ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

MATHICSE

Mathematics Institute of Computational Science and Engineering

School of Basic Sciences - Section of Mathematics

MATHICSE Technical Report

Nr. 39 .2016

December 2016

Analysis of the internodes method for non-conforming discretizations of elliptic equations

Paola Gervasio, Alfio Quarteroni

http://mathicse.epfl.ch

Address: EPFL - SB - MATHICSE (Bâtiment MA) Station 8 - CH-1015 - Lausanne - Switzerland

Phone: +41 21 69 37648

ANALYSIS OF THE INTERNODES METHOD FOR NON-CONFORMING DISCRETIZATIONS OF ELLIPTIC EQUATIONS*

PAOLA GERVASIO† AND ALFIO QUARTERONI‡ §

Abstract. INTERNODES is a general method to deal with non-conforming discretizations of second order partial differential equations on regions partitioned into two or several subdomains. It exploits two intergrid interpolation operators, one for transfering the Dirichlet trace across the interface, the others for the Neumann trace. In this paper we provide several interpretations of the method and we carry out its stability and convergence analysis. In every subdomain the original problem is discretized by the finite element method, using a priori non-matching grids and piecewise polynomials of different degree. Finally, we propose an efficient algorithm for the solution of the corresponding algebraic system.

Key words. domain decomposition, non-conforming approximation, non-conforming grids, interpolation, finite element method, *hp* finite element method

AMS subject classifications. 65N55, 65N30, 65N12, 65D05

1. Introduction.

The INTERNODES (INTERpolation for NOnconforming DEcompositionS) method was introduced in [14] for the *non-conforming* numerical approximations of second order elliptic boundary-value problems. By non-conforming we mean that the computational domain is partitioned into subdomains with non-matching grids at subdomain interfaces or/and different polynomial subspaces are used on the subdomains. The most distinguishing feature of INTERNODES is that it is built on two *interpolation* operators at the subdomain interfaces that allow to exchange information between adjoining subdomains on the problem solution and on its normal fluxes, respectively. Although Lagrange interpolation often represents a natural choice, other interpolation methods can be used as well. For instance in [15] Radial Basis Functions interpolations are successfully employed, especially in cases of non-matching (either straight or curved) interfaces.

The interpolatory construction represents the main difference between INTERNODES and the well known mortar method, the latter being based on a single (L^2) projection operator at subdomain interface. We refer to [14] for a detailed comparison of the implemetation aspects of the INTERNODES and mortar methods.

In spite of being much simpler to implement from a programming point of view, INTERNODES shares the same accuracy than the mortar method, as illustrated in [14] with several numerical results obtained using both $h-$ and $hp-$ finite element approximations.

In this paper we generalize the results of [14] along different strands.

First and above all, we prove theoretically that the INTERNODES method yields a solution that is unique, stable, and convergent with an *optimal* rate of convergence (i.e., that of the best approximation error in every subdomain) in the case of Lagrange interpolation and regular, quasi-uniform and affine triangulations on each subdomain.

*SUBMITTED

‡CMCS, Institute of Mathematics, École Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, Switzerland (alfio.quarteroni@epfl.ch)

†DICATAM, Università degli Studi di Brescia, via Branze 38, I-25123 Brescia, Italy (paola.gervasio@unibs.it)

§MOX, Politecnico di Milano, via Bonardi 9, Milano, 20133, Italy (on leave)

Then, we extend the INTERNODES method to the case of a computational domain split into several (more than two) subdomains with internal cross-points (i.e. boundary points shared by at least three subdomains). Finally, we propose an efficient solution algorithm for the INTERNODES problem after reformulating it as a Schur-complement system depending solely on the interface nodal variables.

An outline of the paper is as follows. In Section 2 we present the differential problem and its two-domain formulation. In Section 3 we recall the two-domain conforming finite element discretizations, while in Section 4 we present the intergrid operators and the INTERNODES method. Sections 5 and 6 are devoted to the algebraic form of INTERNODES: we present an efficient algorithm implementing INTERNODES and we extend the method to decompositions with more than three subdomains and internal cross points. In Section 7 some numerical results are shown for non-conforming *hp*-FEM approximation of second order elliptic boundary-value problems. In Section 8, we compare the algebraic formulation of INTERNODES and that of the unsymmetric mortar method ([10, 21]) and show that the two methods are actually different. Last but not least, in Section 9 we prove the well-posedeness of the INTERNODES problem and carry out its convergence analysis.

2. Problem setting. Let $\Omega \subset \mathbb{R}^{d_\Omega}$, with $d_\Omega = 2,\ 3$, be an open domain with Lipschitz boundary $\partial\Omega$. $\partial\Omega_N$ and $\partial\Omega_D$ are suitable disjoint subsets of $\partial\Omega$ such that $\overline{\partial\Omega_D} \cup \overline{\partial\Omega_N} = \partial\Omega$. We make the following assumption, all along the paper.

ASSUMPTION 2.1. *Let f, α, γ and $\mathbf{b}$ be given functions such that $f \in L^2(\Omega)$, $\alpha \in L^\infty(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\mathbf{b} \in W^{1,\infty}(\Omega)$. Moreover, $\exists \alpha_0 > 0$ such that $\alpha \geq \alpha_0$, $\gamma \geq 0$, $\gamma - \frac{1}{2}\nabla \cdot \mathbf{b} \geq 0$ a.e. in Ω, and $\mathbf{b} \cdot \mathbf{n} \geq 0$ on $\partial\Omega_N$. Finally, if $\partial\Omega_D = \emptyset$, we require that $\gamma - \frac{1}{2}\nabla \cdot \mathbf{b} > 0$ a.e. in Ω.*

Then we look for the solution u of the second order elliptic equation

$$\begin{cases} Lu \equiv -\nabla \cdot (\alpha \nabla u) + \mathbf{b} \cdot \nabla u + \gamma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \partial_L u = 0 & \text{on } \partial\Omega_N, \end{cases} \tag{2.1}$$

where $\partial_L u = \alpha \dfrac{\partial u}{\partial \mathbf{n}}$ and $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. We set

$$V = H^1_{\partial\Omega_D}(\Omega) = \{v \in H^1(\Omega) :\ v = 0 \text{ on } \partial\Omega_D\}. \tag{2.2}$$

The weak form of problem (2.1) is: find $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in V, \tag{2.3}$$

where

$$a(u, v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u) v + \gamma u v) d\Omega, \tag{2.4}$$

while $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$. Under Assumption 2.1 there exists a unique solution of (2.3) (see, e.g., [23]).

We partition Ω into two non-overlapping subdomains Ω_1 and Ω_2 with Lipschitz boundary and such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$. $\Gamma (= \overline{\Gamma}) = \partial\Omega_1 \cap \partial\Omega_2$ is the common interface and, for $k = 1, 2$, we set $\partial\Omega_{D,k} = \partial\Omega_D \cap \partial\Omega_k$ and $\partial\Omega_{N,k} = \partial\Omega_N \cap \partial\Omega_k$.

For $k = 1, 2$ let us introduce the local spaces

$$V_k = \{v \in H^1(\Omega_k) \mid v = 0 \text{ on } \partial\Omega_D \cap \partial\Omega_k\}, \quad V_k^0 = \{v \in V_k \mid v = 0 \text{ on } \Gamma\}, \tag{2.5}$$

and the bilinear forms

$$a_k(u,v) = \int_{\Omega_k} (\alpha \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v + \gamma uv) d\Omega. \tag{2.6}$$

Finally, let Λ be the space of traces of the elements of V on the interface Γ:

$$\Lambda = \{\lambda \in H^{1/2}(\Gamma) : \; \exists v \in V : v|_\Gamma = \lambda\}. \tag{2.7}$$

When $\partial\Gamma \cap \partial\Omega \subset \partial\Omega_N$, $\Lambda = H^{1/2}(\Gamma)$, while when $\partial\Gamma \cap \partial\Omega \subset \partial\Omega_D$, $\Lambda = H^{1/2}_{00}(\Gamma)$; in these cases Λ is endowed with the canonical norm of either $H^{1/2}(\Gamma)$ or $H^{1/2}_{00}(\Gamma)$, respectively ([1]). Intermediate situations can be tackled by suitably defining Λ and its norm (see, e.g., [20, Remark 11.5]).

For $k = 1, 2$, let u_k be the restriction of the solution u of (2.3) to Ω_k, then u_1 and u_2 are the solution of the *transmission problem* (see [13, Ch. VII, Sect. 4])

$$\begin{cases} Lu_k = f & \text{in } \Omega_k, \quad k = 1, 2, \\ u_1 = u_2, \qquad \partial_{L_1} u_1 = \partial_{L_2} u_2 & \text{on } \Gamma, \end{cases} \tag{2.8}$$

where $\partial_{L_k} u_k = \alpha_k \frac{\partial u_k}{\partial \mathbf{n}_k}$ (with $\alpha_k = \alpha_{|\Omega_k}$) denotes the conormal derivative associated with the differential operator L, and $\mathbf{n}_k$ is the outward unit normal vector to $\partial\Omega$ (in particular on Γ, we have $\mathbf{n}_1 = -\mathbf{n}_2$). We denote by $\mathbf{n}_{\Gamma_k}$ the restriction of $\mathbf{n}_k$ to Γ. [1]

More precisely, u_1 and u_2 satisfy the following weak form of the transmission problem (2.8) (see [24, Lemma 1.2.1])): find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\begin{cases} a_k(u_k, v_k^0) = (f, v_k^0)_{L^2(\Omega_k)} & \forall v_k^0 \in V_k^0, \quad k = 1, 2 \\ u_2 = u_1 & \text{on } \Gamma, \\ \displaystyle\sum_{k=1,2} a_k(u_k, \mathcal{R}_k \eta) = \sum_{k=1,2} (f, \mathcal{R}_k \eta)_{L^2(\Omega_k)} & \forall \eta \in \Lambda, \end{cases} \tag{2.9}$$

where

$$\mathcal{R}_k : \Lambda \to V_k, \quad s.t. \quad (\mathcal{R}_k \eta)|_\Gamma = \eta \qquad \forall \eta \in \Lambda \tag{2.10}$$

denotes any possible linear and continuous *lifting operator* from Γ to Ω_k.

REMARK 2.1. *Let $\langle \cdot, \cdot \rangle$ denote the duality between Λ and its dual Λ'. If homogeneous boundary conditions (of either Dirichlet and Neumann type) are given on $\partial\Omega$, by counter-integration by parts, the interface equation $(2.9)_3$ is equivalent to*

$$\langle \partial_{L_1} u_1 + \partial_{L_2} u_2, \eta \rangle = 0 \qquad \forall \eta \in \Lambda, \tag{2.11}$$

and therefore to the transmission condition $(2.8)_3$.

3. Recall on conforming discretization. Let us consider a family of triangulations $\mathcal{T}_h$ of the global domain Ω, depending on a positive parameter (the grid size) $h > 0$. Following standard assumptions we require $\mathcal{T}_h$ to be affine, regular, and quasi-uniform (see [23, Ch. 3]). For any $T \in \mathcal{T}_h$, we assume that $\partial T \cap \partial\Omega$ fully belongs to either $\partial\Omega_D$ or $\partial\Omega_N$. We shall denote by $\mathbb{P}_p$, with p a positive integer, the usual space of algebraic polynomials of total degree less than or equal to p. Let

$$X_h = \{v \in C^0(\overline{\Omega}) : \; v_{|T} \in \mathbb{P}_p, \, \forall T \in \mathcal{T}_h\}, \quad V_h = \{v \in X_h : \; v = 0 \text{ on } \partial\Omega_D\} \tag{3.1}$$

[1] In the entire paper we assume that Γ is sufficiently regular to allow the conormal derivative of u to be well defined. This is certainly the case if Γ is of class $C^{1,1}$ (see [19, Def. 1.2.1.2]).

be the usual finite element spaces associated with $\mathcal{T}_h$. The Galerkin finite element approximation of (2.3) reads: find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h. \tag{3.2}$$

Let us split Ω into two subdomains Ω_1 and Ω_2 and assume that the triangulations $\mathcal{T}_h$ are such that Γ does not cut any element $T \in \mathcal{T}_h$. The triangulations $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,h}$ induced by $\mathcal{T}_h$ on Ω_1 and Ω_2 are therefore compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$).

In each Ω_k $(k = 1, 2)$ we introduce the finite element approximation spaces

$$X_{k,h} = \{v \in C^0(\overline{\Omega}_k) \,:\, v_{|T} \in \mathbb{P}_p,\ \forall T \in \mathcal{T}_{k,h}\}, \tag{3.3}$$

and the finite dimensional subspaces of V_k and V_k^0

$$V_{k,h} = X_{k,h} \cap V_k, \qquad V^0_{k,h} = X_{k,h} \cap V_k^0. \tag{3.4}$$

Moreover, we consider the space of finite dimensional traces on Γ

$$\Lambda_h = \{\lambda = v|_\Gamma,\ v \in V_{1,h} \cup V_{2,h}\} \subset \Lambda. \tag{3.5}$$

For $k = 1, 2$ we define two linear and continuous *discrete lifting operators*

$$\mathcal{R}_{k,h} : \Lambda_h \to V_{k,h}, \quad s.t. \quad (\mathcal{R}_{k,h}\eta_h)|_\Gamma = \eta_h, \quad \forall \eta_h \in \Lambda_h. \tag{3.6}$$

The problem: find $u_{1,h} \in V_{1,h}$ and $u_{2,h} \in V_{2,h}$ such that

$$\begin{cases} a_k(u_{k,h}, v^0_{k,h}) = (f, v^0_{k,h})_{L^2(\Omega_k)} & \forall v^0_{k,h} \in V^0_{k,h}, \quad k = 1, 2 \\ u_{2,h} = u_{1,h} & \text{on } \Gamma, \\ \displaystyle\sum_{k=1,2} a_k(u_{k,h}, \mathcal{R}_{k,h}\eta_h) = \sum_{k=1,2} (f, \mathcal{R}_{k,h}\eta_h)_{L^2(\Omega_k)} & \forall \eta_h \in \Lambda_h. \end{cases} \tag{3.7}$$

is actually equivalent to (3.2), in the sense that $u_{k,h} = u_{h|\Omega_k}$, for $k = 1, 2$ (see [24, Sect. 2.1]). Note that (3.7) is the discrete counterpart of (2.9); in particular, $(3.7)_3$ is the discrete counterpart of $(2.9)_3$.

Defining the *discrete residual functionals* $r_{k,h} \in \Lambda'_h$ by the relations

$$\langle r_{k,h}, \eta_h\rangle = a_k(u_{k,h}, \mathcal{R}_{k,h}\eta_h) - (f, \mathcal{R}_{k,h}\eta_h)_{L^2(\Omega_k)} \quad \text{for any } \eta_h \in \Lambda_h, \tag{3.8}$$

the interface equation $(3.7)_3$ is equivalent to

$$\langle r_{1,h} + r_{2,h}, \eta_h\rangle = 0 \quad \text{for any } \eta_h \in \Lambda_h. \tag{3.9}$$

As seen in Remark 2.1, if homogeneous boundary conditions (of either Dirichlet and Neumann type) are prescribed on $\partial\Omega$, the finite dimensional functionals $r_{k,h}$ represent the approximations of the distributional derivatives $\partial_{L_k} u_k$ on Γ. Then (3.9) can be regarded as the discrete counterpart of (2.11).

4. Non-conforming discretization. Now we consider two a-priori *independent families of triangulations* $\mathcal{T}_{1,h_1}$ in Ω_1 and $\mathcal{T}_{2,h_2}$ in Ω_2, respectively. This means that the meshes in Ω_1 and in Ω_2 can be non-conforming on Γ and characterized by different mesh-sizes h_1 and h_2. Moreover, different polynomial degrees p_1 and p_2 can be used to define the finite element spaces. Inside each subdomain Ω_k we assume that the triangulations $\mathcal{T}_{k,h_k}$ are affine, regular and quasi-uniform ([23, Ch.3]).

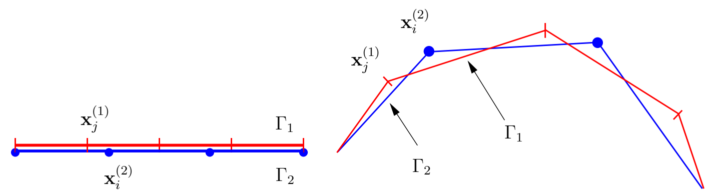

FIG. 4.1. Γ_1 *and* Γ_2 *induced by the triangulations* $\mathcal{T}_{1,h_1}$ *and* $\mathcal{T}_{2,h_2}$

From now on, the finite element approximation spaces are (for $k = 1, 2$):

$$\begin{array}{ll} X_{k,h_k} = \{v \in C^0(\overline{\Omega}_k) \,:\, v_{|T} \in \mathbb{P}_{p_k},\, \forall T \in \mathcal{T}_{k,h_k}\}, & \\ V_{k,h_k} = X_{k,h_k} \cap V_k, & V^0_{k,h_k} = \{v \in V_{k,h_k}, v|_\Gamma = 0\}, \end{array} \tag{4.1}$$

while the spaces of traces on Γ are

$$Y_{k,h_k} = \{\lambda = v|_\Gamma,\ v \in X_{k,h_k}\}, \text{ and } \Lambda_{k,h_k} = \{\lambda = v|_\Gamma,\ v \in V_{k,h_k}\}. \tag{4.2}$$

We set $N_k = \dim(V_{k,h_k})$, $N_k^0 = \dim(V^0_{k,h_k})$, $\overline{n}_k = \dim(Y_{k,h_k})$, and $n_k = \dim(\Lambda_{k,h_k})$.

The space Λ_{k,h_k} takes into account the essential boundary conditions, while Y_{k,h_k} does not. Thus, if $\partial\Omega \cap \partial\Gamma \subset \partial\Omega_N$, then $\Lambda_{k,h_k} = Y_{k,h_k}$ and $n_k = \overline{n}_k$, otherwise $n_k < \overline{n}_k$ because the degrees of freedom associated with the nodes in $\partial\Omega_D \cap \partial\Gamma$ are eliminated.

The Lagrange basis functions of V_{k,h_k} (for $k = 1, 2$) associated with the nodes $\mathbf{x}_i^{(k)}$ of the mesh $\mathcal{T}_{k,h_k}$ are denoted by $\{\varphi_i^{(k)}\}$ for $i = 1, \ldots, N_k$, and they are reordered so that the first $N_k^0 (\leq N_k)$ basis functions span V^0_{k,h_k}.

We denote by Γ_1 and Γ_2 the internal boundaries of Ω_1 and Ω_2, respectively, induced by the triangulations $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$. If Γ is a straight segment, then $\Gamma_1 = \Gamma_2 = \Gamma$, otherwise Γ_1 and Γ_2 can be different (see Fig. 4.1).

For $k = 1, 2$, let $\{\mathbf{x}_1^{(\Gamma_k)}, \ldots, \mathbf{x}_{\overline{n}_k}^{(\Gamma_k)}\} \in \overline{\Gamma}_k$ be the nodes induced by the mesh $\mathcal{T}_{k,h_k}$.

The Lagrange basis functions of Y_{k,h_k} are denoted by $\{\mu_i^{(k)}\}$ for $i = 1, \ldots, \overline{n}_k$ and they are reordered so that the first $n_k (\leq \overline{n}_k)$ basis functions span Λ_{k,h_k}.

4.1. Interpolation and intergrid operators. We introduce two independent operators that exchange information between the two independent grids on the interface Γ.

The first one $\Pi_{12} : Y_{2,h_2} \to Y_{1,h_1}$ is such that

$$(\Pi_{12}\mu_{2,h_2})(\mathbf{x}_i^{(\Gamma_1)}) = \mu_{2,h_2}(\mathbf{x}_i^{(\Gamma_1)}), \quad i = 1, \ldots, \overline{n}_1, \qquad \forall \mu_{2,h_2} \in Y_{2,h_2}. \tag{4.3}$$

The second interpolation operator $\Pi_{21} : Y_{1,h_1} \to Y_{2,h_2}$ is such that

$$(\Pi_{21}\mu_{1,h_1})(\mathbf{x}_i^{(\Gamma_2)}) = \mu_{1,h_1}(\mathbf{x}_i^{(\Gamma_2)}), \quad i = 1, \ldots, \overline{n}_2, \qquad \forall \mu_{1,h_1} \in Y_{1,h_1}. \tag{4.4}$$

The operator Π_{12} is in fact the finite element interpolation operator

$$\mathcal{I}_1 : C^0(\overline{\Gamma}) \to Y_{1,h_1} :\ \forall \eta \in C^0(\overline{\Gamma}) \quad (\mathcal{I}_1\eta)(\mathbf{x}_i^{(\Gamma_1)}) = \eta(\mathbf{x}_i^{(\Gamma_1)}), \quad i = 1, \ldots, \overline{n}_1, \tag{4.5}$$

restricted to the functions of Y_{2,h_2} (instead than operating on the entire $C^0(\overline{\Gamma})$). Similarly Π_{21} is the restriction of

$$\mathcal{I}_2 : C^0(\overline{\Gamma}) \to Y_{2,h_2} :\ \forall \eta \in C^0(\overline{\Gamma}) \quad (\mathcal{I}_2\eta)(\mathbf{x}_i^{(\Gamma_2)}) = \eta(\mathbf{x}_i^{(\Gamma_2)}), \quad i = 1, \ldots, \overline{n}_2 \tag{4.6}$$

to the functions of Y_{1,h_1}.

Remark 4.1. *Using only one intergrid interpolation operator would not guarantee an accurate non-conforming method; this would yield to the so-called* pointwise matching *discussed, e.g., in [5, 3], where both trial and test functions satisfy the relation* $(v_{|\Omega_2})_{|\Gamma} = \Pi_{21}((v_{|\Omega_1})_{|\Gamma})$. *In our approach, the second operator (*Π_{12} *that maps* Y_{2,h_2} *on* Y_{1,h_1}*) matches, in a suitable way, the fluxes across the interface.*

The (rectangular) matrices associated with Π_{21} and Π_{12} are, respectively, $R_{21} \in \mathbb{R}^{\overline{n}_2 \times \overline{n}_1}$ and $R_{12} \in \mathbb{R}^{\overline{n}_1 \times \overline{n}_2}$ and they are defined by

$$\begin{array}{lll} (R_{21})_{ij} & = \mu_j^{(1)}(\mathbf{x}_i^{(\Gamma_2)}) & i = 1, \dots, \overline{n}_2,\ j = 1, \dots, \overline{n}_1, \\ (R_{12})_{ij} & = \mu_j^{(2)}(\mathbf{x}_i^{(\Gamma_1)}) & i = 1, \dots, \overline{n}_1,\ j = 1, \dots, \overline{n}_2. \end{array} \tag{4.7}$$

Remark 4.2. *When* $\Gamma_1 \neq \Gamma_2$ *(geometrical non-conformity) the Rescaled Localized Radial Basis Function (RL-RBF) interpolation (see [15]) represents a very effective alternative to Lagrange interpolation.*

4.2. Formulation of INTERNODES. The *weak form of INTERNODES* reads: find $u_{1,h_1} \in V_{1,h_1}$ and $u_{2,h_2} \in V_{2,h_2}$ such that

$$\begin{cases} a_k(u_{k,h_k}, v^0_{k,h_k}) = (f, v^0_{k,h_k})_{L^2(\Omega_k)} & \forall v^0_{k,h_k} \in V^0_{k,h_k}, \quad k = 1,2 \\ u_{2,h_2} = \Pi_{21} u_{1,h_1} & \text{on } \Gamma_2, \\ r_{1,h_1} + \Pi_{12} r_{2,h_2} = 0 & \text{on } \Gamma_1, \end{cases} \tag{4.8}$$

where $r_{k,h_k} \in \Lambda'_{k,h_k}$, the so called *residuals*, satisfy

$$\langle r_{k,h_k}, \mu_i^{(k)} \rangle = a_k(u_{k,h_k}, \widehat{\mathcal{R}}_k \mu_i^{(k)}) - (f, \widehat{\mathcal{R}}_k \mu_i^{(k)})_{L^2(\Omega_k)} \quad i = 1, \dots, n_k, \tag{4.9}$$

while

$$\widehat{\mathcal{R}}_k : \Lambda_{k,h_k} \to V_{k,h_k}, \quad s.t. \quad (\widehat{\mathcal{R}}_k \eta_{k,h_k})_{|\Gamma} = \eta_{k,h_k}, \quad \forall \eta_{k,h_k} \in \Lambda_{k,h_k} \tag{4.10}$$

are linear and continuous discrete liftings from Γ_k to Ω_k (as, e.g., the finite element interpolant that is zero at all finite element nodes not lying on Γ).

Note the unsymmetrical role played by the domains Ω_1 and Ω_2 in (4.8). In particular the Dirichlet trace on Γ_1 is first interpolated and then transferred to Γ_2. For this reason Ω_1 is named *master* subdomain and Ω_2 *slave* subdomain.

Remark 4.3. *If the discretizations in* Ω_1 *and* Ω_2 *are conforming on* Γ*, then* Π_{21} *and* Π_{12} *are the identity operators and problem (4.8)–(4.9) coincides with (3.7); (4.8)–(4.9) can therefore be regarded as the non-conforming counterpart of (3.7).*

5. Algebraic form of INTERNODES. We start by recalling the algebraic form of the monodomain problem (3.2). Denoting by $\{\varphi_i\}$, for $i = 1, \dots, N$, the Lagrange basis functions of V_h associated with the nodes $\mathbf{x}_i$ of the mesh $\mathcal{T}_h$, and introducing the matrix $A_{ij} = a(\varphi_j, \varphi_i)$, for $i, j = 1, \dots, N$, and the vectors $\mathbf{f} = [(f, \varphi_i)]_{i=1}^N$, $\mathbf{u} = [u_h(\mathbf{x}_i)]_{i=1}^N$, the algebraic form of (3.2) reads

$$A\mathbf{u} = \mathbf{f}. \tag{5.1}$$

Now we derive the algebraic linear system associated with (3.7). For $k = 1, 2$ we define two linear and continuous *lifting operators* $E_k : \Lambda_{k,h_k} \to V_{k,h_k}$, s.t. $(E_k \lambda_{k,h_k})_{|\Gamma} = \lambda_{k,h_k}$, that extend any $\lambda_{k,h_k} \in \Lambda_{k,h_k}$ by setting to zero the values of $E_k \lambda_{k,h_k}$ at all nodes of $\mathcal{T}_{k,h_k}$ not belonging to Γ_k. In particular, if $\lambda_{k,h_k} = \mu_j^{(k)}$ (the jth Lagrange basis function on Γ_k), then $E_k\mu_j^{(k)}$ is the Lagrange basis function of X_{k,h_k} whose restriction on Γ_k coincides with $\mu_j^{(k)}$. For $k=1,2$, we define in a standard way the local stiffness matrices (see, e.g., [25, 24]), i.e. $(A_{kk})_{ij} = a_k(\varphi_j^{(k)}, \varphi_i^{(k)})$, $(A_{\Gamma_k,\Gamma_k})_{ij} = a_k(E_k\mu_j^{(k)}, E_k\mu_i^{(k)})$, $(A_{k,\Gamma_k})_{ij} = a_k(E_k\mu_j^{(k)}, \varphi_i^{(k)})$, and $(A_{\Gamma_k,k})_{ij} = a_k(\varphi_j^{(k)}, E_k\mu_i^{(k)})$.

Recalling the definition of the residuals (4.9), we set

$$r_i^{(k)} = a_k(u_{k,h_k}, E_k\mu_i^{(k)}) - (f, E_k\mu_i^{(k)})_{L^2(\Omega_k)} \qquad \text{for } i=1,\ldots,n_k, \tag{5.2}$$

and

$$\begin{aligned} &\mathbf{f}_k = [(f,\varphi_i^{(k)})_{L^2(\Omega_k)}]_{i=1}^{N_k^0}, \quad \mathbf{f}_{\Gamma_k} = [(f, E_k\mu_i^{(k)})_{L^2(\Omega_k)}]_{i=1}^{n_k}, \\ &\mathbf{u}_k = [u_{k,h_k}(\mathbf{x}_j^{(k)})]_{j=1}^{N_k^0}, \qquad \mathbf{u}_{\Gamma_k} = [u_{k,h_k}(\mathbf{x}_j^{(\Gamma_k)})]_{j=1}^{n_k}, \qquad \mathbf{r}_k = [r_i^{(k)}]_{i=1}^{n_k}. \end{aligned} \tag{5.3}$$

In the case that $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$ are conforming on Γ (in which case $h_1 = h_2$ and $n_1 = n_2$), the algebraic counterpart of the conforming 2-domains problem (3.7) reads

$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma_1} & 0 \\ A_{\Gamma_1,1} & A_{\Gamma_1,\Gamma_1}+A_{\Gamma_2,\Gamma_2} & A_{\Gamma_2,2} \\ 0 & A_{2,\Gamma_2} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_{\Gamma_1} \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_{\Gamma_1}+\mathbf{f}_{\Gamma_2} \\ \mathbf{f}_2 \end{bmatrix}, \tag{5.4}$$

that is equivalent to (5.1), upon setting $\mathbf{u}_{\Gamma_1} = \mathbf{u}_{|\Gamma}$. Notice that we have eliminated the trace $\mathbf{u}_{\Gamma_2}$, since it coincides with $\mathbf{u}_{\Gamma_1}$.

The residual vectors $\mathbf{r}_k$, whose components are defined in (5.2), satisfy

$$\mathbf{r}_k = A_{\Gamma_k,k}\mathbf{u}_k + A_{\Gamma_k,\Gamma_k}\mathbf{u}_{\Gamma_k} - \mathbf{f}_{\Gamma_k}, \quad k=1,2; \tag{5.5}$$

hence the second row of (5.4) can be equivalently written as $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{0}$, and it is the algebraic realization of (3.9).

We write now the algebraic form of the non-conforming problem (4.8)–(4.9). First, we define the *interface mass matrices* M_{Γ_k} as

$$(M_{\Gamma_k})_{ij} = (\mu_j^{(k)}, \mu_i^{(k)})_{L^2(\Gamma)}, \qquad i,j=1,\ldots,\overline{n}_k, \quad k=1,2. \tag{5.6}$$

Then we note that the values $r_i^{(k)}$ are the coefficients of the expansion of r_{k,h_k} with respect to the canonical dual basis $\{\Phi_i^{(k)}\}_{i=1}^{n_k}$ of Λ'_{k,h_k} defined by $\langle \Phi_i^{(k)}, \mu_j^{(k)}\rangle = \delta_{ij}$, $i,j=1,\ldots,n_k$; then (see, e.g., [8]) $\Phi_i^{(k)} = \sum_{j=1}^{n_k}(M_{\Gamma_k}^{-1})_{ji}\mu_j^{(k)}$, meaning that Λ'_{k,h_k} and Λ_{k,h_k} are in fact the same linear space. Then $r_{k,h_k} \in \Lambda'_{k,h_k} = \Lambda_{k,h_k}$ can be expanded also w.r.t. to the Lagrange basis $\{\mu_j^{(k)}\}$ of Λ_{k,h_k} as

$$r_{k,h_k} = \sum_{j=1}^{n_k} z_j^{(k)}\mu_j^{(k)}, \qquad \text{with } z_j^{(k)} \in \mathbb{R}. \tag{5.7}$$

As a matter of fact, we have

$$r_{k,h_k} = \sum_{i=1}^{n_k} r_i^{(k)}\Phi_i^{(k)} = \sum_{i=1}^{n_k} r_i^{(k)} \sum_{j=1}^{n_k}(M_{\Gamma_k}^{-1})_{ji}\mu_j^{(k)} = \sum_{j=1}^{n_k} \underbrace{\left(\sum_{i=1}^{n_k}(M_{\Gamma_k}^{-1})_{ji} r_i^{(k)}\right)}_{z_j^{(k)}} \mu_j^{(k)}.$$

Denoting by $\mathbf{z}_k,\ \mathbf{r}_k \in \mathbb{R}^{n_k}$ the vectors whose entries are the values $z_i^{(k)}$ and $r_i^{(k)}$, respectively, it holds

$$\mathbf{z}_k = M_{\Gamma_k}^{-1}\mathbf{r}_k. \tag{5.8}$$

The intergrid interpolation operator Π_{12} in $(4.8)_3$ applies on the Lagrange expansion (5.7) of r_{2,h_2}, i.e.,

$$\sum_{i=1}^{n_1} z_i^{(1)}\mu_i^{(1)} + \Pi_{12}\left(\sum_{j=1}^{n_2} z_j^{(2)}\mu_j^{(2)}\right) = 0, \tag{5.9}$$

and, thanks to (5.8) and (4.7), the algebraic form of $(4.8)_3$ reads

$$\mathbf{z}_1 + R_{12}\mathbf{z}_2 = \mathbf{0} \quad \text{or, equivalently,} \quad \mathbf{r}_1 + M_{\Gamma_1}R_{12}M_{\Gamma_2}^{-1}\mathbf{r}_2 = \mathbf{0}. \tag{5.10}$$

After defining the *intergrid matrices*

$$Q_{21} = R_{21}, \qquad Q_{12} = M_{\Gamma_1}R_{12}M_{\Gamma_2}^{-1}, \tag{5.11}$$

and by using $\mathbf{u}_{\Gamma_2} = R_{21}\mathbf{u}_{\Gamma_1}$, that is the counterpart of $(4.8)_2$, the algebraic form of (4.8) reads

$$\begin{bmatrix} A_{1,1} & A_{1,\Gamma_1} & 0 \\ A_{\Gamma_1,1} & A_{\Gamma_1,\Gamma_1} + Q_{12}A_{\Gamma_2,\Gamma_2}Q_{21} & Q_{12}A_{\Gamma_2,2} \\ 0 & A_{2,\Gamma_2}Q_{21} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_{\Gamma_1} \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_{\Gamma_1} + Q_{12}\mathbf{f}_{\Gamma_2} \\ \mathbf{f}_2 \end{bmatrix} \tag{5.12}$$

System (5.12) is the form of INTERNODES implemented in practice. By taking $Q_{12} = Q_{21} = I$ we recover the algebraic system (5.4) of the conforming case.

In Section 6 we describe how to treat non-homogeneous Dirichlet boundary conditions and how to solve the algebraic system (5.12) by the Schur-complement approach; then we extend the INTERNODES method to decompositions with more than 2 subdomains.

6. Generalization and algorithmic aspects.

6.1. Non-homogeneous Dirichlet conditions. When non-homogeneous Dirichlet boundary conditions are assigned on $\partial\Omega$, we can recover the homogeneous case by a lifting of the Dirichlet data, so that only the right hand side has to be modified (see, e.g., [23]). However, it is often common practice not to make use of lifting operators. In that case also the Dirichlet boundary nodes become degrees of freedom and the corresponding basis functions have to be extended. In this situation the INTERNODES algebraic form (5.12) has to undergo a slight modification:

$$\begin{aligned} (A_{\Gamma_k,\Gamma_k})_{ij} &= a_k(E_k\mu_j^{(k)}, E_k\mu_i^{(k)}) - \int_{\partial\Omega_{D,k}} \alpha\frac{\partial E_k\mu_j^{(k)}}{\partial \mathbf{n}_k}E_k\mu_i^{(k)}, \quad j = 1,\ldots,n_k, \\ (A_{\Gamma_k,k})_{ij} &= a_k(\varphi_j^{(k)}, E_k\mu_i^{(k)}) - \int_{\partial\Omega_{D,k}} \alpha\frac{\partial \varphi_j^{(k)}}{\partial \mathbf{n}_k}E_k\mu_i^{(k)}, \qquad j = 1,\ldots,N_k^0, \end{aligned} \tag{6.1}$$

where $\mu_i^{(k)}$ is any Lagrange basis function associated with $\mathbf{x}_i^{(\Gamma_k)} \in \partial\Gamma_k \cap \partial\Omega_D$. The subtraction of the boundary integrals in (6.1) is motivated by the fact that, for such $\mu_i^{(k)}$, $E_k\mu_i^{(k)}$ does not satisfy essential boundary conditions on $\partial\Omega_D$. With this change, the residuals (5.2) continue to be the approximations of the normal derivatives at the interface Γ.

6.2. An efficient solution algorithm for system (5.12). After Gaussian elimination of the variables $\mathbf{u}_1$ and $\mathbf{u}_2$, the Schur complement form of (5.12) reads

$$S\mathbf{u}_{\Gamma_1} = \mathbf{b} \tag{6.2}$$

where

$$S = S_1 + Q_{12}S_2Q_{21}, \qquad \mathbf{b} = \mathbf{b}_1 + Q_{12}\mathbf{b}_2. \tag{6.3}$$

$$S_k = A_{\Gamma_k,\Gamma_k} - A_{\Gamma_k,k}A_{k,k}^{-1}A_{k,\Gamma_k}, \qquad k = 1,\ 2, \tag{6.4}$$

are the local Schur complement matrices, while

$$\mathbf{b}_k = \mathbf{f}_{\Gamma_k} - A_{\Gamma_k,k}A_{k,k}^{-1}\mathbf{f}_k \tag{6.5}$$

are the local right hand sides.

System (6.2) can be solved, e.g., by the preconditioned Krylov method, with S_1 as preconditioner. (Notice that matrix $\widetilde{S}_2 = Q_{12}S_2Q_{21}$ is not a good candidate to play the role of precondititioner since it may be singular.)

The sketch of the algorithm is reported in Algorithm 1 for reader's convenience.

Algorithm 1 INTERNODES algorithm for 2 subdomains

for all $k = 1, 2$ **do**
  build the local stiffness matrices $A_{k,k}$, A_{k,Γ_k}, $A_{\Gamma_k,k}$ and A_{Γ_k,Γ_k} (see Sect. 6.1 in the case of non-homogeneous Dirichlet conditions)
  build the right hand sides $\mathbf{f}_k$ and $\mathbf{f}_{\Gamma_k}$ (formula (5.3))
  build the local interface mass matrices M_{Γ_k} (formula (5.6))
end for
build the interpolation matrices R_{21} and R_{12} (formulas (4.7))
build Q_{21} and Q_{12} (formula (5.11)) (only the nodes coordinates on the interfaces are needed in this step)
solve system (5.12) (or (6.2))

6.3. Extension to more than 2 subdomains. INTERNODES can be extended to the case of $M > 2$ subdomains. Let us start with two simple decompositions as in Fig. 6.1, while an example of a more general decomposition is shown in Fig. 6.2, left.

Let us suppose that each Ω_k is convex with Lipschitz boundary $\partial\Omega_k$ (for $k = 1, \ldots, M$), and that any angle between two consecutive edges is less than π. Let $\Gamma_k = \partial\Omega\backslash\partial\Omega$ be the part of the boundary of Ω_k itnernal to Ω, and $\gamma_k^{(i)} \subset \Gamma_k$ be the ith edge of Γ_k (the subindex k identifies the domain, while i denotes the number of the internal edges of $\partial\Omega_k$),

Let $\Gamma_{k\ell} = \Gamma_{\ell k} = \partial\Omega_k \cap \partial\Omega_\ell$ be the interface between the two subdomains Ω_k and Ω_ℓ, and $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$ be the two edges of Ω_k and Ω_ℓ, respectively, whose intersection is $\Gamma_{k\ell}$. In the example depicted in Fig. 6.1, left, we have $\Gamma_{k\ell} = \gamma_k^{(i)} = \gamma_\ell^{(j)}$ for any interface $\Gamma_{k\ell}$ of the decomposition, while in the example depicted in Fig. 6.1, right, we have $\Gamma_{23} = \gamma_2^{(2)} \subset \gamma_3^{(1)}$ and $\Gamma_{13} = \gamma_1^{(1)} \subset \gamma_3^{(1)}$.

Between $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$, one is tagged as *master* and the other as *slave*. We see later on that the union of all the master edges will identify the skeleton $\Gamma^{(m)}$, that in

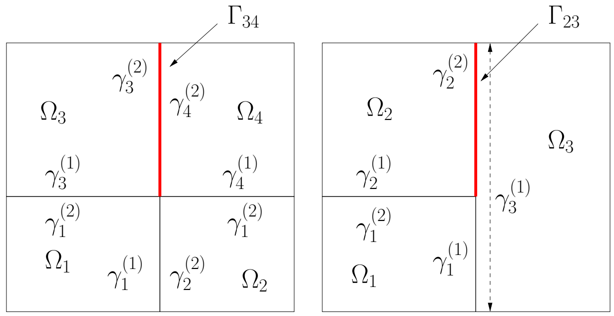

FIG. 6.1. *Two simple decompositions*

the mortar community is named mortar interface. In the example of Fig. 6.1 right, we could tag as master the edge $\gamma_3^{(1)}$ (in which case $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ will be slave), or other way around.

For any $k = 1, \dots, M$, A_k denotes the complete stiffness matrix in Ω_k assembled by considering all the basis functions associated with either the internal nodes of Ω_k or the interface nodes on $\Gamma_k = \partial\Omega_k \setminus \partial\Omega$. Then, by following the notations introduced in Sect. 5, we have

$$A_k = \begin{bmatrix} A_{k,k} & A_{k,\Gamma_k} \\ A_{\Gamma_k,k} & A_{\Gamma_k,\Gamma_k} \end{bmatrix}.$$

Then we build the interface mass as well as the interpolation matrices on each side:

for all $k = 1, \dots, M$ (loop on the subdomains) **do**
  build the stiffness matrix A_k
  for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ (loop on the edges of Ω_k internal to Ω) **do**
    build the local interface mass matrix $M_{\gamma_k^{(i)}}$ on $\gamma_k^{(i)}$
    build the interpolation matrix $R_{(k,i),(\ell,j)}$ from $\gamma_\ell^{(j)}$ to $\gamma_k^{(i)}$
  end for
end for

The column vectors of the matrix $R_{(k,i),(\ell,j)}$ hold the values of the Lagrange basis functions of $\gamma_\ell^{(j)}$ at the nodes of $\gamma_k^{(i)}$. If $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$ are the master and the slave sides, respectively, of $\Gamma_{k\ell}$, then $R_{(\ell,j),(k,i)}$ is the interpolation matrix that maps the master side to the slave one (it plays the role of matrix R_{21} defined in (4.7)), while $R_{(k,i),(\ell,j)}$ is the interpolation matrix from the slave to the master side (as R_{12} in (4.7)).

We warn the reader that, when the measure of $\gamma_\ell^{(j)}$ is larger than that of $\gamma_k^{(i)}$ (as, e.g., $\gamma_3^{(1)}$ and $\gamma_2^{(2)}$ in Fig. 6.1, right), all the basis functions of $\gamma_\ell^{(j)}$ whose support has non-empty intersection with $\gamma_k^{(i)}$ must be taken into account in building $R_{(k,i),(\ell,j)}$, included those basis functions associated with the nodes that do not belong to $\Gamma_{k\ell}$.

The modification presented in Sect. 6.1 for the case of two subdomains with non-homogeneous Dirichlet boundary conditions has to be implemented for the case of $M > 2$ subdomains. In particular, for any interface $\gamma_k^{(i)}$, the nodes of $\partial\gamma_k^{(i)}$ internal to Ω are treated if they were "Dirichlet" boundary points with non-homogenous boundary condition, thus in assembling the local stiffness matrices we use formulas

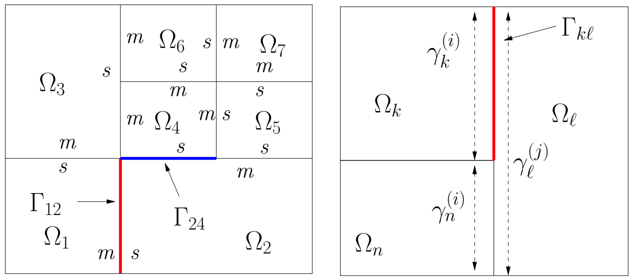

FIG. 6.2. *A partition of* Ω *into 7 subdomains (left figure). The letters* m *and* s *denote the choice made for the master and slave sides. Description of interfaces and edges (right)*

(6.1) instead of $(A_{\Gamma_k,\Gamma_k})_{ij} = a_k(E_k\mu_j^{(k)}, E_k\mu_i^{(k)})$ and $(A_{\Gamma_k,k})_{ij} = a_k(\varphi_j^{(k)}, E_k\mu_i^{(k)})$.

Once the master and slave sides of the interfaces have been fixed, we mark each edge $\gamma_k^{(i)}$ with either the superscript "(m)" (if $\gamma_k^{(i)}$ is a master edge) or "(s)" (otherwise) and we define

$$\Gamma^{(m)} = \bigcup_{k,i} \gamma_k^{(i),(m)}. \tag{6.6}$$

Both the nodes and the degrees of freedom on $\Gamma^{(m)}$ are defined univocally starting from those of the edges $\gamma_k^{(i),(m)}$. The degrees of freedom of the global multidomain problem are the values of u_h at the nodes of $\Gamma^{(m)}$ jointly with the degrees of freedom internal to each Ω_k (as in (6.1)). As done in Section 6.2, we eliminate the degrees of freedom internal to the subdomains Ω_k and solve the Schur complement system (analogous to (6.2))

$$S\mathbf{u}_{\Gamma^{(m)}} = \mathbf{b} \tag{6.7}$$

by, e.g., a Krylov method. The matrix S is never assembled, the kernel subroutine to solve (6.7) (see Algorithm 2) computes the matrix-vector product $\mathbf{w} = S\boldsymbol{\lambda}$, for a given $\boldsymbol{\lambda}$ approximating $\mathbf{u}_{\Gamma^{(m)}}$.

7. Numerical results. Let us consider the Laplace problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = g & \text{on } \partial\Omega. \end{cases}$$

When g is different from zero, by standard arguments we recast the problem into the form (2.1). See [23], Section 6.1 and Section 6.

2D test case. The data f and g are such that the exact solution is $u(x, y) = \sin(xy\pi) + 1$. A decomposition of $\Omega = (0, 2)^2$ in 10 subdomains as in Fig. 7.1 is considered, and independent triangulations in each Ω_k are designed so that on each interface both polynomial non-conformity and geometric non-conformity occur. Either $\mathbb{P}_1$ and quadrilateral *hp*-fem ($\mathbb{Q}_p$) are used to approximate the numerical solution. In order to guarantee full non-conformity on each interface, different polynomial degrees and different element sizes are used inside the subdomains, by setting the polynomial degree equal to either p or $p+1$ on two adjacent domains and the number of elements equal to either N or $N+1$, then we set $h = 1/N$. A non-conforming grid, obtained with $\mathbb{Q}_p$ discretizations in each subdomain, is shown in Fig. 7.1, left.

Algorithm 2 matrix vector product $\mathbf{w} = S\boldsymbol{\lambda}$

INPUT: $\boldsymbol{\lambda}$ (trace on the master interface $\Gamma^{(m)}$)
OUTPUT: $\mathbf{w}$ (residual on the master interface $\Gamma^{(m)}$)
  for all $k = 1, \dots, M$ (loop on the subdomains) **do**
    % extract the trace from $\Gamma^{(m)}$ to Γ_k and store it in $\mathbf{g}_k$
    for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ is a slave edge **do**
      recover the master side $\gamma_\ell^{(j)}$ of Γ_l associated with $\gamma_k^{(i)}$
      extract $\boldsymbol{\lambda}_{|\gamma_\ell^{(j)}}$ from $\boldsymbol{\lambda}$ and interpolate from master to slave:
      $(\mathbf{g}_k)_{|\gamma_k^{(i)}} = R_{(k,i),(\ell,j)}\boldsymbol{\lambda}_{|\gamma_\ell^{(j)}}$
    end for
    for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ is a master edge **do**
      $(\mathbf{g}_k)_{|\gamma_k^{(i)}} = \boldsymbol{\lambda}_{|\gamma_k^{(i)}}$
    end for
    % solve the local problem in Ω_k
    solve $A_k\mathbf{u}_k = \mathbf{f}_k$, with Dirichlet datum $\mathbf{g}_k$ on Γ_k
    % compute the local residual on each internal edge of Γ_k
    for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ (loop on all the edges of Γ_k) **do**
      $\mathbf{r}_{k,i} = (A_k\mathbf{u}_k - \mathbf{f}_k)_{|\gamma_k^{(i)}}$ (keep distinct the contributions of different edges)
    end for
  end for
  % assemble the residuals on Γ_k
  $\mathbf{w} = \mathbf{0}$
  for all $k = 1, \dots, M$ (loop on the subdomains) **do**
    for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ (loop on all the edges of Γ_k) **do**
      if $\gamma_k^{(i)}$ is a master edge **then**
        update $\mathbf{w}_{|\gamma_k^{(i)}} = \mathbf{w}_{|\gamma_k^{(i)}} + \mathbf{r}_{k,i}$
      else
        recover the master edge $\gamma_\ell^{(j)}$ of Ω_ℓ associated with $\gamma_k^{(i)}$
        interpolate from the slave edge $\gamma_k^{(i)}$ to the master edge $\gamma_\ell^{(j)}$ and update:
        $\mathbf{w}_{|\gamma_\ell^{(j)}} = \mathbf{w}_{|\gamma_\ell^{(j)}} + M_{\gamma_\ell^{(j)}} R_{(\ell,j),(k,i)} M^{-1}_{\gamma_k^{(i)}} \mathbf{r}_{k,i}$
      end if
    end for
  end for

In Fig. 7.2, the errors in broken norm (see the next formula (9.30)) are shown, w.r.t. to both h and p (the polynomial degree in the bottom-left subdomain). The error behaviour versus h (see Fig. 7.2 left) agrees with the theoretical estimate of Theorem 9.12, for which we expect $\|u - u_h\|_* \le c(u)h^p$ (in this case $p = 1, 2, 3$, see (9.42)). The convergence rate vs p shown in Fig. 7.2, right, is more than algebraic, as typical in hp-fem. The interested reader can find in [14, 16] a wide collection of numerical results on INTERNODES, even applied to both Navier-Stokes equations and fluid structure interaction problems.

3D test case. The computational domain $\Omega = (0, 2) \times (0, 1) \times (0, 1)$ is decomposed into two subdomains $\Omega_1 = (0, 1)^3$ and $\Omega_2 = (1, 2) \times (0, 1) \times (0, 1)$. The data f and g are set in such a way that the exact solution is $u(x, y, z) = (y^2 - y)(z^2 - z)\sin(xyz\pi)$. In Ω_1 (Ω_2, resp.) we consider a triangulation in $N \times N \times N$ elements $((N - 2) \times (N -$

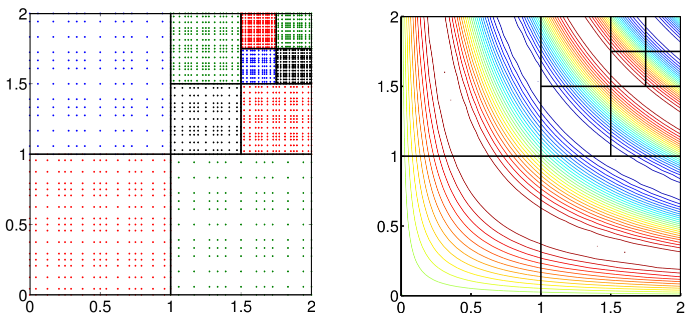

FIG. 7.1. *2D test case. A partition into several subdomains (left picture); the dots are the nodes of the triangulations within the subdomains. The corresponding INTERNODES solution is reported on the right picture*

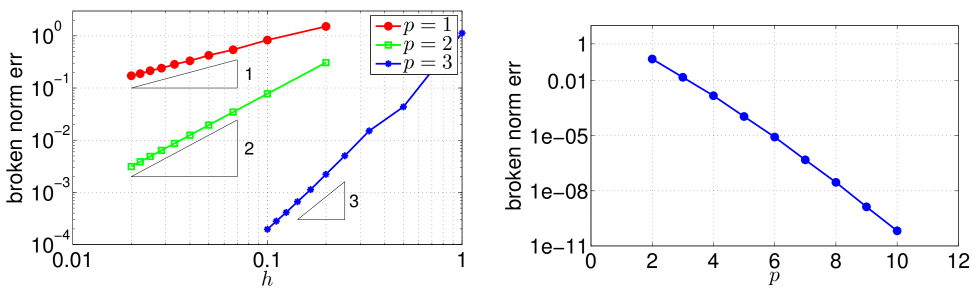

FIG. 7.2. *The broken norm error w.r.t. the mesh-size h with p fixed (left). The broken norm error w.r.t. p (it is the polynomial degree in the bottom-left subdomain), here the mesh size is fixed $h = 1/3$ (right)*

$2) \times (N-2)$, resp.). Then we set $h_1 = 1/N$ and $h_2 = 1/(N-2)$. When $p = 1$ the triangulation is composed by tetrahedra, thus classical $\mathbb{P}_1$ fem are used, while when $p > 1$ the mesh is formed by esahedra and hp-fem with $\mathbb{Q}_p$ local spaces are considered.

In Fig. 7.3, the errors in broken norm are shown, w.r.t. both the mesh size h_1 of Ω_1 and the local polynomial degree $p = p_1 = p_2$. Also in this case the numerical results agree with the theoretical estimate of Theorem 9.12.

8. A comparison between INTERNODES and Mortar method. We follow the notations of [5] for the classical mortar method and those of [21] for the unsymmetric mortar method, a special version of mortar method proposed in [10] in which the cross-domain mass matrices on the interface are computed by suitable quadrature formulas instead of (the computationally heavy) exact integration.

Let $\mu_i^{(k)}$ (for $k = 1, 2$) be the Lagrange basis functions on Γ, and $\psi_i^{(2)}$ the basis functions of the mortar space, being the latter associated with the slave domain Ω_2. Then we set the mortar mass matrices

$$
\begin{aligned}
&\Xi = P^{-1}\Phi, && P_{ij} = \int_\Gamma \mu_j^{(2)}\psi_i^{(2)}, && \Phi_{ij} = \int_\Gamma \mu_j^{(1)}\psi_i^{(2)},\\
&\Xi^{--} = (P^-)^{-1}\Phi^-, && P^-_{ij} = \Sigma_-\mu_j^{(2)}\psi_i^{(2)}, && \Phi^-_{ij} = \Sigma_-\mu_j^{(1)}\psi_i^{(2)},\\
&\Xi^{-+} = (P^-)^{-1}\Phi^+, && && \Phi^+_{ij} = \Sigma_+\mu_j^{(1)}\psi_i^{(2)},
\end{aligned}
$$

being Σ_- (Σ_+, resp.) the quadrature formula on the interface Γ induced by the

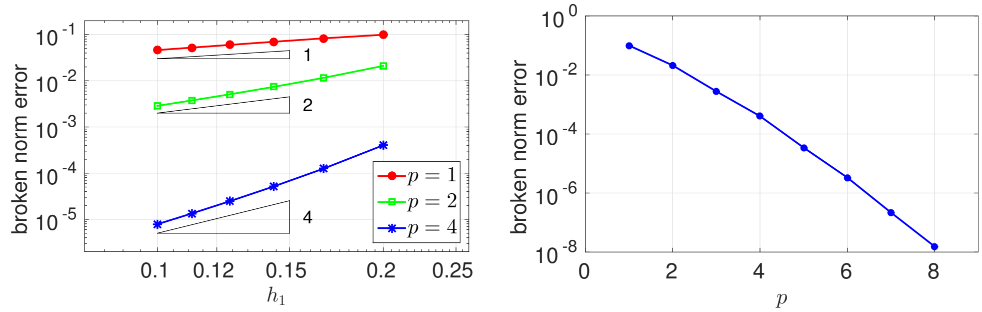

FIG. 7.3. *3D test case. The broken norm error w.r.t. the mesh-size* h_1 *with* $p = p_1 = p_2$ *fixed (left) and w.r.t. the local polynomial degree* $p = p_1 = p_2$*, with fixed mesh size (right)* $h_1 = 1/5$ *and* $h_2 = 1/3$.

discretization in the slave domain Ω_2 (master domain Ω_1, resp.).

Both classical and unsymmetric mortar methods can be recast in the form (5.12) provided that the matrices Q_{12} and Q_{21} are defined as follows:

	classical mortar	unsym. mortar	INTERNODES
Q_{21}	Ξ	Ξ^{--}	R_{21}
Q_{12}	Ξ^T	$(\Xi^{-+})^T$	$M_{\Gamma_1} R_{12} M_{\Gamma_2}^{-1}$

A similarity can however be found between INTERNODES and the unsymmetric mortar method presented in [10, 21]. As a matter of fact, in the case that the quadrature nodes used in Σ_- are a subset of the grid nodes induced on Γ by the discretization inside the slave domain Ω_2, then $\Xi^{--} = R_{21}$. Despite of this, by choosing the basis functions $\psi_i^{(2)}$ of the mortar space as standard (see [5, 21]) we numerically experienced that the matrix $M_{\Gamma_1} R_{12} M_{\Gamma_2}^{-1}$ does not coincide with $(\Xi^{-+})^T$: INTERNODES and unsymmetric mortar are indeed two different methods.

A more thorough comparison between the classical mortar method and INTERNODES can be found in [14]. In the same paper the implementation aspects of the two approaches as well as their convergence rate with respect to the mesh sizes are discussed, concluding that in practice INTERNODES attains the same accuracy as the classical mortar method.

9. Analysis of INTERNODES. In order to analyse INTERNODES for the case of two subdomains, we write an interface formulation of transmission problem (2.8). Along the whole section, c will denote a generic positive constant independent of the mesh sizes h_1 and h_2, but not necessarily the same everywhere.

9.1. Interface formulation of the continuous problem. For $k = 1, 2$, given $\lambda \in \Lambda$ and $f \in L^2(\Omega)$, we consider the non-homogeneous Dirichlet problem

$$\text{find } u_k^{\lambda,f} \in V_k : \quad a_k(u_k^{\lambda,f}, v) = (f, v)_{L^2(\Omega_k)} \quad \forall v \in V_k^0, \quad u_k^{\lambda,f} = \lambda \text{ on } \Gamma. \tag{9.1}$$

Because of the linearity of $a_k(\cdot,\cdot)$, we have $u_k^{\lambda,f} = \overline{u}_k + u_k^\lambda$, where:

$$\overline{u}_k \in V_k^0 : \quad a_k(\overline{u}_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V_k^0 \tag{9.2}$$

and

$$u_k^\lambda \in V_k : \quad a_k(u_k^\lambda, v) = 0 \quad \forall v \in V_k^0, \qquad u_k^\lambda = \lambda \text{ on } \Gamma. \tag{9.3}$$

The following stability estimate holds (see, e.g., [23, Sect. 6.1.2])

$$\|u_k^{\lambda,f}\|_{H^1(\Omega_k)} \le c(\|f\|_{L^2(\Omega_k)} + \|\lambda\|_\Lambda). \tag{9.4}$$

We consider the following *interface formulation* of equation (2.9) with four unknowns: find $\lambda_1 \in \Lambda,\ \lambda_2 \in \Lambda$, and $r_1 \in \Lambda',\ r_2 \in \Lambda'$ such that

$$\begin{cases} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k) - \langle r_k, \mu_k\rangle = (f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\overline{u}_k, \mathcal{R}_k\mu_k) & \forall \mu_k \in \Lambda,\ k = 1,2,\\ \langle t, \lambda_1 - \lambda_2\rangle = 0 & \forall t \in \Lambda' \\ \langle r_1 + r_2, \varphi\rangle = 0 & \forall \varphi \in \Lambda \end{cases} \tag{9.5}$$

where $\mathcal{R}_k$ are defined in (2.10). The multipliers r_k coincide with $\partial_{L_k} u_k$, see Remark 2.1. By eliminating $r_1 = -r_2$ from the last equation and summing up the first two equations we obtain another *interface formulation* of equation (2.9) with three unknowns: find $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$, and $r_2 \in \Lambda'$ s.t.

$$\begin{cases} \displaystyle\sum_{k=1,2} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k) + \langle r_2, \mu_1 - \mu_2\rangle = \sum_{k=1,2} [(f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\overline{u}_k, \mathcal{R}_k\mu_k)] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(\mu_1,\mu_2) \in \Lambda\times\Lambda \\ \langle t, \lambda_1 - \lambda_2\rangle = 0 \qquad\qquad\qquad\qquad\qquad \forall t \in \Lambda'. \end{cases} \tag{9.6}$$

By setting $\boldsymbol{\mu} = (\mu_1, \mu_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{\Lambda} = \Lambda \times \Lambda$ (endowed with the norm $\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} = (\|\lambda_1\|_\Lambda^2 + \|\lambda_2\|_\Lambda^2)^{1/2}$), and $\forall \boldsymbol{\lambda},\ \boldsymbol{\mu} \in \boldsymbol{\Lambda}$, $\forall t \in \Lambda'$,

$$\begin{aligned} &\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1,2} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k), \qquad \mathcal{B}(\boldsymbol{\mu}, t) = \langle t, \mu_1\rangle - \langle t, \mu_2\rangle \\ &\mathcal{F}(\boldsymbol{\mu}) = \sum_{k=1,2} [(f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\overline{u}_k, \mathcal{R}_k\mu_k)], \end{aligned} \tag{9.7}$$

problem (9.6) takes the saddle point form: find $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and $r_2 \in \Lambda'$ s.t.

$$\begin{cases} \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}) + \mathcal{B}(\boldsymbol{\mu}, r_2) = \mathcal{F}(\boldsymbol{\mu}) & \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \\ \mathcal{B}(\boldsymbol{\lambda}, t) = 0 & \forall t \in \Lambda'. \end{cases} \tag{9.8}$$

Lemma 9.1. *The following properties hold:*

1. the bilinear form $\mathcal{A}$ is coercive and continuous on $\boldsymbol{\Lambda}$, i.e., there exist $\alpha_* > 0$ and $C_A > 0$ s.t.

$$\begin{aligned} &\mathcal{A}(\boldsymbol{\mu}, \boldsymbol{\mu}) \ge \alpha_* \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \\ &|\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu})| \le C_A \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} && \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}; \end{aligned} \tag{9.9}$$

2. the bilinear form $\mathcal{B}$ is continuous and satisfies an inf-sup condition, i.e. there exist $C_B > 0$ s.t.

$$\begin{aligned} &|\mathcal{B}(\boldsymbol{\mu}, t)| \le C_B \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda},\ t \in \Lambda', \\ &\inf_{t\in\Lambda'} \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}} \frac{\mathcal{B}(\boldsymbol{\mu}, t)}{\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'}} \ge \sqrt{2}; \end{aligned} \tag{9.10}$$

3. the linear functional $\mathcal{F}$ is continuous, i.e. there exists $C_F > 0$ s.t.

$$|\mathcal{F}(\boldsymbol{\mu})| \le C_F \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \tag{9.11}$$

Proof. 1. By taking $\mathcal{R}_k\mu_k = u_k^{\lambda_k}$, continuity and coercivity of $\mathcal{A}$ are immediate consequence of continuity and coercivity of the bilinear forms a_k (see [23, Sect. 1.2]).

2. The continuity of $\mathcal{B}$ follows from Cauchy-Schwarz inequality. To prove the inf-sup condition, we define the operators $B : \mathbf{\Lambda} \to \Lambda$ and $B^T : \Lambda' \to \mathbf{\Lambda}'$ such that

$$\mathcal{B}(\boldsymbol{\mu}, t) = \langle t, B\boldsymbol{\mu}\rangle = \langle B^T t, \boldsymbol{\mu}\rangle \qquad \forall \boldsymbol{\mu} \in \mathbf{\Lambda},\ \forall t \in \Lambda'.$$

Then, thanks to $(9.7)_1$, it holds $B\boldsymbol{\mu} = \mu_1 - \mu_2$ for any $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbf{\Lambda}$, and $B^T t = [t,\ -t]^T$ for any $t \in \Lambda'$, thus $\|B^T t\|_{\mathbf{\Lambda}'} = \sqrt{2}\|t\|_{\Lambda'}$. It follows that

$$\inf_{t\in\Lambda'} \sup_{\boldsymbol{\mu}\in\mathbf{\Lambda}} \frac{\mathcal{B}(\boldsymbol{\mu}, t)}{\|\boldsymbol{\mu}\|_{\mathbf{\Lambda}}\|t\|_{\Lambda'}} = \inf_{t\in\Lambda'} \sup_{\boldsymbol{\mu}\in\mathbf{\Lambda}} \frac{\langle B^T t, \boldsymbol{\mu}\rangle}{\|\boldsymbol{\mu}\|_{\mathbf{\Lambda}}\|t\|_{\Lambda'}} = \inf_{t\in\Lambda'} \frac{\|B^T t\|_{\mathbf{\Lambda}'}}{\|t\|_{\Lambda'}} \geq \sqrt{2}.$$

3. (9.11) follows by Cauchy-Schwarz inequality and the continuity of the bilinear forms a_k. □

THEOREM 9.2. *Problem (9.6) is well posed. Moreover it is equivalent to (2.9), in the following sense: if* $\{\lambda_1, \lambda_2,\ r_2\}$ *solves (9.6), then* $u_1 = u_1^{\lambda,f}$ *and* $u_2 = u_2^{\lambda,f}$ *(with* $\lambda = \lambda_1 = \lambda_2$*) are the unique solutions of (2.9); conversely, if* $\{u_1,\ u_2\}$ *solves (2.9), then* λ_1, λ_2, *and* r_2 *solve (9.6), with*

$$\lambda_1 = (u_1)_{|\Gamma}, \quad \lambda_2 = (u_2)_{|\Gamma}, \quad \langle r_2, \mu\rangle = a_2(u_2, \mathcal{R}_2\mu) - (f, \mathcal{R}_2\mu)_{L^2(\Omega_2)} \qquad \forall \mu \in \Lambda.$$

Proof. Thanks to Lemma 9.1, the well-posedness of problem (9.6) (existence, uniqueness and stability of the solution) follows by applying stantard results for saddle point problems (see, e.g., [6, Cor. 4.2.1]) to (9.8).

The equivalence between (9.6) and (2.9) can be proved by standard arguments. □

9.2. Interface formulation of the discrete non-conforming problem. Let Λ_{1,h_1} and Λ_{2,h_2} be induced by independent discretizations in Ω_1 and Ω_2 as in Sect. 4. Let $\mathbf{\Lambda}_h = (\Lambda_{1,h_1}, \Lambda_{2,h_2})$ be endowed with the norm of $\mathbf{\Lambda}$, and for $k = 1, 2$, let $\Lambda'_{k,h_k} = (\Lambda_{k,h_k}, \|\cdot\|_{\Lambda'})$ (Λ_{k,h_k} and Λ'_{k,h_k} are identical linear spaces, see Sect. 5).

By applying the conforming finite element approximation introduced in Sect. 3 in each subdomain Ω_k, we can write the finite dimensional counterparts of (9.1)–(9.3): given $f \in L^2(\Omega)$ and $\lambda_{k,h_k} \in \Lambda_{k,h_k}$, for $k = 1, 2$, we denote by $U_k = U_k(\lambda_{k,h_k}, f) \in V_{k,h_k}$ the solution of

$$a_k(U_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V^0_{k,h_k}, \qquad U_k = \lambda_{k,h_k} \text{ on } \Gamma. \tag{9.12}$$

We note that $U_k = \widehat{\mathcal{H}}_k\lambda_{k,h_k} + \overline{U}_k$, where $\overline{U}_k = \overline{U}_k(f) \in V^0_{k,h_k}$ is the solution of

$$a_k(\overline{U}_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V^0_{k,h_k}, \tag{9.13}$$

and $\widehat{\mathcal{H}}_k\lambda_{k,h_k} \in V_{k,h_k}$ is the solution of

$$a_k(\widehat{\mathcal{H}}_k\lambda_{k,h_k}, v) = 0 \quad \forall v \in V^0_{k,h_k}, \qquad \widehat{\mathcal{H}}_k\lambda_{k,h_k} = \lambda_{k,h_k} \text{ on } \Gamma. \tag{9.14}$$

Then similarly to the continuous case,

$$\|U_k\|_{H^1(\Omega_k)} \leq c(\|f\|_{L^2(\Omega_k)} + \|\lambda_{k,h_k}\|_{\Lambda}). \tag{9.15}$$

We introduce the non-conforming counterpart of (9.6), i.e. an interface form of the non-conforming problem (4.8):

find $\lambda_{1,h_1} \in \Lambda_{1,h_1}$, $\lambda_{2,h_2} \in \Lambda_{2,h_2}$, and $r_{2,h_2} \in \Lambda'_{2,h_2}$ s.t.

$$\begin{cases} \displaystyle\sum_{k=1,2} a_k(\widehat{\mathcal{H}}_k\lambda_{k,h_k}, \widehat{\mathcal{R}}_k\mu_{k,h_k}) + \langle \Pi_{12} r_{2,h_2}, \mu_{1,h_1}\rangle - \langle r_{2,h_2}, \mu_{2,h_2}\rangle \\ \qquad = \displaystyle\sum_{k=1,2} \Big[(f, \widehat{\mathcal{R}}_k\mu_{k,h_k})_{L^2(\Omega_k)} - a_k(\overline{U}_k, \widehat{\mathcal{R}}_k\mu_{k,h_k})\Big] \\ \qquad\qquad\qquad\qquad \forall(\mu_{1,h_1}, \mu_{2,h_2}) \in \Lambda_{1,h_1} \times \Lambda_{2,h_2} \\ \langle t_{2,h_2}, \lambda_{2,h_2} - \Pi_{21}\lambda_{1,h_1}\rangle = 0 \qquad\qquad \forall t_{2,h_2} \in \Lambda'_{2,h_2}. \end{cases} \tag{9.16}$$

The discrete lifiting operators $\widehat{\mathcal{R}}_k$ have been introduced in (4.10).

Theorem 9.3. *Problem (9.16) is equivalent to problem (4.8) – (4.9) in the following sense: if $\{\lambda_{1,h_1}, \lambda_{2,h_2}, r_{2,h_2}\}$ solves (9.16), then $\{u_{1,h_1} = U_1,\ u_{2,h_2} = U_2,\ r_{2,h_2}\}$ solves (4.8) – (4.9); conversely, if $\{u_{1,h_1}, u_{2,h_2}, r_{2,h_2}\}$ solves (4.8) – (4.9), then $\{\lambda_{1,h_1} = (u_{1,h_1})_{|\Gamma}, \lambda_{2,h_2} = (u_{2,h_2})_{|\Gamma}, r_{2,h_2}\}$ solves (9.16).*

Proof. Let $\{\lambda_{1,h_1}, \lambda_{2,h_2}, r_{2,h_2}\}$ solve (9.16). Then, $u_{1,h_1} = U_1$ and $u_{2,h_2} = U_2$ (solutions of (9.12)) solve $(4.8)_1$, while, from $(9.16)_2$ it follows $U_2 = \Pi_{21}U_1$ on Γ, i.e., $(4.8)_2$ holds. To prove $(4.8)_3$– (4.9), we set $\mu_{1,h_1} = 0$ and $\mu_{2,h_2} = \mu_i^{(2)}$ in $(9.16)_1$, then $\langle r_{2,h_2}, \mu_i^{(2)}\rangle = a_2(U_2, \widehat{\mathcal{R}}_2\mu_i^{(2)}) - (f, \widehat{\mathcal{R}}_2\mu_i^{(2)})_{L^2(\Omega_2)}$, i.e., r_{2,h_2} and U_2 satisfy (4.9) for $k = 2$. If we set $\mu_{2,h_2} = 0$ and $\mu_{1,h_1} = \mu_i^{(1)}$ in $(9.16)_1$, we have

$$\langle -\Pi_{12} r_{2,h_2}, \mu_i^{(1)}\rangle = a_1(U_1, \widehat{\mathcal{R}}_1\mu_i^{(1)}) - (f, \widehat{\mathcal{R}}_1\mu_i^{(1)})_{L^2(\Omega_1)},$$

then, by setting $r_{1,h_1} = -\Pi_{12}r_{2,h_2}$, r_{1,h_1} and U_1 satisfy (4.9) for $k = 1$, and $(4.8)_3$ holds.

Conversely, let $\{u_{1,h_1}, u_{2,h_2}\}$ solve (4.8) – (4.9) and set $\lambda_{k,h_k} = (u_{k,h_k})_{|\Gamma}$ for $k = 1, 2$, thus $u_{k,h_k} = U_k$. We prove that $\{\lambda_{1,h_1}, \lambda_{2,h_2}, r_{2,h_2}\}$ solves (9.16).

By using (4.9) and (9.12) we have

$$\begin{aligned} \langle r_{k,h_k}, \mu_i^{(k)}\rangle &= a_k(U_k, \widehat{\mathcal{R}}_k\mu_i^{(k)}) - (f, \widehat{\mathcal{R}}_k\mu_i^{(k)})_{L^2(\Omega_k)} \\ &= a_k(U_k, \widehat{\mathcal{R}}_k\mu_i^{(k)}) + a_k(\overline{U}_k, \widehat{\mathcal{R}}_k\mu_i^{(k)}) - (f, \widehat{\mathcal{R}}_k\mu_i^{(k)})_{L^2(\Omega_k)}. \end{aligned} \tag{9.17}$$

Thus, by adding the two equations of (9.17) for $k = 1, 2$ and exploiting $(4.8)_3$, we obtain $(9.16)_1$. Equation $(9.16)_2$ follows from $(4.8)_2$. □

The following result is a consequence of Theorem 9.3 and Remark 4.3.

Corollary 9.4. *If $\Lambda_{1,h_1} = \Lambda_{2,h_2}$, problem (9.16) is equivalent to problem (3.7).*

To study the well-posedness of problem (9.16) in the general case of $\Lambda_{1,h_1} \neq \Lambda_{2,h_2}$, we set $\boldsymbol{\mu}_h = (\mu_{1,h_1}, \mu_{2,h_2})$ for any $\mu_{1,h_1} \in \Lambda_{1,h_1}$ and $\mu_{2,h_2} \in \Lambda_{2,h_2}$ and define:

$$\begin{aligned} &\mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = \sum_{k=1,2} a_k(\widehat{\mathcal{H}}_k\lambda_{k,h_k}, \widehat{\mathcal{R}}_k\mu_{k,h_k}) && \forall \boldsymbol{\lambda}_h,\ \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h, \\ &\mathcal{B}_{1,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle \Pi_{12}t_{2,h_2}, \mu_{1,h_1}\rangle - \langle t_{2,h_2}, \mu_{2,h_2}\rangle && \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h,\ \forall t_{2,h_2} \in \Lambda'_{2,h_2} \\ &\mathcal{B}_{2,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle t_{2,h_2}, \mu_{2,h_2} - \Pi_{21}\mu_{1,h_1}\rangle && \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h,\ \forall t_{2,h_2} \in \Lambda'_{2,h_2} \\ &\mathcal{F}_h(\boldsymbol{\mu}_h) = \sum_{k=1,2} \Big[(f, \widehat{\mathcal{R}}_k\mu_{k,h_k})_{L^2(\Omega_k)} - a_k(U_k, \widehat{\mathcal{R}}_k\mu_{k,h_k})\Big] && \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h. \end{aligned}$$

$\mathcal{A}_h$, $\mathcal{B}_{1,h}$ and $\mathcal{B}_{2,h}$ are bilinear forms, $\mathcal{F}_h$ is a linear functional.

Problem (9.16) takes the following non-symmetric saddle point form (for its analysis in abstract form see [4]): find $\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h$, and $r_{2,h_2} \in \Lambda'_{2,h_2}$ s.t.

$$\left\{ \begin{array}{ll} \mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) + \mathcal{B}_{1,h}(\boldsymbol{\mu}_h, r_{2,h_2}) = \mathcal{F}_h(\boldsymbol{\mu}_h) & \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h \\ \mathcal{B}_{2,h}(\boldsymbol{\lambda}_h, t_{2,h_2}) = 0 & \forall t_{2,h_2} \in \Lambda'_{2,h_2}. \end{array} \right. \tag{9.18}$$

We define the operators B_{1h}, $B_{2h} : \boldsymbol{\Lambda}_h \to \Lambda_{2,h_2}$, and B_{1h}^T, $B_{2h}^T : \Lambda'_{2,h_2} \to \boldsymbol{\Lambda}'_h$ s.t.

$$\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle t_{2,h_2}, B_{kh}\boldsymbol{\mu}_h \rangle = \langle B_{kh}^T t_{2,h_2}, \boldsymbol{\mu}_h \rangle, \quad k = 1, 2,$$

with

$$\begin{array}{ll} B_{1h}\boldsymbol{\mu}_h = \Pi_{12}^* \mu_{1,h_1} - \mu_{2,h_2}, & B_{2h}\boldsymbol{\mu}_h = \mu_{2,h_2} - \Pi_{21}\mu_{1,h_1}, \\ B_{1h}^T t_{2,h_2} = [\Pi_{12} t_{2,h_2},\ -t_{2,h_2}]^T, & B_{2h}^T t_{2,h_2} = [-\Pi_{21}^* t_{2,h_2},\ t_{2,h_2}]^T, \end{array}$$

and, for $i, j = 1, 2$, Π_{ij}^* is the adjoint operator of Π_{ij}, i.e.,

$$\langle \Pi_{ij}^* \mu_i, \mu_j \rangle = \langle \mu_i, \Pi_{ij}\mu_j \rangle. \tag{9.19}$$

In order to prove the continuity of the operators $\mathcal{B}_{k,h}$, the stability of the interpolation operators Π_{12} and Π_{21} is required. This is stated in the next Lemma.

We set $d_\Gamma = d_\Omega - 1$.

Lemma 9.5. *There exist two positive constants c_{12} and c_{21} independent of h_1 and h_2 such that for any $q \in \left]\frac{d_\Gamma}{2}, \frac{3}{2}\right[$ it holds*

$$\|\Pi_{k\ell}\lambda_\ell\|_{H^{1/2}(\Gamma)} \leq c_{k\ell} \left(1 + \left(\frac{h_k}{h_\ell}\right)^q\right)^{1/2} \|\lambda_\ell\|_{H^{1/2}(\Gamma)} \qquad \forall \lambda_\ell \in Y_{\ell,h_\ell}, \tag{9.20}$$

with $k = 1, \ell = 2$, or $k = 2, \ell = 1$.

Proof. We take $k = 2$ and $\ell = 1$ and we first prove that, for any real q such that $\frac{d_\Gamma}{2} < q \leq \sigma < 3/2$,

$$\|\Pi_{21}\lambda_1\|_{L^2(\Gamma)} \leq c\,(1 + (h_2/h_1)^q)\,\|\lambda_1\|_{L^2(\Gamma)} \qquad \forall \lambda_1 \in Y_{1,h_1}. \tag{9.21}$$

Since any $\lambda_1 \in Y_{1,h_1}$ belongs to $H^\sigma(\Gamma)$ for any $\sigma < 3/2$, in view of (10.2) with $s = q$ and by applying (10.1), we have

$$\begin{array}{rl} \|\Pi_{21}\lambda_1\|_{L^2(\Gamma)} & \leq \left[\|\lambda_1 - \Pi_{21}\lambda_1\|^2_{L^2(\Gamma)} + \|\lambda_1\|_{L^2(\Gamma)}\right] \\ & \leq c h_2^q \|\lambda_1\|_{H^q(\Gamma)} + \|\lambda_1\|_{L^2(\Gamma)} \leq c\,(1 + (h_2/h_1)^q)\,\|\lambda_1\|_{L^2(\Gamma)}. \end{array}$$

The stability of Π_{21} in the H^1-norm follows from (10.2) with $s = r = 1$ when $d_\Gamma = 1$, and from (10.5) when $d_\Gamma = 2$. Thus we have $\|\Pi_{21}\lambda_1\|^2_{H^1(\Gamma)} \leq c\|\lambda_1\|^2_{H^1(\Gamma)}$. Now (9.20) follows by interpolation of Sobolev spaces. ◻

LEMMA 9.6.
1. *The bilinear form $\mathcal{A}_h$ is coercive and continuous on $\mathbf{\Lambda}_h$ i.e. there exist $\alpha_* > 0$ and $\widetilde{C}_A > 0$ independent of h_1 and h_2 such that*

$$\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\mu}_h) \geq \alpha_* \|\boldsymbol{\mu}_h\|^2_{\mathbf{\Lambda}} \quad \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h, \tag{9.22}$$

$$\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h) \leq \widetilde{C}_A \|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}} \|\boldsymbol{\psi}_h\|_{\mathbf{\Lambda}} \quad \forall \boldsymbol{\mu}_h, \boldsymbol{\psi}_h \in \mathbf{\Lambda}_h; \tag{9.23}$$

2. *the bilinear forms $\mathcal{B}_{1,h}$ and $\mathcal{B}_{2,h} : \mathbf{\Lambda}_h \to \Lambda_{2,h_2}$ are continuous, i.e., there exist $C_{B1} > 0$ and $C_{B2} > 0$ (depending on the ratio h_1/h_2) such that for $k = 1, 2$*

$$|\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2})| \leq C_{Bk} \|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}} \|t_{2,h_2}\|_{\Lambda'} \qquad \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h, \ \forall t_{2,h_2} \in \Lambda'_{2,h_2}; \tag{9.24}$$

moreover, they satisfy the inf-sup conditions for arbitrary subspaces Λ_{1,h_1} and Λ_{2,h_2}, i.e.

$$\inf_{t_{2,h_2} \in \Lambda'_{2,h_2}} \sup_{\boldsymbol{\mu}_h \in \mathbf{\Lambda}_h} \frac{\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2})}{\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}} \|t_{2,h_2}\|_{\Lambda'}} \geq 1 \quad \textit{for } k = 1, 2; \tag{9.25}$$

3. *the linear functional $\mathcal{F}_h$ is continuous on $\mathbf{\Lambda}_h$.*

Proof. 1. To prove the continuity of $\mathcal{A}_h$ we use the following *finite element uniform extension theorem*: there exists a (discrete harmonic) lifting operator $\widehat{\mathcal{R}}_k : \Lambda_{k,h_k} \to V_{k,h_k}$ s.t.

$$\|\widehat{\mathcal{R}}_k \mu_{k,h_k}\|_{H^1(\Omega_k)} \leq c \|\mu_{k,h_k}\|_\Lambda \quad \forall \mu_{k,h_k} \in \Lambda_{k,h_k}, \tag{9.26}$$

with c independent of h_k (see, e.g. [24, Thm. 4.1.3]). The coercivity of $\mathcal{A}_h$ follows from the coercivity of the form (2.4) and the trace inequality (see [24, Sect. 2.2]).

2. Thanks to Lemma 9.5, for any $q \in]\frac{d_\Gamma}{2}, \frac{3}{2}[$, it holds

$$\begin{aligned} |\mathcal{B}_{2,h}(\boldsymbol{\mu}_h, t_{2,h_2})| &= |\langle t_{2,h_2}, B_{2h}\boldsymbol{\mu}_h\rangle| \leq \|t_{2,h_2}\|_{\Lambda'} (\|\mu_{2,h_2}\|_\Lambda + \|\Pi_{21}\mu_{1,h_1}\|_\Lambda) \\ &\leq c\|t_{2,h_2}\|_{\Lambda'} \left(\|\mu_{2,h_2}\|_\Lambda + (1 + (h_2/h_1)^q)^{1/2} \|\mu_{1,h_1}\|_\Lambda \right). \end{aligned}$$

Estimate (9.24) for $k = 2$ follows by setting $C_{B2} = 2c\,(1 + (h_2/h_1)^q)^{1/2}$. Similarly,

$$\begin{aligned} |\mathcal{B}_{1,h}(\boldsymbol{\mu}_h, t_{2,h_2})| &= |\langle t_{2,h_2}, B_{1h}\boldsymbol{\mu}_h\rangle| \leq \|t_{2,h_2}\|_{\Lambda'} (\|\mu_{2,h_2}\|_\Lambda + \|\Pi^*_{12}\mu_{1,h_1}\|_\Lambda), \\ &\leq c\|t_{2,h_2}\|_{\Lambda'} \left(\|\mu_{2,h_2}\|_\Lambda + (1 + (h_1/h_2)^q)^{1/2} \|\mu_{2,h_2}\|_\Lambda \right), \end{aligned}$$

where we have exploited the property $\|\Pi^*_{12}\| = \|\Pi_{12}\|$ and Lemma 9.5. We conclude that $B_{1,h}$ satisfies (9.24) with $C_{B1} = 2c\,(1 + (h_1/h_2)^q)^{1/2}$.

For any $t_{2,h_2} \in \Lambda'_{2,h_2}$, B^T_{1h} and B^T_{2h} satisfy, respectively,

$$\|B^T_{1h} t_{2,h_2}\|_{\mathbf{\Lambda}'} = \left(\|\Pi_{12} t_{2,h_2}\|^2_{\Lambda'} + \|t_{2,h_2}\|^2_{\Lambda'}\right)^{1/2} \geq \|t_{2,h_2}\|_{\Lambda'},$$

$$\|B^T_{2h} t_{2,h_2}\|_{\mathbf{\Lambda}'} = \left(\|\Pi^*_{21} t_{2,h_2}\|^2_{\Lambda'} + \|t_{2,h_2}\|^2_{\Lambda'}\right)^{1/2} \geq \|t_{2,h_2}\|_{\Lambda'},$$

thus (9.25) is fulfilled for both $k = 1, 2$.

3. $\mathcal{F}_h$ is continuous on $\mathbf{\Lambda}_h$ thanks to both the continuity of the bilinear form (2.4) and the finite element uniform extension theorem (see (9.26)). □

REMARK 9.1. *The constants C_{Bk} do not affect the approximation errors, as we will see in Theorem 9.8.*

THEOREM 9.7. *There exists a unique solution* $(\boldsymbol{\lambda}_h, r_{2,h_2}) \in \boldsymbol{\Lambda}_h \times \Lambda'_{2,h_2}$ *of (9.18) and it satisfies*

$$\|\boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}} \leq \frac{1}{\alpha_*}\|\mathcal{F}_h\|_{\boldsymbol{\Lambda}'}, \quad \|r_{2,h_2}\|_{\Lambda'} \leq \left(1+\frac{C_A}{\alpha_*}\right)\|\mathcal{F}_h\|_{\boldsymbol{\Lambda}'}. \tag{9.27}$$

Moreover, by setting $K_1 = ker(B_{1h})$ *and* $K_2 = ker(B_{2h})$ *there exists* $\widetilde{\alpha} > 0$ *such that*

$$\inf_{\boldsymbol{\mu}_h \in K_2} \sup_{\boldsymbol{\psi}_h \in K_1} \frac{\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h)}{\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}} \geq \widetilde{\alpha}. \tag{9.28}$$

Proof. Thanks to Lemma 9.6, existence and uniqueness of the solution of problem (9.18), as well as inequality (9.27) follow by invoking Corollary 2.2 of [4].

The inequality (9.28) can now be obtained with the following arguments. First we prove that $dim(K_1) = dim(K_2)$. As a matter of fact, let I_{n_2} be the identity matrix of size n_2 and $\mathsf{B}_1,\ \mathsf{B}_2 \in \mathbb{R}^{n_2\times(n_1+n_2)}$ the matrices associated with the operators B_{1h} and B_{2h}, i.e., $\mathsf{B}_1 = \left[R_{12}^T,\ -I_{n_2}\right]$, $\mathsf{B}_2 = \left[R_{21},\ -I_{n_2}\right]$. Then $rank(\mathsf{B}_1) = rank(\mathsf{B}_2) = n_2$; since $dim(ker(A)) + rank(A) = m$ for any $A \in \mathbb{R}^{n\times m}$, we obtain that $dim(ker(\mathsf{B}_1)) = dim(ker(\mathsf{B}_2)) = n_1$.

Now, thanks to [4, Cor. 2.2.] (see also [22, Sect. 4]) the properties (9.22)-(9.25) are sufficient conditions for the existence of a unique solution of problem (9.18); on the other hand, the inf-sup condition (9.28) jointly with (9.23)–(9.25), and the property that $dim(K_1) = dim(K_2)$, are necessary and sufficient conditions for proving the same result. This implicitly guarantees that (9.28) must be satisfied. □

THEOREM 9.8. *Let* $(\boldsymbol{\lambda}, r_2) \in \boldsymbol{\Lambda}\times\Lambda'_2$ *and* $(\boldsymbol{\lambda}_h, r_{2,h_2}) \in \boldsymbol{\Lambda}_h \times \Lambda'_{2,h_2}$ *be the solutions of (9.8) and (9.18), respectively. Then there exists* $c = c(C_A, \widetilde{C}_A, C_B) >$ *s.t.*

$$\begin{aligned}
\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}} + \|r_2 - r_{2,h_2}\|_{\Lambda'} \leq c\left(1+\tfrac{1}{\widetilde{\alpha}}\right)\Big\{ &\inf_{\boldsymbol{\eta}_h\in K_2}\|\boldsymbol{\lambda}-\boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}} \\
&+ \inf_{\boldsymbol{\mu}_h\in\boldsymbol{\Lambda}_h}\left[\|\boldsymbol{\lambda}-\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} + \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{A}-\mathcal{A}_h)(\boldsymbol{\mu}_h,\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right] \\
&+ \inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\left[\|r_2 - t_{2,h_2}\|_{\Lambda'} + \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h, t_{2,h_2})|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right] \\
&+ \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{F}-\mathcal{F}_h)(\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\Big\}.
\end{aligned} \tag{9.29}$$

Proof. It is a direct application of Theorem 2.2 and Corollary 2.3 of [4], thanks to both Lemmas 9.1 and 9.6. □

While the term $\inf_{\boldsymbol{\eta}_h\in K_2}\|\boldsymbol{\lambda}-\boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}}$ depends on the interpolation error of the intergrid operator Π_{21}, the term involving $(\mathcal{B}-\mathcal{B}_{1,h})$ depends on that of Π_{12}. All the other terms only depend on the local finite element approximation in each subdomain.

For any $v \in L^2(\Omega):\ v_{|\Omega_k} \in H^1(\Omega_k),\ k = 1,2$, we define the H^1- broken norm

$$\|v\|_* = \sqrt{\sum_{k=1,2}\|v\|^2_{H^1(\Omega_k)}}. \tag{9.30}$$

If $(u_{1,h_1}, u_{2,h_2}) \in V_{1,h_1}\times V_{2,h_2}$ is the solution of the INTERNODES problem (4.8)-(4.9), we define

$$u_h = \begin{cases} u_{1,h_1} & \text{in } \Omega_1 \\ u_{2,h_2} & \text{in } \Omega_2. \end{cases} \tag{9.31}$$

In order to bound the error between the solution u of problem (2.3) and the INTERNODES solution u_h we need to estimate both the interpolation error due to a double approximation on the interface Γ (from Λ to Λ_{1,h_1} first and then from Λ_{1,h_1} to Λ_{2,h_2}; similarly, by exchanging Λ_{1,h_1} and Λ_{2,h_2}) and an inverse inequality.

THEOREM 9.9. *There exist $c > 0$ and $q \in [1/2, 1[$ independent of h_1 and h_2 s.t.*

$$\|\lambda - \Pi_{21}\mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)} \le c\left[h_2^{\varrho_2-1/2} + h_1^{\varrho_1-1/2}\left((h_2/h_1)^q + 1\right)\right]\|\lambda\|_{H^\sigma(\Gamma)} \quad \forall\lambda \in H^\sigma(\Gamma), \tag{9.32}$$

for any $\sigma > d_\Gamma/2$, where $\varrho_k = \min(\sigma, p_k + 1)$, for $k = 1, 2$.

Proof. Let $\lambda \in H^\sigma(\Gamma)$. We recall that $\Pi_{21}\eta = \mathcal{I}_2\eta$ for any $\eta \in Y_{1,h_1}$ and that $\mathcal{I}_1\lambda \in H^s(\Gamma)$ for any $s < 3/2$. We denote by Id the identity operator, then

$$\|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \le \sum_{k=1,2}\|\lambda - \mathcal{I}_k\lambda\|_{H^{1/2}(\Gamma)} + \|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}.$$

If $d_\Gamma = 1$, in view of (10.2) it holds

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \le ch_2^{1/2}\|\lambda - \mathcal{I}_1\lambda\|_{H^1(\Gamma)}$$

and by applying again (10.2) we have

$$\begin{aligned}\|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} &\le c(h_1^{\varrho_1-1/2} + h_2^{\varrho_2-1/2} + h_2^{1/2}h_1^{\sigma-1})\|\lambda\|_{H^\sigma(\Gamma)}\\ &\le c\left(h_1^{\varrho_1-1/2}\left(1 + (h_2/h_1)^{1/2}\right) + h_2^{\varrho_2-1/2}\right)\|\lambda\|_{H^\sigma(\Gamma)}.\end{aligned}$$

To bound $\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}$ when $d_\Gamma = 2$ we invoke the classical approximation results for general Sobolev spaces (see [12, Thms. 3.1.4, 3.1.5, and 3.1.6]).

Let us assume for now that $\lambda \in W^{t,2+\varepsilon}(\Gamma)$ for some $t \ge p_1 + 1$ and $\varepsilon > 0$. Let $\mathcal{E}_{h_k}$ be the triangulations on Γ induced by the meshes $\mathcal{T}_{k,h_k}$, for $k = 1, 2$. By applying Ciarlet's Theorem 3.1.6 of [12] on each $T \in \mathcal{E}_{h_2}$, thanks to the regularity assumptions on the meshes $\mathcal{T}_{k,h_k}$, for $m = 0, 1$ and any $\varepsilon > 0$ we have

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \le ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}|\lambda - \mathcal{I}_1\lambda|_{W^{1,2+\varepsilon}(\Gamma)}. \tag{9.33}$$

(Notice that all the spaces inclusions required by Theorem 3.1.6 of [12] are satified.)

Now we apply Theorem 3.1.5 of [12] on each $T \in \mathcal{E}_{h_1}$, thus for any $p_1 \ge 1$,

$$|\lambda - \mathcal{I}_1\lambda|_{W^{1,2+\varepsilon}(\Gamma)} \le ch_1^{p_1}\|\lambda\|_{W^{p_1+1,2+\varepsilon}(\Gamma)}, \tag{9.34}$$

and then

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \le ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}h_1^{p_1}\|\lambda\|_{W^{p_1+1,2+\varepsilon}(\Gamma)}. \tag{9.35}$$

The generalization of Ciarlet's theorem provided in [18] for the case of lower regularity, i.e. when $t \in [p_1, p_1 + 1[$, yields, for $\tau = \min(t, p_1 + 1) > 1$,

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \le ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}h_1^{\tau-1}\|\lambda\|_{W^{\tau,2+\varepsilon}(\Gamma)}. \tag{9.36}$$

Thanks to the Sobolev imbedding theorems (see, e.g., [19, (1,4,4,5)]), it holds

$$\|\lambda\|_{W^{\tau,2+\varepsilon}(\Gamma)} \le \|\lambda\|_{W^{\varrho_1,2}(\Gamma)} \qquad \forall\lambda \in W^{\tau,2+\varepsilon}(\Gamma) \cap W^{\varrho_1,2}(\Gamma), \tag{9.37}$$

for any $\varrho_1 \geq \tau$ s.t. $\varrho_1 - 1 = \tau - 2/(2+\varepsilon)$. It is sufficient to choose either $\varepsilon < 2(\varrho_1 - 1)/(2 - \varrho_1)$ if $\varrho_1 \in]1, 2[$, or any $\varepsilon > 0$ if $\varrho_1 \geq 2$ and (9.37) follows.

Thus, by putting $\tau = \varrho_1 - \varepsilon/(2+\varepsilon)$ in (9.36), we conclude that

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^m(\Gamma)} \leq c(h_2/h_1)^{\varepsilon/(2+\varepsilon)} h_2^{1-m} h_1^{\varrho_1 - 1} \|\lambda\|_{H^{\varrho_1}(\Gamma)}. \tag{9.38}$$

Finally, by interpolation of Sobolev spaces (see, e.g., [9, Ch. 14])

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \leq c(h_2/h_1)^{1/2+\varepsilon/(2+\varepsilon)} h_1^{\varrho_1 - 1/2} \|\lambda\|_{H^{\varrho_1}(\Gamma)} \tag{9.39}$$

and the thesis follows with $q = 1/2 + \varepsilon/(2+\varepsilon)$. □

THEOREM 9.10. *Let $\pi_{h_2} : L^2(\Gamma) \to Y_{2,h_2}$ denote the L^2- orthogonal projection operator. Then there exist $c > 0$ and $q \in [1, 3/2[$ independent of both h_1 and h_2 s.t. $\forall r \in H^\nu(\Gamma)$ with $\nu > 1$ and $\zeta_k = \min(\nu, p_k + 1)$ for $k = 1, 2$,*

$$\|\pi_{h_2} r - \Pi_{12}\pi_{h_2} r\|_{L^2(\Gamma)} \leq c\left[h_1^{\zeta_1} + h_2^{\zeta_2}\,(h_1/h_2)^q\right] \|r\|_{H^\nu(\Gamma)}. \tag{9.40}$$

Proof. Since $\nu > 1$ we can interpolate r on Γ and obtain (as $\Pi_{12}\eta_{2,h_2} = \mathcal{I}_1\eta_{2,h_2}$ for any $\eta_{2,h_2} \in Y_{2,h_2}$)

$$\|\pi_{h_2} r - \Pi_{12}(\pi_{h_2} r)\|_{L^2(\Gamma)} \leq \|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} + \|r - \mathcal{I}_1 r\|_{L^2(\Gamma)}.$$

By using (10.2), $\|r - \mathcal{I}_1 r\|_{L^2(\Gamma)} \leq c h_1^{\zeta_1} \|r\|_{H^\nu(\Gamma)}$. For the first term we proceed as follows. If $d_\Gamma = 1$, then

$$\begin{aligned}
\|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} &\leq c h_1 \|r - \pi_{h_2} r\|_{H^1(\Gamma)} && \text{(by (10.2))}\\
&\leq c h_1 (\|r - \mathcal{I}_2 r\|_{H^1(\Gamma)} + \|\mathcal{I}_2 r - \pi_{h_2} r\|_{H^1(\Gamma)})\\
&\leq c h_1 (h_2^{\zeta_2 - 1} \|r\|_{H^\nu(\Gamma)} + h_2^{-1} \|\mathcal{I}_2 r - \pi_{h_2} r\|_{L^2(\Gamma)}) && \text{(by (10.2) and (10.1))}\\
&\leq c(h_1/h_2) h_2^{\zeta_2} \|r\|_{H^\nu(\Gamma)} && \text{(by triangular inequality, (10.2) and (10.3))}
\end{aligned}$$

and the thesis follows with $q = 1$. If $d_\Gamma = 2$ we use the same arguments as before, but applying the interpolation estimates on general Sobolev spaces as in the proof of Theorem 9.9. Thus for a suitable $\varepsilon > 0$ we have

$$\begin{aligned}
\|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} &\leq c h_1^{1+\varepsilon/(2+\varepsilon)} |r - \pi_{h_2} r|_{W^{1,2+\varepsilon}(\Gamma)}\\
&\leq c h_1^{1+\varepsilon/(2+\varepsilon)} h_2^{\zeta_2 - 1 - \varepsilon/(2+\varepsilon)} \|r\|_{H^\nu(\Gamma)} \leq c(h_1/h_2)^{1+\varepsilon/(2+\varepsilon)} h_2^{\zeta_2} \|r\|_{H^\nu(\Gamma)}.
\end{aligned}$$

The thesis follows with $q = 1 + \varepsilon/(2+\varepsilon)$. □

LEMMA 9.11. *For any $i = 1, \ldots, n_1$ let ω_i be the support of the Lagrange basis function $\mu_i^{(1)}$ in Y_{1,h_1}. There exists $c > 0$ independent of h_1 such that*

$$\max_{1 \leq i \leq n_1} \|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c h_1^{(1-d_\Gamma)/2} \|\psi_{1,h_1}\|_{H^{1/2}(\Gamma)} \quad \forall \psi_{1,h_1} \in \Lambda_{1,h_1}. \tag{9.41}$$

Proof. If $d_\Gamma = 1$, in [17] it is proved that

$$\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c\,(1 + \log(diam(\omega_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^{1/2}(\omega_i)}$$

for any piecewise linear function ψ_{1,h_1} and where $diam(\omega_i)$ denotes the diameter of ω_i. By invoking the extension theorem for polynomials proved in [2], the same result can be proved for any $\psi_{1,h_1} \in Y_{1,h_1}$ with local polynomial degree $p_1 > 1$. Since $diam(\omega_i) = 2h_1$, the thesis for $d_\Gamma = 1$ follows.

If $d_\Gamma = 2$, it holds (see [25, Lemma 4.15], whose proof holds for any $p_1 \geq 1$)

$$\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c\,(1+\log(diam(\omega_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^1(\omega_i)}, \quad \forall \psi_{1,h_1} \in Y_{1,h_1}$$

then by applying (10.1), we have

$$\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c h_1^{-1/2}\,(1+\log(diam(\omega_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^{1/2}(\omega_i)}, \quad \forall \psi_{1,h_1} \in Y_{1,h_1}.$$

Since only a finite number of simplices is included in ω_i, the thesis follows also for $d_\Gamma = 2$. □

We can prove now the main result of this section, i.e. the optimal error bound for the INTERNODES method.

Theorem 9.12. *Assume that the solution u of problem (2.3) belongs to $H^s(\Omega)$, for some $s > 3/2$, that $\lambda = u_{|\Gamma} \in H^\sigma(\Gamma)$ for some $\sigma > 1$ and that $r_2 = \partial_{L_2} u_2 \in H^\nu(\Gamma)$ for some $\nu > 0$. Then there exist $q \in [1/2, 1[$, $z \in [3/2, 2[$, and a constant $c > 0$ independent of both h_1 and h_2 s.t.*

$$\begin{aligned}\|u-u_h\|_* \leq\ & c\,\Big\{ \Big(h_1^{\varrho_1-1/2}\,(1+(h_2/h_1)^q) + h_2^{\varrho_2-1/2}\Big)\,\|\lambda\|_{H^\sigma(\Gamma)} \\ & + \sum_{k=1,2} h_k^{\ell_k-1}(\|u_k\|_{H^s(\Omega_k)} + \|u_k^\lambda\|_{H^s(\Omega_k)} + \|\overline{u}_k\|_{H^s(\Omega_k)}) \\ & + \Big[\alpha h_1^{\zeta_1+1/2} + (1+(h_1/h_2)^z)\,h_2^{\zeta_2+1/2}\Big]\,\|r_2\|_{H^\nu(\Gamma)}\Big\},\end{aligned} \tag{9.42}$$

where $\ell_k = \min(s, p_k+1)$ for $k = 1,2$, $\varrho_k = \min(\sigma, p_k+1)$, $\zeta_k = \min(\nu, p_k+1)$, $\alpha = 1$ if $\nu > 1$ and $\alpha = 0$ otherwise.

Proof. For $k = 1,2$ we set $u_k = u_{|\Omega_k}$. Let u_h be the INTERNODES solution defined in (9.31), $\lambda = u_{|\Gamma}$, $\lambda_k = (u_k)_{|\Gamma}$ (notice that $\lambda_1 = \lambda_2 = \lambda$) and $\lambda_{k,h_k} = (u_{k,h_k})_{|\Gamma}$. Then, in view of (9.1) and (9.12) we have $u_k = u_k^{\lambda,f} = u_k^\lambda + \overline{u}_k$ and $u_{k,h_k} = U_k = \widehat{\mathcal{H}}_k \lambda_{k,h_k} + \overline{U}_k$, for $k = 1,2$. Moreover, by standard Galerkin error analysis and (9.15) we have:

$$\begin{aligned}\|u-u_h\|_*^2 &= \sum_{k=1,2} \|u_k - u_{k,h_k}\|^2_{H^1(\Omega_k)} = \sum_{k=1,2} \|u_k^{\lambda,f} - U_k\|^2_{H^1(\Omega_k)} \\ &\leq c \sum_{k=1,2} \Big(\|u_k^{\lambda,f} - U_k\|^2_{H^1(\Omega_k)} + \|\widehat{\mathcal{H}}_k(\lambda - \lambda_{k,h_k})\|^2_{H^1(\Omega_k)}\Big) \\ &\leq c \sum_{k=1,2} \Big(h_k^{\ell_k-1}\|u_k\|_{H^s(\Omega_k)}\Big)^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2_{\boldsymbol{\Lambda}}\end{aligned} \tag{9.43}$$

with $\ell_k = \min(s, p_k+1)$. In order to bound $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}}$ we apply Theorem 9.8 and analyze each term on the right hand side of (9.29).

We have $K_2 = ker(B_{2h}) = \{\boldsymbol{\eta} = (\eta_{1,h_1}, \eta_{2,h_2}) \in \boldsymbol{\Lambda}_h :\ \eta_{2,h_2} = \Pi_{21}\eta_{1,h_1}\}$. If we choose $\eta_{1,h_1} = \mathcal{I}_1\lambda$, using the interpolation error (10.2) and Theorem 9.9 we have

$$\begin{aligned}\inf_{\boldsymbol{\eta}_h \in K_2} \|\boldsymbol{\lambda} - \boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}} &\leq c\left(\|\lambda - \mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)} + \|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}\right) \\ &\leq c\Big(h_1^{\varrho_1-1/2}\,(1+(h_2/h_1)^q) + h_2^{\varrho_2-1/2}\Big)\,\|\lambda\|_{H^\sigma(\Gamma)},\end{aligned}$$

with $\varrho_k = \min(\sigma, p_k+1)$ for $k = 1,2$.

Taking now $\boldsymbol{\mu}_h = (\mathcal{I}_1\lambda, \mathcal{I}_2\lambda)$, still using (10.2) we have

$$\begin{aligned}\inf_{\boldsymbol{\eta}_h\in\boldsymbol{\Lambda}_h}\|\boldsymbol{\lambda}-\boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}} &\leq c\left(\|\lambda-\mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)}+\|\lambda-\mathcal{I}_2\lambda\|_{H^{1/2}(\Gamma)}\right)\\ &\leq c(h_1^{\varrho_1-1/2}+h_2^{\varrho_2-1/2})\|\lambda\|_{H^\sigma(\Gamma)}.\end{aligned}\tag{9.44}$$

With the same choice of $\boldsymbol{\mu}_h = (\mathcal{I}_1\lambda, \mathcal{I}_2\lambda)$ we can bound the error term involving $(\mathcal{A}-\mathcal{A}_h)$ in (9.29) as follows

$$\begin{aligned}|(\mathcal{A}-\mathcal{A}_h)(\boldsymbol{\mu}_h,\boldsymbol{\psi}_h)| &= |\sum_{k=1,2} a_k(u_k^{\mu_{k,h_k}}-\widehat{\mathcal{H}}_k\mu_{k,h_k},\widehat{\mathcal{R}}_k\psi_{k,h_k})|\\ &\leq C_A\sum_{k=1,2}\|u_k^{\mu_{k,h_k}}-\widehat{\mathcal{H}}_k\mu_{k,h_k}\|_{H^1(\Omega_k)}\cdot\|\widehat{\mathcal{R}}_k\psi_{k,h_k}\|_{H^1(\Omega_k)}.\end{aligned}$$

Moreover, since $\mu_{k,h_k} = \mathcal{I}_k\lambda$, by triangular inequality we obtain

$$\begin{aligned}&\|u_k^{\mu_{k,h_k}}-\widehat{\mathcal{H}}_k\mu_{k,h_k}\|_{H^1(\Omega_k)}\\ &\quad\leq \|u_k^\lambda-u_k^{\mathcal{I}_k\lambda}\|_{H^1(\Omega_k)}+\|u_k^\lambda-\widehat{\mathcal{H}}_k\lambda\|_{H^1(\Omega_k)}+\|\widehat{\mathcal{H}}_k\lambda-\widehat{\mathcal{H}}_k(\mathcal{I}_k\lambda)\|_{H^1(\Omega_k)}\\ &\qquad\qquad\text{(by Céa's Lemma on the second term)}\\ &\quad\leq c(\|u_k^{\lambda-\mathcal{I}_k\lambda}\|_{H^1(\Omega_k)}+\|u_k^\lambda-\mathcal{I}_k^\Omega u_k^\lambda\|_{H^1(\Omega_k)}+\|\widehat{\mathcal{H}}_k(\lambda-\mathcal{I}_k\lambda)\|_{H^1(\Omega_k)})\\ &\qquad\qquad\text{(by (9.4), (10.2), and (9.15))}\\ &\quad\leq c\left(h_k^{\varrho_k-1/2}\|\lambda\|_{H^\sigma(\Gamma)}+h_k^{\ell_k-1}\|u_k^\lambda\|_{H^s(\Omega_k)}\right)\end{aligned}$$

where $\mathcal{I}_k^\Omega$ is the Lagrange interpolation operator on $\overline{\Omega}_k$. Thus, thanks to (9.26), we have

$$\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{A}-\mathcal{A}_h)(\boldsymbol{\mu}_h,\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\leq c\sum_{k=1,2}\left[h_k^{\varrho_k-1/2}\|\lambda\|_{H^\sigma(\Gamma)}+h_k^{\ell_k-1}\|u_k^\lambda\|_{H^s(\Omega_k)}\right].\tag{9.45}$$

By using similar arguments,

$$\begin{aligned}|(\mathcal{F}-\mathcal{F}_h)(\boldsymbol{\psi}_h)| &= |\sum_{k=1,2} a_k(\overline{u}_k-\overline{U}_k,\widehat{\mathcal{R}}_k\psi_{k,h_k})|\\ &\leq C_A\sum_{k=1,2}\|\overline{u}_k-\overline{U}_k\|_{H^1(\Omega_k)}\cdot\|\widehat{\mathcal{R}}_k\psi_{k,h_k}\|_{H^1(\Omega_k)}\\ &\leq c\sum_{k=1,2}h_k^{\ell_k-1}\|\overline{u}_k\|_{H^s(\Omega_k)}\cdot\|\psi_{k,h_k}\|_\Lambda,\end{aligned}$$

whence

$$\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{F}-\mathcal{F}_h)(\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\leq c\sum_{k=1,2}h_k^{\ell_k-1}\|\overline{u}_k\|_{H^s(\Omega_k)}.\tag{9.46}$$

We analyze now the error term

$$\mathsf{D}=\inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\left[\|r_2-t_{2,h_2}\|_{\Lambda'}+\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_{2,h_2})|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right].\tag{9.47}$$

We recall that Λ'_{2,h_2} is the dual space of Λ_{2,h_2}, that can be identified to Λ_{2,h_2} endowed with the norm $\|\cdot\|_{\Lambda'}$. Let ω_i denote the support of the Lagrange basis function $\mu_i^{(1)}$

of Y_{1,h_1}. By setting $t_2 = t_{2,h_2}$, we have

$$\begin{aligned}
|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_2)| &= |\langle t_2,\psi_{1,h_1}-\psi_{2,h_2}\rangle - \langle \Pi_{12}t_2,\psi_{1,h_1}\rangle + \langle t_2,\psi_{2,h_2}\rangle| \\
&= |(t_2-\Pi_{12}t_2,\psi_{1,h_1})_{L^2(\Gamma)}| = \left|\sum_{i=1}^{n_1}\psi_{1,h_1}(\mathbf{x}_i^{(\Gamma_1)})(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\Gamma)}\right| \\
&= |\sum_{i=1}^{n_1}\psi_{1,h_1}(\mathbf{x}_i^{(\Gamma_1)})(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}| \leq M\left|\sum_{i=1}^{n_1}(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}\right|,
\end{aligned}$$

with $M = \max_j \|\psi_{1,h_1}\|_{L^\infty(\omega_j)}$.

Let π_{h_1} be the orthogonal projection operator from $L^2(\Gamma)$ onto its subspace Y_{1,h_1}. For any $i = 1,\ldots,n_k$, by setting $\widetilde{\omega}_i = \cup_{j:\omega_j\cap\omega_i\neq\emptyset}\omega_j$, we find

$$\begin{aligned}
|(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}| &= |(\pi_{h_1}t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}| \\
&= \left|\sum_{j:\ \omega_j\cap\omega_i\neq\emptyset}(\pi_{h_1}t_2-\Pi_{12}t_2)(x_j^{(\Gamma_i)})(\mu_j^{(1)},\mu_i^{(1)})_{L^2(\omega_i)}\right| \\
&\leq c\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^\infty(\widetilde{\omega}_i)}\sum_{j:\ \omega_j\cap\omega_i\neq\emptyset}(\mu_j^{(1)},\mu_i^{(1)})_{L^2(\omega_i)} \qquad \text{(by (10.1))} \\
&\leq ch_1^{d_\Gamma/2}\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^2(\widetilde{\omega}_i)}.
\end{aligned}$$

The Lagrange basis functions satisfy the estimate $\|\mu_i^{(1)}\|_{L^2(\omega_i)} \leq ch_1^{d_\Gamma/2}$ and the number of elements in each $\widetilde{\omega}_i$ is finite and independent of h_1. Then

$$\begin{aligned}
\left|\sum_{i=1}^{n_1}(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}\right| &\leq ch_1^{d_\Gamma/2}\sum_{i=1}^{n_1}\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^2(\widetilde{\omega}_i)} \\
&= ch_1^{d_\Gamma/2}\|\pi_{h_1}(t_2-\Pi_{12}t_2)\|_{L^2(\Gamma)} \leq ch_1^{d_\Gamma/2}\|t_2-\Pi_{12}t_2\|_{L^2(\Gamma)}.
\end{aligned}$$

Thus, using the Cauchy-Schwarz inequality and Lemma 9.11 we obtain

$$\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_2)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}} = \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(t_2-\Pi_{12}t_2,\psi_{1,h_1})_{L^2(\Gamma)}|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}} \leq ch_1^{1/2}\|t_2-\Pi_{12}t_2\|_{L^2(\Gamma)}.$$

We choose now $t_2 = t_{2,h_2} = \pi_{h_2}r_2$, being $\pi_{h_2}r_2$ the L^2- orthogonal projection of r_2 on Y_{2,h_2}. Using (10.3) we obtain

$$\inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\|r_2-t_{2,h_2}\|_{\Lambda'} \leq \|r_2-\pi_{h_2}r_2\|_{H^{-1/2}(\Gamma)} \leq ch_2^{\nu+1/2}\|r_2\|_{H^\nu(\Gamma)}. \tag{9.48}$$

If $0 < \nu \leq 1$,

$$\begin{aligned}
\|\pi_{h_2}r_2-\Pi_{12}(\pi_{h_2}r_2)\|_{L^2(\Gamma)} &\leq ch_1\|\pi_{h_2}r_2\|_{H^1(\Gamma)} && \text{(by (10.5))} \\
&\leq (h_1/h_2)h_2^\nu\|\pi_{h_2}r_2\|_{H^\nu(\Gamma)} && \text{(by (10.1))} \\
&\leq (h_1/h_2)h_2^\nu\|r_2\|_{H^\nu(\Gamma)} && \text{(by (10.4))}
\end{aligned}$$

thus, from (9.47), we obtain that

$$\mathsf{D} \leq c\left(1+(h_1/h_2)^{3/2}\right)h_2^{\nu+1/2}\|r_2\|_{H^\nu(\Gamma)}. \tag{9.49}$$

If $\nu > 1$, by using Theorem 9.10 and (9.48) in (9.47), we conclude that there exists $z \in]3/2,2[$ such that

$$\mathsf{D} \leq c\left(h_1^{\zeta_1+1/2}+(1+(h_1/h_2)^z)\,h_2^{\zeta_2+1/2}\right)\|r_2\|_{H^\nu(\Gamma)}. \tag{9.50}$$

By collecting all the intermediate estimates proved thus far we obtain

$$\begin{aligned}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}} \leq \; & c\Big[\Big(h_1^{\varrho_1-1/2}\left(1+(h_2/h_1)^q\right)+h_2^{\varrho_2-1/2}\Big)\|\lambda\|_{H^\sigma(\Gamma)} \\ & +\sum_{k=1,2} h_k^{\ell_k-1}(\|u_k^\lambda\|_{H^s(\Omega_k)}+\|\overline{u}_k\|_{H^s(\Omega_k)}) \\ & +\Big[\alpha h_1^{\zeta_1+1/2}+\left(1+(h_1/h_2)^z\right)h_2^{\zeta_2+1/2}\Big]\|r_2\|_{H^\nu(\Gamma)}\Big],\end{aligned} \tag{9.51}$$

with $\alpha=1$ if $\nu>1$ and $\alpha=0$ otherwise. The thesis follows in view of (9.43). □

10. Appendix. Let $D\subset\mathbb{R}^{d_D}$ with $d_D=1,2,3$ and $\mathcal{T}_h$ be a family of affine, regular and quasi-uniform triangulations in D. Let $X_h=\{v\in C^0(\overline{D}):\ v_{|T}\in\mathbb{P}_p\ \forall T\in\mathcal{T}_h\}$, $\hat{T}$ the reference element, and $\hat{P}$ the polynomial space on $\hat{T}$ ([12]).

For any $q\in[1,+\infty]$ and $m\geq 0$ real, let $W^{m,q}(D)$ denote the generic Sobolev space ([1]); in particular $H^m(D)=W^{m,2}(D)$.

Inverse inequalities for piecewise functions. Let there be given two pairs (ℓ,r) and (m,q) with $\ell,m\geq 0$ and $r,q\in[1,\infty]$ such that $\ell\leq m$ and $\hat{P}\subset W^{m,q}(\hat{T})\cap W^{\ell,r}(\hat{T})$. There exists a positive constant c independent of h such that (see [12, Thm. 3.2.6])

$$\left(\sum_{T\in\mathcal{T}_h}|v|^q_{W^{m,q}(T)}\right)^{1/q}\leq ch^{\ell-m-d_D(1/r-1/q)}\left(\sum_{T\in\mathcal{T}_h}|v|^r_{W^{\ell,r}(T)}\right)^{1/r}\qquad\forall v\in X_h. \tag{10.1}$$

Lagrange interpolation error. Let $\mathcal{I}_h: C^0(\overline{D})\to X_h$ be the Lagrange interpolation operator. For any $r,\ s\in\mathbb{R}$ with $0\leq r\leq 1$, $s>d_D/2$, $\exists c>0$ independent of h s.t.:

$$\|v-\mathcal{I}_h v\|_{H^r(D)}\leq ch^{\ell-r}\|v\|_{H^\ell(D)}\quad\forall v\in H^s(D), \tag{10.2}$$

where $\ell=\min(s,p+1)$ and p denotes the local polynomial degree. For the proof, see, e.g., [23, Thm 3.4.2] if $s\geq 2$ is an integer, and [18, Thm. 2.27] for $1<s<2$. The estimate with $d_D=1$ and $1/2<s<2$ can be proved by following the same arguments used in the cited references.

Projection error. Let $\pi_h: L^2(D)\to X_h$ be the L^2-orthogonal projection operator. For any $r,\ s\in\mathbb{R}$, $\exists c>0$ independent of h s.t.

$$\|v-\pi_h v\|_{(H^r(D))'}\leq ch^{r+\ell}\|v\|_{H^s(D)}\qquad\forall v\in H^s(D) \tag{10.3}$$

with $\ell=\min(s,p+1)$ (see [5, Lemma 2.4]). Moreover (see, e.g., [7])

$$\|\pi_h v\|_{H^s(D)}\leq c_s\|v\|_{H^s(D)}\qquad\forall v\in H^s(D),\ 0\leq s\leq 1, \tag{10.4}$$

with $c_s>0$ depending on s but independent of h.

By using the same arguments adopted to prove Lemma 1 in [11] for $p_k=1$ we can prove for any $p_k\geq 1$,

$$|\mathcal{I}_\ell\eta_k|_{H^1(D)}\leq c|\eta_k|_{H^1(D)},\qquad\|\eta_k-\mathcal{I}_\ell\eta_k\|_{L^2(D)}\leq ch_\ell|\eta_k|_{H^1(D)},\qquad\forall\eta_k\in Y_{k,h_k}, \tag{10.5}$$

with $k=2$ and $\ell=1$, or $k=1$ and $\ell=2$.

Acknowledgments. We are very grateful to Simone Deparis and Davide Forti for many useful discussions and for their advice.

REFERENCES

[1] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM J. Numer. Anal.*, 28(3):624–661, 1991.

[3] C. Bègue, C. Bernardi, N. Debit, Y. Maday, G.E. Karniadakis, C. Mavriplis, and A.T. Patera. Nonconforming spectral element-finite element approximations for partial differential equations. In *Proceedings of the Eighth International Conference on Computing Methods in Applied Sciences and Engineering (Versailles, 1987)*, volume 75, pages 109–125, 1989.

[4] C. Bernardi, C. Canuto, and Y. Maday. Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem. *SIAM J. Numer. Anal.*, 25(6):1237–1271, 1988.

[5] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.

[6] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*. Springer-Verlag, 2013.

[7] J. Bramble, J. Pasciak, and O. Steinbach. On the stability of the L^2 projection in $H^1(\Omega)$. *Math. Comp.*, 71(237):147–156, 2001.

[8] H. J. Brauchli and J. T. Oden. Conjugate approximation functions in finite-element analysis. *Quart. Appl. Math.*, 29:65–90, 1971.

[9] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[10] L. Cazabeau, C. Lacour, and Y. Maday. Numerical quadratures and mortar methods. In *Computational Science for the 21st Century*, pages 119–128. J. Wiley et al. (Eds.), 1997.

[11] T.F. Chan, B.F. Smith, and J. Zou. Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. *Numer. Math.*, 73(2):149–167, 1996.

[12] Ph.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. Classics in applied mathematics 40. Society for Industrial and Applied Mathematics, 2nd edition, 2002.

[13] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology*, volume 2. Springer, Berlin, 2000.

[14] S. Deparis, D. Forti, P. Gervasio, and A. Quarteroni. INTERNODES: an accurate interpolation-based method for coupling the Galerkin solutions of PDEs on subdomains featuring non-conforming interfaces. *Computers & Fluids*, 141:22–41, 2016.

[15] S. Deparis, D. Forti, and A. Quarteroni. A rescaled localized radial basis function interpolation on non-Cartesian and nonconforming grids. *SIAM J. Sci. Comput.*, 36(6):A2745–A2762, 2014.

[16] S. Deparis, D. Forti, and A. Quarteroni. *A fluid-structure interaction algorithm using radial basis function interpolation between non-conforming interfaces*, pages 439–450. Modeling and Simulation in Science, Engineering and Technology. Springer, 2016.

[17] M. Dryja. A method of domain decomposition for three-dimensional finite element elliptic problems. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 43–61. SIAM, Philadelphia, PA, 1988.

[18] M. Feistauer. On the finite element approximation of functions with noninteger derivatives. *Numer. Funct. Anal. Optim.*, 10(1-2):91–110, 1989.

[19] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[20] J.-L. Lions and E. Magenes. *Nonhomogeneous Boundary Value Problems and Applications*. Springer Verlag, Berlin, 1972.

[21] Y. Maday, F. Rapetti, and B.I. Wohlmuth. The influence of quadrature formulas in 2D and 3D mortar element methods. In *Recent developments in domain decomposition methods (Zürich, 2001)*, volume 23 of *Lect. Notes Comput. Sci. Eng.*, pages 203–221. Springer, 2002.

[22] R.A. Nicolaides. Existence, uniqueness and approximation for generalized saddle point problems. *SIAM J. Numer. Anal.*, 19(2):349–357, 1982.

[23] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer Verlag, Heidelberg, 1994.

[24] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, 1999.

[25] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

Recent publications:

MATHEMATICS INSTITUTE OF COMPUTATIONAL SCIENCE AND ENGINEERING
Section of Mathematics

Ecole Polytechnique Fédérale (EPFL)

CH-1015 Lausanne

26.2016 ASSYR ABDULLE, ONDREJ BUDÁČ:
Multiscale model reduction methods for flow in heterogeneous porous media

27.2016 ASSYR ABDULLE, PATRICK HENNING:
Multiscale methods for wave problems in heterogeneous media

28.2016 ASSYR ABDULLE, ORANE JECKER:
On heterogeneous coupling of multiscale methods for problems with and without scale separation

29.2016 WOLFGANG HACKBUSCH, DANIEL KRESSNER, ANDRÉ USCHMAJEW:
Perturbation of higher-order singular values

30.2016 ASSYR ABDULLE, ONDREJ BUDÁČ, ANTOINE IMBODEN:
Multiscale methods and model order reduction for flow problems in three-scale porous media

31.2016 DANIEL KRESSNER, ANA ŠUŠNJARA:
Fast computation of spectral projectors of banded matrices

32.2016 NICCOLO DAL SANTO, SIMONE DEPARIS, ANDREA MANZONI, ALFIO QUARTERONI:
Multi space reduced basis preconditioners for large-scale parametrized PDEs

33.2016 ASSYR ABDULLE, MARTIN HUBER, SIMON LEMAIRE:
An optimization-based numerical method for diffusion problems with sign-changing coefficients

34.2016 ASSYR ABDULLE, ANDREA DI BLASIO:
Numerical homogenization and model order reduction for multiscale inverse problems

35.2016 PETAR SIRKOVIĆ:
A reduced basis approach to large-scale pseudospectra computation

36.2016 ASSYR ABDULLE, GRIGORIOS A. PAVLIOTIS, URBAIN VAES:
Spectral methods for multiscale stochastic differential equations

37.2016 NICOLAS GILLIS, ROBERT LUCE:
A fast gradient method for nonnegative sparse regression with self dictionary

38.2016 FRANCESCO BALLARIN, ELENA FAGGIANO, ANDREA MANZONI, GIANLUIGI ROZZA, ALFIO QUARTERONI, SONIA IPPOLITO, CARLO ANTONA, ROBERTO SCROFANI:
A fast virtual surgery platform for many scenarios haemodynamics of patient-specific coronary artery bypass grafts

39.2016 PAOLA GERVASIO, ALFIO QUARTERONI:
Analysis of the internodes method for non-conforming discretizations of elliptic equations