dist}(S, \partial\Omega) > 0$ for all $t \in [0, T]$. We also assume that $\rho_s = \rho_f$. Then, there exist 2 constants $C > 0$ and $\tau^* > 0$ independents of h and Δt s.t. for all $0 < \Delta t \leq \tau^*$ and for all $h \leq C_0 \, (\Delta t)^2$, we have

$$\sup_{1 \leq k \leq N} \left(|\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + \|\mathbf{u}(t^k) - \mathbf{u}_h^k\|_{L^2(\Omega)} \right) \leq C \Delta t.$$

Remark. For the Navier-Stokes equations, alone without any rigid, the same type of result holds for $h \leq C_0 \Delta t$ ([Pironneau, 1982]) and for $h^2 \leq C_0 \Delta t \leq C_1 h^\sigma$ $(\sigma > 1/2)$ ([Süli, 1988]).

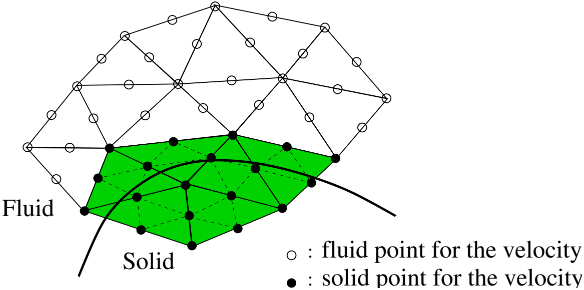
Key ingredient for the proof:
The velocity functions $\mathbf{u}(t^k)$ and $\mathbf{u}_h^k$ **are not rigids at the same place** $\Rightarrow$ difficult to compare them.

We transform

$$\mathbf{u}(t^k) \in \mathcal{K}(\boldsymbol{\xi}(t^k)) \longrightarrow \mathbf{U}^k \in \mathcal{K}(\boldsymbol{\xi}_h^k).$$

We are then able to compare $\mathbf{U}^k$ with $\mathbf{u}_h^k$ since they are both functions in $\mathcal{K}(\boldsymbol{\xi}_h^k)$ i.e. rigid velocities in $S(\boldsymbol{\xi}_h^k)$

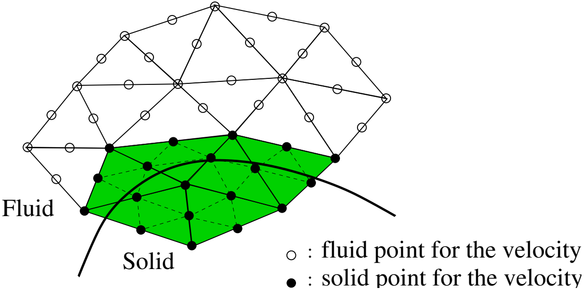
Practical aspects



Rigid basis functions : $\mathbf{l}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{l}_3 = (\mathbf{x} - \xi_h^k)^\perp.$

Pressure and velocity basis functions on Ω : $\varphi_i \in \mathbb{P}_1, \ \ \phi_i \in \mathbb{P}_2$

Practical aspects



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Pressure and velocity basis functions on Ω : $\varphi_i \in \mathbb{P}_1, \;\; \phi_i \in \mathbb{P}_2$

- *Approximate space for the pressure*: $\mathcal{M}_h^k(\xi_h^k) = \text{span}\left\{(\varphi_i)_{i\in\text{fluid}}\right\}$
- *Approximate space for the velocity*:

$$\mathcal{K}_h^k(\xi_h^k) = \text{span}\left\{(\phi_i)_{i\in\text{fluid}}, \; \Big(\sum_{i\in\text{rigid}} \varphi_i \mathbf{l}_j\Big)_{j=1,2,3}\right\} \subset \mathbb{P}_2 \cap \mathcal{K}(\xi_h^k, \theta_h^k)$$

$$\begin{aligned} \mathbf{u}_h^k &= \sum_{i \in \text{fluid}} \mathbf{u}_i \phi_i + \sum_{j=1}^{3} \eta_j \mathbf{l}_j^k \sum_{i \in \text{rigid}} \varphi_i \\ p_h^k &= \sum_{i \in \text{fluid}} p_i \varphi_i \end{aligned}$$

$(\eta_1, \eta_2) = \mathbf{u}_h^k(\boldsymbol{\xi}_h^k) \simeq \boldsymbol{\xi}'(t^k)$ velocity of the mass center

$\eta_3 \simeq \theta'(t^k) =$ angular velocity

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<u>*Linear system*</u>. $U = (\underbrace{\mathbf{u}_i}_{2N_f}, \eta_1, \eta_2, \eta_3)^T \in \mathbb{R}^{2N_f+3}$, $P = (p_i) \in \mathbb{R}^{M_f}$

Basis functions of the rigid space $\mathcal{K}_h^k$: $\psi_i \in \mathbb{P}_2 \Rightarrow \psi_i = \sum_j \mu_{ij}\phi_j$

$$
\begin{aligned}
\mathbf{u}_h^k &= \sum_{i \in \text{fluid}} \mathbf{u}_i \phi_i + \sum_{j=1}^{3} \eta_j \mathbf{l}_j^k \sum_{i \in \text{rigid}} \varphi_i \\
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Basis functions of the rigid space $\mathcal{K}_h^k$: $\psi_i \in \mathbb{P}_2 \Rightarrow \psi_i = \sum_j \mu_{ij} \phi_j$

$$
\Rightarrow \boxed{\mathcal{M} \underbrace{\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}}_{\text{Stokes matrix on } \Omega} \mathcal{M}^T \begin{pmatrix} U \\ P \end{pmatrix} = \mathcal{M} \begin{pmatrix} L \\ 0 \end{pmatrix}}
$$

The transformation matrix $\mathcal{M}$ is of size $(2N_f + 3 + M_f) \times (2N + M)$

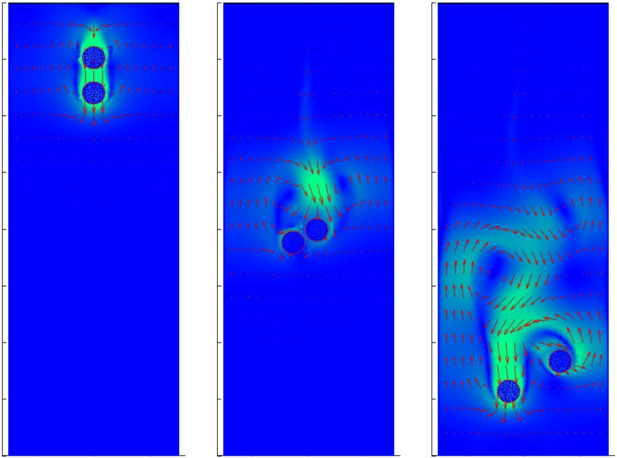
$$
\mathcal{M} = \left(\begin{array}{c|ccc|c|ccc|c|c}
\overbrace{\qquad}^{N_f} & & \overbrace{\qquad}^{N_s} & & \overbrace{\qquad}^{N_f} & & \overbrace{\qquad}^{N_s} & & \overbrace{\qquad}^{M_f} & \overbrace{\qquad}^{M_s} \\
I_2 & & 0 & & 0 & & 0 & & 0 & 0 \\
\hline
0 & & 0 & & I_2 & & 0 & & 0 & 0 \\
\hline
0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 0 \\
0 & \alpha_1 & \cdots & \alpha_1 & 0 & \alpha_2 & \cdots & \alpha_2 & 0 & 0 \\
\hline
0 & & 0 & & 0 & & 0 & & I_1 & 0
\end{array}\right)
$$

I_2 and I_1 are identity matrices of order N_f and M_f respectively and $(\alpha_1, \alpha_2) = (\mathbf{x}(P_i) - \boldsymbol{\xi}_h^k)^{\perp}$.

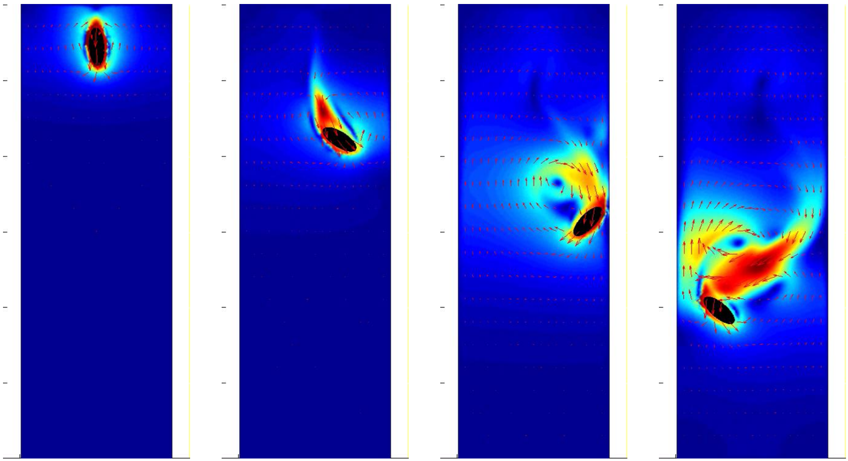
Numerical simulations ($\rho_f = \rho_s$)

Several bodies are easily handled
$\mathbf{f} = \mathbf{0}$ into the fluid, $\mathbf{f} = (0, \ -g)^\top$ into the solid.

- free fall of rigid balls in a fluid (tumbling)



- Freely falling ellipse (unstable vs 'more stable')



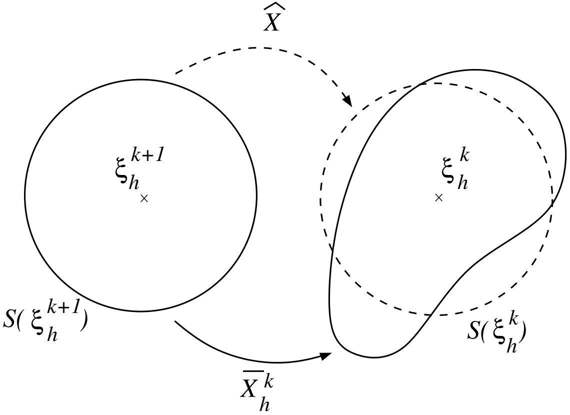
3 - The case of a discontinuous density function

Now, we assume that $\rho_f \neq \rho_s$.

First idea : extend the previous scheme (for $\rho_f = \rho_s$) by approximating the inertial term in the Navier-Stokes equations (material derivative) as:

$$\rho_h^{k+1} \frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t} \quad \text{with } \rho_h^{k+1}(\mathbf{x}) = \begin{cases} \rho_s \text{ if } \mathbf{x} \in S(\boldsymbol{\xi}_h^{k+1}) \\ \rho_f \text{ elsewhere} \end{cases}$$

$\overline{\mathbf{X}}_h^k$ is the approximate characteristics function for the case $\rho_f = \rho_s$.



$$\rho_h^k(\mathbf{x}) = \begin{cases} \rho_s & \text{if } \mathbf{x} \in S(\boldsymbol{\xi}_h^k) \\ \rho_f & \text{elsewhere} \end{cases}$$

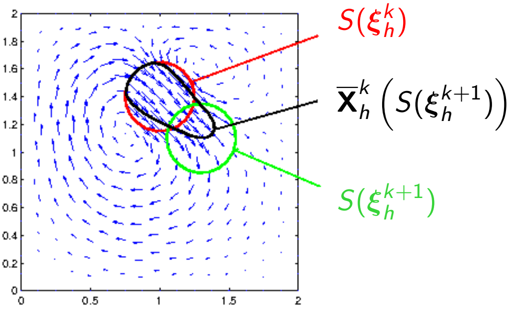
Difficulties :

- The error estimate $\|\mathbf{u}(t^k) - \mathbf{u}_h^k\|_{L^2(\Omega)}$ involves the term

$$\|\rho_h^{k+1} - \rho_h^k \circ \overline{\mathbf{X}}_h^k\|_{L^2(\Omega)} \quad (\text{with } \rho_h^{k+1} = \rho_h^k \circ \widehat{\mathbf{X}})$$

- Generally $\overline{\mathbf{X}}_h^k\left(S(\boldsymbol{\xi}_h^{k+1})\right) \not\subseteq S(\boldsymbol{\xi}_h^k)$ and $\rho_h^{k+1} \neq \rho_h^k \circ \overline{\mathbf{X}}_h^k$.

Numerical example of $\overline{\mathbf{X}}_h^k\left(S(\xi_h^{k+1})\right) \not\subseteq S(\xi_h^k)$



➩ we do not know how to estimate the term $\|\rho_h^{k+1} - \rho_h^k \circ \overline{\mathbf{X}}_h^k\|_{L^2(\Omega)}$ ...

The modified characteristics function

1. We define $\overline{\mathbf{X}}_h^k = \overline{\psi}(t^k; t^{k+1}, \cdot) \simeq \psi(t^k, t^{k+1}, \cdot)$

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\overline{\psi}(t; t^{k+1}, \mathbf{x}) & = & P(\boldsymbol{\xi}_h^k) \left(\mathbf{u}_h^k(\overline{\psi}(t; t^{k+1}, \mathbf{x})) - \mathbf{u}_h^k(\boldsymbol{\xi}_h^k) \right), \\ \overline{\psi}(t^{k+1}; t^{k+1}, \mathbf{x}) & = & \mathbf{x} - \mathbf{u}_h^k(\boldsymbol{\xi}_h^k)\Delta t \end{array} \right.$$

$P(\boldsymbol{\xi})$ is the orthogonal projector on the finite elements space

$$\mathcal{R}_h(\boldsymbol{\xi}) = \{\mathbf{rot}\,\varphi_h,\ \varphi_h \in E_h,\ \varphi_h = 0 \text{ on } \partial\Omega\} \cap \mathcal{K}(\boldsymbol{\xi})$$

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2. The inertial term in Navier-Stokes equations is approximated by

$$\rho_h^{k+1}\frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t} \quad \text{with } \rho_h^{k+1}(\mathbf{x}) = \begin{cases} \rho_s \text{ if } \mathbf{x} \in S(\boldsymbol{\xi}_h^{k+1}) \\ \rho_f \text{ elsewhere} \end{cases}$$

Properties of the modified characteristics function

1. $\overline{\mathbf{X}}_h^k\big(S(\boldsymbol{\xi}_h^{k+1})\big) = S(\boldsymbol{\xi}_h^k)$.

2. If the density function ρ_h^k is extended to ρ_f outside Ω, then $\rho_h^{k+1} = \rho_h^k \circ \overline{\mathbf{X}}_h^k$.

From 2), we have $\|\rho_h^{k+1} - \rho_h^k \circ \overline{\mathbf{X}}_h^k\|_{L^2(\Omega)} = 0$!!!

Convergence result for the modified characteristics with $\rho_f \neq \rho_s$ ([San Martín-S.-Smaranda, 2011])

Theorem. We suppose that Ω is polygonal, convex and that $(\mathbf{u}, p, \boldsymbol{\xi}, \theta)$ is a regular (exact) solution to the fluid-rigid system. We assume that S is a ball and that $\mathrm{dist}(S, \partial\Omega) > 0$ for all $t \in [0, T]$. Let $C_0 > 0$ and $0 < \alpha \leq 1$ be fixed constants. Then, there exist 2 constants $C > 0$ and $\tau^* > 0$ independents of h and Δt s.t. for all $0 < \Delta t \leq \tau^*$ and for all $h \leq C_0 (\Delta t)^{1+\alpha}$, we have

$$\sup_{1 \leq k \leq N} \left(|\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + \|\mathbf{u}(t^k) - \mathbf{u}_h^k\|_{L^2(\Omega)} \right) \leq C \Delta t^{\alpha}.$$

Remark. For $\alpha = 1$, we get the convergence order of the scheme for the case $\rho_f = \rho_s$.

4 - Link with the fictitious domain method

For every $\varphi \in \mathcal{K}(\boldsymbol{\xi})$

$$\int_\Omega \rho \frac{d\mathbf{u}}{dt} \cdot \boldsymbol{\varphi}\, d\mathbf{x} = \rho_f \int_\Omega \frac{d\mathbf{u}}{dt} \cdot \boldsymbol{\varphi}\, d\mathbf{x} + \left(1 - \frac{\rho_f}{\rho_s}\right) \rho_s \int_{S(t)} \frac{d\mathbf{u}}{dt} \cdot \boldsymbol{\varphi}\, d\mathbf{x}$$

Any rigid function $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\xi})$ can be decomposed as $\boldsymbol{\varphi}(\mathbf{x}) = \mathbf{l}_\varphi + \omega_{\boldsymbol{\varphi}}(\mathbf{x} - \boldsymbol{\xi})^\perp$. This implies

$$\int_\Omega \rho \frac{d\mathbf{u}}{dt} \cdot \boldsymbol{\varphi}\, d\mathbf{x} = \rho_f \int_\Omega \frac{d\mathbf{u}}{dt} \cdot \boldsymbol{\varphi}\, d\mathbf{x} + \left(1 - \frac{\rho_f}{\rho_s}\right) \left(m\boldsymbol{\xi}''(t) \cdot \mathbf{l}_\varphi + I\theta''(t)\omega_{\boldsymbol{\varphi}}\right)$$

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Fictitious domain formulation

For all t, find $\mathbf{u}(t) \in \mathcal{K}(\boldsymbol{\xi})$, $p(t) \in \mathcal{M}(\boldsymbol{\xi})$, $\boldsymbol{\xi}(t)$ such that

$$\begin{aligned} \rho_f(\frac{d\mathbf{u}}{dt}, \boldsymbol{\varphi}) + \left(1 - \tfrac{\rho_f}{\rho_s}\right) \left(m\boldsymbol{\xi}''(t) \cdot \mathbf{l}_\varphi + I\theta''(t)\omega_{\boldsymbol{\varphi}}\right) + a(\mathbf{u}, \varphi) + b(\varphi, p) &= (\rho\mathbf{f}, \boldsymbol{\varphi}) \\ b(\mathbf{u}, q) &= 0 \end{aligned}$$

for all $\varphi \in \mathcal{K}(\boldsymbol{\xi})$ *and* $q \in \mathcal{M}(\boldsymbol{\xi})$.

Rewriting the modified characteristics method

Find $\mathbf{u}_h^{k+1} \in \mathcal{K}_h^{k+1}$, $p_h^{k+1} \in \mathcal{M}_h^{k+1}$ s.t.

$$
\begin{aligned}
\left(\rho_h^{k+1} \frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t}, \phi\right) + a(\mathbf{u}_h^{k+1}, \phi) + b(\phi, p_h^{k+1}) &= (\rho_h^{k+1} \mathbf{f}_h^{k+1}, \phi) \\
b(\mathbf{u}_h^{k+1}, q) &= 0
\end{aligned}
$$

for all $\phi \in \mathcal{K}_h^{k+1}$ and $q \in \mathcal{M}_h^{k+1}$ with $\rho_h^k(\mathbf{x}) = \begin{cases} \rho_s \text{ if } \mathbf{x} \in S(\boldsymbol{\xi}_h^k) \\ \rho_f \text{ elsewhere} \end{cases}$

Rewriting the modified characteristics method

Find $\mathbf{u}_h^{k+1} \in \mathcal{K}_h^{k+1}$, $p_h^{k+1} \in \mathcal{M}_h^{k+1}$ s.t.

$$\begin{aligned}\left(\rho_h^{k+1}\frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t}, \phi\right) + a(\mathbf{u}_h^{k+1}, \phi) + b(\phi, p_h^{k+1}) &= (\rho_h^{k+1}\mathbf{f}_h^{k+1}, \phi) \\ b(\mathbf{u}_h^{k+1}, q) &= 0\end{aligned}$$

for all $\phi \in \mathcal{K}_h^{k+1}$ and $q \in \mathcal{M}_h^{k+1}$ with $\rho_h^k(\mathbf{x}) = \begin{cases} \rho_s \text{ if } \mathbf{x} \in S(\boldsymbol{\xi}_h^k) \\ \rho_f \text{ elsewhere} \end{cases}$

We have $\left(\rho_h^{k+1}\frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t}, \phi\right) =$

$$\left(\rho_f\frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t}, \phi\right) + \left(1 - \frac{\rho_f}{\rho_s}\right)\rho_s \int_{S(\boldsymbol{\xi}_h^{k+1})} \frac{(\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\mathbf{X}}_h^k)}{\Delta t} \cdot \phi \, d\mathbf{x}$$

$$\left(\rho_h^{k+1}\frac{(\mathbf{u}_h^{k+1}-\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k)}{\Delta t},\phi\right)=\left(\rho_f\frac{(\mathbf{u}_h^{k+1}-\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k)}{\Delta t},\phi\right)$$

$$+\left(1-\frac{\rho_f}{\rho_s}\right)\left(m\frac{\mathbf{u}_h^{k+1}(\xi_h^{k+1})-\mathbf{u}_h^k(\xi_h^k)}{\Delta t}\cdot\mathbf{l}_\phi+I\,\frac{\omega_{\mathbf{u}_h^{k+1}}-\omega_{\mathbf{u}_h^k}\cos(\theta_k)}{\Delta t}\omega_\phi\right)$$

with $\theta_k=-\omega_{P\mathbf{u}_h^k}\Delta t$.

$$\left(\rho_h^{k+1}\frac{(\mathbf{u}_h^{k+1}-\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k)}{\Delta t},\phi\right)=\left(\rho_f\frac{(\mathbf{u}_h^{k+1}-\mathbf{u}_h^k\circ\overline{\mathbf{X}}_h^k)}{\Delta t},\phi\right)$$

$$+\left(1-\frac{\rho_f}{\rho_s}\right)\left(m\frac{\mathbf{u}_h^{k+1}(\xi_h^{k+1})-\mathbf{u}_h^k(\xi_h^k)}{\Delta t}\cdot\mathbf{l}_\phi+I\,\frac{\omega_{\mathbf{u}_h^{k+1}}-\omega_{\mathbf{u}_h^k}\cos(\theta_k)}{\Delta t}\omega_\phi\right)$$

with $\theta_k=-\omega_{P\mathbf{u}_h^k}\Delta t$.

This is a discretisation of the fictitious domain formulation

$$\rho_f(\frac{d\mathbf{u}}{dt},\boldsymbol{\varphi})+\left(1-\frac{\rho_f}{\rho_s}\right)\left(m\xi''(t)\cdot\mathbf{l}_{\boldsymbol{\varphi}}+I\theta''(t)\omega_{\boldsymbol{\varphi}}\right)$$
$$+a(\mathbf{u},\varphi)+b(\varphi,p)=(\rho\mathbf{f},\boldsymbol{\varphi})$$

B. An extended model for deformable bodies

Self-propulsion by shape changes = ability for a body to move into a fluid by changing its shape (deformation)
The self-propulsion is the result of the interaction between the deformation and the fluid

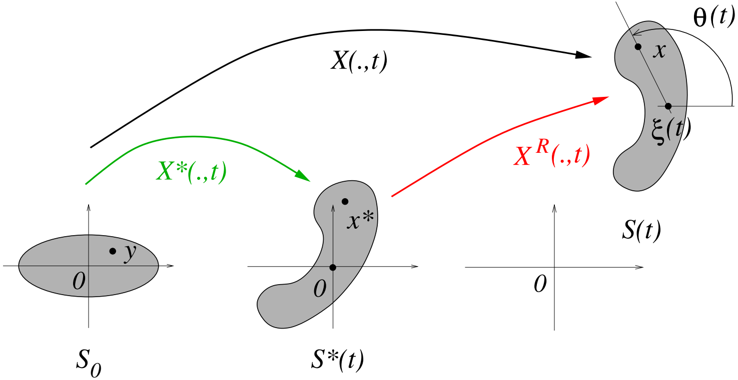
Main hypothesis : The deformation is given once for all and can't be modified by the fluid flow

1- A kinetic model for the deformation (2D)

$\mathcal{S}_0$: reference body.

The transformation X of $\mathcal{S}_0$ is regular

$$\boxed{X(\mathbf{y},t) = X^R(X^*(\mathbf{y},t),t)} \qquad \mathbf{y} \in \mathcal{S}_0, \quad t \geq 0.$$



X^* : deformation of the body (known)

X^R : rigid displacement (**unknown**)

Hypothesis for the deformation X^* :

- The total volume and the total mass of the body are conserved.
- The deformation X^* does not modify the linear and angular momenta of the body. In particular, the mass center of the the deformed body S^* does not change.

The unknowns :

- $\boldsymbol{\xi}(t)$: mass center of the deformable body.
- $\theta(t)$: orientation angle (R_θ : rotation matrix)

The eulerian velocity field of the body $\mathcal{S}(t)$ is:

$$\mathbf{u}_S(\mathbf{x}, t) = \underbrace{\boldsymbol{\xi}'(t) + \theta'(t)(\mathbf{x} - \boldsymbol{\xi}(t))^{\perp}}_{\text{rigid displacement}} + \underbrace{\mathbf{w}(\mathbf{x}, t)}_{\text{deformation}}$$

where

$$\mathbf{w}(\mathbf{x}, t) = R_{\theta(t)}\mathbf{w}^*(\mathbf{x}^*, t), \quad \mathbf{x} = X^R(\mathbf{x}^*, t)$$

and $\mathbf{w}^*(\mathbf{x}^*, t) = \dfrac{\partial X^*}{\partial t}(\mathbf{y}, t)$ is the deformation velocity (**known**).

2- The full system fluid/deformation

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) - \nu \Delta \mathbf{u} + \nabla p = \rho_f \mathbf{f} \quad \text{in } F(t)$$

$$\text{div } \mathbf{u} = 0 \quad \text{in } F(t)$$

$$\mathbf{u} = \boldsymbol{\xi}' + \theta' (\mathbf{x} - \boldsymbol{\xi})^{\perp} + \mathbf{w} \quad \text{on } \partial S(t)$$

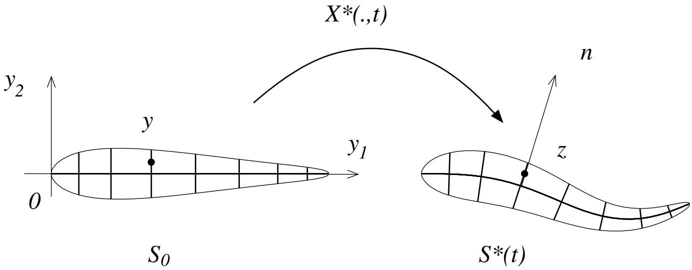
$$m\boldsymbol{\xi}'' = -\int_{\partial S(t)} \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma + \int_{S(t)} \rho_s \mathbf{f}(\mathbf{x}, t) \, d\mathbf{x}$$

$$(I\theta')' = -\int_{\partial S(t)} (\mathbf{x} - \boldsymbol{\xi})^{\perp} \cdot \boldsymbol{\sigma}(\mathbf{u}, p)\mathbf{n} \, d\Gamma + \int_{S(t)} \rho_s (\mathbf{x} - \boldsymbol{\xi})^{\perp} \cdot \mathbf{f}(\mathbf{x}, t) \, d\mathbf{x}$$

$\Rightarrow$ same global formulation as in the rigid body case.

3 - Fish-like swimming simulations

A geometric beam model for X^*: during the deformation, the cross-sections remain perpendicular to the middle line (Kirchhoff's model)



The mean curvature of the middle line is prescribed (carangiform swimming) :

$$K(y_1, t) = (a_2 y_1^2 + a_1 y_1 + a_0) \sin 2\pi(y_1/\lambda - t/T) + a_3$$

Numerical simulations