MATHICSE
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School of Basic Sciences - Section of Mathematics

MATHICSE Technical Report
Nr. 36.2014
August 2014

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Error estimates for finite element approximations of nonlinear monotone elliptic problems with application to numerical homogenization

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Abstract

We consider a finite element method (FEM) with arbitrary polynomial degree for nonlinear monotone elliptic problems. Using a linear elliptic projection, we first give a new short proof of the optimal convergence rate of the FEM in the L^2 norm. We then derive optimal a priori error estimates in the H^1 and L^2 norm for a FEM with variational crimes due to numerical integration. As an application we derive a priori error estimates for a numerical homogenization method applied to nonlinear monotone elliptic problems.

Keywords: nonlinear monotone elliptic problem, high order finite element method, elliptic projection, numerical integration, variational crimes, numerical homogenization.

AMS subject classification (2010): 65N30, 65D30, 74D10.

1 Introduction

We consider a finite element approximation with polynomial degree $l \in \mathbb{N}_{\geq 1}$ of the nonlinear monotone elliptic problem

$$-\operatorname{div}(\mathcal{A}(x, \nabla u)) = f(x) \text{ in } \Omega, \qquad u(x) = 0 \text{ on } \partial\Omega, \tag{1}$$

with $\Omega \subset \mathbb{R}^d$ (for $d \leq 3$) a convex polyhedral domain, $f \in L^2(\Omega)$ and $\mathcal{A}\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. We assume that $\mathcal{A}(\cdot, \xi)\colon \Omega \to \mathbb{R}^d$ is Lebesgue measurable for every $\xi \in \mathbb{R}^d$ and that there exist $C_0, L, \lambda > 0$ such that for almost every (a.e.) $x \in \Omega$:

($\mathcal{A}_0$) boundedness in $\xi = 0$: $|\mathcal{A}(x, 0)| \leq C_0$;

($\mathcal{A}_1$) Lipschitz continuity: $|\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)| \leq L\,|\xi_1 - \xi_2|$, for all $\xi_1, \xi_2 \in \mathbb{R}^d$;

($\mathcal{A}_2$) strong monotonicity: $[\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)] \cdot (\xi_1 - \xi_2) \geq \lambda |\xi_1 - \xi_2|^2$, for all $\xi_1, \xi_2 \in \mathbb{R}^d$.

Solving the nonlinear monotone elliptic problem numerically by a finite element method (FEM) is a standard technique, see [6, Sect. 5] and references therein. While optimal a priori error estimates in the H^1 norm can be proved following the arguments used for linear elliptic problems, see [6, Theorem 5.3.4], sharp estimates in the L^2 norm are harder to derive. In [8], Dendy showed a convergence rate $\min\{l+1, 2l-d/2\}$, see [8, Theorem 2.2], assuming that the numerical solution is bounded in the $W^{1,\infty}$ norm, that the meshes are quasi-uniform and that the regularity $u \in H^{l+1}(\Omega) \cap W^{1,\infty}(\Omega)$, $\mathcal{A} \in \mathcal{C}^2(\Omega \times \mathbb{R}^d)$ holds. This

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result however is non-optimal for low polynomial degree l. For $l = 1$, Frehse and Rannacher obtained in [10] the optimal quadratic convergence as a byproduct of error estimates in the L^∞ norm (the main goal of [10]). They assumed $u \in \mathcal{C}^{2+\alpha}(\Omega)$, $\mathcal{A} \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ (but mentioned that less regularity is sufficient for $\mathcal{A}$), quasi-uniformity of the meshes and used weighted norm techniques (adapted to the nonlinear problem). For two-dimensional problems ($d = 2$), Xu derived in [15] sharp L^2 error estimates for general $l \geq 1$ assuming $u \in H^{l+1}(\Omega) \cap W^{2,2+\epsilon}(\Omega)$, for some $\epsilon > 0$, and $\mathcal{A}$ sufficiently smooth. The essential ingredients in [15] are a linear projection of the exact solution and logarithmic bounds for a discrete Green function. These bounds are however only valid for $d = 2$.

For practical implementation and important applications (e.g., numerical homogenization as described below) of the FEM for (1) numerical integration is used in general, i.e., variational crimes are committed. It is thus important to estimate the effect of the use of quadrature rules on the convergence of the numerical solution – see [7], [6, Sect. 4.1] for linear elliptic problems and [4] for nonlinear nonmonotone problems. For FEM applied to (1), we are only aware of [9], where the effect of numerical integration on the convergence rate in the H^1 norm has been studied for $d = 2$ and polynomial degree $l = 1$.

The main results of this paper are twofold. First we prove optimal convergence rates in the L^2 norm for FEM with general polynomial degree $l \geq 1$ (thus improving the results from [8]). In contrast to [15], our results are valid for $d = 3$, and do not use weighted norm techniques as in [10] thanks to the use of a linear elliptic projection. Second, we prove optimal convergence rates in the H^1 and L^2 norm of the finite element method with arbitrary polynomial degree l and based on a quadrature formula for the nonlinear monotone elliptic problem (1). As an application of the error estimates for FEM with numerical integration for singlescale problems, we present the error analysis for a numerical homogenization method. We consider the finite element heterogeneous multiscale method (FE-HMM) [1] for nonlinear monotone elliptic PDEs with multiple scales. In [11] convergence rates in the H^1 norm have been derived for the FE-HMM for a class of elliptic monotone PDEs (associated to minimization problems). In particular the map $\mathcal{A}(x, \xi)$ of (1) needs to have a scalar potential (w.r.t. ξ) to belong to the class of problems considered in [11]. In contrast we derive optimal error estimates in the H^1 and L^2 norm for general monotone problems. We also note that a posteriori error estimates for elliptic monotone problems have been considered in [12]. We close this introduction by mentioning that similar results for the FE-HMM applied to parabolic monotone problems have been derived in [2, 3].

This paper is structured as follows. We first introduce the FEM for the problem (1) in Section 2. The linear elliptic projection is then defined in Section 3 and the optimal L^2 error estimates for FEM without and with numerical quadrature are proved in Section 4 and 5, respectively. Finally, in Section 6, we present a priori error estimates for the FE-HMM.

2 FEM for nonlinear monotone elliptic PDEs

In this section, we first introduce the weak solution of (1) and then formulate the FEMs –without and with numerical integration– to solve (1) numerically.

Exact solution. We recall that the weak solution of (1) solves the variational problem:

$$\text{find } u \in H_0^1(\Omega) \text{ such that} \qquad B(u; w) = \int_\Omega f\, w\, dx, \qquad \forall\, w \in H_0^1(\Omega), \tag{2}$$

with the map B (nonlinear in the first argument, linear in the second argument) given by

$$B(v;w) = \int_\Omega \mathcal{A}(x,\nabla v)\cdot\nabla w\,dx, \qquad \text{for } v,w\in H_0^1(\Omega). \tag{3}$$

Note that $(\mathcal{A}_0)$ yields the continuity of B in the second variable and $(\mathcal{A}_{1-2})$ ensure that B is strongly monotone and Lipschitz continuous in the first variable. Thus from the nonlinear Lax-Milgram theorem [16, Theorem 25.B] we have that a unique solution $u\in H_0^1(\Omega)$ exists.

FEM without numerical integration. Let $\mathcal{T}_H$ be a simplicial mesh of Ω consisting of open elements $K\in\mathcal{T}_H$ with straight edges. We assume that $\mathcal{T}_H$ is conformal and shape-regular, see [6], and we denote its maximal element diameter by $H=\max_{K\in\mathcal{T}_H}\operatorname{diam} K$. For $l\in\mathbb{N}_{\geq 1}$, we define the finite element space

$$S_0^l(\Omega,\mathcal{T}_H) = \{v^H\in H_0^1(\Omega)\mid v^H|_K\in\mathcal{P}^l(K),\forall\, K\in\mathcal{T}_H\}, \tag{4}$$

where $\mathcal{P}^l(K)$ is the set of polynomials of total degree at most l on the element $K\in\mathcal{T}_H$.

We next define the Galerkin finite element solution u^H of (1):

$$\text{find } u^H\in S_0^l(\Omega,\mathcal{T}_H) \text{ such that} \qquad B(u^H;w^H) = \int_\Omega f\,w^H\,dx, \qquad \forall\, w^H\in S_0^l(\Omega,\mathcal{T}_H). \tag{5}$$

Again, the well-posedness of the problem (5) is shown by [16, Theorem 25.B] using $(\mathcal{A}_{0-2})$.

FEM with numerical integration. Next, we consider the formulation of the finite element method (5) with numerical integration used to evaluate the integral in (3).

Let $\hat{K}$ be the simplicial, d-dimensional reference element and $\{\hat{\omega}_j,\hat{x}_j\}_{j=1}^J$ a quadrature formula on $\hat{K}$ for $J\in\mathbb{N}_{\geq 1}$ (with weights $\hat{\omega}_j$ and nodes $\hat{x}_j$). Assume the conditions:

(**Q1**) $\hat{\omega}_j>0$ for $j=1,\dots,J$; $\sum_{j=1}^J\hat{\omega}_j|\nabla\hat{p}(\hat{x}_j)|^2\geq\hat{\lambda}\|\nabla\hat{p}\|^2_{L^2(\hat{K})}$ on $\mathcal{P}^l(\hat{K})$ for some $\hat{\lambda}>0$;

(**Q2**) $\int_{\hat{K}}\hat{p}(\hat{x})d\hat{x}=\sum_{j=1}^J\hat{\omega}_j\hat{p}(\hat{x}_j)$ for all $\hat{p}\in\mathcal{P}^\sigma(\hat{K})$ with $\sigma=\max\{1,2l-2+\mu\}$, $\mu\in\{0,1\}$.

Remark 2.1. For FEM with numerical integration applied to *linear* elliptic problems optimal L^2 convergence rates have been derived assuming (**Q2**) for $\mu=0$, see [7]. Note, that the more restrictive assumption (**Q2**) for $\mu=1$ is used instead in our analysis to derive optimal L^2 error bounds.

The quadrature formula on $\hat{K}$ induces a quadrature formula $\{\omega_{K_j},x_{K_j}\}_{j=1}^J$ on each element $K\in\mathcal{T}_H$ (via the affine parametrization $F_K\colon\hat{K}\to K$). We thus introduce the map $\tilde{B}_H$ for $v^H,w^H\in S_0^l(\Omega,\mathcal{T}_H)$ by

$$\tilde{B}_H(v^H;w^H) = \sum_{K\in\mathcal{T}_H}\sum_{j=1}^J\omega_{K_j}\mathcal{A}(x_{K_j},\nabla v^H(x_{K_j}))\cdot\nabla w^H(x_{K_j}), \tag{6}$$

and the FEM with numerical integration for the problem (1) then reads as:

$$\text{find } \tilde{u}^H\in S_0^l(\Omega,\mathcal{T}_H) \text{ such that} \quad \tilde{B}_H(\tilde{u}^H;w^H) = \int_\Omega f\,w^H dx, \qquad \forall\, w^H\in S_0^l(\Omega,\mathcal{T}_H). \tag{7}$$

Due to condition (**Q1**) and the hypotheses $(\mathcal{A}_{0-2})$ the form $\tilde{B}_H$ is continuous in the second variable and Lipschitz continuous as well as strongly monotone in the first variable, hence a unique numerical solution $\tilde{u}^H$ exists.

3 A linear elliptic projection

Our analysis is based on the linear elliptic projection u_π^H solving the variational problem:

$$\text{find } u_\pi^H \in S_0^l(\Omega, \mathcal{T}_H) \text{ such that } \quad B_\pi(u_\pi^H, w^H) = B_\pi(u, w^H), \qquad \forall\, w^H \in S_0^l(\Omega, \mathcal{T}_H), \tag{8}$$

where the bilinear form B_π, for $v, w \in H_0^1(\Omega)$, is given by

$$B_\pi(v, w) = \int_\Omega \mathscr{A}(x)\nabla v \cdot \nabla w \, dx, \quad \text{with } \mathscr{A}(x) = D_\xi \mathcal{A}(x, \nabla u(x)) \text{ for a.e. } x \in \Omega. \tag{9}$$

We emphasize that the variational problem (8) is a *linear* elliptic problem. The existence and uniqueness of the elliptic projection is summarized in the following Lemma (for details see [2, Sect. 5.1]).

Lemma 3.1. *Assume that $\mathcal{A}\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $(\mathcal{A}_{1-2})$ and $\mathcal{A}(x, \cdot) \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ for a.e. $x \in \Omega$. Then, $D_\xi \mathcal{A}(x, \xi)$ is uniformly elliptic and bounded. The problem (8) thus admits a unique solution u_π^H.*

As (8) is a linear elliptic problem, classical finite element error bounds are valid for u_π^H, e.g., see [6].

Lemma 3.2. *Assume that $\mathcal{A}$ satisfies $(\mathcal{A}_{1-2})$ and $\mathcal{A}(x, \cdot) \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ for a.e. $x \in \Omega$. Let u be the solution of (2) and u_π^H its elliptic projection (8). If $u \in H^{l+1}(\Omega)$, then*

$$\|\nabla u - \nabla u_\pi^H\|_{L^2(\Omega)} \le CH^l |u|_{H^{l+1}(\Omega)}, \qquad \|u - u_\pi^H\|_{L^2(\Omega)} \le CH^{l+1} |u|_{H^{l+1}(\Omega)},$$

where C is independent of H and where elliptic regularity for the adjoint problem of (8) is needed for the L^2 estimate.

We first show that u_π^H is bounded in several higher order broken norms (see [4, Lemma 1] for a proof).

Lemma 3.3. *Let $\{\mathcal{T}_H\}_{H>0}$ be quasi-uniform (see [6, Condition (3.2.28)]). Under the conditions of Lemma 3.2, we have*

$$\|u_\pi^H\|_{\bar{H}^{l+1}(\Omega)} + \|u_\pi^H\|_{\bar{W}^{l,6}(\Omega)} + \|u_\pi^H\|_{\bar{W}^{l-1,\infty}(\Omega)} \le C\|u\|_{H^{l+1}(\Omega)}, \tag{10}$$

where C is independent of H.

For $k \in \mathbb{N}$ and $1 \le p \le \infty$, the broken norms $\|\cdot\|_{\bar{W}^{k,p}(\Omega)}$ are given by

$$\|v\|_{\bar{W}^{k,p}(\Omega)} = \Big(\textstyle\sum_{K \in \mathcal{T}_H} \|v\|_{W^{k,p}(K)}^p\Big)^{1/p}, \text{ if } p < \infty, \quad \|v\|_{\bar{W}^{k,\infty}(\Omega)} = \max_{K \in \mathcal{T}_H} \|v\|_{W^{k,\infty}(K)},$$

where $W^{k,p}(K)$ and $\|\cdot\|_{W^{k,p}(K)}$ denote the usual Sobolev spaces and norms on K, respectively, and $\bar{W}^{k,2}(\Omega)$ is written as $\bar{H}^k(\Omega)$.

Remark 3.4. Let $u \in H^{l+1}(\Omega)$ and $\mathcal{I}_H u \in S_0^l(\Omega, \mathcal{T}_H)$ be its nodal interpolant. Then, it follows from the interpolation estimates [6, Theorem 3.1.6] and the Sobolev embeddings $H^{l+1}(\Omega) \hookrightarrow W^{l,6}(\Omega), W^{l-1,\infty}(\Omega)$, which hold for $d \le 3$, that the bound (10) holds for $\mathcal{I}_H u$ without any additional assumptions.

Next, we recall the maximum norm error estimate for linear FEM from [5, Sect. 8].

Lemma 3.5. *Assume that $\mathcal{A}$ satisfies $(\mathcal{A}_{1-2})$ and $\mathcal{A}(x,\cdot) \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R}^d)$ for a.e. $x \in \Omega$. Let u be the solution of* (1) *and u_π^H its elliptic projection* (8)*. Further, suppose that*

$$\|u\|_{W^{2,p}(\Omega)} + \|u^*\|_{W^{2,p}(\Omega)} \leq C\|\mathrm{div}(\mathscr{A}\nabla u)\|_{L^p(\Omega)}, \quad \text{for } 1 < p < \sigma \text{ with some } \sigma > d, \tag{11}$$

holds, where $\mathscr{A}$ is given by (9) *and u^* solves the dual problem $B_\pi(w,u^*) = B_\pi(u,w)$ for all $w \in H_0^1(\Omega)$. If $u \in W^{2,\infty}(\Omega)$ and $\mathscr{A}_{ij} \in W^{1,\infty}(\Omega)$, for $1 \leq i,j \leq d$, then for any family $\{\mathcal{T}_H\}_{H>0}$ of quasi-uniform meshes (see [6, Condition (3.2.28)]), there exists $H_0 > 0$ such that for all $H < H_0$*

$$\|u - u_\pi^H\|_{W^{1,\infty}(\Omega)} \leq CH\|u\|_{W^{2,\infty}(\Omega)},$$

where C is independent of H.

If additionally $u \in W^{l+1,\infty}(\Omega)$, for $l \geq 1$, the higher order estimate $\|u - u_\pi^H\|_{W^{1,\infty}(\Omega)} \leq CH^l$ holds. The linear convergence stated in Lemma 3.5 is however sufficient for our analysis.

4 Optimal L^2 error estimate for FEM without quadrature formula

In this section, we prove sharp convergence rates in the L^2 norm for FEM without numerical integration. The ingredients of the proof are Lemma 4.2 and the a priori error bounds for the elliptic projection u_π^H.

Theorem 4.1. *Let u be the exact solution of* (1) *and u^H its finite element approximation given by* (5)*. Assume that $\mathcal{A}$ satisfies $(\mathcal{A}_{0-2})$, the elliptic regularity* (11) *holds and that*

$$\begin{gathered}\mathcal{A}(x,\cdot) \in W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d) \text{ with } \|D_\xi\mathcal{A}(x,\cdot)\|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^{d\times d})} \leq L_\mathcal{A} \text{ for a.e. } x \in \Omega, \\ u \in H^{l+1}(\Omega) \cap W^{2,\infty}(\Omega), \qquad \mathscr{A}_{ij} \in W^{1,\infty}(\Omega), \text{ for } 1 \leq i,j \leq d,\end{gathered} \tag{12}$$

for some $L_\mathcal{A} > 0$ and $\mathscr{A}(x) = D_\xi\mathcal{A}(x,\nabla u(x))$. If $\{\mathcal{T}_H\}_{H>0}$ is a family of quasi-uniform meshes, then there exists an $H_0 > 0$ such that for all $H < H_0$

$$\|u - u^H\|_{L^2(\Omega)} \leq CH^{l+1},$$

where C is independent of H.

Proof. We use the triangle inequality $\|u-u^H\|_{L^2(\Omega)} \leq \|u-u_\pi^H\|_{L^2(\Omega)} + \|u_\pi^H - u^H\|_{L^2(\Omega)}$ and combine Lemma 3.2, the Poincaré inequality and Lemma 4.2, which is shown below. □

We next prove that the elliptic projection u_π^H is close to the numerical solution u^H.

Lemma 4.2. *Under the conditions of Theorem 4.1, there exists $H_0 > 0$ such that for all $H < H_0$*

$$\|\nabla u^H - \nabla u_\pi^H\|_{L^2(\Omega)} \leq CL_\mathcal{A}H^{l+1}\|u\|_{W^{2,\infty}(\Omega)}|u|_{H^{l+1}(\Omega)},$$

where $L_\mathcal{A}$ is the Lipschitz constant of $D_\xi\mathcal{A}(x,\cdot)$ from (12) *and C is independent of H.*

Proof. Let $w^H \in S_0^l(\Omega, \mathcal{T}_H)$. Applying a first order Taylor expansion to $\mathcal{A}(x, \cdot)$ and using the definition (8) of the elliptic projection u_π^H as well as the following Galerkin orthogonality

$$\int_\Omega \left[\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u^H)\right] \cdot \nabla w^H dx = 0, \qquad \forall\, w^H \in S_0^l(\Omega, \mathcal{T}_H),$$

yields

$$\begin{aligned}\int_\Omega \mathcal{A}(x, \nabla u_\pi^H) \cdot \nabla w^H dx &= \int_\Omega \left[\mathcal{A}(x, \nabla u) + D_\xi \mathcal{A}(x, \nabla u)(\nabla u_\pi^H - \nabla u)\right] \cdot \nabla w^H dx \\ &\quad + P(u, u_\pi^H; w^H) \\ &= \int_\Omega \mathcal{A}(x, \nabla u^H) \cdot \nabla w^H dx + P(u, u_\pi^H; w^H), \end{aligned} \tag{13}$$

where $P(u, u_\pi^H; w^H)$ represents the remainder of the Taylor expansion

$$\begin{aligned} P(u, u_\pi^H; w^H) = \int_\Omega \int_0^1 D_\xi \mathcal{A}(x, \nabla u + \tau(\nabla u_\pi^H - \nabla u)) - D_\xi \mathcal{A}(x, \nabla u) d\tau \\ \times (\nabla u_\pi^H - \nabla u) \cdot \nabla w^H dx. \end{aligned} \tag{14}$$

Using the Lipschitz continuity of $D_\xi \mathcal{A}(x, \cdot)$ the term $P(u, u_\pi^H; w^H)$ can be estimated by

$$P(u, u_\pi^H; w^H) \le L_\mathcal{A} \|u - u_\pi^H\|_{W^{1,\infty}(\Omega)} \|\nabla u - \nabla u_\pi^H\|_{L^2(\Omega)} \|\nabla w^H\|_{L^2(\Omega)}. \tag{15}$$

Combining $(\mathcal{A}_2)$, the identity (13), the estimate (15) and Lemmas 3.2 and 3.5 we conclude

$$\begin{aligned} \lambda \|\nabla u_\pi^H - \nabla u^H\|_{L^2(\Omega)}^2 &\le \int_\Omega \left[\mathcal{A}(x, \nabla u_\pi^H) - \mathcal{A}(x, \nabla u^H)\right] \cdot (\nabla u_\pi^H - \nabla u^H) dx \\ &= P(u, u_\pi^H; u_\pi^H - u^H) \\ &\le C L_\mathcal{A} H^{l+1} \|u\|_{W^{2,\infty}(\Omega)} |u|_{H^{l+1}(\Omega)} \|\nabla u_\pi^H - \nabla u^H\|_{L^2(\Omega)}. \end{aligned} \tag{16}$$

□

5 A priori error estimate for FEM with quadrature formula

While Theorem 4.1 holds for FEM without numerical integration, we now show optimal convergence rates in the H^1 and L^2 norm taking into account the variational crimes due to quadrature formulas. At the end of the section, we present numerical results to illustrate the convergence results of Theorem 5.1.

Theorem 5.1. *Let u be the exact solution of* (1) *and $\tilde{u}^H \in S_0^l(\Omega, \mathcal{T}_H)$ be its approximation obtained by the FEM with numerical integration, see* (7)*. Let $\mu \in \{0, 1\}$. Assume either that $u \in H^{l+1}(\Omega)$ (if $\mu = 0$) or the hypotheses of Theorem 4.1 (if $\mu = 1$). If the quadrature formula satisfies* (**Q1**), (**Q2**) *for the given μ and $\mathcal{A}$ has the regularity*

$$\begin{aligned} &\mathcal{A}(\cdot, \xi) \in W^{1+\mu,\infty}(\Omega), \ \|\mathcal{A}(\cdot, \xi)\|_{W^{1+\mu,\infty}(\Omega)} \le C(L_0 + |\xi|), \ \forall\, \xi \in \mathbb{R}^d, && \text{if } l = 1, \\ &\mathcal{A} \in W^{l+\mu,\infty}(\Omega \times B_R(0); \mathbb{R}^d), \ \|\mathcal{A}\|_{W^{l+\mu,\infty}(\Omega \times B_R(0); \mathbb{R}^d)} \le C(L_0 + R), \ \forall\, R > 0, && \text{if } l \ge 2, \end{aligned} \tag{17}$$

for some $L_0 > 0$, then, we have the a priori error estimates

$$\|u - \tilde{u}^H\|_{H^1(\Omega)} \le C H^l, \quad \text{if } \mu = 0, \qquad \|u - \tilde{u}^H\|_{L^2(\Omega)} \le C H^{l+1}, \quad \text{if } \mu = 1,$$

where C is independent of H.

We first estimate the quadrature error –a key ingredient for the proof of Theorem 5.1.

Lemma 5.2. *Let $\mu \in \{0,1\}$. Assume* (**Q2**) *and the regularity* (17) *of $\mathcal{A}$ for the given μ. Then, we have*

$$\Big|B_H(v^H;w^H) - \tilde{B}_H(v^H;w^H)\Big| \le CH^{l+\mu}\mathcal{Q}(v^H)\big\|\nabla w^H\big\|_{L^2(\Omega)}, \qquad \forall\, v^H, w^H \in S_0^l(\Omega,\mathcal{T}_H),$$

where C is independent of H and $\mathcal{Q}(v^H)$ is given by

$$\mathcal{Q}(v^H) = L_0 + \big\|\nabla v^H\big\|_{L^2(\Omega)}, \qquad \text{if } l=1,$$

$$\mathcal{Q}(v^H) = \Big(L_0 + \big|\nabla v^H\big|_{L^\infty(\Omega)}\Big)\Big(1+\big\|v^H\big\|^{l+\mu}_{\bar{W}^{l-1,\infty}(\Omega)}\Big)\left(1+\sum_{\kappa=1}^{\kappa_l}\big|v^H\big|^{\kappa}_{\bar{W}^{l,2\kappa}(\Omega)}\right), \qquad \text{if } l \ge 2,$$

where L_0 is the constant from (17) *and $\kappa_2 = 2+\mu$, $\kappa_3 = 1+\mu$ and $\kappa_l = 1$ for $l \ge 4$.*

Proof. As the proof for $l=1$ can be found in [2, Lemma 5.9], we only consider $l \ge 2$. For $v^H, w^H \in S_0^l(\Omega,\mathcal{T}_H)$, we define the local (componentwise) quadrature error

$$E_K^i(\mathcal{A},v^H,w^H) = \int_K \mathcal{A}_i(x,\nabla v^H)\partial_{x_i} w^H dx - \sum_{j=1}^J \omega_{K_j}\mathcal{A}_i(x_{K_j},\nabla v^H(x_{K_j}))\partial_{x_i} w^H(x_{K_j}),$$

for $K \in \mathcal{T}_H$ and $1 \le i \le d$. As $v^H, w^H \in \mathcal{P}^l(K)$, we consider from now on

$$E_K^i(\mathcal{A},\mathbf{p},q) = \int_K \mathcal{A}_i(x,\mathbf{p}(x))q(x)dx - \sum_{j=1}^J \omega_{K_j}\mathcal{A}_i(x_{K_j},\mathbf{p}(x_{K_j}))q(x_{K_j})$$

with $\mathbf{p}(x) = (p_1(x),\dots,p_d(x))^T \in (\mathcal{P}^{l-1}(K))^d$ and $q \in \mathcal{P}^{l-1}(K)$. We further define $R = |\mathbf{p}|_{L^\infty(K)}$.

Transferring the local quadrature error E_K^i back onto the reference element $\hat{K}$ via the affine parametrization $F_K\colon \hat{K} \to K$ one gets

$$E_K^i(\mathcal{A},\mathbf{p},q) = |\det \partial F_K|\hat{E}(\hat{\mathcal{A}}_i,\hat{\mathbf{p}},\hat{q}), \tag{18}$$

with $\hat{\mathcal{A}}_i(\hat{x},\xi) = \mathcal{A}_i(F_K(\hat{x}),\xi)$ (for $\xi \in \mathbb{R}^d$), $\hat{\mathbf{p}}(\hat{x}) = \mathbf{p}(F_K(\hat{x}))$, $\hat{q}(\hat{x}) = q(F_K(\hat{x}))$ and

$$\hat{E}(\hat{\mathcal{A}}_i,\hat{\mathbf{p}},\hat{q}) = \int_{\hat{K}} \hat{\mathcal{A}}_i(\hat{x},\hat{\mathbf{p}}(\hat{x}))\hat{q}(\hat{x})d\hat{x} - \sum_{j=1}^J \hat{\omega}_j\hat{\mathcal{A}}_i(\hat{x}_j,\hat{\mathbf{p}}(\hat{x}_j))\hat{q}(\hat{x}_j).$$

By construction we have that $\hat{\mathbf{p}} \in (\mathcal{P}^{l-1}(\hat{K}))^d$ and $\hat{q} \in \mathcal{P}^{l-1}(\hat{K})$. In what follows, we omit the index i for $\mathcal{A}_i$ and $\hat{\mathcal{A}}_i$. From (**Q2**) and the Bramble-Hilbert Lemma, see [6, Theorem 4.1.3], we obtain that

$$\Big|\hat{E}(\hat{\mathcal{A}},\hat{\mathbf{p}},\hat{q})\Big| \le C\Big|\hat{\mathcal{A}}(\cdot,\hat{\mathbf{p}}(\cdot))\Big|_{W^{l+\mu,\infty}(\hat{K})}\|\hat{q}\|_{L^2(\hat{K})}. \tag{19}$$

Using the multivariate Faà-di-Bruno fromula, we obtain that $|\hat{\mathcal{A}}(\cdot,\hat{\mathbf{p}})|_{W^{l+\mu,\infty}(\hat{K})}$ is bounded by a sum of terms of the type

$$\Big\|\partial_{\hat{x}}^\lambda \partial_\xi^\nu \hat{\mathcal{A}}(\cdot,\hat{\mathbf{p}}(\cdot))\Big\|_{L^\infty(\hat{K})}\prod_{k=1}^{|\nu|}|\hat{p}_{j_k}|_{W^{r_k,\infty}(\hat{K})}, \ \text{ where } 1 \le r_k \le l-1 \text{ and } |\lambda| + \sum_{k=1}^{|\nu|} r_k = l+\mu, \tag{20}$$

with multi-indices $\lambda, \nu \in \mathbb{N}^d$ and integers $j_k \in \{1, \dots, d\}$, $r_k \in \mathbb{N}$, for $1 \le k \le |\nu|$ (without loss of generality, we assume that $r_k \ge r_{k+1}$). Some care is needed for the terms in (20) with $r_k = l-1$, the highest order derivative. Let us denote by $\kappa \in \mathbb{N}$ the number of factors in (20) with $r_k = l-1$ and by $\kappa_l \in \mathbb{N}$ the maximal value of κ attained for a given $l \ge 2$. When setting $\lambda = 0$, one derives from (20) that $\kappa_l = \lfloor (l+\mu)/(l-1) \rfloor$, i.e., $\kappa_2 = 2+\mu$, $\kappa_3 = 1+\mu$ and $\kappa_l = 1$ for $l \ge 4$. We then bound the terms of (20) by

$$\left\| \partial_{\hat{x}}^{\lambda} \partial_{\xi}^{\nu} \hat{\mathcal{A}} \right\|_{L^\infty(\hat{K}\times B_R(0))} \times \begin{cases} \prod_{k=1}^{|\nu|} |\hat{\mathbf{p}}|_{W^{r_k,\infty}(\hat{K})}, & \text{if } \kappa = 0, \\ C|\hat{\mathbf{p}}|^{\kappa}_{W^{l-1,2\kappa}(\hat{K})} \prod_{k=\kappa+1}^{|\nu|} |\hat{\mathbf{p}}|_{W^{r_k,\infty}(\hat{K})}, & \text{if } 1 \le \kappa \le 3, \end{cases} \tag{21}$$

where we used $|\hat{\mathbf{p}}|_{L^\infty(\hat{K})} = R$ and the equivalence of norms on $(\mathcal{P}^{l-1}(\hat{K}))^d$. Next, we recall the bounds

$$\begin{aligned} \left\| \partial_{\hat{x}}^{\lambda} \partial_{\xi}^{\nu} \hat{\mathcal{A}} \right\|_{L^\infty(\hat{K}\times B_R(0))} &\le C H_K^{|\lambda|} \left\| \partial_{x}^{\lambda} \partial_{\xi}^{\nu} \mathcal{A} \right\|_{L^\infty(K\times B_R(0))}, && \text{for } 0 \le |\lambda| + |\nu| \le l+\mu, \\ |\hat{v}|_{W^{j,q}(\hat{K})} &\le C H_K^{j} |\det \partial F_K|^{-1/q} |v|_{W^{j,q}(K)}, && \text{for } v \in W^{j,q}(K), j \in \mathbb{N}, \end{aligned} \tag{22}$$

where $\hat{v}(\hat{x}) = v(F_K(\hat{x}))$ on $\hat{K}$ and $1 \le q \le \infty$ (with $1/q$ set to 0, if $q = \infty$), see [6, Theorem 3.1.2]. Estimating (21) by using (22) and the equality from (20) then yields

$$\begin{aligned} (20) \le{}& C H_K^{l+\mu} \left\| \partial_{x}^{\lambda} \partial_{\xi}^{\nu} \mathcal{A} \right\|_{L^\infty(K\times B_R(0))} \\ &\times \begin{cases} \prod_{k=1}^{|\nu|} |\mathbf{p}|_{W^{r_k,\infty}(K)}, & \text{if } \kappa = 0, \\ |\det \partial F_K|^{-1/2} |\mathbf{p}|^{\kappa}_{W^{l-1,2\kappa}(K)} \prod_{k=\kappa+1}^{|\nu|} |\mathbf{p}|_{W^{r_k,\infty}(K)}, & \text{if } 1 \le \kappa \le 3. \end{cases} \end{aligned} \tag{23}$$

Next, we observe that

$$\prod_{k=\kappa+1}^{|\nu|} |\mathbf{p}|_{W^{r_k,\infty}(K)} \le \|\mathbf{p}\|^{|\nu|-\kappa}_{W^{l-2,\infty}(K)} \le 1 + \|\mathbf{p}\|^{l+\mu}_{W^{l-2,\infty}(K)}. \tag{24}$$

As from (22) it holds that $\|\hat{q}\|_{L^2(\hat{K})} \le C|\det \partial F_K|^{-1/2} \|q\|_{L^2(K)}$, using (19), (23) and (24) we obtain

$$\begin{aligned} \left| \hat{E}(\hat{\mathcal{A}}, \hat{\mathbf{p}}, \hat{q}) \right| \le{}& C H_K^{l+\mu} |\det \partial F_K|^{-1} \|\mathcal{A}\|_{W^{l+\mu,\infty}(K\times B_R(0))} \left(1 + \|\mathbf{p}\|^{l+\mu}_{W^{l-2,\infty}(K)}\right) \\ &\times \left(|\det \partial F_K|^{1/2} + \sum_{\kappa=1}^{\kappa_l} |\mathbf{p}|^{\kappa}_{W^{l-1,2\kappa}(K)} \right) \|q\|_{L^2(K)}. \end{aligned} \tag{25}$$

Let us then take $q = \partial_{x_i} w^H|_K$, $\mathbf{p} = \nabla v^H|_K$ and $R = |\nabla v^H|_{L^\infty(\Omega)}$. We note that $|\det \partial F_K| = C|K|$ and combine (18) and (25) to estimate $|E_K^i(\mathcal{A}, v^H, w^H)|$. The proof is then concluded by summing over $K \in \mathcal{T}_H$ and $1 \le i \le d$ and using the bound from (17) for the derivatives of $\mathcal{A}$. □

Proof of Theorem 5.1. We use $\|u - \tilde{u}^H\|_{H^{1-\mu}(\Omega)} \le \|u - \mathcal{U}^H\|_{H^{1-\mu}(\Omega)} + C\|\nabla\theta^H\|_{L^2(\Omega)}$, where $\mathcal{U}^H \in S_0^l(\Omega, \mathcal{T}_H)$, $\theta^H = \tilde{u}^H - \mathcal{U}^H$ and the constant C depends on the Poincaré constant. To estimate the second term we apply the bound (used for both $\mu = 0, 1$)

$$\begin{aligned} \lambda \left\| \nabla\theta^H \right\|^2_{L^2(\Omega)} &\le \tilde{B}_H(\tilde{u}^H; \theta^H) - \tilde{B}_H(\mathcal{U}^H; \theta^H) \\ &= B(u; \theta^H) - B(\mathcal{U}^H; \theta^H) + B(\mathcal{U}^H, \theta^H) - \tilde{B}_H(\mathcal{U}^H; \theta^H), \end{aligned} \tag{26}$$

where we used the strong monotonicity of $\tilde{B}_H$ (due to (**Q1**)) and the variational problems (1) and (7).

Consider first $\mu = 0$ and let $\mathcal{U}^H = \mathcal{I}_H u \in S_0^l(\Omega, \mathcal{T}_H)$ be the nodal interpolant of $u \in H^{l+1}(\Omega)$. We thus have $\|u - \mathcal{U}^H\|_{H^1(\Omega)} \leq CH^l |u|_{H^{l+1}(\Omega)}$, see [6, Theorem 3.2.1]. Combining the inequality (26), the Lipschitz continuity of B and Lemma 5.2 yields

$$\begin{aligned}\lambda \left\|\nabla\theta^H\right\|^2_{L^2(\Omega)} &\leq L\|\nabla u - \nabla \mathcal{I}_H u\|_{L^2(\Omega)} \left\|\nabla\theta^H\right\|_{L^2(\Omega)} + |B(\mathcal{I}_H u; \theta^H) - \tilde{B}_H(\mathcal{I}_H u; \theta^H)| \\ &\leq CH^l \Big(|u|_{H^{l+1}(\Omega)} + \mathcal{Q}(\mathcal{I}_H u)\Big) \left\|\nabla\theta^H\right\|_{L^2(\Omega)}.\end{aligned}$$

Using Remark 3.4 the term $\mathcal{Q}(\mathcal{I}_H u)$ from Lemma 5.2 can be bounded by $C\|u\|_{H^{l+1}(\Omega)}$ for $l \geq 1$. We thus have the desired estimate $\|\nabla\theta^H\|_{L^2(\Omega)} \leq CH^l$.

Consider now the case $\mu = 1$ and let $\mathcal{U}^H = u_\pi^H$ be the elliptic projection (8). Using the Taylor remainder P from (14), we obtain from (26)

$$\begin{aligned}\lambda \left\|\nabla\theta^H\right\|^2_{L^2(\Omega)} &\leq |P(u, u_\pi^H; \theta^H)| + |B(u_\pi^H; \theta^H) - \tilde{B}_H(u_\pi^H; \theta^H)| \\ &\leq CH^{l+1} \Big(\|u\|_{W^{2,\infty}(\Omega)} |u|_{H^{l+1}(\Omega)} + \mathcal{Q}(u_\pi^H)\Big) \left\|\nabla\theta^H\right\|_{L^2(\Omega)},\end{aligned}$$

where we used the estimate (16) and Lemma 5.2. Using Lemma 3.3 to bound $\mathcal{Q}(u_\pi^H)$ yields $\mathcal{Q}(u_\pi^H) \leq C\|u\|_{H^{l+1}(\Omega)}$ for arbitrary $l \geq 1$. We conclude the proof by using Lemma 3.2. ☐

Numerical results. To illustrate the convergence rates from Theorem 5.1 we consider the model problem (1) on $\Omega = (0,1)^2$ with data

$$\mathcal{A}(x, \xi) = \left(1 + \frac{1}{x_1^3 + 0.05} + \frac{1 + x_1 x_2}{(1 + |\xi|^2)^{1/4}}\right)\xi,$$

and a right-hand side function $f(x)$ chosen such that $u(x) = 8\sin(\pi x_1)x_2(1 - x_2)$ is the exact solution of (1). We then discretize Ω by uniform simplicial meshes $\mathcal{T}_H$ with $N = 2^k$ elements in each spatial dimension (with $k = 2, \ldots, 9$) and use FE spaces (4) with polynomial degree $l = 1, 2, 3$. For the implementation we apply a quadrature formula satisfying (**Q2**) with $\mu = 0$ (the right-hand side functional $\int_\Omega f\, w^H dx$ is evaluated using the same quadrature formula). The nonlinear algebraic equations are solved by a Newton method. For our test, around 6 iterations are sufficient for convergence up to machine precision. The relative error measured in the L^2 and H^1 norm are plotted in Figure 1.(a) and 1.(b), respectively.

In Figure 1, we observe that for a given polynomial degree l the errors in the L^2 and H^1 norm decrease at rate $l + 1$ and l, respectively, with respect to $H \sim 1/N$ as predicted by Theorem 5.1. Note that although the applied quadrature rule satisfies (**Q2**) only for $\mu = 0$, instead of $\mu = 1$ assumed for the L^2 estimate in Theorem 5.1, it still leads to the optimal convergence rate in the L^2 norm for the considered test problem.

6 Application: FE-HMM for nonlinear monotone elliptic homogenization problems

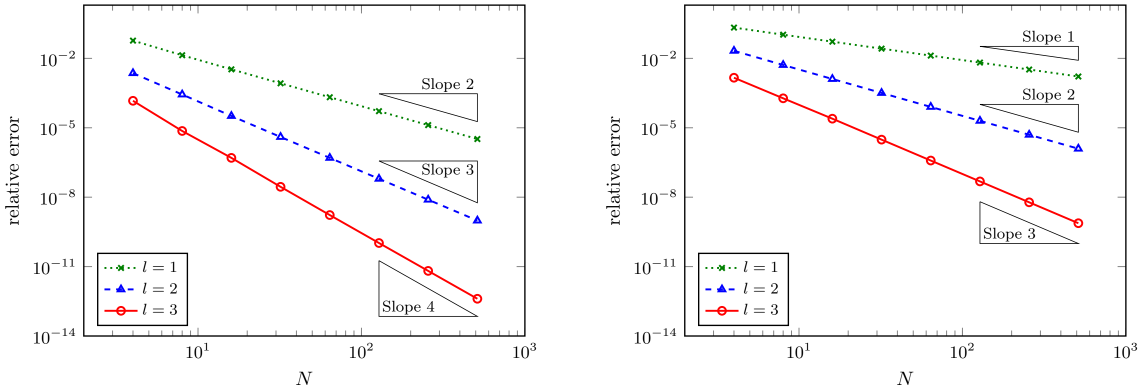
In this section we consider a problem of type (1) with a small scale $\varepsilon > 0$

$$-\operatorname{div}(\mathcal{A}^\varepsilon(x, \nabla u^\varepsilon)) = f \text{ in } \Omega, \qquad u^\varepsilon = 0 \text{ on } \partial\Omega, \tag{27}$$

where $\mathcal{A}^\varepsilon$ satisfies $(\mathcal{A}_{0-2})$ uniformly in ε. Using homogenization theory the multiscale problem (27) can be upscaled, see [14]. For $\varepsilon \to 0$, the solutions u^ε converge in the sense of G-convergence to the homogenized solution u^0 solving the effective problem

$$-\operatorname{div}(\mathcal{A}^0(x, \nabla u^0)) = f \text{ in } \Omega, \qquad u^0 = 0 \text{ on } \partial\Omega, \tag{28}$$

where $\mathcal{A}^0$ is the homogenized map satisfying again $(\mathcal{A}_{0-2})$.



(a) Relative error measured in L^2 norm.

(b) Relative error measured in H^1 norm.

Figure 1: Convergence test for FEM with numerical integration. Relative error as a function of N (the number of elements per spatial dimension of the mesh) for different polynomial degrees l of the FE space, in particular, $l = 1$ (dotted line), $l = 2$ (dashed line) and $l = 3$ (solid line).

FE-HMM. We study a numerical homogenization method that aims at approximating the homogenized solution u^0 by solving (28) for which the unknown map $\mathcal{A}^0$ is locally approximated by numerical upscaling.

Let $\mathcal{T}_H$ be a macroscopic mesh of Ω, i.e., with mesh size $H \gg \varepsilon$, and consider the finite element space $S_0^l(\Omega, \mathcal{T}_H)$, see (4), for $l \in \mathbb{N}_{\geq 1}$. For $K \in \mathcal{T}_H$, let $\{\omega_{K_j}, x_{K_j}\}_{j=1}^J$ be quadrature weights and nodes induced by a quadrature formula on the reference element satisfying (**Q1**) and (**Q2**), see Section 2. For $\delta \geq \varepsilon$, we define the microscopic sampling domains $K_{\delta_j} = x_{K_j} + \delta(-1/2, 1/2)^d$ centered around the quadrature nodes x_{K_j} of the macro element $K \in \mathcal{T}_H$. On each K_{δ_j} and for $q \in \mathbb{N}_{\geq 1}$, we then consider the micro FE-space $S^q(K_{\delta_j}, \mathcal{T}_h) \subset W(K_{\delta_j})$, see (4) with boundary conditions prescribed by $W(K_{\delta_j})$, on a microscopic simplicial mesh $\mathcal{T}_h$. For the space $W(K_{\delta_j})$ we either choose $W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}) = \{v \in H^1_{per}(K_{\delta_j}) \mid \int_{K_{\delta_j}} v\, dx = 0\}$ for periodic coupling or $W(K_{\delta_j}) = H^1_0(K_{\delta_j})$ for Dirichlet coupling.

The FE-HMM approximation u^H of the homogenized solution u^0 is defined by:

$$\text{find } u^H \in S_0^l(\Omega, \mathcal{T}_H) \text{ such that} \qquad B_H(u^H; w^H) = \int_\Omega f\, w^H dx, \quad \forall\, w^H \in S_0^l(\Omega, \mathcal{T}_H), \tag{29}$$

where the modified macroscopic map $B_H(v^H; w^H)$ for $v^H, w^H \in S_0^l(\Omega, \mathcal{T}_H)$ –nonlinear in v^H and linear in w^H– is defined as

$$B_H(v^H; w^H) = \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \frac{\omega_{K_j}}{|K_{\delta_j}|} \int_{K_{\delta_j}} \mathcal{A}^\varepsilon(x, \nabla v^h_{K_j}) dx \cdot \nabla w^H(x_{K_j}), \tag{30}$$

with the micro function $v^h_{K_j}$ solving the constrained nonlinear micro problem: find $v^h_{K_j} - v^H_{lin,j} \in S^q(K_{\delta_j}, \mathcal{T}_h)$ such that

$$\int_{K_{\delta_j}} \mathcal{A}^\varepsilon(x, \nabla v^h_{K_j}) \cdot \nabla z^h dx = 0, \quad \forall z^h \in S^q(K_{\delta_j}, \mathcal{T}_h), \tag{31}$$

where $v^H_{lin,j} = v^H(x_{K_j}) + (x - x_{K_j}) \cdot \nabla v^H(x_{K_j})$ is the linearization of v^H at the quadrature point x_{K_j}. Note that the following energy equivalence holds, see [2, Lemma 3.2],

$$\|\nabla v^H_{lin,j}\|_{L^2(K_{\delta_j})} \leq \left\|\nabla v^h_{K_j}\right\|_{L^2(K_{\delta_j})} \leq \frac{L}{\lambda}\left(\sqrt{|K_{\delta_j}|}\frac{C_0}{L} + \|\nabla v^H_{lin,j}\|_{L^2(K_{\delta_j})}\right). \tag{32}$$

Similarly to $\tilde{B}_H$, we obtain from the hypotheses (**Q1**), ($\mathcal{A}_{0-2}$) and the energy equivalence (32) that the form B_H is continuous in the second variable and Lipschitz continuous as well as strongly monotone in the first variable (arguments analagous to the linear case, see [1, Prop. 2]). Thus, there exists a unique FE-HMM approximation u^H defined by (29).

Error analysis. We now sketch the a priori error analysis for the FE-HMM (for details we refer to [13]). Let us define the additional error contributions due to the HMM upscaling error decomposed into modeling error r_{mod} and micro error r_{mic} evaluated at $v^H \in S^l_0(\Omega, \mathcal{T}_H)$ by

$$\begin{aligned} r_{mod}(v^H) &= \sup_{w^H \in S^l_0(\Omega,\mathcal{T}_H)} \left|\tilde{B}_H(v^H; w^H) - \bar{B}_H(v^H; w^H)\right| \|\nabla w^H\|^{-1}_{L^2(\Omega)}, \\ r_{mic}(v^H) &= \sup_{w^H \in S^l_0(\Omega,\mathcal{T}_H)} |\bar{B}_H(v^H; w^H) - B_H(v^H; w^H)| \|\nabla w^H\|^{-1}_{L^2(\Omega)}, \end{aligned} \tag{33}$$

with the form $\tilde{B}_H$ given as above by (6) but with $\mathcal{A}$ replaced by $\mathcal{A}^0$ and the form $\bar{B}_H$ obtained by replacing the numerical micro solutions $v^h_{K_j}$ in (30) by the exact micro solutions v_{K_j} solving (31) in the Sobolev space $W(K_{\delta_j})$.

Then, the following fully discrete a priori error estimates hold.

Theorem 6.1. *Let u^0 be the exact homogenized solution and u^H its FE-HMM approximation given by* (29)*. Assume that the conditions of Theorem 5.1 hold with $\mu \in \{0, 1\}$ for the homogenized problem* (28)*. Then, we have*

$$\begin{aligned} \|u^0 - u^H\|_{H^1(\Omega)} &\leq C(H^l + r_{mod}(\mathcal{I}_H u^0) + r_{mic}(\mathcal{I}_H u^0)), && \text{if } \mu = 0, \\ \|u^0 - u^H\|_{L^2(\Omega)} &\leq C(H^{l+1} + r_{mod}(u^H_\pi) + r_{mic}(u^H_\pi)), && \text{if } \mu = 1, \end{aligned}$$

with C independent of H, h, δ and ε and where the modeling and micro erro r_{mod} and r_{mic} from (33) *are evaluated at either the nodal interpolant $\mathcal{I}_H u^0$ or the elliptic projection u^H_π of u^0 solving* (8) *with $\mathscr{A}(x) = D_\xi \mathcal{A}^0(x, \nabla u^0)$.*

Proof. Following the lines of the proof of Theorem 5.1 with either $\mathcal{U}^H = \mathcal{I}_H u^0$ (if $\mu = 0$) or $\mathcal{U}^H = u^H_\pi$ (if $\mu = 1$), we obtain $\|\nabla u^H - \nabla \mathcal{U}^H\|_{L^2(\Omega)} \leq C(H^{l+\mu} + r_{mod}(\mathcal{U}^H) + r_{mic}(\mathcal{U}^H))$. □

If additionally the exact micro solutions v_{K_j} are smooth enough we can give an error bound with a decay rate for the micro error. For piecewise linear micro FE ($q = 1$ in (31)), we have the bound from [2]

$$r_{mic}(\mathcal{U}^H) \leq C(\tfrac{h}{\varepsilon})^2.$$

The modeling error r_{mod} can be estimated if $\mathcal{A}^\varepsilon$ is locally periodic, i.e., $\mathcal{A}^\varepsilon(x, \xi) = \mathcal{A}(x, x/\varepsilon, \xi)$ where $\mathcal{A}(x, y, \xi)$ is $(0, 1)^d$-periodic in y and sufficiently smooth. If we further replace $\mathcal{A}^\varepsilon(x, \xi)$ by $\mathcal{A}(x_{K_j}, x/\varepsilon, \xi)$ in (30) and (31), we have, see [2],

$$\begin{aligned} r_{mod}(\mathcal{U}^H) &= 0, && \text{if } W(K_{\delta_j}) = W^1_{per}(K_{\delta_j}), \tfrac{\delta}{\varepsilon} \in \mathbb{N}_{\geq 1}, \\ r_{mod}(\mathcal{U}^H) &\leq C(\tfrac{\varepsilon}{\delta})^{1/2}, && \text{if } W(K_{\delta_j}) = H^1_0(K_{\delta_j}), \delta > \varepsilon. \end{aligned}$$

Acknowledgements. This work was supported in part by the Swiss National Science Foundation under Grant 200021_134716/1.

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