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From the free surface flow of a viscoelastic fluid towards the elastic deformation of a solid

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From the free surface flow of a viscoelastic fluid towards the elastic deformation of a solid

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Résumé

La déformation d'un solide élastique, à partir de l'écoulement à frontière libre d'un fluide viscoélastique. *Pour citer cet article : A. Name1, A. Name2, C. R. Mecanique 333 (2005).*

Abstract

The free surface flow of an Oldroyd-B viscoelastic fluid is considered, following [1]. When removing a term in the extra-stress constitutive relation, the description of an elastic incompressible solid is obtained, in Eulerian coordinates, with the velocity field as unknown, rather than the usual deformation field.

Two simulations are proposed, a bouncing ball and an oscillating beam.

Key words: Viscoelastic fluid; Free surface; Elastic solid

Mots-clés : Fluide viscoélastique; Frontière libre; Solide élastique

Version française abrégée

L'écoulement à frontière libre d'un fluide viscoélastique d'Oldroyd-B est considéré, selon l'article [1]. En supprimant un terme dans la loi constitutive de l'extra-contrainte, nous décrivons un solide élastique incompressible, en coordonnées euleriennes, avec le champ de vitesse comme inconnue, plutôt que le champ de déformation.

Deux simulations sont proposées, le rebond d'une balle et une poutre vibrante.

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1. Introduction

2. The model

We follow [1]. Consider a cavity Λ of $\mathbb{R}^d$, $d = 2$ or 3, partially filled with a viscoelastic incompressible fluid, for instance a Newtonian solvent with non-interacting polymer chains. We are interested in computing the fluid shape between time 0 and time T. Let $\Omega(t) \subset \Lambda$ be the liquid region at time t and let Ω_T be the space-time domain containing the fluid. The velocity $u : \Omega_T \to \mathbb{R}^d$, pressure $p : \Omega_T \to \mathbb{R}$ and extra-stress $\sigma : \Omega_T \to \mathbb{R}^{d\times d}$ satisfy the mass and momentum equations

$$\rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u - 2\eta_s \text{div } \epsilon(u) + \nabla p - \text{div } \sigma = \rho g, \tag{1}$$

$$\text{div } u = 0, \tag{2}$$

supplemented with the Oldroyd-B constitutive equation

$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} + u\cdot\nabla\sigma - \nabla u\sigma - \sigma\nabla u^T\right) = 2\eta_p\epsilon(u). \tag{3}$$

Hereabove ρ is the fluid density, η_s the solvent viscosity, $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the symmetric part of the velocity gradient, g the gravity, λ the polymer relaxation time, η_p the polymer viscosity. Let $\varphi : \Lambda \times (0,T)$ be the characteristic function of the liquid. Then, the domain containing the fluid at time t is defined by

$$\Omega(t) = \{x \in \Lambda; \varphi(x,t) = 1\}.$$

Assuming the fluid particles move with the fluid velocity, φ must satisfy

$$\frac{\partial\varphi}{\partial t} + u\cdot\nabla\varphi = 0 \qquad \text{in } \Lambda\times(0,T) \tag{4}$$

so that

$$\varphi(X(t),t) = \varphi(X(0),0) \qquad 0 \le t \le T,$$

with $\dot{X}(t) = u(X(t),t)$ and $X(0) \in \Omega(0)$. Concerning the initial conditions, $\varphi(0)$ or equivalently $\Omega(0)$ must be provided at time 0, so as $u(0)$ and $\sigma(0)$ in $\Omega(0)$. Concerning the boundary conditions, it is assumed that no external force applies on the fluid's free surface (the set of points where φ jumps from 0 to 1), thus

$$\Big(-pI + 2\eta_s\epsilon(u) + \sigma\Big)n = 0, \tag{5}$$

where n is the unit outer normal of the free surface. On the boundary of the fluid being in contact with the walls, either slip, imposed or no-slip boundary conditions apply.

3. The elastic limit

In order to consider the elastic limit of the Oldroyd-B constitutive equation, we shall replace (3) by

$$\alpha\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} + u\cdot\nabla\sigma - \nabla u\sigma - \sigma\nabla u^T\right) = 2\eta_p\epsilon(u), \tag{6}$$

where $\alpha = 0$ or 1. Three cases can be considered:

- $\alpha = 1$, $\lambda = 0$, $\eta_s \ge 0$, $\eta_p \ge 0$: an incompressible Newtonian fluid with viscosity $\eta_s + \eta_p$.
- $\alpha = 1$, $\lambda > 0$, $\eta_s \ge 0$, $\eta_p > 0$: an incompressible Oldroyd-B viscoelastic fluid.
- $\alpha = 0$, $\lambda > 0$, $\eta_s = 0$, $\eta_p > 0$: an incompressible elastic solid formulated in Eulerian variables, with velocity u as unknown, instead of the usual deformation field.

3.1. *The mesoscopic counterpart*

It is well known [2] that the Oldroyd-B model has an equivalent mesoscopic formulation where the non-interacting polymer chains are modeled as dumbbells, that is two beads connected with a Hookean spring. The springs elongation q is a kinetic variable and the density probability $f(x,q,t)$ must satisfy a Fokker-Planck equation. When (3) is replaced by (6), the Fokker-Planck equation of the dumbbells is

$$\frac{\partial f}{\partial t} + \mathrm{div}_x(uf) + \mathrm{div}_q\Big((\nabla_x u)qf - \frac{\alpha}{2\lambda}qf\Big) = \frac{\alpha}{2\lambda}\mathrm{div}_q(\nabla_q f). \tag{7}$$

the extra stress being defined by

$$\sigma(x,t) = \frac{\eta_p}{\lambda}\Big(\int_{q\in\mathbb{R}^d} qq^T f(x,t,q)dq - I\Big).$$

The physical interpretation of (7) is the following. The second term indicates that the center of mass of a dumbbell moves with the fluid velocity. The third term takes into account the drag force due to the beads, the last term in the left hand side corresponds to the spring force, the term in the right-hand side to random thermal fluctuations. When $\alpha = 0$, both the spring force and the random fluctuations disappear in the model.

3.2. *Integral formulation*

The integral formulation corresponding to (6) is the following, see also [3]. Let $X(t;x)$ be the position at time t of the fluid particle that left $x \in \Omega(0)$ at time 0, that is the solution of $\dot{X}(t) = u(X(t),t)$, $X(0) = x$, or equivalently

$$X(t;x) = x + \int_0^t u(X(s;x),s)ds. \tag{8}$$

Let F be the deformation tensor defined by $F(x,t) = \nabla X(t;x)$ that is

$$F_{ij}(x,t) = \frac{\partial X_i}{\partial x_j}(t;x) \quad i,j = 1,...,d.$$

Taking the derivative of (8) with respect to x we have

$$F(x,t) = I + \int_0^t \nabla u(X(s;x),s)F(x,s)ds,$$

so that taking the derivative with respect to t, we obtain

$$\frac{\partial F}{\partial t}(x,t) = \nabla u(X(t;x),t)F(x,t).$$

We can then check (taking the derivative with respect to t) that

$$\sigma(X(t;x),t) = \frac{\eta_p}{\lambda^2}\left(\int_0^t e^{-(t-s)/\lambda}F(x,t)F^{-1}(x,s)F^{-T}(x,s)F^T(x,t)ds + \lambda(e^{-t/\lambda}F(x,t)F^T(x,t) - I)\right)$$

satisfies (6) when $\alpha = 1$ (Oldroyd-B), whereas

$$\sigma(X(t;x),t) = \frac{\eta_p}{\lambda}\left(F(x,t)F^T(x,t) - I\right)$$

satisfies (6) when $\alpha = 0$ (elastic solid). This model ($\alpha = 0$) coincides with the Eulerian formulation of an incompressible Neo-Hookean material described in [4]. It should be stressed that in [4] the unknown is the deformation $X(t;x) - x$, whereas here the unknown is the velocity u.

4. The numerical method

The numerical method is the one presented in [1] for viscoelastic fluids; it stems from the one advocated in [5,6] for Newtonian flows. An implicit, order one, splitting method is used for the time discretization, in order to decouple advection and diffusion phenomena. Advection - transport of φ, u and σ - is performed using a forward characteristics method on a structured grid. Diffusion - eq. (1) (2) and (6) without advection terms - is solved on a finite element mesh with continuous, piecewise linear stabilized finite elements. In order to reduce numerical diffusion of the volume fraction of liquid φ, the size h of the structured grid is three to five times smaller than the finite element mesh size H. The CFL number is between one and ten.

5. Numerical experiments

5.1. *Bouncing ball*

All physical data are given in the international system of units. Consider the cavity $\Lambda = [-0.2, 0.2] \times [-0.2, 0.2] \times [0, 0.3]$ partially filled with a viscoelastic fluid in the elastic limit - $\alpha = 0$ in (6) and $\eta_s = 0$ in (1) - having density $\rho = 1000$, polymer viscosity $\eta_p = 0.1$, elastic coefficient $\lambda = 0.005$ and subject to gravity $g = (0, 0, -9.81)$ in the vertical direction. In order to allow bouncing when the ball touches the bottom wall of the cavity, Signorini-like boundary conditions are enforced, see Section 2.2 in [5] for details.

At initial time, the fluid is the ball centered at $(0, 0, 0.15)$ with radius 0.1 and has velocity $(0, 0, -0.1)$. The finite element mesh is uniform, produced by the gmsh software [7], the requested mesh size is $H = 0.005$, has 485485 vertices and 2823660 tets. The size of the cells is $h = 0.001$, the time step is 0.025. The (sequential) CPU time on a i7-2820QM CPU @ 2.30GHz with 16 Gb RAM is less than four hours. As reported in Figure 1, the ball bounces on the bottom wall of the cavity.

5.2. *Oscillating beam*

The same fluid is considered with the same numerical parameters but with a different inital condition. At initial time, the fluid is a beam located at $[-0.2, 0.1] \times [-0.05, 0.05] \times [0.1, 0.2]$ and still has velocity $(0, 0, -0.1)$. At the plane $x = 0$, the fluid is in contact with the cavity and has zero velocity, thus zero deformation. As reported in Figure 2, the beam bends towards the bottom of the cavity, bounces, and finally reaches a stationary position.

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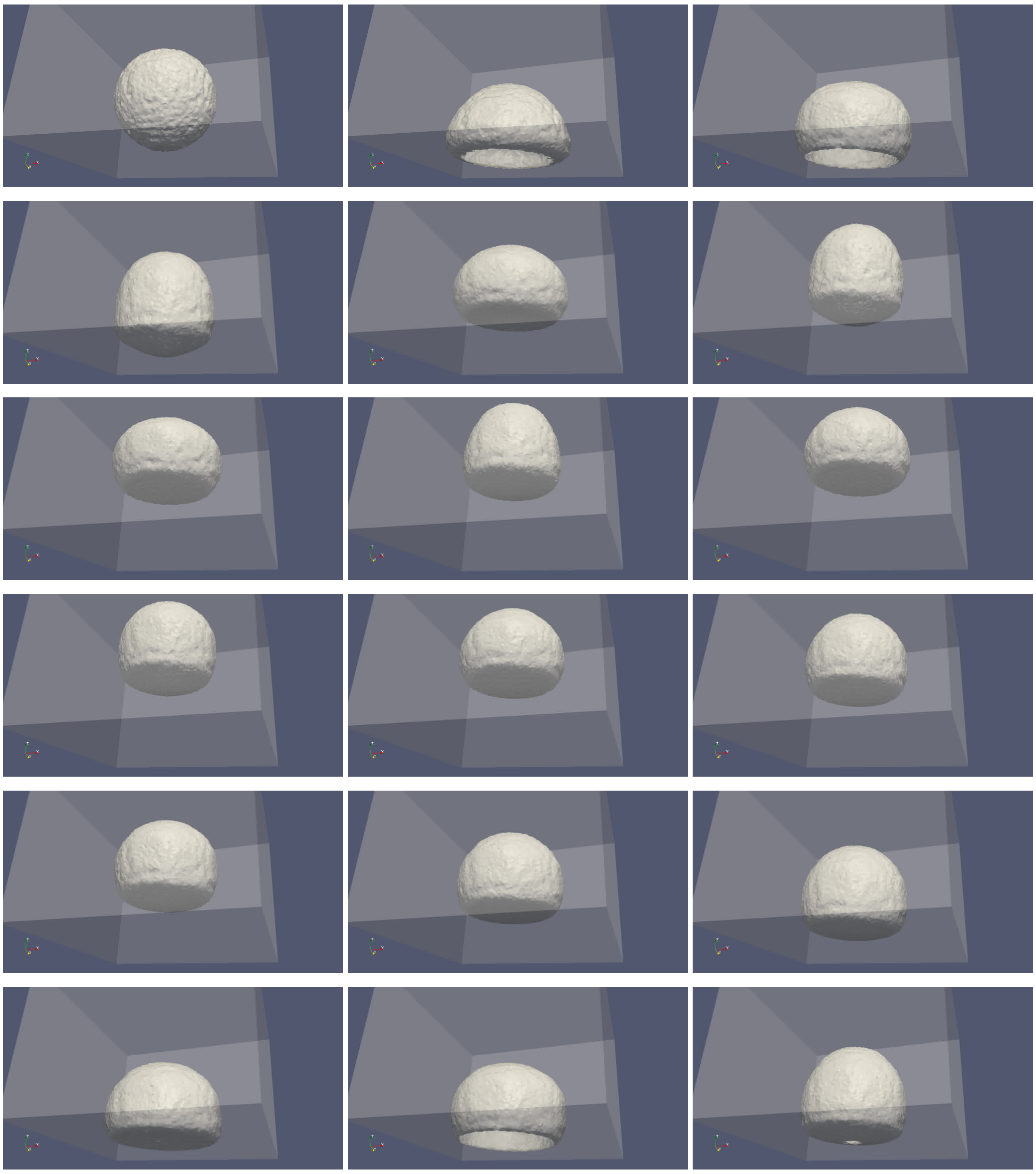

Figure 1. Ball bouncing. Shape of the deformed ball at time steps 3, 6, 9, ..., 54.

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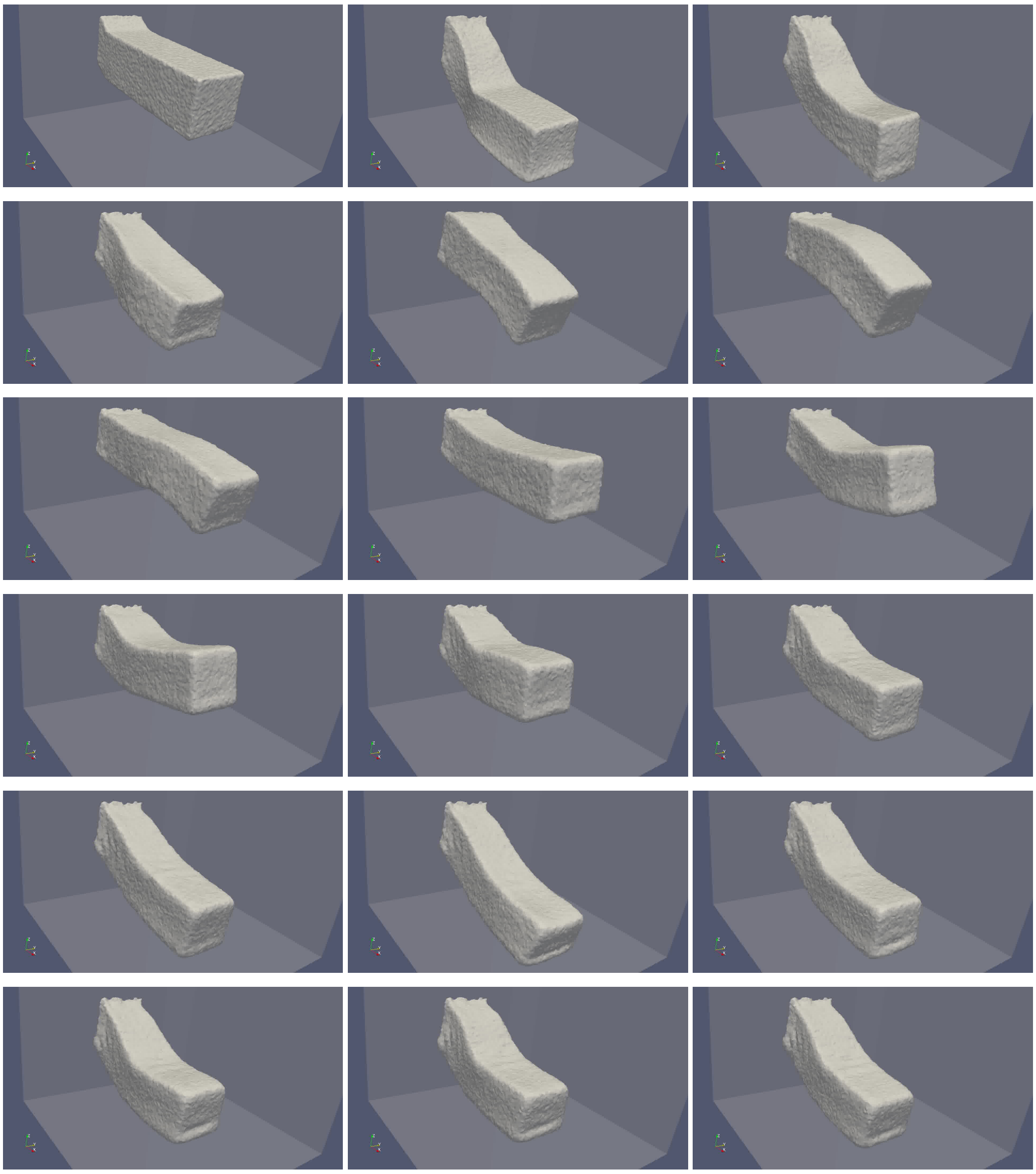

Figure 2. Oscillating beam. Shape of the beam at time steps 3, 6, 9, ..., 54.

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