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AN OPTIMIZATION BASED COUPLING METHOD FOR MULTISCALE PROBLEMS

ASSYR ABDULLE*, ORANE JECKER†, AND ALEXANDER SHAPEEV ‡

Abstract. A new multiscale coupling method is proposed for elliptic problems with highly oscillatory coefficients with a continuum of scales in a subset of the computational domain and scale separation in complementary regions of the computational domain. A discontinuous Galerkin (DG) finite element heterogeneous multiscale method (FE-HMM) is used in the region with scale separation while a continuous standard finite element method is used in the region without scale separation. The use of a DG FE-HMM method allows for a flexible meshing of the different models in the overlapping region. The unknown boundary conditions at the interfaces are obtained by minimizing the error of the two models in the overlap region. We prove the well-posedness of both the continuous and discrete coupling problems and establish convergence of the multiscale method towards the fine scale solution. As in the region with scale separation we obtain an approximation at a cost independent of the smallest scale in the problem, the computational cost of the multiscale method is significantly smaller than a fine scale solver over the whole computational domain, while the algorithm allows to treat situation for which standard numerical homogenization methods do not apply.

Key words. optimization based coupling, virtual control, homogenization, multiscale problem, HMM, discontinuous Galerkin

AMS subject classifications. 65N30, 35J15, 35B27,49J20

1 Introduction

Partial differential equations (PDEs) with multiple scales are used to model a wide range of physical systems with numerous applications, ranging from material and natural sciences to problems in engineering or biology. When the ratio of the smallest scale in the problem to the size of the computational domain is very large, the numerical approximation of such problems with classical numerical methods can become computationally prohibitive as the smallest scales in the problem have to be resolved leading to discretization with very large degrees of freedom. Numerous multiscale methods have been developed in the past decade. Without attempting to be exhaustive we recall two important approaches that we will contrast later with the new multiscale method proposed and analyzed in this paper. We will focus on linear elliptic problems, but note that some methods described below have been proposed also for other types of PDEs.

We first mention methods based on coarse oscillatory basis functions that encode the high variation of the data in the multiscale PDE. In this class of methods we have for example the multiscale finite element method (see the references in e.g., [20]) and the recently proposed local orthogonal decomposition (LOD) (see [30, 25]). In principle these methods can be applied to problems with general coefficients (e.g., without structural assumption on the coefficients) and convergence has indeed been proved for rough coefficients for the LOD in [30, 25]. While these methods are quite general, they come also with a high computational cost to precompute the coarse basis functions, as the original fine scale problem has to be solved on localized coarse elements whose union is a partition of the computational domain of interest.

The next class of multiscale methods that we mention are methods supplementing macroscopic data (computed through micro computations) for the solution of an effective equation solved by a macroscopic solver. This approach widely used by engineers

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(see e.g., the references in [21]) has been developed into a general framework in the heterogeneous multiscale method (HMM) [18, 2]. When finite element methods (FEMs) are used (at the micro and macro scales) these methods are called finite element heterogeneous multiscale methods (FE-HMM). The theoretical justification of these methods is that of the homogenization theory: given a family of PDEs indexed by a parameter ε, the theory of $H-$convergence establish the convergence of a subsequence of solutions to an effective PDE under quite general assumptions (e.g., boundedness and ellipticity of the diffusion tensor of an elliptic problems and right-hand side in the dual of the Hilbert space considered in the weak formulation). In a numerical approach such as the FE-HMM, the micro computations are usually done in sampling domains of size much smaller than the mesh width used for the macroscopic solver. Hence to extract the effective data a computational cost independent of the small scales can be achieved when indeed the small scales can be localized, i.e., when the problem features scale separation. Rigorous convergence analysis has been established for locally periodic coefficients or random stationary coefficients [1, 19, 3, 2].

In this paper we are interested in problems in which the scales are separated in a subset of the computational domain with possibly a continuum of scales in the complementary domain. Our aim is to couple numerical homogenization methods such as the FE-HMM in part of the computational domain with a fine scale solver. Such problems arise in many situations, for example heterogeneous composite materials whose effective properties can be well captured by assuming a (locally) periodic microstructure that can however not be valid near defects. In our modeling the smallest scale is supposed to be still discretized at the continuum level but for some applications atomistic scale should be considered.

Algorithms that couple numerical homogenization methods with a fine scale solver have appeared in the literature. We mention the goal-oriented method [34] in which the unknown boundary conditions for the fine scale subregions are provided by a precomputed homogenized solution. Recently in [8] the authors propose a local-global solution based on the L^2 projection of the homogenized solution onto the solutions of fine scale local problems.

In this paper we propose and analyze a new coupling strategy inspired by virtual control method pioneered in [24, 29, 22] (see also [17] for recent developments). Our method shares also some similarities with the recent work atomistic-to-continuum coupling [35]. The method that we propose relies on a decomposition of the computational domain Ω into a region without scale separation ω where the homogenized model is not valid, an overlapping region ω_0 where both the fine scale and the homogenized models are valid and a region ω_2 where the homogenized solution describes adequately the physical problem. Thus, we decompose the domain into a family of overlapping domains and introduce virtual (interface) controls as boundary conditions. The interface controls will act as unknown traces or fluxes and the problem is reformulated as a minimization problem with states equations as constraints. The optimal boundary controls of two overlapping domains are found by a heterogeneous optimization problem that is based on minimizing the discrepancy between the two models on the overlap region. It is shown that by using a Caccioppoli inequality, the minimization can be performed for an L^2 norm. As is the region with scale separation an energy approximation towards the fine scale problems can also be obtained through the use of a locally periodic corrector we also obtain an H^1 convergence rates towards the fine scale solution over the whole computational domain. In order to allow flexibility in the mesh used in the coarse and fine scale regions, we use the Discontinuous Galerkin

FE-HMM [4] for the numerical homogenization. The method analyzed in this paper has first been announced in [5]. In this paper we give a more general framework for the method presented in [5] and give the first full analysis for both the continuous and the discrete coupling algorithms.

We finally relate our method with the recently developed numerical homogenization of periodic microstructure with defect proposed in [10]. There, the highly oscillatory coefficients is assumed to be the sum of a periodic function and a localized perturbation. The goal is to compute an approximation of the fine scale solution that relies on homogenization but uses a non-periodic corrector on a domain that accounts for the defect. We will further compare these two approaches each of which is interesting in its own right in our numerical experiments.

The outline of this article is as follows. In Section 2 we describe our optimization based multiscale method and prove the well-posedness of the optimization problem. A priori error estimates of the continuous version of the optimization algorithm is proved in Section 3, while the fully discrete optimization based method is described in Section 4. In Section 5 we state and prove fully discrete error estimates of the multiscale optimization based method towards the fine scale solution. Numerical experiments that verify the theoretical convergence rates and and comparisons with other coupling strategies are provided in Section 6.

Notations. In what follows, $C > 0$ is used to denote a generic constant independent of ε. We consider the usual Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega) \mid D^r u \in L^2(\Omega), |r| \leq 1\}$, where $r \in \mathbb{N}^d, |r| = r_1 + \ldots + r_d$ and $D^r = \partial_1^{r_1} \ldots \partial_d^{r_d}$. The notation $|\cdot|$ stands for the standard Euclidean norm in $\mathbb{R}^d$. Let Y denote the unit cube $(0,1)^d$ and define $W^1_{\mathrm{per}}(Y) := \{v \in H^1_{\mathrm{per}}(Y) \mid \int_Y v \mathrm{d}y = 0\}$ where the set $H^1_{\mathrm{per}}(Y)$ is the closure of $\mathcal{C}^\infty_{\mathrm{per}}(Y)$.

2 Optimization based method Let Ω be a convex, polygonal domain in $\mathbb{R}^d, d = 1,2,3$, with a boundary $\Gamma = \Gamma_D \cup \Gamma_N$; where Dirichlet conditions are imposed on Γ_D and Neumann conditions on Γ_N. Further, assume that $\Gamma_D \cap \Gamma_N = \emptyset$ and that Γ_D has positive measure. Let $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$, and consider the following second-order elliptic problem

$$\begin{aligned} -\mathrm{div}\,(a^\varepsilon \nabla u^\varepsilon) &= f, \quad \text{in } \Omega, \\ u^\varepsilon &= g_D, \text{ on } \Gamma_D, \\ n \cdot (a^\varepsilon \nabla u^\varepsilon) &= g_N, \text{ on } \Gamma_N, \end{aligned} \tag{2.1}$$

where the $a^\varepsilon \in (L^\infty(\Omega))^{d\times d}$ are highly oscillatory, bounded coefficients with scale separation only in some subregions of Ω. Further, a^ε is uniformly elliptic, that is

$$\exists \quad 0 < \alpha, \beta < +\infty, \text{ such that } \alpha|\xi|^2 \leq \xi^\top a^\varepsilon(x)\xi \leq \beta|\xi|^2, \qquad \xi \in \mathbb{R}^d. \tag{2.2}$$

Thanks to Lax-Milgram Lemma, problem (2.1) is well-posed.

Let ω denote a subregion of Ω, without scale separation and consider two overlapping subdomains, ω_1 and ω_2, with $\omega_1 \cap \omega_2 = \omega_0$, $\omega \cup \omega_2 = \Omega$, and $\omega \subset \omega_1$; Figure 1 illustrates possible domain decompositions. Assume that the tensor a^ε is given by $a^\varepsilon(x) = a^\varepsilon_\omega(x)\mathbb{1}_\omega(x) + a^\varepsilon_2(x)\mathbb{1}_{\omega_2}(x)$, where $\mathbb{1}_\omega$ denotes the characteristic function associated to the subdomain ω. Further, assume that the tensor a^ε_2 has scale separation, e.g. $a_2(x, x/\varepsilon)$, and is locally periodic in the fast variable. Following the homogenization theory, a tensor a^0_2 can be derived from a^ε_2. On the contrary, in the tensor a^ε_ω,

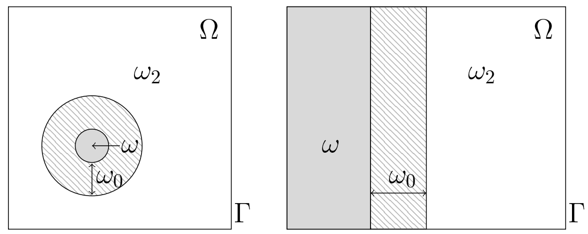


FIGURE 1. *Illustration for* ω_1, ω_2, *and* ω_0.

the scales are not well separated which prevents the use of numerical homogenization methods. The heterogeneities can also be present in the right hand side f, and following homogenization theory, the smooth part of f converges to a function f^0, when the size of the heterogeneities goes to zero, see [14].

Let $\Gamma_1 = \partial\omega_1 \setminus \Gamma$ and $\Gamma_2 = \partial\omega_2 \setminus \Gamma$ be Lipschitz continuous boundaries. We consider the following minimization problem: find $u_1^\varepsilon \in H^1(\omega_1)$ and $u_2^0 \in H^1(\omega_2)$, such that $\frac{1}{2}\|u_1^\varepsilon - u_2^0\|^2_{\mathrm{L}^2(\omega_0)}$ is minimized under the following constraints, for $i = 1, 2$,

$$\begin{aligned} -\mathrm{div}\,(a_i \nabla u_i) &= f, && \text{in } \omega_i, \\ u_i &= \theta_i, && \text{on } \Gamma_i, \\ u_i &= g_D, && \text{on } \partial\omega_i \cap \Gamma_D, \\ n_i \cdot (a_i \nabla u_i) &= g_N, && \text{on } \partial\omega_i \cap \Gamma_N, \end{aligned} \tag{2.3}$$

where the boundary conditions θ_i, which we refer to as virtual controls, are to be determined. Here and in what follows, we will sometimes use the short hand notations

$$\begin{aligned} a_1 &= a_1^\varepsilon = a_\omega^\varepsilon \mathbb{1}_\omega + a_2^\varepsilon \mathbb{1}_{\omega_0}, \quad & u_1 &= u_\varepsilon^1, \\ a_2 &= a_2^0, & u_2 &= u_2^0, \end{aligned}$$

and $u_i(\theta_i)$ to emphasize the dependency on θ_i. One could also consider Neumann boundary controls instead of Dirichlet controls, and follow the theory with some adjustments.

The strategy is to solve a minimization problem in a space of admissible controls, where the cost function to minimize is

$$J(\theta_1, \theta_2) = \frac{1}{2}\|u_1^\varepsilon(\theta_1) - u_2^0(\theta_2)\|^2_{\mathrm{L}^2(\omega_0)}. \tag{2.4}$$

The existence and uniqueness of the solution will be proved following Lions [27].

Following the virtual control method exposed in [22], we split the solutions in two parts as

$$u_1^\varepsilon(\theta_1) = u_{1,0}^\varepsilon + v_1^\varepsilon(\theta_1), \qquad u_2^0(\theta_2) = u_{2,0}^0 + v_2^0(\theta_2),$$

and call (v_1^ε, v_2^0) the state variables that satisfy, for $i = 1, 2$,

$$\left\{ \begin{aligned} -\mathrm{div}\,(a_i \nabla v_i) &= 0, && \text{in } \omega_i, \\ v_i &= \theta_i, && \text{on } \Gamma_i, \\ v_i &= 0, && \text{on } \partial\omega_i \cap \Gamma_D, \\ n_i \cdot (a_i \nabla u_i) &= 0, && \text{on } \partial\omega_i \cap \Gamma_N, \end{aligned} \right. \tag{2.5}$$

where $v_1 = v_1^\varepsilon$, and $v_2 = v_2^0$.

The space of admissible Dirichlet controls on Γ_i, $i=1,2$, is defined by

$$\mathcal{U}_i^D = \{\mu_i \in H^{1/2}(\Gamma_i) \mid \exists u \in H^1(\omega_i), u = \mu_i \text{ on } \Gamma_i, u_i = 0 \text{ on } \partial\omega_i \cap \Gamma_D, \text{ and } n_i \cdot (a_i \nabla u_i) = 0 \text{ on } \partial\omega_i \cap \Gamma_N\}.$$

For simplicity, we set $\mathcal{U} := \mathcal{U}_1^D \times \mathcal{U}_2^D$. We define for $i=1,2$

$$\begin{aligned} H_D^1(\omega_i) &= \{u \in H^1(\omega_i) \mid u = 0 \text{ on } \partial\omega_i \cap \Gamma_D\}, \\ H_{D,\Gamma_i}^1(\omega_i) &= \{u \in H^1(\omega_i) \mid u = 0 \text{ on } \partial\omega_i \cap \Gamma_D \text{ and } \Gamma_i\}. \end{aligned}$$

Let $\gamma_D : H^1(\Omega) \to H^{1/2}(\Gamma_D)$ denote a linear continuous map, called the trace map. As g_D is in $H^{1/2}(\Gamma_D)$, there exists $R_{g_D} \in H^1(\Omega)$, called a lifting of the boundary data g_D, such that $\gamma_D(R_{g_D}) = g_D$. Further, there exist a constant $C(\Omega)$ depending on Ω such that

$$\|R_{g_D}\|_{\mathrm{H}^1(\Omega)} \le C(\Omega)\|g_D\|_{\mathrm{H}^{1/2}(\Gamma_D)}.$$

The function $u_{i,0} \in H_{D,\Gamma_i}^1(\omega_i)$ satisfies, for all test functions $v \in H_{D,\Gamma_i}^1(\omega_i)$,

$$\begin{aligned} B_i(u_{i,0}, v) &:= \int_{\omega_i} a_i \nabla u_{i,0} \cdot \nabla v \mathrm{d}x \\ &= \int_{\omega_i} f v \mathrm{d}x - \int_{\omega_i} a_i \nabla R_{g_D} \cdot \nabla v \mathrm{d}x + \int_{\partial\omega_i \cap \Gamma_N} g_N v \mathrm{d}s =: F_i(v). \end{aligned} \tag{2.6}$$

The state solution $v_i \in H_D^1(\omega_i)$ verifies, for $i=1,2$,

$$B_i(v_i, v) = 0, \qquad \forall v \in H_{D,\Gamma_i}^1(\omega_i).$$

Thanks to the Lax-Milgram Lemma, the solutions $u_{1,0}^\varepsilon$ and $u_{2,0}^0$ exist and are unique. Moreover, if the virtual controls θ_1 and θ_2 are given, the solutions v_1^ε and v_2^0 can be uniquely determined. The solutions $u_{1,0}^\varepsilon$ and $u_{2,0}^0$ can be computed before the coupling as they are independent of the virtual controls (θ_1, θ_2).

Homogenization method. As mentioned at the beginning of this section, homogenization method can be used in ω_2 to capture the effective behavior u_2^0. With additional information on the structure of the tensor a_2^ε, such as $a_2^\varepsilon(x) = a_2(x, x/\varepsilon) = a_2(x,y)$ is Y-periodic in y, where $Y = (0,1)^d$, the homogenized tensor a_2^0 can be explicitly computed

$$a_2^0(x) = \frac{1}{|Y|}\int_Y a_2^\varepsilon(x)\left(I + \nabla\chi\right)\mathrm{d}y,$$

where $\nabla\chi = (\nabla\chi^1, \ldots, \nabla\chi^d)$ and I denotes the $d \times d$ identity matrix. Let $(e_i)_{i=1}^d$ be the canonical basis of $\mathbb{R}^d$. The functions $\chi^j \in W_{per}^1(Y)$ are called the first order correctors and, for $j = 1, \ldots, d$, χ^j is solution of the cell problem

$$\int_Y a_2^\varepsilon(x)\nabla\chi^j \cdot \nabla v \mathrm{d}y = -\int_Y a_2^\varepsilon(x) e_j \nabla v \mathrm{d}y, \quad \forall v \in W_{per}^1(Y), \tag{2.7}$$

with periodic boundary conditions.

This homogenized solution u_2^0 will be a good approximation of u^ε in the L^2 norm but will fail in the H^1 norm. However we can correct the homogenized solution and prove convergence in H^1 norm in a subregion of ω_2. Let u^0 be the homogenized solution corresponding to u^ε in ω_2 with, for all $x \in \Gamma_2$, $u^0(x) = u^\varepsilon(x)$ in the sense of the trace. Then, u^0 can be corrected using the periodic correctors χ^j and we obtain convergence to u^ε in the H^1 norm, on ω_2,

$$\|u^\varepsilon - (u^0 + \varepsilon w(x, x/\varepsilon))\|_{\mathrm{H}^1(\omega_2)} \leq C\varepsilon^{1/2},$$

where the corrector term $w(x, x/\varepsilon)$ is given by

$$w(x, x/\varepsilon) = \sum_{j=1}^{d} \chi^j(x, x/\varepsilon) \frac{\partial u^0(x)}{\partial x_j}, \quad x \in \omega_2.$$

This will be explained in detail during the analysis in Section 3. For classical results and proofs in homogenization see, among others, [9, 26].

Non-overlapping domain decomposition. The strategy relies on overlapping domain decomposition, but one could treat the problem using a domain decomposition method without overlapping domains, [36]. Let n stands for the outer normal derivative at the interface Γ_2. The problem will be as follows: find $u_1^\varepsilon \in H^1(\omega)$, $u_2^0 \in H^1(\omega_2)$ satisfying

$$\begin{aligned} -\mathrm{div}\,(a_\omega^\varepsilon \nabla u_1^\varepsilon) &= f, && \text{in } \omega,\\ u_1^\varepsilon &= u_2^0, && \text{on } \Gamma_2,\\ n\cdot(a_\omega^\varepsilon \nabla u_1^\varepsilon) &= n\cdot(a_2^0 \nabla u_2^0), && \text{on } \Gamma_2,\\ -\mathrm{div}\,(a_2^0 \nabla u_2^0) &= f, && \text{in } \omega_2, \end{aligned}$$

with the boundary conditions on Γ_D and Γ_N inherited from problem (2.1).

The Euler-Lagrange variational formulation. We recall that the cost is given by

$$J(\mu_1, \mu_2) = \frac{1}{2}\|u_1^\varepsilon(\mu_1) - u_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)}, \qquad (\mu_1, \mu_2) \in \mathcal{U},$$

and using the splitting into $v_i(\mu_i)$ and $u_{i,0}$, it can be written as

$$\begin{aligned} J(\mu_1, \mu_2) &= \frac{1}{2}\|v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)} + \frac{1}{2}\|u_{1,0}^\varepsilon - u_{2,0}^0\|^2_{\mathrm{L}^2(\omega_0)}\\ &\quad + \int_{\omega_0} \big(v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\big)\big(u_{1,0}^\varepsilon - u_{2,0}^0\big)\,\mathrm{d}x\\ &= \frac{1}{2}\pi\big((\mu_1, \mu_2), (\mu_1, \mu_2)\big) - F(\mu_1, \mu_2) + \frac{1}{2}\|u_{1,0}^\varepsilon - u_{2,0}^0\|^2_{\mathrm{L}^2(\omega_0)}, \end{aligned}$$

where $\pi : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ is given by

$$\pi\big((\theta_1, \theta_2), (\mu_1, \mu_2)\big) = \int_{\omega_0} \big(v_1^\varepsilon(\theta_1) - v_2^0(\theta_2)\big)\big(v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\big)\,\mathrm{d}x, \tag{2.8}$$

and $F : \mathcal{U} \to \mathbb{R}$,

$$F(\mu_1, \mu_2) = -\int_{\omega_0} \big(v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\big)\big(u_{1,0}^\varepsilon - u_{2,0}^0\big)\,\mathrm{d}x, \tag{2.9}$$

for (θ_1, θ_2), $(\mu_1, \mu_2) \in \mathcal{U}$. Following [27], the existence and uniqueness of the optimal controls hold when the form π is a scalar product on the space of admissible controls.

To prove the coercivity of the form π, we need a strong version of the Cauchy-Schwarz inequality. The proof is given in the Appendix A.1.

LEMMA A.3 (Strong Cauchy-Schwarz). *Let $v_1^\varepsilon \in H^1_D(\omega_1)$ and $v_2^0 \in H^1_D(\omega_2)$ be solutions of (2.5), for $i = 1, 2$, respectively. Then, there exist an $\varepsilon_0 > 0$ and a positive constant $C_s < 1$ such that for all $\varepsilon \leq \varepsilon_0$, it holds*

$$\int_{\omega_0} v_1^\varepsilon v_2^0 \mathrm{d}x \leq C_s \|v_1^\varepsilon\|_{\mathrm{L}^2(\omega_0)} \|v_2^0\|_{\mathrm{L}^2(\omega_0)}.$$

LEMMA 2.1. *Let v_1^ε and v_2^0 be solutions of (2.5), for $i = 1, 2$, respectively. The following bounds hold*

$$\|v_1^\varepsilon\|_{\mathrm{L}^2(\omega)} \leq \frac{C}{\tau} \|v_1^\varepsilon\|_{\mathrm{L}^2(\omega_0)},$$
$$\|v_2^0\|_{\mathrm{L}^2(\Omega\setminus\omega_1)} \leq \frac{C}{\tau} \|v_2^0\|_{\mathrm{L}^2(\omega_0)},$$

where τ is the width of the overlap and C is a constant depending on α, β, and the Poincaré constant associated to ω_1 and ω_2, respectively.

Proof. We prove the Lemma for the function v_1^ε. Let η be a cutoff function such that $\eta = 1$ in $\overline{\omega}$, $\eta = 0$ in $\Omega \setminus \omega_1$ and $|\nabla\eta| \leq 1/\tau$. Further, we have $\mathrm{supp}(\nabla\eta) \subset \omega_0$. Then, $\eta v_1 \in H^1_0(\omega_1)$ and due to (2.5) it is a-harmonic (see Appendix A.1)

$$\|v_1^\varepsilon\|_{\mathrm{L}^2(\omega)} \leq \|\eta v_1^\varepsilon\|_{\mathrm{L}^2(\omega_1)} \leq C_{\omega_1} \|\nabla(\eta v_1^\varepsilon)\|_{\mathrm{L}^2(\omega_1)},$$

using Poincaré inequality. The proof follows from the Caccioppoli inequality A.2, as

$$\|\nabla(\eta v_1^\varepsilon)\|^2_{\mathrm{L}^2(\omega_1)} \leq \frac{\beta}{\alpha\tau^2} \|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_0)}.$$

We obtain

$$\|v_1^\varepsilon\|_{\mathrm{L}^2(\omega)} \leq C_{\omega_1} \sqrt{\frac{\beta}{\alpha}} \frac{1}{\tau} \|v_1^\varepsilon\|_{\mathrm{L}^2(\omega_0)}.$$

The proof is similar for v_2^0. ☐

LEMMA 2.2. *Let ε_0 be given by the strong Cauchy-Schwarz Lemma A.3, and assume that $\varepsilon \leq \varepsilon_0$. Then, the form π defines an inner product on $\mathcal{U}$.*

Proof. The bilinearity, symmetry, and positivity are clear. We prove that the form is definite, i.e., $\pi\big((\mu_1,\mu_2),(\mu_1,\mu_2)\big) = 0$ if and only if $(\mu_1,\mu_2) = (0,0)$.
On the one hand, if the virtual controls are zero traces or fluxes, the state functions v_1 and v_2 must be zero everywhere, as they are solutions of boundary value problems with zero right hand side and boundary conditions. Thus $\pi\big((\mu_1,\mu_2),(\mu_1,\mu_2)\big) = 0$.
On the other hand, using the strong Cauchy Schwarz Lemma A.3

$$\begin{aligned}
0 = \pi((\mu_1,\mu_2),(\mu_1,\mu_2)) &= \|v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)} \\
&= \|v_1^\varepsilon(\mu_1)\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)} - 2\int_{\omega_0} v_1^\varepsilon(\mu_1) v_2^0(\mu_2)\mathrm{d}x \\
&\geq \|v_1^\varepsilon(\mu_1)\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)} - 2C_s \|v_1^\varepsilon(\mu_1)\|_{\mathrm{L}^2(\omega_0)} \|v_2^0(\mu_2)\|_{\mathrm{L}^2(\omega_0)} \\
&\geq (1 - C_s) \left(\|v_1^\varepsilon(\mu_1)\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0(\mu_2)\|^2_{\mathrm{L}^2(\omega_0)} \right).
\end{aligned}$$

As $C_S < 1$, it holds that $\|v_1^\varepsilon(\mu_1)\|_{\mathrm{L}^2(\omega_0)} = \|v_2^0(\mu_2)\|_{\mathrm{L}^2(\omega_0)} = 0$ which implies that $v_1^\varepsilon = v_2^0 = 0$, a. e. in ω_0. By Lemma 2.1, we have then that $\|v_1^\varepsilon(\mu_1)\|_{\mathrm{L}^2(\omega)} = 0$ and $\|v_2^0(\mu_2)\|_{\mathrm{L}^2(\Omega\setminus\omega_1)} = 0$, thus $v_i = 0$ a.e. in ω_i, for $i = 1, 2$. Then, we obtain, for $i = 1, 2$,

$$\|\mu_i\|_{\mathrm{H}^{1/2}(\Gamma_i)} \leq C_1\|v_i(\mu_i)\|_{\mathrm{H}^1(\omega_i)} = 0,$$

where the constants depends on ω_i, and the trace operators $\gamma_i : H^{1/2}(\Gamma_i) \to H^1(\omega_i)$. Thus, $\mu_i = 0$ on Γ_i and the form π is an inner product on $\mathcal{U}$.
□

We can then define a norm on $\mathcal{U}$ induced by the inner product π. For a pair $(\mu_1, \mu_2) \in \mathcal{U}$, we set

$$\|(\mu_1, \mu_2)\|_{\mathrm{L}^\star(\mathcal{U})} := \|v_1^\varepsilon(\mu_1) - v_2^0(\mu_2)\|_{\mathrm{L}^2(\omega_0)}. \tag{2.10}$$

The space $\mathcal{U}$ might not be complete with respect to this norm, but we can construct a completion of $\mathcal{U}$, and solve the minimization problem in the completed space. Let us denote the completed control space by $\hat{\mathcal{U}}$. Using the Hahn-Banach theorem, the inner product π and the functional F can be continuously extended in a unique way on $\hat{\mathcal{U}}$ and and we denote by $\hat{\pi}$ and $\hat{F}$, these extensions. The form $\hat{\pi}$ is continuous, symmetric, and coercive in $\hat{\mathcal{U}}$. The existence and uniqueness of the optimal pair in $\hat{\mathcal{U}}$ is given in the next Theorem.

Theorem 2.3. *The minimization problem* (2.4) *has a unique solution* $(\theta_1, \theta_2) \in \hat{U}$*, that satisfies the Euler-Lagrange equation*

$$\hat{\pi}\big((\theta_1, \theta_2), (\mu_1, \mu_2)\big) = \hat{F}(\mu_1, \mu_2), \qquad \forall(\mu_1, \mu_2) \in \hat{U}, \tag{2.11}$$

where $\hat{\pi}$ *and* $\hat{F}$ *are the continuous extensions of* π *and* F *given by* (2.8) *and* (2.9).

Proof. The existence and uniqueness of $(\theta_1, \theta_2) \in \hat{\mathcal{U}}$ follows from [27, Theorem I. 1.1], as the form $\hat{\pi}$ is symmetric, continuous, and coercive, and $\hat{F}$ is continuous. □

The optimal pair $(\theta_1, \theta_2) \in \hat{\mathcal{U}}$ minimizes the cost function, but in general there exists no functions $u_i \in H^1(\omega_i)$ that satisfy (2.3). However, there exists an embedding $\sigma : \mathcal{U} \to \hat{\mathcal{U}}$ such that $\sigma(\mathcal{U})$ is dense in $\hat{\mathcal{U}}$. Further, we can identify $\mathcal{U}$ with $\sigma(\mathcal{U})$ and conclude that (θ_1, θ_2) is the limit of a sequence $(\theta_{1n}, \theta_{2n})_{n\in\mathbb{N}}$ with $u_i(\theta_{in}) \in H^1(\omega_i)$ satisfying (2.3). In the sequel, for simplicity, we assume that the optimal pair is in $\mathcal{U}$ and hence $u_i(\theta_i) \in H^1(\omega_i)$, for $i = 1, 2$ (we then also have $v_i(\theta_i) \in H^1(\omega_i)$).

Optimality system. The state solutions and the optimal controls $(\theta_1, \theta_2) \in \mathcal{U}$ are obtained by solving an optimality system, derived from the minimization problem. The boundary value problems on ω_1 and ω_2 act as constraints. Let λ_i, $i = 1, 2$, be Lagrange multipliers associated to the constraints in ω_i, and consider the critical point of the Lagrangian functional

$$\begin{aligned}\mathcal{L}(u_1^\varepsilon, \lambda_1, \theta_1, u_2^0, \lambda_2, \theta_2) = \frac{1}{2}\|u_1^\varepsilon - u_2^0\|^2_{\mathrm{L}^2(\omega_0)} + \langle f + \operatorname{div}(a_1^\varepsilon\nabla u_1^\varepsilon), \lambda_1\rangle_{H^{-1},H^1} \\ + \langle f + \operatorname{div}(a_2^0\nabla u_2^0), \lambda_2\rangle_{H^{-1},H^1},\end{aligned}$$

with $u_i \in H^1_D(\omega_i)$ and $\lambda_i \in H^2(\omega_i)$ with $\lambda_i = 0$ on $\partial\omega_i \cap \Gamma_D$ and Γ_i, and $n_i \cdot (a_i \nabla\lambda_i) = 0$ on $\partial\omega_i \cap \Gamma_N$, for $i = 1, 2$. Using the transposition method [28], we can write the right hand side of the Langrangian in terms of the state, Lagrange multipliers, and controls variables. Computing the Gâteau derivatives for each of the unknowns leads to the optimality system.
We note that the optimality system can also be derived by using the adjoint problems of (2.3).

3 A priori error analysis of the continuous coupled problem In this section, we give an a priori error analysis of the optimization based method. The analysis is separated into fine and coarse scale error estimates. The solution of the minimization problem with constraints (2.3) gives us a fine scale solution in ω_1 and a coarse scale solution in ω_2. Looking at the error between the solution of the coupling and the exact fine scale solution u^ε on either ω_1 or ω_2, obliges to estimate terms on the boundary Γ_1 or Γ_2, respectively. In order to avoid such additional error terms, we introduce an intermediate domain ω^+ with $\omega \subset \omega^+ \subset \omega_1$. Then given $u_1^\varepsilon(\theta_1)$ and $u_2^0(\theta_2)$, the solutions of the optimization based coupling method, we define

$$\bar{u}^\varepsilon = \begin{cases} u_1^\varepsilon(\theta_1), & \text{in } \omega^+, \\ u_2^{rec}(\theta_2), & \text{in } \Omega \setminus \omega^+, \end{cases} \tag{3.1}$$

where u_2^{rec} stands for a correction to the homogenized solution $u_2^0(\theta_2)$ given below. The main convergence results are

$$\begin{aligned} \|u^\varepsilon - \bar{u}^\varepsilon\|_{\mathrm{H}^1(\omega^+)} &\leq C\varepsilon, \\ \|u^\varepsilon - \bar{u}^\varepsilon\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} &\leq C\varepsilon^{1/2}, \end{aligned}$$

where the constants depends on the width of ω^+ and the ellipticity constants of a_2^ε. For the analysis, we consider the classical locally periodic correctors χ^j solutions of (2.7), but other post-processing procedure could be used. The correction $u_2^{rec}(\theta_2)$ is given by

$$u_2^{rec}(x) = u_2^0(x) + \varepsilon \sum_{j=1}^{d} \chi^j(x, x/\varepsilon) \frac{\partial u_2^0(x)}{\partial x_j}, \qquad x \in \Omega \setminus \omega^+,$$

where $u_2^0 = u_2^0(\theta_2)$. We sometimes use $u_2^0(\theta_2)$ and $u_2^{rec}(\theta_2)$ to emphasize the dependency on θ_2. We will however avoid the heavy notation $u_2^0(\theta_2)(x)$ and drop the dependency on θ_2 when writing such map as function of x.

A priori error estimates to the fine scale solver in ω^+. The coupled solution restricted to the subregion ω^+ is given by the fine scale solution $u_1^\varepsilon(\theta_1)$, hence the error becomes $\|u^\varepsilon - \bar{u}^\varepsilon\|_{\mathrm{H}^1(\omega^+)} = \|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{H}^1(\omega^+)}$.
Let τ denote the width of the overlap ω_0 and recall that the heterogeneous tensor a_2^ε satisfies the ellipticity condition (2.2). Further, we denote by τ^+ the distance between $\partial\omega^+$ and ω; it holds $\tau^+ < \tau$. Moreover, we suppose that there exists $\varepsilon_0 > 0$, such that the strong Cauchy-Schwarz Lemma A.3 holds, for all $\varepsilon \leq \varepsilon_0$.
Let $\gamma_i : H^1(\omega_i) \to H^{1/2}(\Gamma_i)$, $i = 1, 2$, be trace operators and consider the solution u^ε restricted to the domain ω_2,

$$\begin{aligned} -\mathrm{div}\,(a_2^\varepsilon \nabla u^\varepsilon) &= f, && \text{in } \omega_2, \\ u^\varepsilon &= \gamma_2(u^\varepsilon), && \text{on } \Gamma_2, \\ u^\varepsilon &= g_D, && \text{on } \partial\omega_2 \cap \Gamma_D, \\ n_2 \cdot (a_2^\varepsilon \nabla u^\varepsilon) &= g_N, && \text{on } \partial\omega_2 \cap \Gamma_N. \end{aligned}$$

Further, for a fixed $\varepsilon \leq \varepsilon_0$, we introduce $u^0 \in H^1(\omega_2)$, homogenized solution of

$$\begin{aligned} -\mathrm{div}\,(a_2^0 \nabla u^0) &= f, && \text{in } \omega_2, \\ u^0 &= \gamma_2(u^\varepsilon), && \text{on } \Gamma_2, \\ u^0 &= g_D, && \text{on } \partial\omega_2 \cap \Gamma_D, \\ n_2 \cdot (a_2^0 \nabla u^0) &= g_N, && \text{on } \partial\omega_2 \cap \Gamma_N. \end{aligned} \tag{3.2}$$

We assume that strong convergence in the L^2 norm is available [26, Sect. 1.4], i.e.,

$$\|u^\varepsilon - u^0\|_{\mathrm{L}^2(\omega_2)} \le C\varepsilon. \tag{3.3}$$

Remark 3.1. *The error estimate* (3.3) *holds if* $a_2(\cdot, y) \in W^{1,\infty}(Y)$, *and* $u^0 \in H^2(\omega_2)$. *This can be seen by following the lines of the proof in [26]. Thanks to the regularity of* a_2^ε, *we have* $\chi^j \in W^{1,\infty}(Y)$. *The regularity on the tensor can be relaxed to* $a_2(\cdot, y) \in W^{1,p}(Y)$ *for* $p > 2$, *and* $\chi^j \in W^{1,p}(Y) \cap \mathcal{C}^{1,s}(\overline{Y})$ *for* $s = 1 - d/p$. *For the proof of* (3.3), *we refer to [26, 31].*

We follow the framework introduced in [35] and define an operator $P : \mathcal{U} \to H^1(\omega_1) \times H^1(\Omega \setminus \omega_1)$ by

$$(\mu_1, \mu_2) \mapsto P(\mu_1, \mu_2) = \begin{cases} u_{1,0}^\varepsilon + v_1^\varepsilon(\mu_1), & \text{in } \omega_1, \\ u_{2,0}^0 + v_2^0(\mu_2), & \text{in } \Omega \setminus \omega_1, \end{cases} \tag{3.4}$$

where v_i are solutions of (2.5), for $i = 1, 2$. We note that for the traces $(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon))$ of the exact solution u^ε, we obtain

$$P(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon)) = \begin{cases} u^\varepsilon, & \text{in } \omega_1, \\ u^0, & \text{in } \Omega \setminus \omega_1. \end{cases}$$

The operator P can be split into $P(\mu_1, \mu_2) = U_0 + Q(\mu_1, \mu_2)$, for $(\mu_1, \mu_2) \in \mathcal{U}$, where we define

$$U_0 = \begin{cases} u_{1,0}^\varepsilon, & \text{in } \omega_1, \\ u_{2,0}^0, & \text{in } \Omega \setminus \omega_1, \end{cases} \quad \text{and} \quad Q(\mu_1, \mu_2) = \begin{cases} v_1^\varepsilon(\mu_1), & \text{in } \omega_1, \\ v_2^0(\mu_2), & \text{in } \Omega \setminus \omega_1. \end{cases} \tag{3.5}$$

Theorem 3.2. *Let* u^ε *be the solution of* (2.1) *and* $\bar{u}^\varepsilon$ *be given by* (3.1). *Suppose that* u^0 *and* χ^j *are regular enough so that* (3.3) *holds. Let* ε_0 *be given by the strong Cauchy-Schwarz Lemma A.3, and assume that* $\varepsilon \le \varepsilon_0$. *Then, we have*

$$\|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{H}^1(\omega^+)} \le C\varepsilon,$$

where the constant C *depends on* τ, τ^+, α, β, *and on the domains* ω_1 *and* ω_2.

Proof. The difference $u^\varepsilon - u_1^\varepsilon(\theta_1) \in H_D^1(\omega_1)$ is a^ε-harmonic in ω_1 and Caccioppoli inequality A.1 can be applied; that is

$$\|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{H}^1(\omega^+)} \le \frac{C}{\tau - \tau^+} \|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{L}^2(\omega_1)},$$

where the constant C depends on the ellipticity constants of the tensor a^ε. Let us focus on the L^2 norm; recalling that $u_1^\varepsilon(\theta_1) = P(\theta_1, \theta_2)$, it holds that

$$\begin{aligned} &\|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{L}^2(\omega_1)} = \|u^\varepsilon - P(\theta_1, \theta_2)\|_{\mathrm{L}^2(\omega_1)} \\ &\le \|u^\varepsilon - P(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon))\|_{\mathrm{L}^2(\omega_1)} + \|P(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon)) - P(\theta_1, \theta_2)\|_{\mathrm{L}^2(\omega_1)}. \end{aligned}$$

By the definitions of P and u^ε, the first L^2 error is zero and it remains

$$\begin{aligned} \|u^\varepsilon - u_1^\varepsilon(\theta_1)\|_{\mathrm{L}^2(\omega_1)} &\le \|P(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon)) - P(\theta_1, \theta_2)\|_{\mathrm{L}^2(\omega_1)} \\ &= \|U_0 - Q(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon)) - U_0 + Q(\theta_1, \theta_2)\|_{\mathrm{L}^2(\omega_1)} \\ &\le \|Q\| \|(\gamma_1(u^\varepsilon), \gamma_2(u^\varepsilon)) - (\theta_1, \theta_2)\|_{\mathrm{L}^*(\mathcal{U})}, \end{aligned}$$

where the norm $\|\cdot\|_{\mathrm{L}^*(\mathcal{U})}$ is induced by the inner product π and defined in (2.10). Using Lemmas 3.3 and 3.4 given below proves the result. ∎

Lemma 3.3. *Let u^ε and u^0 solve* (2.1) *and* (3.2) *respectively, and let $(\theta_1,\theta_2)\in\mathcal{U}$ be the optimal virtual controls. Then*

$$\|\big(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)\big)-(\theta_1,\theta_2)\|_{\mathrm{L}^*(\mathcal{U})}\leq\|u^\varepsilon-u^0\|_{\mathrm{L}^2(\omega_0)}.$$

Proof. By definition, we have

$$\begin{aligned}&\|\big(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)\big)-(\theta_1,\theta_2)\|_{\mathrm{L}^*(\mathcal{U})}=\\&\sup_{(\mu_1,\mu_2)\in\mathcal{U}}\frac{|\pi\big((\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)),(\mu_1,\mu_2)\big)-\pi\big((\theta_1,\theta_2),(\mu_1,\mu_2)\big)|}{\|(\mu_1,\mu_2)\|_{\mathrm{L}^*(\mathcal{U})}}.\end{aligned}$$

We look at the numerator. As the pair (θ_1,θ_2) minimizes the cost function J, the Euler-Lagrange formulation (2.11) holds and

$$\begin{aligned}&\pi\big((\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)),(\mu_1,\mu_2)\big)-\pi((\theta_1,\theta_2),(\mu_1,\mu_2))=\\&=\int_{\omega_0}\big(v_1^\varepsilon(\gamma_1(u^\varepsilon))-v_2^0(\gamma_2(u^\varepsilon))\big)\big(v_1^\varepsilon(\mu_1)-v_2^0(\mu_2)\big)\mathrm{d}x\\&\quad+\int_{\omega_0}\big(v_1^\varepsilon(\mu_1)-v_2^0(\mu_2)\big)(u_{1,0}^\varepsilon-u_{2,0}^0)\mathrm{d}x\\&=\int_{\omega_0}\Big(\big(v_1^\varepsilon(\gamma_1(u^\varepsilon))+u_{1,0}^\varepsilon\big)-\big(v_2^0(\gamma_2(u^\varepsilon))+u_{2,0}^0\big)\Big)\big(v_1^\varepsilon(\mu_1)-v_2^0(\mu_2)\big)\mathrm{d}x\\&=\int_{\omega_0}(u^\varepsilon-u^0)\big(v_1^\varepsilon(\mu_1)-v_2^0(\mu_2)\big)\mathrm{d}x\leq\|u^\varepsilon-u^0\|_{\mathrm{L}^2(\omega_0)}\|(\mu_1,\mu_2)\|_{\mathrm{L}^*(\mathcal{U})}.\end{aligned}$$

The result follows. ∎

To complete the a priori error analysis in the continuous case, we need to bound the norm of the operator Q.

Lemma 3.4. *Let ε_0 be given by the strong Cauchy-Schwarz Lemma A.3, and assume that $\varepsilon\leq\varepsilon_0$. The operator Q, defined by* (3.5), *is bounded*

$$\|Q\|\leq C,$$

where the constant C depends on ω_1, ω_2, τ, and the strong Cauchy-Schwarz constant, see Lemma A.3.

Proof. By definition, the norm of the operator Q is given by

$$\|Q\|:=\sup_{(\mu_1,\mu_2)\in\mathcal{U}}\frac{\|Q(\mu_1,\mu_2)\|_{\mathrm{L}^2(\Omega)}}{\|(\mu_1,\mu_2)\|_{\mathrm{L}^*(\mathcal{U})}}.$$

For $(\mu_1,\mu_2)\in\mathcal{U}$, we show the existence of a positive constant such that

$$\|Q(\mu_1,\mu_2)\|^2_{\mathrm{L}^2(\Omega)}\leq C\|(\mu_1,\mu_2)\|^2_{\mathrm{L}^*(\mathcal{U})}.$$

For simplicity, we set $v_i=v_i(\mu_i)$, $i=1,2$. Using Lemma 2.1, we have

$$\begin{aligned}\|Q(\mu_1,\mu_2)\|^2_{\mathrm{L}^2(\Omega)}&=\|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_1)}+\|v_2^0\|^2_{\mathrm{L}^2(\Omega\setminus\omega_1)}\\&\leq\frac{C(\omega_1;\omega_2)}{\tau^2}\left(\|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_0)}+\|v_2^0\|^2_{\mathrm{L}^2(\omega_0)}\right).\end{aligned}$$

Next, using the strong Cauchy-Schwarz Lemma A.3, yields

$$\begin{aligned}\|(\mu_1,\mu_2)\|^2_{\mathrm{L}^*(\mathcal{U})} &= \|v_1^\varepsilon - v_2^0\|^2_{\mathrm{L}^2(\omega_0)} = \|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0\|^2_{\mathrm{L}^2(\omega_0)} - 2\int_{\omega_0} v_1^\varepsilon v_2^0 \mathrm{d}x \\ &\geq \|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0\|^2_{\mathrm{L}^2(\omega_0)} - 2C_s\|v_1^\varepsilon\|_{\mathrm{L}^2(\omega_0)}\|v_2^0\|_{\mathrm{L}^2(\omega_0)} \\ &\geq (1-C_s)\left(\|v_1^\varepsilon\|^2_{\mathrm{L}^2(\omega_0)} + \|v_2^0\|^2_{\mathrm{L}^2(\omega_0)}\right).\end{aligned}$$

Summarizing, this gives

$$\|Q(\mu_1,\mu_2)\|^2_{\mathrm{L}^2(\Omega)} \leq \frac{C(\omega_1;\omega_2)}{\tau^2(1-C_s)}\|(\mu_1,\mu_2)\|^2_{\mathrm{L}^*(\mathcal{U})}.$$

□

A priori error estimates to the reconstructed coarse scale solver in $\Omega\setminus\omega^+$. In this section, we give an a priori error estimate in the coarse scale region $\Omega\setminus\omega^+$. The coupled solution restricted to the subregion $\Omega\setminus\omega^+$ is given by $u_2^{rec}(\theta_2)$.

Lemma 3.5. *Let* u^ε *and* u_2^0 *be the solutions of problems* (2.1) *and* (2.3), *respectively. Assume that* (3.3) *holds, we obtain*

$$\|u^\varepsilon - u_2^0(\theta_2)\|_{\mathrm{L}^2(\omega_2)} \leq C\varepsilon.$$

Proof. Let us define an operator $P:\mathcal{U}\to H^1(\omega)\times H^1(\omega_2)$ by

$$P(\mu_1,\mu_2) = \begin{cases} u_{1,0}^\varepsilon + v_1^\varepsilon(\mu_1), & \text{in } \omega,\\ u_{2,0}^0 + v_2^0(\mu_2), & \text{in } \omega_2,\end{cases}$$

and consider the decomposition $P = U_0 + Q$, following (3.5). It holds $u_2^0(\theta_2) = P(\theta_1,\theta_2)|_{\omega_2}$, and

$$\begin{aligned}\|u^\varepsilon - u_2^0(\theta_2)\|_{\mathrm{L}^2(\omega_2)} &\leq \|u^\varepsilon - P(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon))\|_{\mathrm{L}^2(\omega_2)} \\ &\quad + \|P(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)) - P(\theta_1,\theta_2)\|_{\mathrm{L}^2(\omega_2)}.\end{aligned}$$

The term $P(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon))$, restricted on ω_2, is equal to $u_2^0(\gamma_2(u^\varepsilon))$, which is defined as the homogenized solution u^0 obtained in (3.2). Using (3.3), we have

$$\begin{aligned}\|u^\varepsilon - u_2^0(\theta_2)\|_{\mathrm{L}^2(\omega_2)} &\leq \|u^\varepsilon - u^0\|_{\mathrm{L}^2(\omega_2)} + \|P(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)) - P(\theta_1,\theta_2)\|_{\mathrm{L}^2(\omega_2)} \\ &\leq C\varepsilon + \|Q\|\|(\gamma_1(u^\varepsilon),\gamma_2(u^\varepsilon)) - (\theta_1,\theta_2)\|_{\mathrm{L}^*(\mathcal{U})}.\end{aligned}$$

Following the proof of Lemma 3.4, we can show that $\|Q\|$ is bounded, and using Lemma 3.3, we obtain

$$\|u^\varepsilon - u_2^0(\theta_2)\|_{\mathrm{L}^2(\omega_2)} \leq C_1\varepsilon + C_2\|u^\varepsilon - u^0\|_{\mathrm{L}^2(\omega_0)} \leq C\varepsilon.$$

□

Theorem 3.6. *Let* u^ε *be solution of* (2.1) *and* $u_2^0(\theta_2)$ *be given by* (3.1). *Let* $a_2(x,y)\in\mathcal{C}(\overline{\omega}_2;L^\infty_{per}(Y))$ *and* $\chi^j\in W_{per}(Y)$, $j=1,\ldots,d$. *If in addition,* $u^\varepsilon\in H^2(\Omega)$, $u_2^0(\theta_2)\in H^2(\omega_2)$, *and* $\chi^j\in W^{1,\infty}(Y)$, $j=1,\ldots,d$, *it holds*

$$\|u^\varepsilon - u_2^{rec}(\theta_2)\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} \leq C\varepsilon^{1/2},$$

where the constant C is independent of ε, but depends on τ, τ^+, and the ellipticity constants of a_2^ε.

Proof. Recall that u^0 is the homogenized solution of (3.2), and using the periodic corrector χ, we have a reconstructed solution $u^{0,rec}$ given by

$$u^{0,rec}(x) = u^0(x) + \varepsilon \sum_{j=1}^{d} \chi^j(x, x/\varepsilon) \frac{\partial u^0(x)}{\partial x_j}.$$

Using the triangular inequality with $u^{0,rec}$, we have

$$\|u^\varepsilon - u_2^{rec}(\theta_2)\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} \leq \|u^\varepsilon - u^{0,rec}\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} + \|u^{0,rec} - u_2^{rec}(\theta_2)\|_{\mathrm{H}^1(\Omega\setminus\omega^+)}.$$

The first norm is bounded by $C\varepsilon^{1/2}$; this follows from [26]. The second norm can be bounded by

$$\|u^{0,rec} - u_2^{rec}(\theta_2)\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} \leq \|u^0 - u_2^0(\theta_2)\|_{\mathrm{H}^1(\Omega\setminus\omega^+)} + \varepsilon \| \sum_{j=1}^{d} \chi^j(x, x/\varepsilon) \left(\frac{\partial u^0(x)}{\partial x_j} - \frac{\partial u_2^0(x)}{\partial x_j} \right) \|_{\mathrm{H}^1(\Omega\setminus\omega^+)}.$$

Each of the term can be bounded by $C\varepsilon$, using the Caccioppoli inequality on the difference $u^0 - u_2^0(\theta_2)$ and Lemma 3.5. □

4 Fully discrete optimization-based coupling method In this section, we derive a numerical method to solve the optimization based fine scale and coarse scale problems. To fully resolve the fine scales in ω_1, we need a triangulation with mesh size that resolves the fine scale, whereas the triangulation of $\Omega \setminus \omega_1$ can be coarse and independent of the smallest scale thanks to numerical homogenization techniques. In order to allow for flexible meshing, we do not impose continuity of the numerical homogenization method on Γ_1. Here we choose to use a discontinuous Galerkin method on ω_2 and continuous FE method on ω_1.

In what follows, we restrict the family of problems (2.1) to homogeneous Dirichlet problems, i.e., we set $g_D = 0$ and $\Gamma_N = \{\emptyset\}$. We denote by $H_D^1(\omega_i)$ the set of functions in $H^1(\omega_i)$ that vanish on $\partial\omega_i \cap \Gamma_D$, for $i = 1, 2$.

Further, we assume that the strong Cauchy-Schwarz Lemma A.3 and its discrete version A.3 hold.

Numerical method for the fine scale problem. Let $\mathcal{T}_{\tilde{h}}$ be a partition of ω_1, in simplicial or quadrilateral elements, with mesh size $\tilde{h} \ll \varepsilon$ where $\tilde{h} = \max_{K\in\mathcal{T}_{\tilde{h}}} h_K$, and h_K is the diameter of the element K. In addition, we suppose that the partition is admissible and shape regular [13],

(T1) **admissible.** $\overline{\omega}_1 = \cup_{K\in\mathcal{T}_h} K$ and the intersection of two elements is either empty, a vertex, or a common face.

(T2) **shape regular.** There exists $\sigma > 0$ such that $h_K/\rho_K \leq \sigma$, where ρ_K is the diameter of the largest circle contained in the element K.

For simplicity, we consider a piecewise FE in ω_1, given by

$$V_D^1(\omega_1, \mathcal{T}_{\tilde{h}}) = \{w \in H_D^1(\omega_1) \mid w_{|K} \in \mathcal{R}^1(K), \ \ \forall K \in \mathcal{T}_{\tilde{h}}\},$$

where $\mathcal{R}^1$ is the space of piecewise polynomials on K. Further, we denote by $V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$ the functions in $V_D^1(\omega_1, \mathcal{T}_{\tilde{h}})$ that vanish on $\partial\omega_1$.

Let $u_{1,\tilde{h}}$ be the numerical approximation of u_1^ε satisfying (2.3) for $i = 1$. We can decomposed $u_{1,\tilde{h}}$ into $u_{1,\tilde{h}} = u_{1,0,\tilde{h}} + v_{1,\tilde{h}}$, where $v_{1,\tilde{h}} \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}})$ is obtained by the optimization method and $u_{1,0,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$ is the solution of

$$B_1(u_{1,0,\tilde{h}}, w_{1,\tilde{h}}) = F_1(w_{1,\tilde{h}}), \quad \forall w_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}), \tag{4.1}$$

where the right hand side F_1 is given by

$$F_1(w_{1,\tilde{h}}) = \int_{\omega_1} f w_{1,\tilde{h}} \mathrm{d}x.$$

Thanks to the Poincaré inequality, the bilinear form B_1 is coercive and bounded over $V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$; the existence and uniqueness of $u_{1,0,\tilde{h}}$ follows. We note that a quadrature formula should be considered for the bilinear form B_1 and for the right hand side F_1.

Discontinuous Galerkin (DG) method for the coarse scale problem. Let $\mathcal{T}_H$ be a mesh over ω_2, with discontinuity in Γ_1 and mesh size $H = \max_{K\in\mathcal{T}_H} h_K$; further we assume that $\mathcal{T}_H$ is admissible (T1) and shape regular (T2). We denote by E the set of $(d-1)$ dimensional elements of $\mathcal{T}_H$ that form the boundary Γ_1 — it will be edges (for $d = 2$) or faces (for $d = 3$). Further, assume that the set E is composed of the smallest common interface between two elements K_+ and K_- of $\mathcal{T}_H$, with intersection in Γ_1; that is e is in E if $e = \min K_+ \cap K_-$ and $e \subset \Gamma_1$. As the solutions of problem (2.3), for $i = 2$, are assumed to be continuous in $\omega_2 \setminus \Gamma_1$, we construct a piecewise FE space as

$$V_D^1(\omega_2, \mathcal{T}_H) = \{v \in H_D^1(\omega_2 \setminus \Gamma_1) \cap L^2(\omega_2) \mid v_{|K} \in \mathcal{R}^1(K), \ \ \forall K \in \mathcal{T}_H\},$$

we denote by $V_0^1(\omega_2, \mathcal{T}_H)$ the set of functions of $V_D^1(\omega_2, \mathcal{T}_H)$ that vanish over $\partial\omega_2$. For $v \in V^1(\omega_2, \mathcal{T}_H)$, we consider its average $\{\cdot\}$ and its jump $[\![\cdot]\!]$ given by

$$\{v\} = \frac{1}{2}(v_+ + v_-), \quad \text{and} \quad [\![v]\!] = v_+ n_+ + v_- n_-,$$

where $v_\pm := v|_{K_\pm}$ denotes the trace of v from within $K_\pm$ and $n_\pm$ stands for the unit outward normal in $K_\pm$.

Quadrature formula. For piecewise FE spaces, a quadrature formula is given by the pair $(x_K, |K|)$, where x_K is the barycenter of K. The sampling domain of size δ around each points x_K is denoted by $K_\delta = x_K + \delta[-1/2, 1/2]^2$.

The numerically homogenized tensor $a_2^{0,h}(x_K)$, around the quadrature point x_K, is obtained using numerical solutions of micro problems defined in the sampling domains K_δ; we note that a numerical approximation of f^0 can be obtained similarly. Let us consider a partition $\mathcal{T}_h$ of K_δ in simplicial or quadrilateral elements K of diameter h_K; the mesh size is $h = \max_{K\in\mathcal{T}_h} h_K$ and, as the fine scales should be resolved in K_δ, we impose $h < \varepsilon$. The piecewise micro FE space is given by

$$S^1(K_\delta, \mathcal{T}_h) = \{v^h \in W(K_\delta) \mid v^h_{|K} \in \mathcal{R}^1(K), \ \ \forall K \in \mathcal{T}_h\},$$

where $W(K_\delta)$ depends on the boundary conditions imposed on the micro problems; $W(K_\delta) = H_0^1(K_\delta)$ for Dirichlet coupling, or $W(K_\delta) = W_{\mathrm{per}}^1(K_\delta)$ for periodic coupling. We introduce discrete micro problems: find $\psi_{K_\delta}^{i,h} \in S^1(K_\delta, \mathcal{T}_h)$, $i = 1, \ldots, d$, solution of

$$\int_{K_\delta} a_2^\varepsilon(x) \nabla\psi_{K_\delta}^{i,h} \cdot \nabla v_j^h \mathrm{d}x = -\int_{K_\delta} a_2^\varepsilon(x) e_i \cdot \nabla v_j^h \mathrm{d}x, \qquad \forall v_j^h \in S^1(K_\delta, \mathcal{T}_h). \tag{4.2}$$

The numerically homogenized tensor at a quadrature point x_K in a macro element K, is computed by

$$a_2^{0,h}(x_K) = \frac{1}{|K_\delta|}\int_{K_\delta} a_2^\varepsilon(x)\left(I + \nabla\psi_{K_\delta}^h\right)\mathrm{d}x,$$

where $\nabla\psi_{K_\delta}^h = (\nabla\psi_{K_\delta}^{1,h}, \dots, \nabla\psi_{K_\delta}^{d,h})$. Following [4], we define a DG macro bilinear form $B_{2,H}(\cdot,\cdot)$ over $V_D^1(\omega_2, \mathcal{T}_H) \times V_D^1(\omega_2, \mathcal{T}_H)$ by

$$\begin{aligned} B_{2,H}(v_{2,H}, w_{2,H}) = &\sum_{K\in\mathcal{T}_H} |K| a_2^{0,h}(x_K)\nabla v_{2,H}(x_K)\cdot\nabla w_{2,H}(x_K) \\ &+\sum_{e\in E}\int_e \mu_e [\![v_{2,H}]\!][\![w_{2,H}]\!]\mathrm{d}s \\ &-\sum_{e\in E}\int_e \left(\{a_2^{0,h}(x_K)\nabla v_{2,H}(x_K)\}[\![w_{2,H}]\!]\right. \\ &\left.+\{a_2^{0,h}(x_K)\nabla w_{2,H}(x_K)\}[\![v_{2,H}]\!]\right)\mathrm{d}s, \end{aligned} \tag{4.3}$$

where the functions μ_e stand for weighting functions that penalize the jumps of $v_{2,H}$ and $w_{2,H}$ over the element e in E. It is given by $\mu_e = \kappa h_e^{-1}$, with $\kappa > 0$, and h_e is the size of the interface e.
The numerical homogenized solution $u_{2,H}$ is split into $u_{2,H} = u_{2,0,H} + v_{2,H}$, where $v_{2,H} \in V_D^1(\omega_2, \mathcal{T}_H)$ is given by the coupling and $u_{2,0,H} \in V_0^1(\omega_2, \mathcal{T}_H)$ by solving

$$B_{2,H}(u_{2,0,H}, w_{2,H}) = F_2(w_{2,H}), \quad \forall w_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H). \tag{4.4}$$

The right hand side F_2 is given by

$$F_2(w_{2,H}) = \sum_{K\in\mathcal{T}_H} |K| f(x_K) w_{2,H}(x_K).$$

Remark 4.1. *Considering non-homogeneous Dirichlet boundary condition $g_D \neq 0$ on Γ_D and Neumann condition on $\Gamma_N \neq \{0\}$ leads to some additional terms in the right hand sides F_1 and F_2 of problems* (4.1) *and* (4.4)*, respectively. In particular, one should construct a lifting of the Dirichlet data as explained Section 2.*

Remark 4.2. *Higher order FE spaces can be considered and we note that the macro FEM over ω_1 and the micro FEM over the sampling domains can be easily generalized to higher order FEM. For the DG-FE-HMM some work needs to be done on the average of the fluxes, and we refer to [4].*

4.1 Numerical Algorithm In this section, we state the discrete coupling, give the algorithm, and present the main convergence results. The well-posedness and the proofs of the errors estimates are given in the next sections.
The solution $(u_{1,\tilde{h}}, u_{2,H}) \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_D^1(\omega_2, \mathcal{T}_H)$ satisfies

$$\min_{\mu_{1,\tilde{h}},\mu_{2,H}} \frac{1}{2}\|u_{1,\tilde{h}}(\mu_{1,\tilde{h}}) - u_{2,H}(\mu_{2,H})\|^2_{\mathrm{L}^2(\omega_0)} \text{ subject to } \begin{cases} B_1(u_{1,\tilde{h}}, w_{1,\tilde{h}}) &= F_1(w_{1,\tilde{h}}), \\ B_{2,H}(u_{2,H}, w_{2,H}) &= F_2(w_{2,H}), \end{cases}$$

for all $w_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$ and $w_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H)$. Introducing discrete Lagrange multipliers $\lambda_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$ and $\lambda_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H)$ for each of the constraint, leads to a discrete optimality system: find $(v_{1,\tilde{h}}, \lambda_{1,\tilde{h}}, v_{2,H}, \lambda_{2,H}) \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_D^1(\omega_2, \mathcal{T}_H) \times V_0^1(\omega_2, \mathcal{T}_H)$ satisfying

$$\int_{\omega_0} (v_{1,\tilde{h}} - v_{2,H}) w_{1,\tilde{h}} \mathrm{d}x - B_1(w_{1,\tilde{h}}, \lambda_{1,\tilde{h}}) = -\int_{\omega_0} \left(u_{1,0,\tilde{h}} - u_{2,0,H} \right) w_{1,\tilde{h}} \mathrm{d}x, \tag{4.5}$$

$$B_1(v_{1,\tilde{h}}, \xi_{1,\tilde{h}}) = F_1(\xi_{1,\tilde{h}}) - B_1(u_{1,0,\tilde{h}}, \xi_{1,\tilde{h}}), \tag{4.6}$$

$$\int_{\omega_0} (v_{2,H} - v_{1,\tilde{h}}) w_{2,H} \mathrm{d}x - B_{2,H}(w_{2,H}, \lambda_{2,H}) = \int_{\omega_0} (u_{1,0,\tilde{h}} - u_{2,0,H}) w_{2,H} \mathrm{d}x, \tag{4.7}$$

$$B_{2,H}(v_{2,H}, \xi_{2,H}) = F_2(\xi_{2,H}) - B_{2,H}(u_{2,0,H}, \xi_{2,H}), \tag{4.8}$$

for all $w_{1,\tilde{h}} \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}})$, $\xi_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$, $w_{2,H} \in V_D^1(\omega_2, \mathcal{T}_H)$, and $\xi_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H)$. The optimality system (4.5) to (4.8) can be written in matrix form, for the unknown vector $U = (v_{1,\tilde{h}}, v_{2,H}, \lambda_{1,\tilde{h}}, \lambda_{2,H})^\top$, as

$$\begin{pmatrix} M & -B^\top \\ B & 0 \end{pmatrix} U = G. \tag{4.9}$$

The algorithm for the numerical coupling method is given below.

1. Find $u_{1,0,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$ satisfying
$$B_1(u_{1,0,\tilde{h}}, w_{1,\tilde{h}}) = F_1(w_{1,\tilde{h}}), \quad \forall w_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}). \tag{4.10}$$
2. Find $u_{2,0,H} \in V_0^1(\omega_2, \mathcal{T}_H)$ satisfying
$$B_{2,H}(u_{2,0,H}, w_{2,H}) = F_2(w_{2,H}), \quad \forall w_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H). \tag{4.11}$$
3. Find $v_{1,\tilde{h}} \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}})$ and $v_{2,H} \in V_D^1(\omega_2, \mathcal{T}_H)$ by solving the saddle point problem (4.9).

We state the two main convergence results for the fully discrete coupling. The optimization based method relies on the DG-FE-HMM, thus one should expect to find the DG-FE-HMM error in the a priori estimates. The DG-FE-HMM is split into a macro, micro and modeling error

$$e_{HMM} \leq e_{MAC} + e_{MIC} + e_{MOD}.$$

The macro and micro errors correspond to FE errors due to the choice of macro and micro FE methods respectively. The modeling error is due to the upscaling procedure, and will be influenced by the choice of boundary conditions for (4.2), δ the size of the sampling domain, and whether we consider collocation. Details about the DG-FE-HMM error are given in the Section 5. Let $(\theta_{1,\tilde{h}}, \theta_{2,H})$ be the discrete couple of boundary conditions given by the minimization problem (2.4). We recall the notations

$$\begin{aligned} u_{1,\tilde{h}}(\theta_{1,\tilde{h}}) &\text{ denotes the fine scale numerical solution in } \omega_1, \\ u_{2,H}(\theta_{2,H}) &\text{ denotes coarse scale numerical solution in } \omega_2. \end{aligned}$$

The coupling solution, denoted by $\bar{u}_{\tilde{h}H}$, is defined as

$$\bar{u}_{\tilde{h}H} = \begin{cases} u_{1,\tilde{h}}(\theta_{1,\tilde{h}}), & \text{in } \omega^+, \\ u_{2,H}^{rec}(\theta_{2,H}), & \text{in } \Omega \setminus \omega^+, \end{cases} \tag{4.12}$$

where $u_{2,H}^{rec}(\theta_{2,H})$ corresponds to the reconstructed coarse scale solution $u_{2,H}(\theta_{2,H})$ and is defined by

$$u_{2,H}^{rec}(x) = u_{2,H}(x) + \sum_{j=1}^{d} \psi_{K_\varepsilon}^{j,h}(x) \frac{\partial u_{2,H}}{\partial x_j}(x), \qquad x \in K,$$

where $\psi_{K_\varepsilon}^{j,h}$ are the micro solutions of (4.2). As the reconstructed numerical solution might be discontinuous across elements in ω_2, we consider a broken H^1 semi-norm,

$$\|v\|_{\bar{\mathrm{H}}^1(\Omega)}^2 := \sum_{K\in\mathcal{T}_h(\omega^+)} \|\nabla v\|_{\mathrm{L}^2(K)}^2 + \sum_{K\in\mathcal{T}_H(\Omega\setminus\omega^+)} \|\nabla v\|_{\mathrm{L}^2(K)}^2.$$

We next stat our main convergence result for the optimization based numerical solution. We first have an error estimate in the fine scale region.

THEOREM 4.3 (A priori error analysis in ω^+). *Let ε_0 be given by the strong Cauchy-Schwarz Lemma A.3 and consider $\varepsilon \le \varepsilon_0$. Let u^ε and u^0 be the exact solutions of problems* (2.1) *and* (3.2)*, respectively, and $\bar{u}_{\tilde{h}H}$ be the numerical solution of the coupling* (4.12)*. Assume $u^\varepsilon \in H^{s+1}(\Omega)$, with $s \le 1$, $u^0 \in H^2(\omega_2)$, and assume that* (3.3) *holds, then*

$$\|u^\varepsilon - u_{1,\tilde{h}}(\theta_{1,\tilde{h}})\|_{\bar{\mathrm{H}}^1(\omega^+)} \le C_1 \tilde{h}^s |u^\varepsilon|_{H^{s+1}(\omega_1)} + \frac{C_2}{\tau-\tau^+}\left(\tilde{h}^{s+1}|u^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + e_{HMM,L^2}\right),$$

where the constants are independent of ε, H, $\tilde{h}$, and h. The DG-FE-HMM error is given in Lemmas 5.2, 5.3, and 5.5.

Next, we state an error estimates in the coarse scale region for the optimization based numerical solution with correctors.

THEOREM 4.4 (Error estimates in $\Omega\setminus\omega^+$). *Let u^ε be the exact solution of problem* (2.1) *and $\bar{u}_{\tilde{h}H}$ be the numerical solution of the coupling* (4.12)*. Let $a_2^\varepsilon(x) = a_2(x, x/\varepsilon)$, where $a_2(x,y)$ is Y-periodic in y and satisfies $a_2(x,y) \in \mathcal{C}(\overline{\omega}_2; L^\infty_{per}(Y))$. Let $\psi_{K_\varepsilon}^j(x) \in W^1_{per}(K_\varepsilon)$, $j = 1,\ldots,d$. If in addition, $u^\varepsilon \in H^2(\Omega)$, $u_2^0(\theta_2) \in H^2(\omega_2)$, $u_1^\varepsilon \in H^{s+1}(\omega_1)$, with $s \le 1$, and $\psi_{K_\varepsilon}^j(x) \in W^{1,\infty}(K_\varepsilon)$, $j = 1,\ldots,d$. It holds,*

$$\begin{aligned}\|u_2^{rec}(\theta_2) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\setminus\omega^+)} \le{}& C_1\varepsilon^{1/2} + C_2\left(\frac{h}{\varepsilon}\right) + C_3 H |u_2^0|_{H^2(\omega_2)} \\ &+ \frac{C_4}{\tau^+}\left(\tilde{h}^{s+1}|u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2|u_2^0|_{H^2(\omega_2)}\right).\end{aligned}$$

where the constants are independent of $H, \tilde{h}, h$, and ε.

Both theorems will be proved in Section 5. We first discuss the well-posedness of the numerical method.

4.2 Well-posedness In this subsection, we prove the well-posedness of the discrete coupling problem. The well-posedness of the DG optimization based coupling method can be established using Brezzi's theory [11] and the well-posedness of problems (4.10) and (4.11). The Lax-Milgram Lemma implies the existence and uniqueness of $u_{1,0,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}})$.

Due to the discontinuity in $\mathcal{T}_H$, the space $V_0^1(\omega_2, \mathcal{T}_H)$ is not a subspace of $H_0^1(\omega_2)$, however, it will lie in the piecewise Sobolev space

$$H^2(\mathcal{T}_H) := \prod_{K\in\mathcal{T}_H} H^2(K) = \{v \in L^1(\omega_2) \mid v_{|K} \in H^2(K),\ \forall K \in \mathcal{T}_H\}.$$

Suppose that the exact solution $u_{2,0}$ of problem (2.6) is in the space $H_0^1(\omega_2)\cap H^2(\omega_2)$, we define the proper space for the analysis as $V(\omega_2) := V_0^1(\omega_2,\mathcal{T}_H) + H_0^1(\omega_2)\cap H^2(\omega_2) \subset H^2(\mathcal{T}_H)$, see discussions in [7, 16]. The space $V(\omega_2)$ is equipped with the norm

$$\|\|v\|\|_{\omega_2} := \left(\|\nabla v\|^2_{\mathrm{L}^2(\omega_2)} + \sum_{K\in\mathcal{T}_H} h_K^2 |v|^2_{2,K} + |v|^2_* \right)^{1/2}, \tag{4.13}$$

where

$$\|\nabla v\|^2_{\mathrm{L}^2(\omega_2)} = \sum_{K\in\mathcal{T}_H} |v|^2_{1,K},\quad |v|^2_{2,K} = \sum_{|r|=2} \|\partial^r v\|^2_{\mathrm{L}^2(K)},\quad \text{and } |v|^2_* = \sum_{e\in E} \|\mu_e^{1/2} [\![v]\!]\|^2_{\mathrm{L}^2(e)}.$$

One can prove that (4.13) is a norm over $V(\omega_2)$, using the discrete Poincaré-Friedrich inequality [11],

$$\|v\|^2_{\mathrm{L}^2(\omega_2)} \le C(\|\nabla v\|^2_{\mathrm{L}^2(\omega_2)} + |v|^2_*), \tag{4.14}$$

Thanks to local inverse inequalities [13], restricting $V(\omega_2)$ to $V_0^1(\omega_2,\mathcal{T}_H)$, reduces the norm (4.13) to

$$\|\|v\|\|_{\omega_2} = \left(\|\nabla v\|^2_{\mathrm{L}^2(\omega_2)} + |v|^2_* \right)^{1/2}.$$

Proposition 4.5. *There exists a value κ_0, that depends only on* (2.2)*, the shape regularity of $\mathcal{T}_H$, and the dimension d, such that for all $\kappa \ge \kappa_0$, the bilinear form $B_{2,H}$* (4.3) *is stable in $V_0^1(\omega_2,\mathcal{T}_H)$,*

$$B_{2,H}(v_H,v_H) \ge C_1 \|\|v_H\|\|^2, \qquad \forall v_H \in V_0^1(\omega_2,\mathcal{T}_H).$$

Furthermore, the bilinear form is bounded

$$B_{2,H}(v_H,w_H) \le C_2 \|\|v_H\|\| \|\|w_H\|\|, \qquad \forall v_H, w_H \in V_0^1(\omega_2,\mathcal{T}_H).$$

The constants C_1 and C_2 are independent of $H,\tilde{h},h$, and ε.

Proof. See [4, Lemmas 4.3, 4.4, and 5.18].

Theorem 4.6. *Let assumption* (2.2) *hold. Then there exists a unique solution $u_{1,0,\tilde{h}} \in V_0^1(\omega_1,\mathcal{T}_{\tilde{h}})$ of problem* (4.10) *which satisfies*

$$\|u_{1,0,\tilde{h}}\|_{\mathrm{H}^1(\omega_1)} \le C_1 \|F_1\|_{\mathrm{H}^{-1}(\omega_1)},$$

with a constant C_1 independent of $H,\tilde{h}$, and ε.
Moreover, let κ_0 be given by Proposition 4.5. Then, the problem (4.11) *admits a unique solution $u_{2,0,H} \in V_0^1(\omega_2,\mathcal{T}_H)$ and it holds*

$$\|\|u_{2,0,H}\|\| \le C_2 \|F_2\|_{\mathrm{H}^{-1}(\omega_2)},$$

where the constant C_2 is independent of $H,h,\tilde{h}$, and ε.

Proof. The existence and uniqueness of $u_{1,0,\tilde{h}}$ and $u_{2,0,H}$ follows from Lax-Milgram Lemma and Proposition 4.5. □

We introduce $V^1(\Gamma_i)$ as the set of functions $\mu_i \in \mathcal{U}^i$ that are piecewise polynomials on the elements over Γ_i, $i = 1,2$. Let us write system (4.5) to (4.8) in term of the discrete virtual controls $\theta_{1,\tilde{h}}$ and $\theta_{2,H}$: find $(\theta_{1,\tilde{h}}, \lambda_{1,\tilde{h}}, \theta_{2,H}, \lambda_{2,H}) \in V^1(\Gamma_1) \times V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V^1(\Gamma_2) \times V_0^1(\omega_2, \mathcal{T}_H)$ satisfying

$$\pi((\theta_{1,\tilde{h}}, \theta_{2,H}), (\mu_{1,\tilde{h}}, \mu_{2,H})) - B((\mu_{1,\tilde{h}}, \mu_{2,H}), (\lambda_{1,\tilde{h}}, \lambda_{2,H})) = G(\mu_{1,\tilde{h}}, \mu_{2,H}), \tag{4.15}$$

$$B((\theta_{1,\tilde{h}}, \theta_{2,H}), (\xi_{1,\tilde{h}}, \xi_{2,H})) = 0, \tag{4.16}$$

for all $(\mu_{1,\tilde{h}}, \mu_{2,H}) \in V^1(\Gamma_1) \times V^1(\Gamma_2)$ and $(\xi_{1,\tilde{h}}, \xi_{2,H}) \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_0^1(\omega_2, \mathcal{T}_H)$. The forms π, B, and G are defined by

$$\pi((\theta_{1,\tilde{h}}, \theta_{2,H}), (\mu_{1,\tilde{h}}, \mu_{2,H})) = \int_{\omega_0} (v_{1,\tilde{h}}(\theta_{1,\tilde{h}}) - v_{2,H}(\theta_{2,H}))(v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) - v_{2,H}(\mu_{2,H}))\mathrm{d}x,$$

$$B((\theta_{1,\tilde{h}}, \theta_{2,H}), (\xi_{1,\tilde{h}}, \xi_{2,H})) = B_1(\theta_{1,\tilde{h}}, \xi_{1,\tilde{h}}) + B_{2,H}(\theta_{2,H}, \xi_{2,H}),$$

$$G(\theta_{1,\tilde{h}}, \theta_{2,H}) = -\int_{\omega_0} (u_{1,0,\tilde{h}} - u_{2,0,H})(v_{1,\tilde{h}}(\theta_{1,\tilde{h}}) - v_{2,H}(\theta_{2,H}))\mathrm{d}x.$$

To prove the well-posedness of system (4.15) – (4.16), we need to show that

- The form π is continuous and coercive on $V^1(\Gamma_1) \times V^1(\Gamma_2)$ equipped with the inner product π.
- The form B is continuous and satisfies an inf-sup condition.

The coercivity of π is clear as π is an inner product on $V^1(\Gamma_1) \times V^1(\Gamma_2)$, see Lemma 2.2. The continuity can be easily obtained with Cauchy-Schwarz and the discrete Poincaré inequality (4.14). We prove the inf-sup condition for the bilinear form B.

Lemma 4.7. *The form B satisfies*

$$\sup_{(\mu_{1,\tilde{h}}, \mu_{2,H})} \frac{B((\mu_{1,\tilde{h}}, \mu_{2,H}), (\xi_{1,\tilde{h}}, \xi_{2,H}))}{\|(\mu_{1,\tilde{h}}, \mu_{2,H})\|_{\mathrm{L}^*(\hat{U})}} \geq C \left(\|\xi_{1,\tilde{h}}\|_{\mathrm{H}^1(\omega_1)} + |||\xi_{2,H}|||_{\omega_2} \right),$$

for all $(\xi_{1,\tilde{h}}, \xi_{2,H}) \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_0^1(\omega_2, \mathcal{T}_H)$. The constant C is independent of ε.

Proof. Let $(\xi_{1,\tilde{h}}, \xi_{2,H}) \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_0^1(\omega_2, \mathcal{T}_H)$. By the definition of B, we have

$$B((\mu_{1,\tilde{h}}, \mu_{2,H}), (\xi_{1,\tilde{h}}, \xi_{2,H})) = B_1(\mu_{1,\tilde{h}}, \xi_{1,\tilde{h}}) + B_{2,H}(\mu_{2,H}, \xi_{2,H}).$$

Take $(\mu_{1,\tilde{h}}, \mu_{2,H})$ such that $v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) = \xi_{1,\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_h)$ and $v_{2,H}(\mu_{2,H}) = \xi_{2,H} \in V_0^1(\omega_2, \mathcal{T}_H)$. Then,

$$B_1(\mu_{1,\tilde{h}}, \xi_{1,\tilde{h}}) = \int_{\omega_1} a_1^\varepsilon \nabla v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) \cdot \nabla \xi_{1,\tilde{h}} \mathrm{d}x = \int_{\omega_1} a_1^\varepsilon \nabla \xi_{1,\tilde{h}} \cdot \nabla \xi_{1,\tilde{h}} \mathrm{d}x \geq C \|\xi_{1,\tilde{h}}\|^2_{\mathrm{H}^1(\omega_1)}.$$

Similarly, by the coercivity of $B_{2,H}$, it holds

$$B_{2,H}(\mu_{2,H}, \xi_{2,H}) \geq C |||\xi_{2,H}|||^2_{\omega_2}.$$

Thus,

$$B((\mu_{1,\tilde{h}}, \mu_{2,H}), (\xi_{1,\tilde{h}}, \xi_{2,H})) \geq C \left(\|\xi_{1,\tilde{h}}\|_{\mathrm{H}^1(\omega_1)} + |||\xi_{2,H}|||_{\omega_2} \right)^2,$$

where the constant is independent of H, h, $\tilde{h}$, and ε. We can conclude as

$$\begin{aligned} \|(\mu_{1,\tilde{h}}, \mu_{2,H})\|_{\mathrm{L}^*(\hat{U})} &\leq \|v_{1,\tilde{h}}(\mu_{1,\tilde{h}})\|_{\mathrm{L}^2(\omega_1)} + \|v_{2,H}(\mu_{2,H})\|_{\mathrm{L}^2(\omega_2)} \\ &\leq C \left(\|v_{1,\tilde{h}}(\mu_{1,\tilde{h}})\|_{\mathrm{H}^1(\omega_1)} + |||v_{2,H}(\mu_{2,H})|||_{\omega_2} \right) \\ &= C \left(\|\xi_{1,\tilde{h}}\|_{\mathrm{H}^1(\omega_1)} + |||\xi_{2,H}|||_{\omega_2} \right). \end{aligned}$$

∎

5 Fully discrete error estimates In this section, we derive error estimates for the fully discrete optimization-based method. A post-processing procedure is used on the coarse solution $u_{2,H}(\theta_{2,H})$, in order to reach convergence to the exact solution u^ε. The norm considered is a broken H^1 semi-norm as we allow the corrected solution to be discontinuous across elements of $\Omega\backslash\omega$. The fully discrete analysis is then conducted for the error

$$\|u^\varepsilon - \bar{u}_{\tilde{h}H}\|_{\bar{\mathrm{H}}^1(\Omega)} = \sum_{K\in\mathcal{T}_h(\omega^+)} \|\nabla(u^\varepsilon - \bar{u}_{\tilde{h}H})\|^2_{\mathrm{L}^2(K)} + \sum_{K\in\mathcal{T}_H(\Omega\backslash\omega^+)} \|\nabla(u^\varepsilon - \bar{u}_{\tilde{h}H})\|^2_{\mathrm{L}^2(K)}.$$

where the numerical solution of the coupling $\bar{u}_{\tilde{h}H}$ is given by (4.12). In the fully discrete analysis of the DG-FE-HMM method, the error between the homogenized solution and its approximation is decomposed into a macro, micro, and modeling error [2]. These errors will contribute to the a priori estimates of our method.

Remark 5.1. *In Section 3, the error estimates depend on the bound of the operator Q (3.5). This bound was obtained in Lemma 3.4 using Caccioppoli inequalities. In the fully discrete case, we introduce a discrete operator $Q^{\tilde{h},H}$, which is a discrete version of the operator Q and the estimates will depend on $\|Q^{\tilde{h},H}\|$. For conforming FE spaces the norm of $Q^{\tilde{h},H}$ is bounded independently of the mesh sizes $\tilde{h}, h$, and H; this can be seen by following the lines of Lemma 3.4. For non-conforming meshes, we will assume that $\|Q^{\tilde{h},H}\|$ is bounded independently of $\tilde{h}, h$, and H. In what follows, we will use the notations P, U_0, and Q, previously used in the continuous analysis, to denote the operators in the discrete analysis.*

Preliminaries. We recall that u^0, solution of (3.2), denotes a homogenized solution over ω_2 with boundary condition on Γ_2 given by the trace of the physical solution u^ε, for a fixed ε. The DG-FE-HMM method gives us an approximation $u^H \in V_0^1(\omega_2, \mathcal{T}_H)$ of the homogenized solution u^0. We state here the main results needed to bound $\|u^0 - u^H\|_{\mathrm{L}^2(\omega_2)}$, for further details we refer to [1, 2, 3], and the references therein. We decompose the DG-FE-HMM error into the macro, micro, and modeling errors

$$\|u^0 - u^H\|_{\mathrm{L}^2(\omega_2)} \leq e_{MAC} + e_{MIC} + e_{MOD}.$$

Macro Error. We define $u_H^0 \in V_0^1(\omega_2, \mathcal{T}_H)$ as the FEM approximation of the homogenized problem (3.2), i.e.,

$$B^0_{2,H}(u^0_H, v_H) = F_2(v_H), \qquad \forall v_H \in V_0^1(\omega_2, \mathcal{T}_H), \tag{5.1}$$

where the bilinear form is given by

$$\begin{aligned} B^0_{2,H}(u_H, v_H) = &\sum_{K\in\mathcal{T}_H} |K| a_2^0(x_K)\nabla u^H \nabla v^H + \sum_{e\in E}\int_e \mu_e [\![u_H]\!][\![v_H]\!]\mathrm{d}s \\ &- \sum_{e\in E}\int_e \left(\{a_2^0\nabla u_H\}[\![v_H]\!] + \{a_2^0 \nabla v_H\}[\![u_H]\!]\right)\mathrm{d}s, \quad \forall u_H, v_H \in V_0^1(\omega_2, \mathcal{T}_H). \end{aligned} \tag{5.2}$$

The error can be formulated as

$$\begin{aligned} \|u^0 - u^H\|_{\mathrm{L}^2(\omega_2)} &\leq \|u^0 - u^0_H\|_{\mathrm{L}^2(\omega_2)} + \|u^0_H - u^H\|_{\mathrm{L}^2(\omega_2)} \\ &\leq \|u^0 - u^0_H\|_{\mathrm{L}^2(\omega_2)} + |||u^0_H - u^H|||_{\omega_2}, \end{aligned}$$

where the first norm is the macro error and the second norm stands for the modeling and micro errors.

To simplify the analysis we make the following assumptions on the structure of the tensor a_2^ε,

(H1) $a_2^\varepsilon(x) = a_2(x, x/\varepsilon) = a_2(x, y)$ is Y-periodic in y and $a_2(\cdot, y)|_K$ is constant within each $K \in \mathcal{T}_H$.

LEMMA 5.2 (Macro error). *Let u^0 and u_H^0 be the solutions of problems* (3.2) *and* (5.1) *respectively. Assume that* (2.2) *and (H1) hold, and that $u^0 \in H^2(\omega_2)$. Then,*

$$\|u^0 - u_H^0\|_{L^2(\omega_2)} \leq CH^2,$$

where the constant C is independent of $H, \tilde{h}, h$, and ε, but depends on the stability constant of the bilinear form $B_{2,H}^0$.

Proof. See [7]. □

Micro and modeling Errors. For the micro and modeling errors, we follow [4, Section 5]. We assume the following regularity on $\psi_{K_\delta}^i$, the non-discretized micro solutions of problem (4.2), in $W(K_\delta)$

(H2) $|\psi_{K_\delta}^i|_{H^2(K_\delta)} \leq C\varepsilon^{-1}\sqrt{|K_\delta|}, \quad$ for $i = 1, \ldots, d$.

LEMMA 5.3 (Micro error). *Let u_H^0 be the solution of* (5.1) *and u^H be the DG-FE-HMM approximation of u^0. Assume that* (2.2) *holds, then*

$$|||u_H^0 - u^H|||_{\omega_2} \leq C \sup_{K \in \mathcal{T}_H} \|a_2^0(x_K) - a_2^{0,h}(x_K)\|_F |||u^H|||,$$

where the constant C is independent of H, $\tilde{h}$, h, and ε. Further, assuming (H2), the Frobenius norm is bounded by

$$\sup_{K \in \mathcal{T}_H} \|a_2^0(x_K) - a_2^{0,h}(x_K)\|_F \leq C\left(\frac{h}{\varepsilon}\right)^2 + e_{MOD}.$$

Proof. Follows from [4, Section 5]. □

REMARK 5.4. *Higher order micro error $\left(\frac{h}{\varepsilon}\right)^{2q}$ can be obtained for higher order micro FEM, provided higher order regularity of the micro functions,*

$$|\psi_{K_\delta}^i|_{H^{q+1}(K_\delta)} \leq C\varepsilon^{-q}\sqrt{|K_\delta|} \quad \textit{for } i = 1, \ldots, d.$$

The modeling error will depend on the choice of boundary condition on the micro problems.

LEMMA 5.5 (Modeling error). *The modeling error is given by*

$$e_{MOD} = \begin{cases} 0, & S^1(K_\delta, \mathcal{T}_h) \subset W_{per}^1(K_\delta), \delta/\varepsilon \in \mathbb{N}, \textit{ and collocation}, \\ C_1\delta, & S^1(K_\delta, \mathcal{T}_h) \subset W_{per}^1(K_\delta), \delta/\varepsilon \in \mathbb{N}, \\ C_2\frac{\varepsilon}{\delta}, & S^1(K_\delta, \mathcal{T}_h) \subset H_0^1(K_\delta), \delta/\varepsilon \notin \mathbb{N}, \textit{ and collocation}, \\ C_3\left(\delta + \frac{\varepsilon}{\delta}\right), & S^1(K_\delta, \mathcal{T}_h) \subset H_0^1(K_\delta), \delta/\varepsilon \notin \mathbb{N}. \end{cases}$$

Proof. see [1, 2]. □

5.1 A priori error estimates in the fine scale region In this section, we will prove Theorem 4.3.

Proof of Theorem 4.3. Let $u_{\tilde{h}} \in V_D^1(\omega_1, \mathcal{T}_{\tilde{h}})$ be the FE approximation of the physical solution u^ε over the mesh $\mathcal{T}_{\tilde{h}}$, i.e. $u_{\tilde{h}} = u_{1,0,\tilde{h}} + v_{1,\tilde{h}}(I^{\tilde{h}}\gamma_1(u))$, where $I^{\tilde{h}}$ is the Lagrange interpolant on Γ_1. Classical FE estimates hold

$$\|u^\varepsilon - u_{\tilde{h}}\|_{\mathrm{H}^1(\omega^+)} \leq C\tilde{h}^s |u^\varepsilon|_{H^{s+1}(\omega_1)},$$

where the constant C is independent of $H, h, \tilde{h}$, and ε. Applying a triangular inequality, we obtain

$$\|\nabla(u^\varepsilon - \bar{u}_{\tilde{h}H})\|_{\mathrm{L}^2(\omega^+)} \leq C\tilde{h}^s |u^\varepsilon|_{H^{s+1}(\omega_1)} + \|\nabla(u_{\tilde{h}} - \bar{u}_{\tilde{h}H})\|_{\mathrm{L}^2(\omega^+)}.$$

The numerical solution $\bar{u}_{\tilde{h}H}$ over ω^+ is equal to the numerical fine scale solution $u_{1,\tilde{h}}(\theta_{1,\tilde{h}})$, it holds

$$B_1(u_{\tilde{h}} - u_{1,\tilde{h}}, v_{\tilde{h}}) = 0, \qquad \forall v_{\tilde{h}} \in V_0^1(\omega_1, \mathcal{T}_{\tilde{h}}),$$

i.e., the difference $u_{\tilde{h}} - u_{1,\tilde{h}}(\theta_{1,\tilde{h}})$ is a-harmonic in ω_1 and thus the Caccioppoli inequality A.1 can be applied,

$$\|\nabla(u_{\tilde{h}} - u_{1,\tilde{h}}(\theta_{1,\tilde{h}}))\|_{\mathrm{L}^2(\omega^+)} \leq \frac{C}{(\tau - \tau^+)}\|u_{\tilde{h}} - u_{1,\tilde{h}}(\theta_{1,\tilde{h}})\|_{\mathrm{L}^2(\omega_1)},$$

where the constant $C > 0$ is independent of $H, \tilde{h}, h$, and ε, but depends on the ellipticity constants of the tensor a^ε. Consider an operator $P : V^1(\Gamma_1) \times V^1(\Gamma_2) \to V_D^1(\omega_1, \mathcal{T}_{\tilde{h}}) \times V_D^1(\Omega \setminus \omega_1, \mathcal{T}_H)$ defined as

$$P(\mu_{1,\tilde{h}}, \mu_{2,H}) = \begin{cases} u_{1,0,\tilde{h}} + v_{1,\tilde{h}}(\mu_{1,\tilde{h}}), & \text{in } \omega_1, \\ u_{2,0,H} + v_{2,H}(\mu_{2,H}), & \text{in } \Omega \setminus \omega_1. \end{cases}$$

As in the continuous case, we decompose the operator P as $P = U_0 + Q$. Over ω_1, it holds $u_{1,\tilde{h}}(\theta_{1,\tilde{h}}) = P(\theta_{1,\tilde{h}}, \theta_{2,H})$ and $u_{\tilde{h}} = P(I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon))$. Then,

$$\begin{aligned} \|u_{\tilde{h}} - u_{1,\tilde{h}}(\theta_{1,\tilde{h}})\|_{\mathrm{L}^2(\omega_1)} &= \|P(I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)) - P(\theta_{1,\tilde{h}}, \theta_{2,H})\|_{\mathrm{L}^2(\omega_1)} \\ &\leq \|Q\|\|(I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)) - (\theta_{1,\tilde{h}}, \theta_{2,H})\|_{\mathrm{L}^*(\mathcal{U})}. \end{aligned}$$

As $(\theta_{1,\tilde{h}}, \theta_{2,H})$ are the discrete optimal virtual controls, they satisfy

$$\begin{aligned} &\int_{\omega_0} \big(v_{1,\tilde{h}}(\theta_{1,\tilde{h}}) - v_{2,H}(\theta_{2,H})\big)\big(v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) - v_{2,H}(\mu_{2,H})\big)\mathrm{d}x \\ &= -\int_{\omega_0} \big(v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) - v_{2,H}(\mu_{2,H})\big)(u_{1,0,\tilde{h}} - u_{2,0,H})\mathrm{d}x, \end{aligned}$$

for all $(\mu_{1,\tilde{h}}, \mu_{2,H}) \in V^1(\Gamma_1) \times V^1(\Gamma_2)$. Then,

$$\begin{aligned} &\|(I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)) - (\theta_{1,\tilde{h}}, \theta_{2,H})\|_{\mathrm{L}^*(\mathcal{U})} \\ &= \sup_{(\mu_{1,\tilde{h}}, \mu_{2,H})} \frac{|\pi\big((I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)\big) - \pi\big((\theta_1, \theta_2), (\mu_{1,\tilde{h}}, \mu_{2,H})\big)|}{\|(\mu_{1,\tilde{h}}, \mu_{2,H})\|_{\mathrm{L}^*(\mathcal{U})}}, \end{aligned}$$

and following the proof of Lemma 3.3,

$$
\begin{aligned}
\pi\big((I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)\big) - \pi\big((\theta_1,\theta_2),(\mu_{1,\tilde{h}},\mu_{2,H})\big) \\
&= \int_{\omega_0} (u_{\tilde{h}} - u^H)\big(v_{1,\tilde{h}}(\mu_{1,\tilde{h}}) - v_{2,H}(\mu_{2,H})\big)\mathrm{d}x \\
&\le \|u_{\tilde{h}} - u^H\|_{\mathrm{L}^2(\omega_0)}\|(\mu_{1,\tilde{h}},\mu_{2,H})\|_{\mathrm{L}^*(\mathcal{U})},
\end{aligned}
$$

where $u^H = u_{2,0,H} + v_{2,H}(I^H\gamma_2(u))$. We obtain that

$$
\|(I^{\tilde{h}}\gamma_1(u^\varepsilon), I^H\gamma_2(u^\varepsilon)) - (\theta_{1,\tilde{h}},\theta_{2,H})\|_{\mathrm{L}^*(\mathcal{U})} \le \|u_{\tilde{h}} - u^H\|_{\mathrm{L}^2(\omega_0)},
$$

and summarizing, we have

$$
\|\nabla(u_{\tilde{h}} - u_{1,\tilde{h}}(\theta_{1,\tilde{h}}))\|_{\mathrm{L}^2(\omega^+)} \le C\|u_{\tilde{h}} - u^H\|_{\mathrm{L}^2(\omega_0)}.
$$

Then, we decompose the error into

$$
\|u_{\tilde{h}} - u^H\|_{\mathrm{L}^2(\omega_0)} \le \|u_{\tilde{h}} - u^\varepsilon\|_{\mathrm{L}^2(\omega_0)} + \|u^\varepsilon - u^0\|_{\mathrm{L}^2(\omega_0)} + \|u^0 - u^H\|_{\mathrm{L}^2(\omega_0)}, \tag{5.3}
$$

provided that the solutions u^ε and u^0 are smooth enough, standard FE estimates and (3.3) can be applied to bound the first two quantities in (5.3), i.e.,

$$
\|u_{\tilde{h}} - u^H\|_{\mathrm{L}^2(\omega_0)} \le C\tilde{h}^{s+1}|u^\varepsilon|_{H^{s+1}(\omega_1)} + C\varepsilon + \|u^0 - u^H\|_{\mathrm{L}^2(\omega_0)}.
$$

We bound the error in ω_0 by the error in ω_2

$$
\|u^0 - u^H\|_{\mathrm{L}^2(\omega_0)} \le \|u^0 - u^H\|_{\mathrm{L}^2(\omega_2)} \le \|u^0 - u^0_H\|_{\mathrm{L}^2(\omega_2)} + \|u^0_H - u^H\|_{\mathrm{L}^2(\omega_2)}.
$$

The two norms corresponds to the DG-FE-HMM error in the L^2 norm and are given by Lemmas 5.2, 5.3, and 5.5.

5.2 A priori error estimates in the scale separated region We prove an a priori error bound between u^ε and $\bar{u}_{\tilde{h}H}$ in $\Omega \setminus \omega^+$, where $\bar{u}_{\tilde{h}H}$ is defined in (4.12). For simplicity, we assume that $\delta = \varepsilon$ and choose periodic coupling conditions between the macro and micro problem. We recall that the reconstructed homogenized solution u_2^{rec}, and its numerical approximation $u_{2,H}^{rec}$, are given by

$$
u_2^{rec}(x) = u_2^0(x) + \varepsilon\sum_{j=1}^{d}\chi^j(x, x/\varepsilon)\frac{\partial u_2^0(x)}{\partial x_j}, \tag{5.4}
$$

$$
u_{2,H}^{rec}(x) = u_{2,H}(x) + \sum_{j=1}^{d}\psi_{K_\varepsilon}^{j,h}(x)\frac{\partial u_{2,H}(x)}{\partial x_j}, \tag{5.5}
$$

where $u_2^0 = u_2^0(\theta_2)$ and $u_{2,H} = u_{2,H}(\theta_2^H)$ are the exact solution and numerical solution of the coupling in ω_2, respectively, and $\psi_{K_\varepsilon}^{j,h}$ are the micro solutions of (4.2). We sometimes use $u_2^{rec}(\theta_2)$ and $u_{2,H}^{rec}(\theta_2^H)$ to emphasize the dependence on θ_2 and θ_2^H, respectively.

We introduce the discrete micro problems on K_ε; find u^h such that $u^h - u_{2,H} \in S^1(K_\varepsilon, \mathcal{T}_h)$ and

$$
\int_{K_\varepsilon} a_2^\varepsilon(x)\nabla v^h \cdot \nabla z^h \mathrm{d}x = 0, \qquad \forall z^h \in S^1(K_\varepsilon, \mathcal{T}_h). \tag{5.6}
$$

From assumption (H1), the tensor a_2^ε is constant in each macro element $K \in \mathcal{T}_H$. This simplifies the analysis as the modeling error is zero. We introduce a semi-discrete problem over ω_2: find $\bar{u}_{2,H} \in V_D^1(\omega_2, \mathcal{T}_H)$ the solution of

$$\begin{aligned} \bar{B}_{2,H}(\bar{u}_{2,H}, v_H) &= F_2(v_H), \qquad \forall v_H \in V^1(\omega_2, \mathcal{T}_H), \\ \bar{u}_{2,H} &= \theta_2^H, \text{ on } \Gamma_2, \end{aligned}$$

where the bilinear form $\bar{B}_{2,H}$ on $V^1(\omega_2, \mathcal{T}_H) \times V^1(\omega_2, \mathcal{T}_H)$ is given by

$$\begin{aligned} \bar{B}_{2,H}(v_H, w_H) = &\sum_{K \in \mathcal{T}_H} \frac{|K|}{|K_\varepsilon|} \int_{K_\varepsilon} a_2^\varepsilon(x) \nabla v \cdot \nabla w \mathrm{d}x + \sum_{e \in E} \int_e \mu_e [\![v_H]\!] [\![w_H]\!] \mathrm{d}s \\ &- \sum_{e \in E} \int_e \left(\overline{\{a_2^\varepsilon \nabla v\}} [\![w_H]\!] + \overline{\{a_2^\varepsilon \nabla w\}} [\![v_H]\!] \right) \mathrm{d}s. \end{aligned}$$

where v and w are solutions of (5.6) in the exact Sobolev space $W(K_\varepsilon)$.
For a vector valued function η, we define the average of the multiscale fluxes as

$$\overline{\{\eta\}} = \frac{1}{2} \left(\frac{1}{|K_\varepsilon^+|} \int_{K_\varepsilon^+} \eta_+ \mathrm{d}x + \frac{1}{|K_\varepsilon^-|} \int_{K_\varepsilon^-} \eta_- \mathrm{d}x \right).$$

We can then define $\bar{u}_{2,H}^{rec}$ by

$$\bar{u}_{2,H}^{rec}(x) = \bar{u}_{2,H}(x) + \sum_{j=1}^d \psi_{K_\varepsilon}^j(x) \frac{\partial \bar{u}_{2,H}(x)}{\partial x_j}, \qquad x \in K, \tag{5.7}$$

where $\bar{u}_{2,H} = \bar{u}_{2,H}(\theta_2^H)$. We use $\bar{u}_{2,H}^{rec}(\theta_2^H)$ to denote the dependence on θ_2^H.
We now give the proof of Theorem 4.4.
Proof of Theorem 4.4. We decompose the error into

$$\|u^\varepsilon - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} \leq \|u^\varepsilon - u_2^{rec}(\theta_2)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} + \|u_2^{rec}(\theta_2) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)}.$$

From Theorem 3.6, it holds that $\|u^\varepsilon - u_2^{rec}(\theta_2)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} \leq C_1 \varepsilon^{1/2}$. We focus on $\|u_2^{rec}(\theta_2) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)}$ and follow [2, Section 3.3.3]. Using the triangular inequality, we obtain

$$\begin{aligned} \|u^\varepsilon - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} &\leq C_1 \varepsilon^{1/2} + \|u_2^{rec}(\theta_2) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} \\ &\leq C_1 \varepsilon^{1/2} + \|u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{\mathrm{rec}}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} \\ &\quad + \|\bar{u}_{2,H}^{\mathrm{rec}}(\theta_2^H) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)}. \end{aligned}$$

Lemma 5.7 gives us

$$\begin{aligned} \|u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{\mathrm{rec}}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} &\leq C_3 H |u_2^0|_{H^2(\omega_2)} \\ &\quad + \frac{C_4}{\tau^+} \left(\tilde{h}^{s+1} |u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2 |u_2^0|_{H^2(\omega_2)} \right). \end{aligned}$$

Further, Lemma 5.9 provides us with

$$\|\bar{u}_{2,H}^{\mathrm{rec}}(\theta_2^H) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega \setminus \omega^+)} \leq C_2 \left(\frac{h}{\varepsilon} \right).$$

Collecting the previous results give

$$\|u_2^{rec}(\theta_2) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\backslash\omega^+)} \leq C_1\varepsilon^{1/2} + C_2\left(\frac{h}{\varepsilon}\right) + C_3 H|u_2^0|_{H^2(\omega_2)} + \frac{C_4}{\tau^+}\left(\tilde{h}^{s+1}|u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2|u_2^0|_{H^2(\omega_2)}\right).$$

Remark 5.6. *Theorem 4.4 can be adapted for general tensor $a_2^\varepsilon(x)$ without a two-scale structure. In that case, the modeling error is present in the last term of the error.*

Recall that we assumed periodic coupling with $\delta = \varepsilon$ and that (H1) and (H2) hold. Further, we assume Lipschitz continuity of the tensor in the first variable, i.e. $a_2(x, y) \in W^{1,\infty}(\overline{\omega}_2, L^\infty(Y))$.

Lemma 5.7. *Let $u_2^{rec}(\theta_2)$ and $\bar{u}_{2,H}^{rec}(\theta_2^H)$ be given by* (5.4) *and* (5.7)*. Assume that $u_2^0 \in H^2(\omega_2)$, $u_1^\varepsilon \in H^{s+1}(\omega_1)$, with $s \leq 1$, and that the exact solutions of the micro problem*(4.2) *verify (H2). Then*

$$\|u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\backslash\omega^+)} \leq C_1 H|u_2^0|_{H^2(\omega_2)} + \frac{C_2}{\tau^+}\left(\tilde{h}^{s+1}|u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2|u_2^0|_{H^2(\omega_2)}\right),$$

where the constants are independent of $H, \tilde{h}, h$, and ε.

Proof. Using the definitions of $u_2^{rec}(\theta_2)$ and $\bar{u}_{2,H}^{rec}(\theta_2^H)$, it holds

$$\|u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{rec}(\theta_2^H)\|^2_{\bar{\mathrm{H}}^1(\Omega\backslash\omega^+)} = \sum_{K\in\mathcal{T}_H(\Omega\backslash\omega^+)} \|\nabla(u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{rec}(\theta_2^H))\|^2_{\mathrm{L}^2(K)}$$

$$\leq \sum_{K\in\mathcal{T}_H(\Omega\backslash\omega^+)} \|\nabla(u_2^0 - \bar{u}_{2,H})\|^2_{\mathrm{L}^2(K)} + \sum_{K\in\mathcal{T}_H(\Omega\backslash\omega^+)} \|\sum_{j=1}^d \nabla\big(\varepsilon\chi^j(x, x/\varepsilon)\frac{\partial u_2^0}{\partial x_j} - \psi^j_{K_\varepsilon}(x)\frac{\partial \bar{u}_{2,H}}{\partial x_j}\big)\|^2_{\mathrm{L}^2(K)}.$$

Thanks to (H1), it holds $\varepsilon\chi^j(x, x/\varepsilon) = \psi^j_{K_\varepsilon}(x)$, then the second norm is zero. We recall the bilinear form (5.2) for the problem (2.3) with a quadrature formula,

$$B_{2,H}^0(v_H, w_H) = \sum_{K\in\mathcal{T}_H} |K|a_2^0(x_K)\nabla v_H\cdot\nabla w_H + \sum_{e\in E}\int_e \mu_e[\![v_H]\!][\![w_H]\!]\mathrm{d}s - \sum_{e\in E}\int_e \left(\{a_2^0(x_K, x/\varepsilon)v\}[\![w_H]\!] + \{a_2^0(x_K, x/\varepsilon)w\}[\![v_H]\!]\right)\mathrm{d}s,$$

and define $\hat{u}_{2,H}(\theta_2^H) \in V_D^1(\omega_2, \mathcal{T}_H)$ solution of

$$B_{2,H}^0(\hat{u}_{2,H}, w_H) = F_2(w_H), \qquad \forall w_H \in V_0^1(\omega_2, \mathcal{T}_H).$$

By [2, Proposition 14], it holds that $\bar{u}_{2,H} = \hat{u}_{2,H}$. By hypothesis $u_2^0(\theta_2)$ and $\bar{u}_{2,H}(\theta_2^H)$ have zero boundary conditions on $\partial\omega_2 \cap \partial\Omega$, and we can use [12, Lemmas 4.1, 4.2],

$$\|u_2^0(\theta_2) - \bar{u}_{2,H}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\backslash\omega^+)} \leq C_1 \inf_{w\in V_D^1(\omega_2,\mathcal{T}_H), w=I^H\theta_2 \text{ on } \Gamma_2} \|u_2^0(\theta_2) - w\|_{\bar{\mathrm{H}}^1(\omega_2)} + \frac{C_2}{\tau^+}\|u_2^0(\theta_2) - \bar{u}_{2,H}(\theta_2^H)\|_{\mathrm{L}^2(\omega_2)}.$$

The first norm can be bounded by

$$\inf_{w} \|u_2^0(\theta_2) - w\|_{\bar{\mathrm{H}}^1(\omega_2)} \le \|u_2^0(\theta_2) - u_{2,H}(I^H\theta_2)\|_{\bar{\mathrm{H}}^1(\omega_2)} \le C_1 H |u_2^0|_{H^2(\omega_2)},$$

where $u_{2,H}(I^H\theta_2)$ is the FEM solution with an interpolation of θ_2 on Γ_2. Following the proof of Theorem 4.3, the second part is bounded by

$$\begin{aligned}\|u_2^0(\theta_2) - \bar{u}_{2,H}(\theta_2^H)\|_{\mathrm{L}^2(\omega_2)} &\le \|u_2^0(\theta_2) - u_{2,H}(I^H\theta_2)\|_{\mathrm{L}^2(\omega_2)} \\ &\quad + \|u_{2,H}(I^H\theta_2) - \bar{u}_{2,H}(\theta_2^H)\|_{\mathrm{L}^2(\omega_2)} \\ &\le C_1 H^2 |u_2^0|_{H^2(\omega_2)} + \|Q(I^{\tilde{h}}\theta_1, I^H\theta_2) - Q(\theta_1^{\tilde{h}}, \theta_2^H)\|_{\mathrm{L}^2(\omega_2)} \\ &\le C_1 H^2 |u_2^0|_{H^2(\omega_2)} + C_2 \|u_{1,\tilde{h}}(I^{\tilde{h}}\theta_1) - \bar{u}_{2,H}(I^H\theta_2)\|_{\mathrm{L}^2(\omega_0)},\end{aligned}$$

where we have used that $(\theta_1^{\tilde{h}}, \theta_2^H)$ is the optimal couple of the discrete minimization problem and that Q is bounded. Finally using the triangular inequality, we have

$$\begin{aligned}\|u_{1,\tilde{h}}(I^{\tilde{h}}\theta_1) - \bar{u}_{2,H}(I^H\theta_2)\|_{\mathrm{L}^2(\omega_0)} &\le \|u_{1,\tilde{h}}(I^{\tilde{h}}\theta_1) - u_1^\varepsilon(\theta_1)\|_{\mathrm{L}^2(\omega_0)} \\ &\quad + \|u_1^\varepsilon(\theta_1) - u_2^0(\theta_2)\|_{\mathrm{L}^2(\omega_0)} \\ &\quad + \|u_2^0(\theta_2) - \bar{u}_{2,H}(I^H\theta_2)\|_{\mathrm{L}^2(\omega_0)} \\ &\le C\left(\tilde{h}^{s+1}|u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2|u_2^0|_{H^2(\omega_2)}\right).\end{aligned}$$

Summarizing,

$$\begin{aligned}\|u_2^{rec}(\theta_2) - \bar{u}_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\setminus\omega^+)} &\le C_1 H |u_2^0|_{H^2(\omega_2)} \\ &\quad + \frac{C_2}{\tau^+}\left(\tilde{h}^{s+1}|u_1^\varepsilon|_{H^{s+1}(\omega_1)} + \varepsilon + H^2|u_2^0|_{H^2(\omega_2)}\right).\end{aligned}$$

The result of the Lemma follows. □

REMARK 5.8. *The proof of Lemma 5.7, can be generalized for functions with non homogeneous boundary conditions. This can be done by splitting the solutions into a function depending of the controls and a function independent of the controls. The proof follows the same lines.*

LEMMA 5.9. *Let $\bar{u}_{2,H}^{rec}(\theta_2^H)$ and $u_{2,H}^{rec}(\theta_2^H)$ be defined by* (5.7) *and* (5.5)*, respectively. Then*

$$\|\bar{u}_{2,H}^{rec}(\theta_2^H) - u_{2,H}^{rec}(\theta_2^H)\|_{\bar{\mathrm{H}}^1(\Omega\setminus\omega^+)} \le C\left(\frac{h}{\varepsilon}\right).$$

Proof. Follows from [2, Section 3.3.3]. □

6 Numerical experiments In this section we present various numerical experiments to illustrate the convergence rates and the performance of our method. In the first two examples, we compare our coupling method with the classical global-local method [34], where the homogenized solution is considered as the boundary condition on Γ_1. To facilitate the numerical comparison, we assume that the meshes $\mathcal{T}_{\tilde{h}}$ and $\mathcal{T}_H$ have the same triangulation in the overlap ω_0. The implementations can be adapted to the case where the meshes are not equal in ω_0, using interpolations between the two meshes.

6.1 A domain with a crack. Consider an elliptic boundary value problem in $\Omega = [0,1]^2$,

$$-\text{div}\,(a^\varepsilon \nabla u) = 0, \text{ in } \Omega,$$

with Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$, where $\varphi \in [0, 2\pi]$ is the angle measured counterclockwise from the axis $\{(x, 0.5) : x \geq 0\}$. We add free Neumann boundary condition on the crack $\{x \in \Omega : x_1 \geq 0, x_2 = 0.5\}$. The homogenization model might not be accurate around the crack. A mesh refinement of the coarse model around the crack may lead to coarse meshes with mesh size smaller than ε, hence it requires more work around the crack than the FEM with scale resolution. For the treatment of crack problem with the FE-HMM, we cite [6]. We take a tensor a^ε – represented in Figure 2(a) for $\varepsilon = 1/10$ – with separation of scale and locally periodic in Y,

$$a^\varepsilon(x_1, x_2) = \left(\frac{1}{(1.1 + \cos(2\pi x_1/\varepsilon))^2} + \frac{1}{(1.1 + \cos(2\pi x_2/\varepsilon))^2} \right)^{1/2}.$$

Let $x_c = [1/2, 1/2]$ be the center of Ω. The classical global-local numerical solution is the approximation of the following problem; consider $\omega_1 = x_c + \frac{1}{15}[-1, 1]^2$ and $\Gamma_1 = \partial\omega_1$,

$$\begin{aligned} -\text{div}\,(a^\varepsilon \nabla u) &= f, && \text{in } \omega_1, \\ u &= u^0, && \text{on } \Gamma_1, \end{aligned} \tag{6.1}$$

where u^0 is the homogenized solution. Recall that $\omega \subset \omega_1$ where $\omega = x_c + \frac{1}{30}[-1, 1]^2$. We compute the numerical homogenized solution u^H over Ω on the coarse initial mesh, and use the value of u^H as Dirichlet boundary condition on Γ_1 and solve problem (6.1) with a fine scale FEM.

We refine uniformly in ω_1 and as the mesh size in ω should be small enough to capture the microscopic scales of the problems, it would be prohibitive to compute the numerical homogenized solution at each iteration. The coupling and the classical global-local method are both performed on the same mesh, where the coarse mesh in $\Omega \setminus \omega_1$ is left unchanged. We then compare the numerical solution with a reference solution obtained with a FEM on a very fine mesh. The reference solution is shown in figure 2(d) and the numerical optimization based coupling solution in figure 2(c). We plot the H^1 semi-norm for the two methods in figure 2(b). We see that the global-local method reaches a threshold value, as expected due to the use of the numerical homogenized function u^H as Dirichlet data on Γ_1.

6.2 Singular source term. In this experiment, we consider an elliptic problem with a singular source term given by random peaks. The tensor is assumed to have scale-separation and be Y-periodic,

$$a^\varepsilon(x) = \frac{1}{6} \left(\frac{1.1 + \sin(2\pi x_1/\varepsilon x_2/\varepsilon)}{1.1 + \sin(2\pi x_2/\varepsilon)} + \sin(4x_1^2 x_2^2) + 2 \right).$$

Depending on the location of the random peaks, the numerical homogenized right-had side f^0 can be wrong, leading to an inaccurate approximation of u^0. As in the crack experiments, we compute a numerical approximation of u^0 on a coarse initial mesh and then use it as boundary condition on Γ_1. In figure 3(a) we show the tensor for $\varepsilon = 1/25$. Let $x_c = [1/2, 1/2]$ be the center of Ω, we set $\omega = x_c + \frac{1}{12}[-1, 1]^2$ and

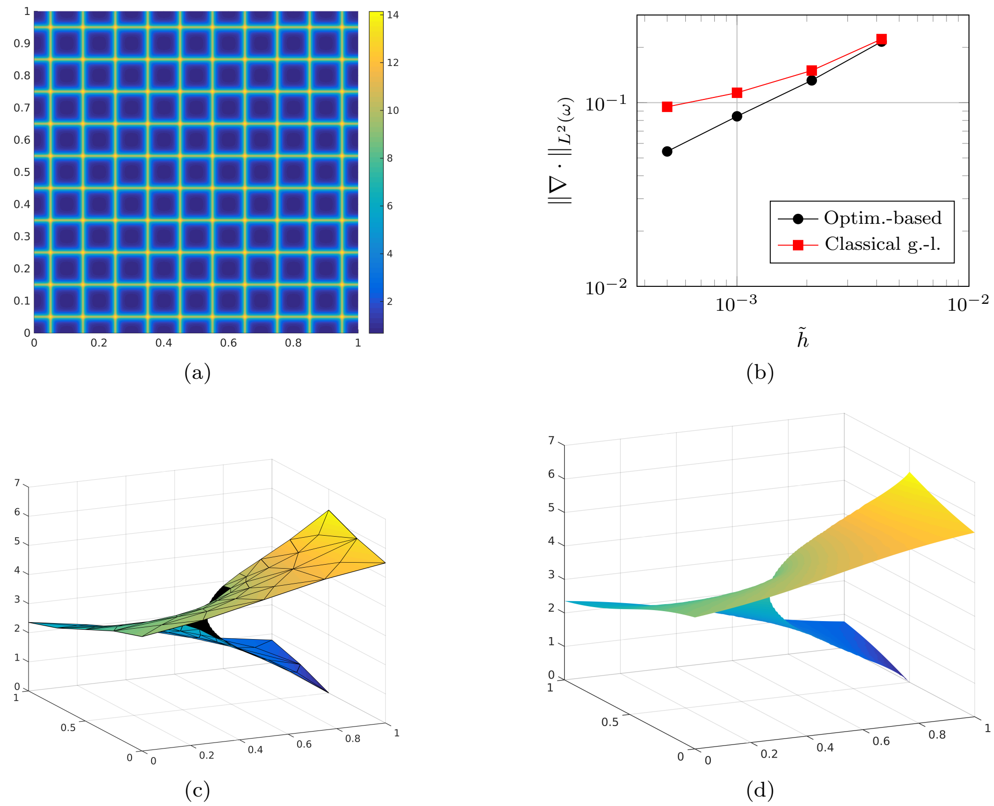


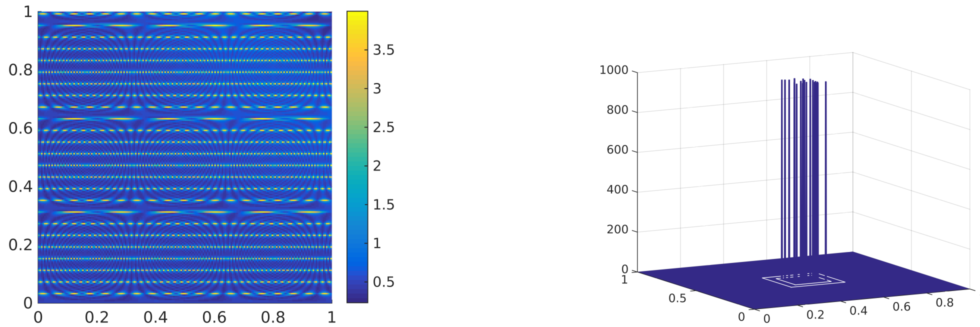
FIGURE 2. *Crack experiment: (a) tensor for $\varepsilon = 1/10$, (b) H^1 semi-norm in ω, for the optimization based coupling (black) and the classical coupling (red), (c) numerical optimization-based solution, and (d) reference solution.*

$\omega_1 = x_c + \frac{1}{4}[-1,1]^2$. In Figure 3(b), we illustrate the random source term f with 20 peaks. Figures 4(a) and 4(b) illustrate the reference solution and the optimization based solutions with the fine scale solution in ω and the coarse scale solution in $\Omega \setminus \omega$. The H^1 error to the reference solution, for $\varepsilon = 1/10$ and 100 random peaks, is shown in figure 5, for the classical global-local method (in red) and the coupling (in black). While we observe a linear convergence rate for the optimization based method as predicted by Theorem 4.3, we see that the classical coupling leads to saturation in the error decay. This is due to inaccurate boundary conditions for the fine scale problems.

6.3 A domain with a defect. We consider an homogenization problem with a local perturbation in the tensor, treated in [10]. The PDE is

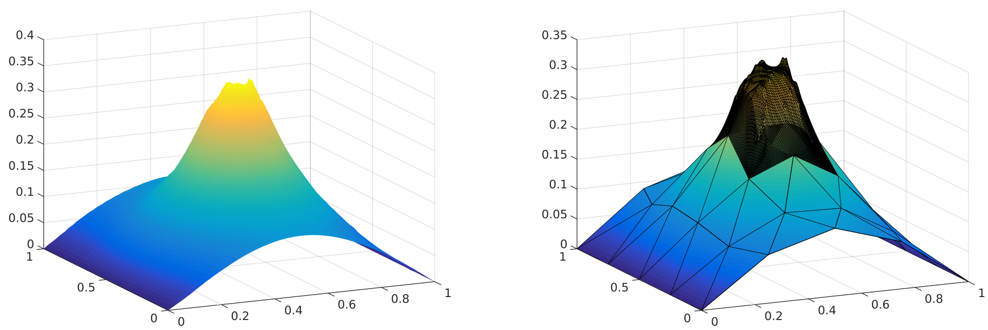
$$\begin{aligned} -\mathrm{div}\,(a^\varepsilon \nabla u^\varepsilon) &= f, \text{ in } \Omega, \\ u^\varepsilon &= 0, \text{ on } \partial\Omega, \end{aligned}$$

where the tensor is of the form $a^\varepsilon = a^\varepsilon_{per}(x) + b^\varepsilon(x)$, with $a^\varepsilon_{per}(x) = a_{per}(x, x/\varepsilon)$ is (locally) periodic and $b^\varepsilon \in L^2(\Omega)^2$ is a local perturbation of size ε. A numerical homogenized solution u^H can be obtained with FE-HMM and produces a good approximation of u^ε in the L^2 norm. To obtain good approximation in the H^1 norm



(a) (b)

FIGURE 3. *Singular source term experiment: (a) tensor for* $\varepsilon = 1/25$*, (b) right hand side with* 20 *random peaks.*



(a) (b)

FIGURE 4. *Singular source term experiment: (a) reference solution, (b) optimization based numerical solution.*

one needs to add correctors. However, the usual periodic cell problems are not valid as a_2^ε is not periodic. One could compute the periodic correctors corresponding to the tensor a_{per}^ε, and use them to correct the homogenized solution. This will be a good approximation at the large scale but will fail at the fine scale close to the defect. Following the approach in [10], a new corrector can be computed by adding a term to the periodic correctors as follows. Let $\chi^j \in W_{per}^1(Y)$ be the classical periodic correctors that satisfy the cell problems

$$\int_Y a_{per}^\varepsilon(x)\nabla\chi^j \cdot \nabla v \mathrm{d}y = -\int_Y a_{per}^\varepsilon(x) e_j \nabla v \mathrm{d}y, \qquad \forall v \in W_{per}^1(Y).$$

Then, the additional term will be the solution of a Dirichlet boundary value problem in $K_n = [-n\varepsilon, n\varepsilon]^2$, where n is large enough so that the effect of the defect are negligible at the boundary of K_n. The problem reads: find $\chi_b^j \in H_0^1(K_n)$

$$\int_{K_n} a^\varepsilon(x)\nabla\chi_b^j \cdot \nabla v \mathrm{d}x = -\int_{K_n} b^\varepsilon(x)(e_j + \nabla\chi^j)\cdot\nabla v \mathrm{d}x, \qquad \forall v \in H_0^1(K_n).$$

On can extend χ^j periodically to K_n and obtain a corrector $\tilde{\chi}^j(x) = \chi^j(x) + \chi_b^j(x)$ for all $x \in K_n$.

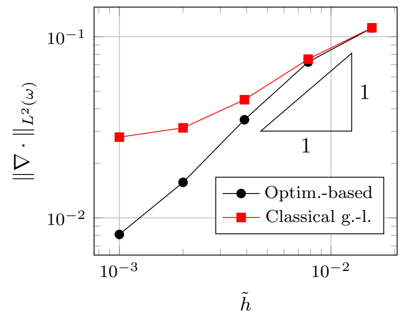


FIGURE 5. *Singular source term experiment: H^1 semi-norm in ω for the optimization based coupling (black) and the classical coupling (red).*

In this numerical example, we compute the FE-HMM solution and add to it either the periodic correctors χ or the modified correctors $\tilde{\chi}$. We then compare these two solutions with the optimization based solution presented in this paper. We will take the same oscillatory data as given in[10, Section 4.]. Let $\Omega = [-1,1]^2$ and define

$$\begin{aligned} a^{\varepsilon}_{per}(x_1,x_2) &= 3+\cos(2\pi x_1/\varepsilon)+\sin(2\pi x_2/\varepsilon),\\ b^{\varepsilon}(x_1,x_2) &= 10\exp(-(x_1^2/\varepsilon^2+x_2^2/\varepsilon^2)),\\ f(x_1,x_2) &= \sin(\pi x_1)\cos(\pi x_2). \end{aligned}$$

We use a uniform triangular mesh and compute a reference solution on a very fine mesh. We compute the periodic correctors on $\mathcal{T}_h(Y)$ and extend it to $[-n\varepsilon, n\varepsilon]^2$ where n is sufficiently large. The terms χ_b are then computed on $[-n\varepsilon, n\varepsilon]^2$ with Dirichlet boundary conditions and adding them to χ, we obtain the non periodic correctors $\tilde{\chi}$. In each macro element K we defined a mesh $\mathcal{T}_{h_{rec}}(K)$, obtained by uniform refinement of K until the mesh size h_{rec} is smaller or equal to h. The two reconstructed solutions read

$$\begin{aligned} u^{\varepsilon,rec}_H(x) &= u^H(x)+\sum_{i=1}^{d}\varepsilon\chi^{i,h}(x,x/\varepsilon)\frac{\partial u^H(x)}{\partial x_i},\\ \tilde{u}^{\varepsilon,rec}_H(x) &= u^H(x)+\sum_{i=1}^{d}\varepsilon\tilde{\chi}^{i,h}(x,x/\varepsilon)\frac{\partial u^H(x)}{\partial x_i}, \end{aligned}$$

where both correctors are defined on $[-n\varepsilon, n\varepsilon]^2$ with mesh size h and interpolated to $\mathcal{T}_{h_K}(K)$. In the coupling method, the fine scale region ω_1 will be centered around the defect, as its size is ε, we set $\omega = [-1/4, 1/4]^2$ and $\omega_1 = [-1/2, 1/2]^2$. The mesh size in ω_1 is equal to h_{rec} and the mesh size in the coarse region $\Omega \setminus \omega_1$ is H. We recall that the fine scale reference solution is given by

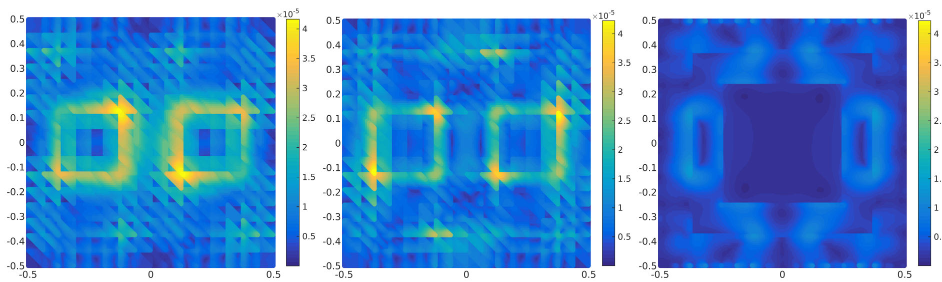
$$\bar{u}_{\tilde{h}H} = \begin{cases} u_{1,\tilde{h}}, & \text{in } \omega_+,\\ u^{rec}_{2,H}, & \text{in } \Omega\setminus\omega_+, \end{cases}$$

where we have chosen $\omega_+ = [-3/8, 3/8]^2$. We compute the error between the reference solution and the numerical solutions $u^{\varepsilon,rec}_H, \tilde{u}^{\varepsilon,rec}_H$, and $\bar{u}_{\tilde{h}H}$ in ω_1 and in $[-\varepsilon,\varepsilon]^2$. We first take $\varepsilon = 1/5$, $H = 1/16$ and a micro degree of freedom of $N_{micro} = \frac{1}{32^2}$.

	Method	Rel. error in ω_1	Rel. error in $[-\varepsilon, \varepsilon]^2$
$\varepsilon = 1/5$	periodic correctors	0.436	1.589
	non-periodic correctors	0.396	0.992
	optimization based coupling	0.119	0.030
$\varepsilon = 1/10$	periodic correctors	0.281	1.076
	non-periodic correctors	0.260	0.720
	optimization based coupling	0.039	0.006

TABLE 1

Relative error in ω_1 and $[-\varepsilon, \varepsilon]^2$, with $\varepsilon = 1/5$ and $\varepsilon = 1/10$, between the reference solution and the periodic, non-periodic reconstructed solution, and for the optimization based solution.



(a) (b) (c)

FIGURE 6. *Error in ω_1 between the reference solution and the numerical fine scale solution obtained with periodic correctors (a), non periodic correctors (b), and by the coupling (c).*

We look at the relative error between the reference solution and the reconstructed solution $u_H^{\varepsilon,rec}$ (resp. $\tilde{u}_H^{\varepsilon,rec}$) for the periodic correctors (resp. non periodic),

$$\frac{\|\nabla(u^\varepsilon - u_H^{\varepsilon,rec})\|_{L^2(\omega_1)}}{\|\nabla u^\varepsilon\|_{L^2(\omega_1)}}.$$

As expected (see e.g. [10]), the errors with the two reconstructed solutions are similar in the far field, and one should look at the error around the defects to see the advantage of the correctors $\tilde{\chi}$. In Table 1, we see the relative errors for the three methods for $\varepsilon = 1/5$ and $\varepsilon = 1/10$. In Figure 6, we display the error in ω_1 between the reference solution and the numerical fine scale solutions obtained with the periodic correctors 6(a), non-periodic correctors 6(b), and the optimization based method 6(c). While the errors between the periodic and non-periodic methods are similar in ω_1, the difference is more important in $[-\varepsilon, \varepsilon]^2$, near the defect. There is however a significant improvement when the optimization based coupling method is used. This is to be expected as a fine scale solver is used in ω_1 and is coupled with a coarse scale solver. The strength of the method is that it produces a good H^1 approximation of the fine scale solution on Ω, but allows for a large mesh size H in $\Omega \setminus \omega_1$. We note that in [10], the same macro and micro degrees of freedom were used, with macro mesh size of 1/1000 leading to a smaller discretization error and a larger difference between the periodic correctors and the non-periodic correctors. Setting H to such a small value is not necessary in our experiments as we only need a fine mesh in ω_1 and want to take full advantage of the homogenization techniques in the region with scale separation.

Acknowledgments. The work of A. Abdulle and O. Jecker was supported in part by the Swiss National Science Foundation.

Appendix. Inequalities.

A.1 Continuous inequalities Let us start by recalling the Caccioppoli inequality [23]. Let $\omega \subset \omega_1$ be subdomains of Ω with $\tau = \mathrm{dist}(\partial\omega, \partial\omega_1)$ and set $\Gamma = \partial\Omega$. For a tensor a, we define the set of a-harmonic functions by $\mathcal{H}(\omega_1)$, which consists of functions $u \in L^2(\omega_1) \cap H^1_{\mathrm{loc}}(\omega_1)$ such that

$$B_1(u, v) = \int_{\omega_1} a\nabla u \cdot \nabla v \mathrm{d}x = 0, \qquad \forall v \in \mathcal{C}_0^\infty(\omega_1),$$

where H^1_{loc} is defined by

$$H^1_{\mathrm{loc}}(\omega_1) := \{u \in H^1(O) \mid \text{ for any open set } O \text{ with } \overline{O} \subset \omega_1\}.$$

If the domains have shared boundaries, i.e., $\partial\omega_1 \cap \Gamma \neq \emptyset$, we construct the space of a-harmonic functions by $\mathcal{H}_0(\omega_1)$, which consists of functions $u \in \mathcal{H}(\omega_1)$ with zero boundary condition on $\partial\omega_1 \cap \Gamma$. We recall that $\Gamma_1 = \partial\omega_1 \setminus \Gamma$.

Theorem A.1 (Caccioppoli inequality [23]). *Let $u \in \mathcal{H}(\omega_1)$, then*

$$\|\nabla u\|_{\mathrm{L}^2(\omega)} \leq \frac{2\beta^{1/2}}{\alpha^{1/2}\tau}\|u\|_{\mathrm{L}^2(\omega_1)}.$$

We note that elliptic problem with a non null right hand side can also be considered and we refer to [23] for details. We generate next the above result in order to have only the overlapping domain in the right hand side.

Lemma A.2. *Let $\omega_0 = \omega_1 \setminus \overline{\omega}$. Let $u \in \mathcal{H}(\omega_1)$, then*

$$\|\nabla u\|_{\mathrm{L}^2(\omega)} \leq \frac{2\beta^{1/2}}{\alpha^{1/2}\tau}\|u\|_{\mathrm{L}^2(\omega_0)}.$$

Proof. Let $\eta \in \mathcal{C}^1_0(\omega_1)$ be a cutoff function with $\eta = 1$ in $\overline{\omega}$, $\eta = 0$ in $\partial\omega_1$, and $|\nabla\eta| \leq 1/\tau$. Further, $\eta = 0$ on Γ_1 and $\mathrm{supp}(\nabla\eta) \subset \omega_0$. Then, it holds that $\eta^2 u \in H^1_0(\omega_1)$ and

$$\int_{\omega_1} a\nabla u \cdot \nabla(\eta^2 u)\mathrm{d}x = 0.$$

Then,

$$0 = \int_{\omega_1} a\nabla u \cdot \nabla(\eta^2 u)\mathrm{d}x = 2\int_{\omega_1} a\nabla u \cdot \nabla\eta\eta u\mathrm{d}x + \int_{\omega_1} a\nabla u \cdot \nabla u\eta^2\mathrm{d}x.$$

By definition, it holds

$$\alpha\|\nabla u\|_{\mathrm{L}^2(\omega)} \leq \int_{\omega_1} a\nabla(\eta u) \cdot \nabla(\eta u)\mathrm{d}x,$$

and

$$\begin{aligned}\int_{\omega_1} a\nabla(\eta u)\cdot\nabla(\eta u)\mathrm{d}x &= \int_{\omega_1} a\nabla(\eta u)\cdot\nabla(\eta u)\mathrm{d}x - \int_{\omega_1} a\nabla u\cdot\nabla(\eta^2 u)\mathrm{d}x \\ &= \int_{\omega_1} a\nabla(\eta u)\cdot\nabla(\eta u)\mathrm{d}x - 2\int_{\omega_1} a\nabla u\cdot\nabla\eta\eta u\mathrm{d}x - \int_{\omega_1} a\nabla u\cdot\nabla u\eta^2\mathrm{d}x \\ &= \int_{\omega_1} a\nabla\eta\cdot\nabla\eta u^2\mathrm{d}x \\ &= \int_{\omega_0} a\nabla\eta\cdot\nabla\eta u^2\mathrm{d}x \\ &\leq \frac{\beta}{\tau^2}\int_{\omega_0} u^2\mathrm{d}x = \frac{\beta}{\tau^2}\|u\|^2_{\mathrm{L}^2(\omega_0)}.\end{aligned}$$

□

In the next lemma, we prove a strong version of the Cauchy-Schwarz inequality.

We recall the problems for the state variables: find $v_i \in H^1_D(\omega_i)$ solution of

$$\text{(A.1)}\qquad \begin{aligned} -\mathrm{div}\,(a_i\nabla v_i) &= 0, && \text{in } \omega_i,\\ v_i &= \theta_i, && \text{on } \Gamma_i,\\ v_i &= 0, && \text{on } \partial\omega_i\cup\Gamma_D,\\ n_i\cdot(a_i\nabla v_i) &= 0, && \text{on } \partial\omega_i\cap\Gamma_N,\end{aligned}$$

where $a_1 = a_1^\varepsilon$ and $a_2 = a_2^0$.

Lemma A.3 (Strong Cauchy-Schwarz). *Let $v_1^\varepsilon \in H^1_D(\omega_1)$ and $v_2^0 \in H^1_D(\omega_2)$ be solutions of (A.1) for $i = 1, 2$. Then, there exist an $\varepsilon_0 > 0$ and a positive constant $C_s < 1$ such that for all $\varepsilon \leq \varepsilon_0$, it holds*

$$\int_{\omega_0} v_1^\varepsilon v_2^0\mathrm{d}x \leq C_s\|v_1^\varepsilon\|_{\mathrm{L}^2(\omega_0)}\|v_2^0\|_{\mathrm{L}^2(\omega_0)}.$$

Proof. We reason by contradiction. Suppose that there exist a sequence of $\{\varepsilon_n\}_{n\geq 1}$ that tends to zero such that

$$\int_{\omega_0} v_1^{\varepsilon_n} v_2^0\mathrm{d}x > C_n\|v_1^{\varepsilon_n}\|_{\mathrm{L}^2(\omega_0)}\|v_2^0\|_{\mathrm{L}^2(\omega_0)}, \qquad \forall n\geq 1,$$

for all sequence $\{C_n\}_{n\geq 1}$ that tends to 1, with $C_n < 1$. Without loss of generality, we can normalize the vectors $v_1^{\varepsilon_n}$ and v_2, and obtain

$$\|v_1^{\varepsilon_n}\|_{\mathrm{L}^2(\omega_0)} = 1, \quad \|v_2^0\|_{\mathrm{L}^2(\omega_0)} = 1 \quad \text{and} \quad (v_1^{\varepsilon_n}, v_2^0)_{\mathrm{L}^2(\omega_0)} := \int_{\omega_0} v_1^{\varepsilon_n} v_2^0\mathrm{d}x \to 1.$$

As the sequence of tensors $\{a_1^{\varepsilon_n}\}_{n\geq 1} \in (L^\infty(\omega_1))^{d\times d}$ is bounded, and uniformly elliptic, by the H-convergence, there exists a subsequence of $\{\varepsilon_n\}_{n\geq 1}$ still denoted by $\{\varepsilon_n\}_{n\geq 1}$ and a tensor $a_1^0 \in (L^\infty(\omega_1))^{d\times d}$ bounded, and uniformly elliptic such that $\{a_1^{\varepsilon_n}\}_{n\geq 1}$ H-converge to a_1^0. By definition of the H-convergence, the solution $v_1^{\varepsilon_n}$ of (A.1) — for the subsequence $\{\varepsilon_n\}$ — is such that

i) $v_1^{\varepsilon_n} \rightharpoonup v_1^0$ in $H^1(\omega_1)$ and,
ii) $a_1^{\varepsilon_n}\nabla v_1^{\varepsilon_n} \rightharpoonup a_1^0\nabla v_1^0$ in $L^2(\omega_1)^d$,

where v_1^0 is the unique solution of

$$\begin{aligned} -\mathrm{div}\,(a_1^0\nabla v_1^0) &= 0, && \text{in } \omega_1,\\ v_1^0 &= \theta_1, && \text{on } \Gamma_1,\\ v_1^0 &= 0, && \text{on } \partial\omega_1 \cup \Gamma_D,\\ n_1\cdot(a_1^0\nabla v_1^0) &= 0, && \text{on } \partial\omega_1 \cap \Gamma_N. \end{aligned}$$

As $H^1(\omega_1)$ is compactly embedded in $L^2(\omega_1)$, strong convergence in L^2 of $v_1^{\varepsilon_n}$ to v_1^0, for a subsequence of $\{\varepsilon_n\}_{n\geq 1}$, is achieved, i.e.,

$$v_1^{\varepsilon_n} \to v_1^0 \text{ in } L^2(\omega_1).$$

By the continuity of the norm, we have that

$$\lim_{n\to\infty}(v_1^{\varepsilon_n}, v_2)_{\mathrm{L}^2(\omega_0)} = (v_1^0, v_2)_{\mathrm{L}^2(\omega_0)}, \qquad \|v_1^0\|_{\mathrm{L}^2(\omega_0)} \leq 1 \quad \text{and} \quad (v_1^0, v_2)_{\mathrm{L}^2(\omega_0)} = 1.$$

As

$$1 = (v_1^0, v_2)_{\mathrm{L}^2(\omega_0)} \leq \|v_1^0\|_{\mathrm{L}^2(\omega_0)}\|v_2\|_{\mathrm{L}^2(\omega_0)} \leq 1,$$

we must have that $\|v_1^0\|_{\mathrm{L}^2(\omega_0)}\|v_2\|_{\mathrm{L}^2(\omega_0)} = 1$ and hence $\|v_1^0\|_{\mathrm{L}^2(\omega_0)} = 1$. The previous inequalities become equalities, i.e.,

$$1 = (v_1^0, v_2)_{\mathrm{L}^2(\omega_0)} = \|v_1^0\|_{\mathrm{L}^2(\omega_0)}\|v_2\|_{\mathrm{L}^2(\omega_0)}.$$

An equality in Cauchy-Schwarz is possible if and only if v_1^0 and v_2 are linearly dependent, that is there exist a constant $c > 0$ such that $v_1^0 = cv_2$ a.e. in ω_0. As the norms of v_1^0 and v_2 are equal to 1, we can easily conclude that $c = \pm 1$ and that $v_1^0 = \pm v_2$ a.e. in ω_0. Finally, as $(v_1^0, v_2)_{\mathrm{L}^2(\omega_0)} = 1$ it holds that $v_1^0 = v_2$.
Both v_1^0 and v_2 are solutions of a homogenized equation and are equal on the overlap, so we can combine them into an homogenized solution on the entire domain Ω. Further, the tensor a_2^0 and a_1^0 are equals in ω_0. Indeed, let us continuously extend the tensors a_2^ε and a_1^ε to the domain Ω. The tensor a_1^ε H-converge to the tensor a_1^0 and the tensor a_2^ε H-converge to a_2^0, in Ω. It holds that $a_2^\varepsilon = a_1^\varepsilon$ in ω_0, and using the locality of H-convergence [32, 14], we can conclude that $a_2^0 = a_1^0$ in ω_2. Thus they are equal in the overlap.
Let us split ω_0 into two disjoint rings $\omega_0 = \omega_0^1 \cup \omega_0^2$. As the solutions v_1^0 and v_2 are equal in ω_0, we can construct a smooth function $\bar{v}$ over Ω as

$$\bar{v}(x) = \begin{cases} v_1^0(x), & \text{if } x \in \omega \cup \omega_0^1,\\ v_2(x), & \text{if } x \in \omega_2 \setminus \omega_0^1. \end{cases}$$

The function $\bar{v}$ is in $H_D^1(\Omega)$, has zero Neumann boundary condition on Γ_N, and satisfies

$$\int_\Omega \bar{a}^0\nabla\bar{v}\cdot\nabla w\mathrm{d}x = 0, \qquad \forall w \in H_D^1(\Omega),$$

where the tensor $\bar{a}^0$ is given by

$$\bar{a}^0 = \begin{cases} a_1^0 & \text{in } \omega \cup \omega_0^1,\\ a_2^0 & \text{in } \omega_2 \setminus \omega_0^1. \end{cases}$$

The solution $\bar{v}$ must be zero everywhere in Ω, i.e., $\bar{v} \equiv 0$, which is a contradiction with $\|\bar{v}\|_{\mathrm{L}^2(\omega_0)} = 1$. $\square$

A.2 Discrete inequalities. Let $\omega \subset \omega_1 \subset \Omega$, with $\tau = \mathrm{dist}(\partial\omega_1, \partial\omega)$ and consider a partition $\mathcal{T}_h$ of Ω in simplicial or quadrilateral elements K, with diameter h_K and denote $h = \max_{K\in\mathcal{T}_h} h_K$. Further, we assume that h is smaller than τ and that $\mathcal{T}_h$ is admissible (T1) and shape regular (T2). The inequalities are given for general FE spaces of degree $p \geq 1$.
We give a discrete Caccioppoli inequality for functions $v^h \in V^p(\omega_1, \mathcal{T}_h)$ solution of

$$B_1(v^h, w^h) := \int_{\omega_1} a\nabla v^h \cdot \nabla w^h \mathrm{d}x = 0, \quad \forall w^h \in V_0^p(\omega_1, \mathcal{T}_h). \tag{A.2}$$

Let us denote by I_h the Lagrange interpolant, and state a super approximation useful in the proof of the discrete Caccioppoli inequality.

Lemma A.4. *Let $\eta \in \mathcal{C}^1(\omega_1)$ with $|\nabla\eta| \leq C\tau^{-1}$. Then for each $v^h \in V^p(\omega_1, \mathcal{T}_h)$ and $K \in \mathcal{T}_h$, with $h_K \leq \tau$, it holds,*

$$\|\eta^2 v^h - I_h(\eta^2 v^h)\|_{\mathrm{H}^1(\omega_1)} \leq C\left(\frac{h_K}{\tau}\|\nabla(\eta v^h)\|_{\mathrm{L}^2(K)} + \frac{h_K}{\tau^2}\|v^h\|_{\mathrm{L}^2(K)}\right).$$

Proof. See [15, Theorem 2.1]. □
We recall that local inverse inequalities are valid for functions $v^h \in V^p(\omega_1, \mathcal{T}_h)$; that is

$$\|\nabla v^h\|_{\mathrm{L}^2(K)} \leq C h_K^{-1}\|v^h\|_{\mathrm{L}^2(K)}, \tag{A.3}$$

where the constant C is independent of h_K.

Lemma A.5 (Discrete Caccioppoli inequality for interior domains, [33]). *Let $v^h \in V^p(\omega_1, \mathcal{T}_h)$ satisfy equation* (A.2) *for all $w^h \in V_0^p(\omega_1, \mathcal{T}_h)$; it holds*

$$\|\nabla v^h\|_{\mathrm{L}^2(\omega)} \leq C\frac{1}{\tau}\|v^h\|_{\mathrm{L}^2(\omega_1)},$$

where the constant C is independent of h.

Proof. Let $\eta \in \mathcal{C}_0^1(\omega_1)$ be a cutoff function with $|\nabla\eta| \leq C\tau^{-1}$. We have that η satisfies $\eta \equiv 0$ in $\Omega \setminus \omega_1$, $\eta \equiv 1$ in ω, and $|\nabla\eta| \leq 1/\tau$ for points in ω_0. By the uniform ellipticity of the tensor a, it holds

$$\alpha\|\nabla v^h\|^2_{\mathrm{L}^2(\omega)} \leq \int_{\omega_1} a\nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x.$$

Using $\eta^2 v^h$ as a test function in (A.2), and expanding the integral, we obtain

$$\int_{\omega_1} a\nabla v^h \cdot \nabla(\eta^2 v^h)\mathrm{d}x = \int_{\omega_1} a\nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x + 2\int_{\omega_1} a\eta\nabla v^h \cdot \nabla\eta v^h \mathrm{d}x,$$

and thus

$$\begin{aligned}
\int_{\omega_1} a\nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x &\leq \int_{\omega_1} a\nabla v^h \cdot \nabla(\eta^2 v^h)\mathrm{d}x - 2\int_{\omega_1} (\eta a^{1/2}\nabla v^h)\cdot(v^h a^{1/2}\nabla\eta)\mathrm{d}x \\
&\leq \int_{\omega_1} a\nabla v^h \cdot \nabla(\eta^2 v^h)\mathrm{d}x + 2\int_{\omega_1} (\eta a^{1/2}\nabla v^h)\cdot(v^h a^{1/2}\nabla\eta)\mathrm{d}x \\
&\leq B_1(v^h, \eta^2 v^h) + \zeta\int_{\omega_1} a\nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x + \frac{1}{\zeta}\int_{\omega_1} a v^h \nabla\eta \cdot \nabla\eta v^h \mathrm{d}x \\
&\leq B_1(v^h, \eta^2 v^h) + \zeta\int_{\omega_1} a\nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x + \frac{\beta}{\zeta\tau^2}\|v^h\|^2_{\mathrm{L}^2(\omega_1)}.
\end{aligned}$$

The last step is to bound the quantity $B_1(v^h, \eta^2 v^h)$. Let us consider $I_h(\eta^2 v^h) \in V^p(\omega_1, \mathcal{T}_h)$, it holds

$$B_1(v^h, I(\eta^2 v^h)) = 0,$$

and then

$$\begin{aligned}
B_1(v^h, \eta^2 v^h) = B_1(v^h, \eta^2 v^h - I(\eta^2 v^h)) &= \int_{\omega_1} a\nabla v^h \nabla(\eta^2 v^h - I(\eta^2 v^h))\mathrm{d}x \\
&\le \beta \|\nabla v^h\|_{\mathrm{L}^2(\omega_1)} \|\nabla(\eta^2 v^h - I(\eta^2 v^h))\|_{\mathrm{L}^2(\omega_1)} \\
&= \beta \sum_{K\in\mathcal{T}_h} \|\nabla v^h\|_{\mathrm{L}^2(K)} \|\nabla(\eta^2 v^h - I(\eta^2 v^h))\|_{\mathrm{L}^2(K)}.
\end{aligned}$$

Using the local inverse inequality (A.3) and Lemma A.4, we obtain

$$\begin{aligned}
B_1(v^h, \eta^2 v^h) &\le C\beta \sum_{K\in\mathcal{T}_h} \frac{1}{h_K} \|v^h\|_{\mathrm{L}^2(K)} \left(\frac{h_K}{\tau} \|\nabla(\eta v^h)\|_{\mathrm{L}^2(K)} + \frac{h_K}{\tau^2}\|v^h\|_{\mathrm{L}^2(K)} \right) \\
&= \beta \sum_{K\in\mathcal{T}_h} \|v^h\|_{\mathrm{L}^2(K)} \frac{C}{\tau} \|\nabla(\eta v^h)\|_{\mathrm{L}^2(K)} + \frac{C}{\tau^2} \|v^h\|^2_{\mathrm{L}^2(K)} \\
&\le \beta \sum_{K\in\mathcal{T}_h} \frac{C}{\tau^2}\left(\frac{1}{\zeta}+1\right) \|v^h\|^2_{\mathrm{L}^2(K)} + \zeta \|\nabla(\eta v^h)\|^2_{\mathrm{L}^2(K)} \\
&\le \frac{C\beta}{\tau^2}\left(\frac{1}{\zeta}+1\right) \|v^h\|^2_{\mathrm{L}^2(\omega_1)} + \beta\zeta\|\eta\nabla v^h\|^2_{\mathrm{L}^2(\omega_1)} + \beta\zeta \|v^h\nabla\eta\|^2_{\mathrm{L}^2(\omega_1)} \\
&\le \beta\left(\frac{C}{\tau^2}\left(\frac{1}{\zeta}+1+\zeta\right) \|v^h\|^2_{\mathrm{L}^2(\omega_1)} + \zeta\|\eta\nabla v^h\|^2_{\mathrm{L}^2(\omega_1)} \right).
\end{aligned}$$

Recalling that

$$\|\eta\nabla v^h\|^2_{\mathrm{L}^2(\omega_1)} = \int_{\omega_1} \nabla v^h \cdot \nabla v^h \eta^2 \mathrm{d}x \le \frac{1}{\alpha}\int_{\omega_1} a\nabla v^h\cdot\nabla v^h\eta^2\mathrm{d}x,$$

and collecting the previous bounds, it holds

$$\begin{aligned}
\int_{\omega_1} a\nabla v^h\cdot\nabla v^h\eta^2\mathrm{d}x \le C\frac{\beta}{\tau^2}\left(\frac{2}{\zeta}+1+\zeta\right)\|v^h\|^2_{\mathrm{L}^2(\omega_1)} \\
+ \zeta\left(\frac{\beta}{\alpha}+1\right)\int_{\omega_1} a\nabla v^h\cdot\nabla v^h\eta^2\mathrm{d}x.
\end{aligned}$$

This gives, for $\zeta \ne 1/(\beta/\alpha+1)$,

$$(1-\zeta(\beta/\alpha+1))\int_{\omega_1} a\nabla v^h\cdot\nabla v^h\eta^2\mathrm{d}x \le C\frac{\beta}{\tau^2}\left(\frac{2}{\zeta}+1+\zeta\right)\|v^h\|^2_{\mathrm{L}^2(\omega_1)},$$

and finally

$$\|\nabla v^h\|^2_{\mathrm{L}^2(\omega)} \le \frac{C}{(1-\zeta(\beta/\alpha+1))}\left(\frac{2}{\zeta}+1+\zeta\right)\frac{\beta}{\alpha\tau^2}\|v^h\|^2_{\mathrm{L}^2(\omega_1)}.$$

□ Assume now that $\partial\omega \cap \Gamma \neq \emptyset$. A discrete Caccioppoli inequality can be proved.

Lemma A.6 (Discrete Caccioppoli inequality for domains with shared boundaries). *Let $v^h \in V^p(\omega_1, \mathcal{T}_h)$ satisfy equation* (A.2) *for all $w^h \in V_0^p(\omega_1, \mathcal{T}_h)$. Further assume that $v^h = 0$ on $\partial\omega_1 \cap \Gamma$. Then it holds*

$$\|\nabla v^h\|_{\mathrm{L}^2(\omega)} \leq C\frac{1}{\tau}\|v^h\|_{\mathrm{L}^2(\omega_1)},$$

where the constant C is independent of h.

Proof. We consider now a cutoff function η such that $\eta \equiv 1$ in $\overline{\omega}$, $\eta \equiv 0$ in $\Omega \setminus \omega_1$, and with $\eta \equiv 0$ on $\partial\omega_1 \cap \Omega$. We can then follow the proof of Lemma A.5, as

$$B_1(v^h, \eta^2 v^h) = \int_{\omega_1} a\nabla v^h \cdot \nabla(\eta^2 v^h)\mathrm{d}x = 0.$$

□

We now show that the strong Cauchy-Schwarz inequality A.3 is still valid for discrete functions. For simplicity in the notations, we omit the ε dependency in v_1.

Lemma A.7. *Let $\varepsilon < \varepsilon_0$ and $C_s < 1$ be given by the strong Cauchy-Schwarz Lemma A.3, and let $v_{1,\tilde{h}} \in V_D^p(\omega_1, \mathcal{T}_{\tilde{h}})$ and $v_{2,H} \in V_D^p(\omega_2, \mathcal{T}_H)$ be numerical solutions of* (4.9)*. There exist $\tilde{h}_0 > 0$ and $H_0 > 0$ such that*

$$\int_{\omega_0} v_{1,\tilde{h}} v_{2,H}\mathrm{d}x \leq C_s\|v_{1,\tilde{h}}\|_{\mathrm{L}^2(\omega_0)}\|v_{2,H}\|_{\mathrm{L}^2(\omega_0)}, \qquad \forall \tilde{h} < \tilde{h}_0, H < H_0.$$

Proof. Let $\{\tilde{h}_n, H_n\}_{n\geq 1}$ be a sequence of mesh sizes converging to zero. We have strong convergence in L^2, for a subsequence of $\{\tilde{h}_n, H_n\}_{n\geq 1}$ still denoted by $\{\tilde{h}_n, H_n\}_{n\geq 1}$, of the numerical solutions $v_{1,\tilde{h}_n}$ and v_{2,H_n} to the exact solutions v_1 and v_2 respectively. Thus

$$\lim_{n\to\infty}\int_{\omega_0} v_{1,\tilde{h}_n} v_{2,H_n}\mathrm{d}x = \int_{\omega_0} v_1 v_2 \mathrm{d}x \quad \text{and} \quad \begin{array}{ll}\lim_{n\to\infty}\|v_{1,\tilde{h}_n}\|_{\mathrm{L}^2(\omega_0)} & = \|v_1\|_{\mathrm{L}^2(\omega_0)}, \\ \lim_{n\to\infty}\|v_{2,H_n}\|_{\mathrm{L}^2(\omega_0)} & = \|v_2\|_{\mathrm{L}^2(\omega_0)}.\end{array}$$

We recall that the strong Cauchy-Schwarz is valid for v_1 et v_2; there exists an ε_0 and a constant $C_s < 1$, such that for all $\varepsilon \leq \varepsilon_0$, it holds

$$\int_{\omega_0} v_1 v_2 \mathrm{d}x \leq C_s\|v_1\|_{\mathrm{L}^2(\omega_0)}\|v_2\|_{\mathrm{L}^2(\omega_0)}.$$

Then, using the strong Cauchy-Schwarz inequality for v_1 and v_2, it holds

$$\begin{aligned}\lim_{n\to\infty}\int_{\omega_0} v_{1,\tilde{h}_n} v_{2,H_n}\mathrm{d}x &= \int_{\omega_0} v_1 v_2 \mathrm{d}x \\ &\leq C_s\|v_1\|_{\mathrm{L}^2(\omega_0)}\|v_2\|_{\mathrm{L}^2(\omega_0)} \\ &= \lim_{n\to\infty} C_s\|v_{1,\tilde{h}_n}\|_{\mathrm{L}^2(\omega_0)}\|v_{2,H_n}\|_{\mathrm{L}^2(\omega_0)}.\end{aligned}$$

Then, there exist an $\varepsilon_0 > 0$ and a constant $C_s < 1$, such that for all $\varepsilon \leq \varepsilon_0$, there exist $\tilde{h}_0 > 0$ and $H_0 > 0$, such that

$$\int_{\omega_0} v_{1,\tilde{h}} v_{2,H}\mathrm{d}x \leq C_s\|v_{1,\tilde{h}}\|_{\mathrm{L}^2(\omega_0)}\|v_{2,H}\|_{\mathrm{L}^2(\omega_0)}, \qquad \forall \tilde{h} \leq \tilde{h}_0, H \leq H_0.$$

□

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