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### Structured canonical forms for products of (skew-)symmetric matrices and the matrix equation $XAX = B$

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**STRUCTURED CANONICAL FORMS FOR  
PRODUCTS OF (SKEW-)SYMMETRIC MATRICES  
AND THE MATRIX EQUATION  $XAX = B$**

DANIEL KRESSNER\* AND XIN LIU†

**Abstract.** The contragredient transformation  $A \mapsto P^{-1}AP^{-\top}$ ,  $B \mapsto P^{\top}BP$  of two matrices  $A, B$  effects simultaneous similarity transformations of the products  $AB$  and  $BA$ . This work provides structured canonical forms under this transformation for symmetric or skew-symmetric  $A, B$ . As an application, these forms are used to study the quadratic matrix equation  $XAX = B$ , where both  $A, B$  are skew-symmetric or symmetric matrices. Necessary and sufficient conditions for the existence of a (nonsingular) symmetric solution  $X$  are formulated in terms of the structured canonical form.

**Key words.** Skew-symmetric matrix, symmetric matrix, matrix product, quadratic matrix equation.

**AMS subject classifications.** 15A21, 15A24, 15B57.

**1. Introduction.** This work has been motivated by the quadratic matrix equation

$$XAX = B \tag{1.1}$$

for complex skew-symmetric matrices  $A, B \in \mathbb{C}^{n \times n}$ . Such a matrix equation appears, for example, in a study of second order pullback equations [3].

A calculation reveals that  $X$  is a symmetric solution to (1.1) if and only if  $\tilde{X} = P^{\top}XP$  is a symmetric solution to

$$\tilde{X}P^{-1}AP^{-\top}\tilde{X} = P^{\top}BP$$

for any nonsingular  $P \in \mathbb{C}^{n \times n}$ , where  $P^{\top}, P^{-\top}$  denote the transpose of  $P$  and the inverse of  $P^{\top}$ . Hence, we can replace  $A, B$  by their simultaneous contragredient transformation

$$A \mapsto P^{-1}AP^{-\top}, \quad B \mapsto P^{\top}BP, \tag{1.2}$$

with  $P^{-\top} := (P^{-1})^{\top}$ .

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In this work, we will derive structured canonical forms under the transformation (1.2). Rao and Mitra [15, Ch. 6] as well as Vander Beek [20] derived structured canonical forms under the transformation  $A \mapsto P^{-1}AP^{-*}$ ,  $B \mapsto P^*BP$ , where  $*$  denotes the conjugate transpose and  $A, B$  are Hermitian. However, we are not aware of structured canonical forms under (1.2) for complex (skew-)symmetric matrices  $A, B$ . This is quite surprising when considering the abundance of results on the closely related congruence transformation of a (skew-)symmetric pencil, see [13, 19] and the references therein. One contribution of this work is to fill this gap.

Note that the contragredient transformation (1.2) effects simultaneous similarity transformations  $AB \mapsto P^{-1}ABP$  and  $BA \mapsto P^T BAP^{-T}$ . This connects (1.2) to eigenvalue problems for products of (skew-)symmetric matrices, which have also been studied intensively [1, 4, 5, 7, 12, 16, 18].

The rest of this work is organized as follows. In Section 2, we derive structured canonical forms for symmetric or skew-symmetric  $A, B$ . As an application, based on the canonical forms we derive, necessary and sufficient conditions for the existence of a symmetric solution to  $XAX = B$ ,  $A = \pm A^T$ ,  $B = \pm B^T$  are given in Sections 3 and 4.

**2. Structured canonical form.** In this section, we derive a canonical form under  $A \mapsto P^{-1}AP^{-T}$ ,  $B \mapsto P^T BP$  for a matrix pair  $(A, B)$  with  $A, B \in \mathbb{C}^{n \times n}$  symmetric or skew-symmetric. Our approach closely follows the approach by Thompson [19] for deriving canonical forms under congruence transformations. Let us first recall *general* contragredient equivalence transformations; see [8, 11, 17].

**DEFINITION 2.1.** *Let  $A, C \in \mathbb{C}^{m \times n}$  and  $B, D \in \mathbb{C}^{n \times m}$ . We say that  $(A, B)$  is contragrediently equivalent to  $(C, D)$ , and we write  $(A, B) \sim (C, D)$ , if there are nonsingular  $X \in \mathbb{C}^{m \times m}$  and  $Y \in \mathbb{C}^{n \times n}$  such that  $XAY^{-1} = C$ ,  $YBX^{-1} = D$ .*

We next recall two useful results from [8, 11, 17]. The first result gives a useful rank characterization and the second result summarizes the canonical form under *general* contragredient equivalence transformations.

**LEMMA 2.2.** *Two pairs  $(A, B)$  and  $(C, D)$  are contragrediently equivalent if and only if*

- (i)  $AB$  is similar to  $CD$ ; and
- (ii)  $\text{rank } A = \text{rank } C$ ,  $\text{rank } BA = \text{rank } DC$ ,  $\dots$ ,  $\text{rank}(BA)^l = \text{rank}(DC)^l$ ;  
where  $l := \min\{m, n\}$ .

**THEOREM 2.3.** *Consider a matrix pair  $(A, B)$  with  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{n \times m}$ . Then there are nonsingular matrices  $P \in \mathbb{C}^{m \times m}$ ,  $Q \in \mathbb{C}^{n \times n}$  such that  $(P^{-1}AQ, Q^{-1}BP)$  can be written as the direct sum of pairs taking one of the following forms:*

- |   |                          |
|---|--------------------------|
| (i) $(I_k, J_k(\lambda))$ with $\lambda \in \mathbb{C} \setminus \{0\}$ , | (nonzero Jordan pair)    |
| (ii) $(I_k, J_k(0))$ ,  | (right zero Jordan pair) |
| (iii) $(J_k(0), I_k)$ ,   | (left zero Jordan pair)  |
| (iv) $(F_k, G_k^\top)$ ,  | (right singular pair)    |
| (v) $(F_k^\top, G_k)$ ,   | (left singular pair)     |

where

$$J_k(\lambda) = \underbrace{\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}}_{k \times k}, \quad F_k = \underbrace{\begin{bmatrix} 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix}}_{(k-1) \times k}, \quad G_k = \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \end{bmatrix}}_{(k-1) \times k}.$$

This decomposition is uniquely determined up to permutation of the summands.

It is easy to see that a zero or nonzero Jordan pair induces a Jordan block of the same size in the Jordan canonical forms of  $AB$  and  $BA$ . A right singular pair  $(F_k, G_k^\top)$  induces  $(k-1) \times (k-1)$  and  $k \times k$  zero Jordan blocks in  $AB$  and  $BA$ , respectively. A left singular pair  $(F_k^\top, G_k)$  induces  $k \times k$  and  $(k-1) \times (k-1)$  zero Jordan blocks in  $AB$  and  $BA$ , respectively. Hence, while there is a one-to-one correspondence between nonzero Jordan blocks in  $AB$  or  $BA$  and nonzero Jordan pairs in  $(A, B)$ , the same cannot be said about zero Jordan blocks. This is closely connected to the fact that the Jordan canonical forms of  $AB$  and  $BA$  may differ only on the zero Jordan blocks [6, 14].

For the rest of this paper, we assume that  $A, B$  are symmetric or skew-symmetric. As shown in the following sections, this assumption imposes certain restrictions on the Jordan and singular pairs of Theorem 2.3. We will make frequent use of the *flip permutation matrix* with ones on the anti-diagonal and zeros everywhere else:

$$\Pi_k = \underbrace{\begin{bmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{bmatrix}}_{k \times k}. \quad (2.1)$$

## 2.1. Nonzero Jordan pairs.

**Both  $A$  and  $B$  are skew-symmetric.** The following classical result [18] completely characterizes the Jordan structure of a skew-symmetric product; see also [10, 4.4.P33].

**THEOREM 2.4.** *A matrix  $C \in \mathbb{C}^{n \times n}$  can be written as the product of two skew-symmetric matrices if and only if the following two statements hold for the Jordan canonical form of  $C$ :*

- (i) Each  $k \times k$  Jordan block associated with a nonzero eigenvalue of  $C$  occurs an even number of times.
- (ii) Let  $k_1 \geq k_2 \geq k_3 \geq \dots$  denote the sizes of the Jordan blocks associated with the zero eigenvalue of  $C$ . Then

$$k_{2i-1} - k_{2i} \leq 1$$

for  $i = 1, 2, \dots$

The first part of this result implies that we can group the nonzero Jordan pairs in the canonical form of  $(A, B)$  for skew-symmetric  $A, B$  into pairs

$$\left( \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} J_k(\lambda) & 0 \\ 0 & J_k(\lambda) \end{bmatrix} \right).$$

Such a pair is contragrediently equivalent to the skew-symmetric pair

$$\left( \begin{bmatrix} 0 & \Pi_k \\ -\Pi_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{bmatrix} \right). \quad (2.2)$$

**$A$  is skew-symmetric and  $B$  is symmetric.** In this case, the similarity of  $AB$  and  $(AB)^T = -BA$  implies that Jordan blocks of  $AB$  always come in pairs  $J_k(\lambda)$  and  $J_k(-\lambda)$  for  $\lambda \neq 0$ . This allows us to group the nonzero Jordan pairs in the canonical form of  $(A, B)$  into

$$\begin{aligned} \left( \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} J_k(\lambda) & 0 \\ 0 & J_k(-\lambda) \end{bmatrix} \right) &\sim \left( \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} J_k(\lambda) & 0 \\ 0 & -J_k(\lambda) \end{bmatrix} \right) \\ &\sim \left( \begin{bmatrix} 0 & \Pi_k \\ -\Pi_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{bmatrix} \right). \end{aligned}$$

We have used that  $J_k(-\lambda)$  and  $-J_k(\lambda)$  are similar. Note that the last pair is skew-symmetric / symmetric.

**Both  $A$  and  $B$  are symmetric.** In this case, it follows that  $(I_k, J_k(\lambda))$  is contragrediently equivalent to the symmetric pair  $(\Pi_k, \Pi_k J_k(\lambda))$ .

**2.2. Zero Jordan pairs.** The Jordan structure of  $AB$  does not determine the structure of zero Jordan and singular pairs for  $(A, B)$ . To make this determination, we recall the construction of these pairs from [11, Sec. 2]. It turns out that the sizes of zero Jordan and singular pairs can be determined uniquely from the tuple

$$\begin{aligned} \phi(A, B) = & (\text{rank}(A), \text{rank}(BA), \text{rank}(ABA), \dots, \text{rank}(A(BA)^{n-1}), \text{rank}((BA)^n), \\ & \text{rank}(B), \text{rank}(AB), \text{rank}(BAB), \dots, \text{rank}(B(AB)^{n-1}), \text{rank}((AB)^n)). \end{aligned}$$

Let  $k$  be the smallest integer such that  $(BA)^k = (AB)^k = 0$ . Define

$$\ell_1 = \text{rank}(A(BA)^{k-1}), \quad \ell_2 = \text{rank}(B(AB)^{k-1}).$$

Then, according to [11],  $\ell_1$  is the number of right zero Jordan pairs  $(I_k, J_k(0))$  and  $\ell_2$  is the number of left zero Jordan pairs  $(J_k(0), I_k)$ .

We now combine these findings with the (skew-)symmetric structure of  $A$  and  $B$ .

**Both  $A$  and  $B$  are skew-symmetric.** The skew-symmetry of  $A, B$  implies that both  $A(BA)^{k-1}$  and  $B(AB)^{k-1}$  are skew-symmetric. For example,

$$(A(BA)^{k-1})^\top = (A^\top B^\top)^{k-1} A^\top = -A(BA)^{k-1}.$$

Since the rank of a skew-symmetric matrix is even,  $\ell_1$  and  $\ell_2$  are even. Thus, we may combine the right zero Jordan pairs of size  $k$  into  $\ell_1/2$  pairs of size  $2k$ :

$$\begin{aligned} \left( \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right], \left[ \begin{array}{cc} J_k(0) & 0 \\ 0 & J_k(0) \end{array} \right] \right) &\sim \left( \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right], \left[ \begin{array}{cc} -J_k(0) & 0 \\ 0 & -J_k(0) \end{array} \right] \right) \\ &\sim \left( \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right] \right). \end{aligned}$$

Analogously, the left zero Jordan pairs of size  $k$  can be combined into  $\ell_2/2$  pairs of size  $2k$ :

$$\begin{aligned} \left( \left[ \begin{array}{cc} J_k(0) & 0 \\ 0 & J_k(0) \end{array} \right], \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right] \right) &\sim \left( \left[ \begin{array}{cc} -J_k(0) & 0 \\ 0 & -J_k(0) \end{array} \right], \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right] \right) \\ &\sim \left( \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right] \right). \end{aligned}$$

**$A$  is skew-symmetric and  $B$  is symmetric.** This case requires to distinguish between even and odd  $k$ .

- *Even  $k$ :* In this subcase,  $B(AB)^{k-1}$  is skew-symmetric, so the number  $\ell_2 = \text{rank}(B(AB)^{k-1})$  of left zero Jordan blocks  $(J_k(0), I_k)$  is also even. Hence, these blocks can be grouped into

$$\begin{aligned} &\left( \left[ \begin{array}{cc} J_k(0) & 0 \\ 0 & J_k(0) \end{array} \right], \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right] \right) \\ &\sim \left( \left[ \begin{array}{cc} -J_k(0) & 0 \\ 0 & J_k(0) \end{array} \right], \left[ \begin{array}{cc} I_k & 0 \\ 0 & I_k \end{array} \right] \right) \\ &\sim \left( \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k \\ \Pi_k & 0 \end{array} \right] \right). \end{aligned}$$

For even  $k$ , each individual *right zero Jordan block*  $(I_k, J_k(0))$  can be transformed into a skew-symmetric / symmetric pair:

$$(I_k, J_k(0)) \sim \left( I_k, \begin{bmatrix} I_{\frac{k}{2}} & 0 \\ 0 & -I_{\frac{k}{2}} \end{bmatrix} J_k(0) \right) \sim \left( \begin{bmatrix} 0 & \Pi_{\frac{k}{2}} \\ -\Pi_{\frac{k}{2}} & 0 \end{bmatrix}, \Pi_k J_k(0) \right).$$

- *Odd  $k$* : In this subcase,  $A(BA)^{k-1}$  is skew-symmetric, so the number  $\ell_1 = \text{rank}(A(BA)^{k-1})$  of *right zero Jordan blocks*  $(I_k, J_k(0))$  is even. Once again, these blocks can be grouped into

$$\begin{aligned} & \left( \begin{bmatrix} I_k & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} J_k(0) & 0 \\ 0 & J_k(0) \end{bmatrix} \right) \\ & \sim \left( \begin{bmatrix} 0 & \Pi_k \\ -\Pi_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Pi_k J_k(0) \\ \Pi_k J_k(0) & 0 \end{bmatrix} \right). \end{aligned}$$

For odd  $k$ , each individual *left zero Jordan block*  $(J_k(0), I_k)$  can be transformed into a skew-symmetric / symmetric pair:

$$(J_k(0), I_k) \sim \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Pi_{\frac{(k-1)}{2}} \\ 0 & -\Pi_{\frac{(k-1)}{2}} & 0 \end{bmatrix}, \Pi_k \right).$$

**Both  $A$  and  $B$  are symmetric.** Left and right zero Jordan pairs can be symmetrized as follows:

$$(I_k, J_k(0)) \sim (\Pi_k, \Pi_k J_k(0)), \quad (J_k(0), I_k) \sim (\Pi_k J_k(0), \Pi_k).$$

**2.3. Singular pairs.** The treatment of singular pairs is identical for all pairs  $(A, B)$  such that  $A^\top = \varepsilon_A A$  and  $B^\top = \varepsilon_B B$  with  $\varepsilon_A, \varepsilon_B \in \{-1, +1\}$ . Let us define

$$\ell_3 = \text{rank}((AB)^{k-1}), \quad \ell_4 = \text{rank}((BA)^{k-1}).$$

According to [11],  $\ell_3 - \ell_1 - \ell_2$  is the number of left singular pairs  $(F_k^\top, G_k)$  and  $\ell_4 - \ell_1 - \ell_2$  is the number of right singular pairs  $(F_k, G_k^\top)$ . The skew-symmetry or symmetry of  $A, B$  imply  $\ell_3 = \ell_4$ , allowing us to form  $\ell_3 - \ell_1 - \ell_2$  groups of the form

$$\left( \begin{bmatrix} F_k^\top & 0 \\ 0 & F_k \end{bmatrix}, \begin{bmatrix} G_k & 0 \\ 0 & G_k^\top \end{bmatrix} \right) \sim \left( \begin{bmatrix} 0 & F_k \\ \varepsilon_A F_k^\top & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_k \\ \varepsilon_B G_k^\top & 0 \end{bmatrix} \right).$$

To summarize, there are nonsingular matrices  $P_1, Q_1$  such that

$$(P_1^{-1} A Q_1, Q_1^{-1} B P_1) = \left( \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A} \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & \tilde{B} \end{bmatrix} \right),$$



where  $A_1$  and  $B_1$  inherit the (skew-)symmetry of  $A$  and  $B$ , respectively, and contain their zero Jordan blocks of size  $k \times k$  as well as their singular blocks of size  $k \times (k-1)$  or  $(k-1) \times k$ .

The described process of splitting off zero Jordan blocks and singular blocks is now repeated with the remaining pair  $(\tilde{A}, \tilde{B})$ . For this purpose, we derive all required properties from the fact that  $\phi(\tilde{A}, \tilde{B})$  is given by the element-wise difference between  $\phi(A, B) = \phi(P_1^{-1}AQ_1, Q_1^{-1}BP_1)$  and  $\phi(A_1, B_1)$ . In particular, the parity properties of  $\phi(A, B)$  induced by the (skew-)symmetry of  $A$  and  $B$  are inherited by  $\phi(\tilde{A}, \tilde{B})$ . According to the construction of Section 2.2, these parity properties are all we need to process the  $\tilde{k} \times \tilde{k}$  zero Jordan blocks of  $(\tilde{A}, \tilde{B})$ , where  $\tilde{k} < k$  is the smallest integer such that  $(\tilde{B}\tilde{A})^{\tilde{k}} = (\tilde{A}\tilde{B})^{\tilde{k}} = 0$ . Moreover,

$$\begin{aligned}\tilde{\ell}_3 &= \text{rank}((\tilde{A}\tilde{B})^{\tilde{k}-1}) = \ell_3 - \text{rank}((A_1B_1)^{\tilde{k}-1}) \\ &= \ell_4 - \text{rank}((B_1A_1)^{\tilde{k}-1}) = \text{rank}((\tilde{B}\tilde{A})^{\tilde{k}-1}) = \tilde{\ell}_4.\end{aligned}$$

Thus, the  $\tilde{k} \times (\tilde{k}-1)$  and  $(\tilde{k}-1) \times \tilde{k}$  singular blocks of  $(\tilde{A}, \tilde{B})$  can be paired as described in Section 2.3. Consequently, there are nonsingular matrices  $P_2, Q_2$  such that

$$(P_2^{-1}AQ_2, Q_2^{-1}BP_2) = \left( \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix}, \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix} \right),$$

where  $A_2$  and  $B_2$  inherit the (skew-)symmetry of  $A$  and  $B$ , respectively, and contain their zero Jordan blocks of size  $\tilde{k} \times \tilde{k}$  as well as their singular blocks of size  $\tilde{k} \times (\tilde{k}-1)$  or  $(\tilde{k}-1) \times \tilde{k}$ . The process is repeated until the remaining parts in both  $A$  and  $B$  are nonsingular, at which point all zero Jordan and singular pairs have been found.

**2.4. Main result.** THEOREM 2.5. *Let  $A, B \in \mathbb{C}^{n \times n}$  be skew-symmetric or symmetric. Then there exists a nonsingular  $P \in \mathbb{C}^{n \times n}$  such that  $(P^{-1}AP^{-\top}, P^{\top}BP)$  is the direct sum of pairs taking one of the following forms:*

(a) *When  $A, B \in \mathbb{C}^{n \times n}$  are both symmetric:*

(ai)  $(\Pi_k, \Pi_k J_k(\lambda))$  with  $\lambda \in \mathbb{C} \setminus \{0\}$ , (nonzero Jordan pair)

(aia)  $(\Pi_k, \Pi_k J_k(0))$ , (right zero Jordan pair)

(aiaa)  $(\Pi_k J_k(0), \Pi_k)$ , (left zero Jordan pair)

(aiv)  $\left( \begin{bmatrix} 0 & F_k \\ F_k^{\top} & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_k \\ G_k^{\top} & 0 \end{bmatrix} \right)$ . (singular pair)

(b) *When  $A \in \mathbb{C}^{n \times n}$  is skew-symmetric and  $B \in \mathbb{C}^{n \times n}$  is symmetric:*

(bi)  $\left( \begin{bmatrix} 0 & \Pi_k \\ -\Pi_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{bmatrix} \right)$  with  $\lambda \in \mathbb{C} \setminus \{0\}$ , (nonzero Jordan pair)

$$\begin{aligned}
& (bii) \left( \left[ \begin{array}{cc} 0 & \Pi_{\frac{k}{2}} \\ -\Pi_{\frac{k}{2}} & 0 \end{array} \right], \Pi_k J_k(0) \right), \quad (\text{right zero Jordan pair when } k \text{ is even}) \\
& (biii) \left( \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ \Pi_k J_k(0) & 0 \end{array} \right] \right), \quad (\text{right zero Jordan pair} \\
& \quad \text{when } k \text{ is odd}) \\
& (biv) \left( \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k \\ \Pi_k & 0 \end{array} \right] \right), \quad (\text{left zero Jordan pair} \\
& \quad \text{when } k \text{ is even}) \\
& (bv) \left( \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \Pi_{\frac{(k-1)}{2}} \\ 0 & -\Pi_{\frac{(k-1)}{2}} & 0 \end{array} \right], \Pi_k \right), \quad (\text{left zero Jordan pair when } k \text{ is} \\
& \quad \text{odd}) \\
& (bvi) \left( \left[ \begin{array}{cc} 0 & F_k \\ -F_k^\top & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & G_k \\ G_k^\top & 0 \end{array} \right] \right). \quad (\text{singular pair}) \\
& (c) \text{ When } A, B \in \mathbb{C}^{n \times n} \text{ are both skew-symmetric:} \\
& \quad (ci) \left( \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & -\Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{array} \right] \right) \text{ with } \lambda \in \mathbb{C} \setminus \{0\}, \text{ (nonzero} \\
& \quad \text{Jordan pair)} \\
& \quad (cii) \left( \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right] \right), \quad (\text{right zero Jordan pair}) \\
& \quad (ciii) \left( \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right] \right), \quad (\text{left zero Jordan pair}) \\
& \quad (civ) \left( \left[ \begin{array}{cc} 0 & F_k \\ -F_k^\top & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & G_k \\ -G_k^\top & 0 \end{array} \right] \right). \quad (\text{singular pair})
\end{aligned}$$

Here,  $F_k, G_k, J_k(\lambda)$  are defined as in Theorem 2.3 and  $\Pi_k$  is the  $k \times k$  flip permutation matrix defined in (2.1). This decomposition is uniquely determined up to permutation of the summands.

*Proof.* Theorem 2.3 and the construction in Sections 2.1, 2.2 and 2.3 show that there exist nonsingular matrices  $P, Q$  such that  $(M, N) = (P^{-1}AQ, Q^{-1}BP)$  has the stated form. In particular,  $M$  and  $N$  inherit the (skew-)symmetry of  $A$  and  $B$ , respectively. We complete the proof by a modification of the arguments given in [19, page 339].

By replacing  $A \leftarrow Q^\top A Q$  and  $B \leftarrow Q^{-1} B Q^{-\top}$ , we may assume without loss of generality that  $Q = I_n$ . Since  $A = PM$  is skew-symmetric or symmetric, we have  $PM = MP^\top$ . This implies  $P^2 M = PMP^\top = M(P^2)^\top$  or, more generally,  $P^d M = M(P^d)^\top$  for every integer  $d \geq 0$ . This shows that  $g(P)M = Mg(P)^\top$  for every polynomial  $g$  and, analogously,  $g(P)^\top N = Ng(P)$ . Every primary matrix

function  $f(P)$  can be expressed as a matrix polynomial  $g(P)$  [9, Chap. 6] and hence

$$f(P)M = Mf(P)^\top, \quad f(P)^\top N = Nf(P).$$

In particular, this holds for the primary matrix function  $f(P) = P^{-1/2}$ . Setting  $R = P^{1/2}$  thus gives

$$\begin{aligned} R^{-1}AR^{-\top} &= P^{-1/2}PMP^{-\top/2} = P^{-1/2}PP^{-1/2}M = M, \\ R^\top BR &= P^{\top/2}NP^{-1}P^{1/2} = NP^{1/2}P^{-1}P^{1/2} = N. \end{aligned}$$

This completes the proof, using the fact that the uniqueness of the structured canonical form is inherited from the general canonical form in Theorem 2.3.  $\square$

**3. Symmetric solution to  $XAX = B$  for skew-symmetric  $A, B$ .** The following theorem contains the main result of this section.

**THEOREM 3.1.** *Let  $A, B \in \mathbb{C}^{n \times n}$  be skew-symmetric and let  $\ell_k, r_k$  denote the number of left and right zero Jordan pairs of size  $k$  in the canonical form of  $(A, B)$ , respectively. Then:*

- (i) *There exists a symmetric solution  $X$  to the matrix equation  $XAX = B$  if and only if  $\ell_k \leq r_k$  for all  $k$ .*
- (ii) *There exists a nonsingular symmetric solution  $X$  to the matrix equation  $XAX = B$  if and only if  $\ell_k = r_k$  for all  $k$ .*

**REMARK 3.2.** *From the construction described in Section 2.2, it follows that*

$$d_j := \text{rank}(B(AB)^{j-1}) - \text{rank}(A(BA)^{j-1}) = \sum_{k \geq j} (\ell_k - r_k).$$

*Hence, Theorem 3.1 (i) is equivalent to requiring that the sequence  $d_1, d_2, \dots$  decreases monotonically. Theorem 3.1 (ii) is equivalent to requiring that all  $d_j = 0$ .*

The rest of this section is concerned with the proof of Theorem 3.1. By the discussion in the introduction, we may assume without loss of generality that  $A$  and  $B$  are in the structured canonical form described by Theorem 2.5.

**3.1. Sufficiency of condition for solvability.** To prove that  $\ell_k \leq r_k$  is sufficient for the existence of a symmetric solution, we study the solvability for individual pairs in the structured canonical form of  $(A, B)$ ; see Theorem 2.5 (c).

**Nonzero Jordan pair.** When assuming  $X = \text{diag}(X_{11}, X_{22})$ , the equation

$$X \begin{bmatrix} 0 & \Pi_k \\ -\Pi_k & 0 \end{bmatrix} X = \begin{bmatrix} 0 & -\Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{bmatrix} \quad (3.1)$$

reduces to  $X_{11}\Pi_k X_{22} = -\Pi_k J_k(\lambda)$ . Premultiplying with  $\Pi_k$  and setting  $\tilde{X}_{11} = \Pi_k X_{11} \Pi_k$  gives  $\tilde{X}_{11} X_{22} = -J_k(\lambda)$ . Theorem 1 in [2] ensures that this equation has complex symmetric solutions  $\tilde{X}_{11}, X_{22}$ . Thus, there always exists a symmetric solution for a nonzero Jordan pair.

**Right zero Jordan pair.** By the same arguments, a symmetric solution  $X$  to (3.1) also exists for  $\lambda = 0$ , which corresponds to a right zero Jordan pair. Note that any solution is necessarily singular.

**Singular pair.** When assuming  $X = \text{diag}(X_{11}, X_{22})$ , the equation

$$X \begin{bmatrix} 0 & F_k \\ -F_k^\top & 0 \end{bmatrix} X = \begin{bmatrix} 0 & G_k \\ -G_k^\top & 0 \end{bmatrix}$$

reduces to  $X_{11} F_k X_{22} = G_k$ . The matrices  $X_{11} = \Pi_{k-1}$  and  $X_{22} = \Pi_k$  give a symmetric solution to this equation.

**Combinations of left and right zero Jordan pairs.** It remains to prove the existence of a solution for the case  $\ell_k \leq r_k$ . Since we already know the solvability for individual right zero Jordan pairs, we need to consider only the combination of a single  $k \times k$  right zero Jordan pair with a single  $k \times k$  left zero Jordan pair:

$$A = \begin{bmatrix} 0 & \Pi_k & 0 & 0 \\ -\Pi_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \Pi_k J_k(0) \\ 0 & 0 & -\Pi_k J_k(0) & 0 \end{bmatrix}, \quad (3.2)$$

$$B = \begin{bmatrix} 0 & \Pi_k J_k(0) & 0 & 0 \\ -\Pi_k J_k(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Pi_k \\ 0 & 0 & -\Pi_k & 0 \end{bmatrix}. \quad (3.3)$$

LEMMA 3.3. *Let  $A, B$  be defined as in (3.2)–(3.3). Then there exists a symmetric solution to  $XAX = B$ .*

*Proof.* We have

$$(A, B) \sim (\tilde{A}, \tilde{B}) := \left( \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & J_k(0) & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & J_k(0) \end{bmatrix}, \begin{bmatrix} J_k(0) & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & J_k(0) & 0 \\ 0 & 0 & 0 & I_k \end{bmatrix} \right).$$

Let us introduce the  $2k \times 2k$  matrix

$$S_{2k} := \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix} + i \begin{bmatrix} & & & -1 & 0 \\ & \ddots & & 0 & 1 \\ -1 & \ddots & \ddots & & \\ 0 & 1 & & & \end{bmatrix}, \quad (3.4)$$

where  $i$  denotes the imaginary unit. By [11, Proposition 41],

$$\left( \begin{bmatrix} I_k & 0 \\ 0 & J_k^\top(0) \end{bmatrix}, \begin{bmatrix} J_k^\top(0) & 0 \\ 0 & I_k \end{bmatrix} \right) \sim (S_{2k}, S_{2k}).$$

It follows that

$$\begin{aligned} (A, B) \sim (\tilde{A}, \tilde{B}) &\sim \left( \begin{bmatrix} S_{2k} & 0 \\ 0 & S_{2k} \end{bmatrix}, \begin{bmatrix} S_{2k} & 0 \\ 0 & S_{2k} \end{bmatrix} \right) \\ &\sim \left( \begin{bmatrix} 0 & S_{2k} \\ -S_{2k} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -S_{2k} \\ S_{2k} & 0 \end{bmatrix} \right). \end{aligned}$$

Therefore, analogous to the argument given for Theorem 2.5, there exists a nonsingular  $P$  such that

$$P \begin{bmatrix} 0 & S_{2k} \\ -S_{2k} & 0 \end{bmatrix} P^\top = A, \quad P^{-\top} \begin{bmatrix} 0 & -S_{2k} \\ S_{2k} & 0 \end{bmatrix} P^{-1} = B.$$

Since  $Y \begin{bmatrix} 0 & S_{2k} \\ -S_{2k} & 0 \end{bmatrix} Y = \begin{bmatrix} 0 & -S_{2k} \\ S_{2k} & 0 \end{bmatrix}$  has the solution  $Y = \begin{bmatrix} 0 & I_{2k} \\ I_{2k} & 0 \end{bmatrix}$ , it follows that  $X = P^{-\top} \begin{bmatrix} 0 & I_{2k} \\ I_{2k} & 0 \end{bmatrix} P^{-1}$  is a symmetric solution to  $XAX = B$ .  $\square$

The preceding results show the sufficiency of the condition in Theorem 3.1 (i). Moreover, except for individual right zero Jordan pairs, all constructed solutions are nonsingular. This shows the sufficiency of the condition in Theorem 3.1 (ii).

**3.2. Necessity of condition for solvability.** In this subsection, we show that the existence of a symmetric solution  $X$  to  $XAX = B$  implies  $\ell_k \leq r_k$ . For this purpose, we first transform the skew-symmetric / symmetric pair  $(A, X)$  to structured canonical form; see Theorem 2.5 (b). We then have to verify only that the condition holds for each of the blocks in the canonical form.

**LEMMA 3.4.** *Let the skew-symmetric / symmetric pair  $(A, X) \in \mathbb{C}^{n \times n}$  coincide with any of the canonical pairs in Theorem 2.5 (bi)–(bvi). Then  $\ell_k \leq r_k$  for all  $k$ , where  $\ell_k, r_k$  denote the number of left and right zero Jordan pairs in the canonical form of the skew-symmetric pair  $(A, B) := (A, XAX)$ , respectively.*

*Proof.* The statement of the lemma holds for all canonical pairs with nonsingular  $A$ , that is, for all canonical pairs in Theorem 2.5 (bi)–(biii). It remains to discuss the cases (biv)–(bvi).

*Case (biv).* For the case of a left zero Jordan pair when  $k$  is even, we have

$$(A, B) = (A, XAX) = \left( \left[ \begin{array}{cc} 0 & \Pi_k J_k(0) \\ -\Pi_k J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & -J_k(0)\Pi_k \\ J_k(0)\Pi_k & 0 \end{array} \right] \right) \\ \sim \left( \left[ \begin{array}{cc} 0 & J_k(0) \\ -J_k(0) & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & -J_k(0) \\ J_k(0) & 0 \end{array} \right] \right).$$

This allows us to compute

$$\begin{aligned} \text{rank}((AB)^{\frac{k}{2}}) &= \text{rank}((BA)^{\frac{k}{2}}) = 0, \\ \text{rank}(B(AB)^{\frac{k}{2}-1}) &= \text{rank}(A(BA)^{\frac{k}{2}-1}) = 2, \\ \text{rank}((AB)^{\frac{k}{2}-1}) &= \text{rank}((BA)^{\frac{k}{2}-1}) = 4. \end{aligned}$$

Thus the canonical form of the pair  $(A, B)$  contains two right zero blocks  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$  and two left zero blocks  $(J_{\frac{k}{2}}(0), I_{\frac{k}{2}})$ .

*Case (bv).* For the case of a left zero Jordan pair when  $k$  is odd, we have  $(A, X) \sim (J_k(0), I_k)$ , and therefore  $(A, B) = (A, XAX) = (J_k(0), J_k(0))$ . This gives  $\text{rank}(AB)^l = k - 2l$  for  $l \leq \frac{k}{2}$ . It follows that

$$\begin{aligned} \text{rank}((AB)^{\frac{k+1}{2}}) &= \text{rank}((BA)^{\frac{k+1}{2}}) = 0, \\ \text{rank}(B(AB)^{\frac{k-1}{2}}) &= \text{rank}(A(BA)^{\frac{k-1}{2}}) = 0, \\ \text{rank}((AB)^{\frac{k-1}{2}}) &= \text{rank}((BA)^{\frac{k-1}{2}}) = 1. \end{aligned}$$

This means that the canonical form of the pair  $(A, B)$  consists of the singular blocks  $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}})$  and  $(F_{\frac{k+1}{2}}, G_{\frac{k+1}{2}}^\top)$ .

*Case (bvi).* In this case, we have

$$(A, B) = \left( \left[ \begin{array}{cc} 0 & F_k \\ -F_k^\top & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & -G_k F_k^\top G_k \\ G_k^\top F_k G_k^\top & 0 \end{array} \right] \right)$$

and therefore

$$AB = \left[ \begin{array}{cc} J_{k-1}^\top(0)^2 & 0 \\ 0 & J_k(0)^2 \end{array} \right].$$

Therefore, for even  $k$ , we calculate

$$\begin{aligned} \text{rank}((AB)^{\frac{k}{2}}) &= \text{rank}((BA)^{\frac{k}{2}}) = 0, \\ \text{rank}(B(AB)^{\frac{k}{2}-1}) &= 0, \text{rank}(A(BA)^{\frac{k}{2}-1}) = 2, \\ \text{rank}((AB)^{\frac{k}{2}-1}) &= \text{rank}((BA)^{\frac{k}{2}-1}) = 3. \end{aligned}$$

This implies that the canonical form of the pair  $(A, B)$  consists of two copies of  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$  as well as the singular blocks  $(F_{\frac{k}{2}}^\top, G_{\frac{k}{2}})$  and  $(F_{\frac{k}{2}}, G_{\frac{k}{2}}^\top)$ .

For odd  $k$ , a similar calculation shows that the canonical form of the pair  $(A, B)$  consists of two copies of  $(I_{\frac{k-1}{2}}, J_{\frac{k-1}{2}}(0))$  as well as the singular blocks  $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}}^\top)$  and  $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}})$ .  $\square$

Lemma 3.4 completes the proof of Theorem 3.1 (i). The following lemma yields the necessary condition for *nonsingular* solutions  $X$ .

LEMMA 3.5. *With the same notation as in Lemma 3.4, suppose that  $X$  is nonsingular. Then  $\ell_k = r_k$  for all  $k$ .*

*Proof.* Among the canonical pairs in Theorem 2.5 (b), only the cases (bi), (biv), and (bv) correspond to nonsingular  $X$ . The proof of Lemma 3.4 reveals that the desired statement holds for (biv) and (bv). It remains to discuss the case (bi), that is,

$$(A, X) = \left( \left[ \begin{array}{cc} 0 & \Pi_k \\ -\Pi_k & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & \Pi_k J_k(\lambda) \\ \Pi_k J_k(\lambda) & 0 \end{array} \right] \right)$$

for  $\lambda \neq 0$ . Since  $A$  and  $B = XAX$  are nonsingular, the canonical form of  $(A, B)$  does not contain *any* zero Jordan blocks, that is,  $\ell_k = r_k = 0$  in this case.  $\square$

Lemma 3.5 completes the proof of Theorem 3.1 (ii).

**4. Symmetric solution to  $XAX = B$  for symmetric  $A, B$ .** Without too much effort, the approach from the previous section can be adapted to study the existence of (nonsingular) symmetric solutions  $X$  to  $XAX = B$  for *symmetric*  $A$  and  $B$ . In the following, we describe the key steps only.

The following result extends Lemma 3.4 and yields necessary conditions for the existence of  $X$ .

LEMMA 4.1. *Let the symmetric pair  $(A, X) \in \mathbb{C}^{n \times n}$  coincide with any of the canonical pairs in Theorem 2.5 (ai)-(aiv). Then the canonical pairs in the canonical form of  $(A, XAX)$  are given by:*

- (ai):  $(I_k, J_k(\lambda^2))$ ;
- (aii):  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$  and  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$  when  $k$  is even; or  
 $(I_{\frac{k-1}{2}}, J_{\frac{k-1}{2}}(0))$  and  $(I_{\frac{k+1}{2}}, J_{\frac{k+1}{2}}(0))$  when  $k$  is odd;
- (aiii):  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$  and  $(J_{\frac{k}{2}}(0), I_{\frac{k}{2}})$  when  $k$  is even; or  
 $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}}^\top)$  and  $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}})$  when  $k$  is odd;
- (aiv):  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$ ,  $(I_{\frac{k}{2}}, J_{\frac{k}{2}}(0))$ ,  $(F_{\frac{k}{2}}, G_{\frac{k}{2}}^\top)$ , and  $(F_{\frac{k}{2}}^\top, G_{\frac{k}{2}})$  when  $k$  is even; or

$(I_{\frac{k-1}{2}}, J_{\frac{k-1}{2}}(0)), (I_{\frac{k-1}{2}}, J_{\frac{k-1}{2}}^2(0)), (F_{\frac{k+1}{2}}, G_{\frac{k+1}{2}}^\top),$  and  $(F_{\frac{k+1}{2}}^\top, G_{\frac{k+1}{2}})$  when  $k$  is odd.

*Proof.* In the cases (ai) and (aii), we have  $A = \Pi_k$  and  $X = \Pi_k J_k(\lambda)$ . This implies  $(A, XAX) \sim (I_k, J_k^2(\lambda))$  and, therefore, the results follow from the Jordan canonical form of  $J_k^2(\lambda)$ ; see [9, Theorem 6.2.25]. Case (aiii) follows from the proof of Lemma 3.4 for the cases (biv) and (bv). Case (aiv) follows from the proof of Lemma 3.4 for the case (bvi).  $\square$

Lemma 4.1 shows that  $\ell_k \leq r_k$  is again a necessary condition on the pair  $(A, B)$  for the existence of a symmetric solution  $X$  of  $XAX = B$ . However, the lemma imposes additional conditions on the remaining  $r_k - \ell_k$  right zero Jordan pairs of size  $k \times k$ , which are not grouped with left zero Jordan pairs. Briefly speaking, we need to be able to arrange these right zero Jordan pairs into groups of two such that their sizes differ at most by one:

$$\{(I_k, J_k(0)), (I_k, J_k(0))\} \quad \text{or} \quad \{(I_{k-1}, J_{k-1}(0)), (I_k, J_k(0))\}.$$

To cover  $k = 1$ , we also allow for individual pairs  $(I_1, J_1(0)) = (1, 0)$ . It turns out that we arrive at exactly the same conditions characterizing the existence of a square root for a matrix with  $r_k - \ell_k$  zero Jordan blocks of size  $k \times k$ . We therefore refrain from a detailed description and refer to Section 6.4 in [9] instead. We say that a pair  $(A, B)$  has the *square root property* if these conditions hold. Note that the cases (aii) and (aiv) in Lemma 4.1 do not arise for nonsingular  $X$ . Hence, there is no need to additionally require the square root property if the solution is assumed to be nonsingular.

To show that  $\ell_k \leq r_k$  and the square root property are also sufficient, we proceed as in Section 3.1 and assume without loss of generality that the symmetric pair  $(A, B)$  is in the structured canonical form of Theorem 2.5 (a). The existence of a (nonsingular) symmetric solution  $X$  then follows from Lemma 4.1, after grouping the canonical pairs appropriately. To summarize, we have established the following result.

**THEOREM 4.2.** *Let  $A, B \in \mathbb{C}^{n \times n}$  be symmetric and let  $\ell_k, r_k$  denote the number of left and right zero Jordan pairs of size  $k$  in the canonical form of  $(A, B)$ , respectively. Then:*

- (i) *There exists a symmetric solution  $X$  to the matrix equation  $XAX = B$  if and only if  $\ell_k \leq r_k$  for all  $k$  and  $(A, B)$  has the square root property.*
- (ii) *There exists a nonsingular symmetric solution  $X$  to the matrix equation  $XAX = B$  if and only if  $\ell_k = r_k$  for all  $k$ .*

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## REFERENCES

- [1] B. D. O. Anderson. Orthogonal decomposition defined by a pair of skew-symmetric forms. *Linear Algebra and Appl.*, 8:91–93, 1974.
- [2] A. J. Bosch. The factorization of a square matrix into two symmetric matrices. *Amer. Math. Monthly*, 93(6):462–464, 1986.
- [3] G. Csató, B. Dacorogna, and O. Kneuss. The second order pullback equation. *Calc. Var. Partial Differential Equations*, 2013. To appear.
- [4] D. Ž. Djokovic. On the product of two alternating matrices. *Amer. Math. Monthly*, 98(10):935–936, 1991.
- [5] M. P. Drazin. A note on skew-symmetric matrices. *Math. Gaz.*, 36:253–255, 1952.
- [6] H. Flanders. Elementary divisors of  $AB$  and  $BA$ . *Proc. Amer. Math. Soc.*, 2:871–874, 1951.
- [7] R. Gow and T. J. Laffey. Pairs of alternating forms and products of two skew-symmetric matrices. *Linear Algebra Appl.*, 63:119–132, 1984.
- [8] O. Holtz. Applications of the duality method to generalizations of the Jordan canonical form. *Linear Algebra Appl.*, 310(1-3):11–17, 2000.
- [9] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.
- [10] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.
- [11] R. A. Horn and D. I. Merino. Contragredient equivalence: a canonical form and some applications. *Linear Algebra Appl.*, 214:43–92, 1995.
- [12] Kh. D. Ikramov and Kh. Fassbender. On the product of two skew-Hamiltonian or two skew-symmetric matrices. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 359(Chislennye Metody i Voprosy Organizatsii Vychislenii. XXI):45–51, 217–218, 2008.
- [13] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.
- [14] R. A. Lippert and G. Strang. The Jordan forms of  $AB$  and  $BA$ . *Electron. J. Linear Algebra*, 18:281–288.
- [15] C. R. Rao and S. K. Mitra. *Generalized inverse of matrices and its applications*. John Wiley & Sons, Inc., New York-London-Sydney, 1971.
- [16] L. Rodman. Products of symmetric and skew-symmetric matrices. *Linear and Multilinear Algebra*, 43(1-3):19–34, 1997.
- [17] V. V. Sergeichuk. Canonical matrices for linear matrix problems. *Linear Algebra Appl.*, 317(1-3):53–102, 2000.
- [18] H. Stenzel. Über die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischen Matrizen, als Produkt von zwei alternierenden Matrizen und als Produkt von einer symmetrischen und einer alternierenden Matrix. *Math. Z.*, 15(1):1–25, 1922.
- [19] R. C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.
- [20] J. Van der Beek. *Isconjugativity of Hermitian Matrices*. PhD thesis, Oregon State Univ., Corvallis, 1971. Available from <http://hdl.handle.net/1957/17114>.

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