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WEIGHTED REDUCED BASIS METHOD FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH ELLIPTIC PDE CONSTRAINT

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Abstract. In this paper we develop and analyze an efficient computational method for solving stochastic optimal control problems constrained by elliptic partial differential equation (PDE) with random input data. We first prove both existence and uniqueness of the optimal solution. Regularity of the optimal solution in the stochastic space is studied in view of the analysis of stochastic approximation error. For numerical approximation, we employ finite element method for the discretization of physical variables and stochastic collocation method for the discretization of random variables. In order to alleviate the computational effort, we develop a model order reduction strategy based on a weighted reduced basis method. A global error analysis of the numerical approximation is carried out and several numerical tests are performed to verify our analysis.

Key words. uncertainty quantification, stochastic optimal control, saddle point formulation, stochastic regularity, stochastic collocation method, weighted reduced basis method, error estimate

AMS subject classifications. 35J20, 49N10, 65K10, 65C30, 60H15

1. Introduction.

Optimal control problems are often associated to partial differential equations (PDEs) when modeling physical processes in several fields of applied sciences. The cost functional to minimize or maximize expresses the discrepancy between the optimal solution of the PDE model and suitable experimental measurements or observations with a possible additional regularization term of the control function. In practical applications, uncertainties arising from various sources, for instance the PDE coefficients, boundary conditions, external loadings, experimental measurements, may have a significant impact on the optimal solution. Mathematical theories and computational methods have been developed for many years in dealing with deterministic optimal control problems without taking the uncertainties into account [17, 12, 32], while stochastic optimal control problems with PDE constraints and random inputs have gained substantial attention only recently [14, 13, 27, 16, 31, 7].

When solving stochastic optimal control problems, several challenging issues deserve attention. Among those we mention the following that will be addressed in this work: the well-posedness of the stochastic optimal problem, including proof of existence, uniqueness and regularity of the stochstic optimal solution, the set up of efficient approximation methods in both physical space and probability space, especially when the latter is in high dimensions leading to the "curse-of-dimensionality", the ill conditioning and coupled nature of the optimality system derived from the optimal control problems that make its numerical solution very involved.

About well-posedness, following Lions' theory [17] for optimal control problems in deterministic cases [12, 32], existence of optimal solution of stochastic optimal control problems can be obtained in many different problem settings, see [14, 13, 16]. In addition, Brezzi-Rappaz-Raviart theory (see [14, 13]) has also been invoked to prove existence of a Lagrangian multiplier for stochastic optimal control problems constrained by steady diffusion equations with deterministic distributed control function

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and Neumann boundary control function, respectively. In [7], the authors formulated the stochastic optimal control problem constrained by an advection-diffusion equation with a random advection field and boundary control function into a stochastic saddle point problem and employed Brezzi theorem [4] to obtain both existence and uniqueness properties of the stochastic optimal solution. We mention that the uniqueness of optimal solution for stochastic optimal control problems in more general settings, e.g. with parabolic or nonlinear PDE constraints, boundary or distributive control functions, various extensions of cost functional [31], are still open.

When it comes to the numerical approximation of the stochastic optimal control problems, one may directly apply some well developed methods for deterministic approximation in physical space, for instance finite element method with appropriate preconditioning techniques [22, 32]. As for stochastic approximation in the probability space, besides the application of the fast convergent stochastic Galerkin method [13, 27], the recent works [27, 31, 16, 7] made use of the stochastic collocation method [33, 1, 21], which is a non-intrusive method featuring both easy implementation and fast convergence, thus avoiding solving a tensor-product large algebraic system encountered by the intrusive stochastic Galerkin method. Finally, sequential quadratic programming and trust-region algorithm were applied together with stochastic collocation method in [31] and [16] to alleviate computational cost in solving the optimality system. However, when the optimality system becomes very expensive to solve, more efficient techniques are needed in order to solve the optimality system for many times. In this perspective, model order reduction techniques such as proper orthogonal decomposition or reduced basis method are promising, see [18, 19] for the application of the latter method in solving parametrized optimal control problems.

In this paper, we consider the stochastic optimal control problem constrained by a linear elliptic equation with distributed stochastic control function. We provide an analysis of well-posedness, that is existence, uniqueness and stability of the stochastic optimal solution of this stochastic optimal control problem for the first time. We use finite element method with (optimal) preconditioning techniques for deterministic approximation of the optimal solution in physical space and stochastic collocation method for stochastic approximation in the probability space. To reduce the computational cost of solving a considerable number of optimality systems, we develop a model order reduction strategy based on a weighted reduced basis method, leading to a reduced optimality system that enables many-query solutions with a posteriori error estimate. Analysis of a global error of the stochastic optimal solution and its statistics is carried out by studying error contributions from different sources: the finite element approximation, the stochastic collocation approximation, and the weighted reduced basis approximation. Convergence results and computational efficiency and accuracy are verified and illustrated by numerical tests in multi-dimensional probability space.

The paper is organized as follows: in section 2 we state the stochastic optimal control problems with elliptic PDE constraints and random inputs. Section 3 is devoted to the study of the mathematical properties of the stochastic optimal control problems Analytic regularity of the stochastic optimal solution in probability space is obtained in section 4 via recursively using Brezzi's theorem for stability estimate. In section 5, we present numerical approximation to solve the stochastic optimality system, followed by section 6 for both separate and global error estimates of the proposed numerical approximations. Numerical tests for verification and illustration of our method are reported in section 7. We close the paper by drawing some conclusions and indicating possible future developments in the last section 8.

2. Problem statement. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space, where Ω is a set of outcomes $\omega \in \Omega$, $\mathfrak{F}$ is a σ-algebra of events and $P : \mathfrak{F} \to [0, 1]$, $P(\Omega) = 1$, is a probability measure. Let D be an open and bounded physical domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary ∂D. Let $v : D \times \Omega \to \mathbb{R}$ represent a *random field*, which is a real-valued random function defined in D for each outcome $\omega \in \Omega$. We define the product Hilbert space $\mathcal{H}^s(D) := L^2(\Omega) \otimes H^s(D), s \in \mathbb{R}$ equipped with the norm

$$||v||_{\mathcal{H}^s(D)} := \left(\int_\Omega ||v(\cdot, \omega)||^2_{H^s(D)} dP(\omega)\right)^{1/2} < \infty, \tag{2.1}$$

where $H^s(D)$ is the Hilbert space of functions defined in the physical domain D [24, 22]. When $s = 0$, we denote $H^0(D) \equiv L^2(D)$, and thus $\mathcal{H}^0(D) \equiv \mathcal{L}^2(D)$ by convention. Similar to its deterministic counterpart, we define the stochastic inner product

$$(w, v) = \int_\Omega \int_D wv dx dP(\omega) \quad \forall w, v \in \mathcal{L}^2(D). \tag{2.2}$$

2.1. Karhunen-Loève expansion. Suppose that the random field v has a continuous and bounded covariance function defined as

$$\mathbb{C}[v](x, x') := \mathbb{E}[(v(x, \cdot) - \mathbb{E}[v](x)))(v(x', \cdot) - \mathbb{E}[v](x')))], \quad \forall x, x' \in D, \tag{2.3}$$

where the expectation $\mathbb{E}[v]$ of the random field v is given by

$$\mathbb{E}[v](x) := \int_\Omega v(x, \omega) dP(\omega), \quad \forall x \in D. \tag{2.4}$$

Then by Mercer's theorem [26], for almost every (a.e.) $\omega \in \Omega$, the following Karhunen-Loève expansion holds

$$v(x, \omega) = \mathbb{E}[v](x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} v_n(x) y_n(\omega), \quad \forall x \in D, \tag{2.5}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the positive eigenvalues of the covariance function $\mathbb{C}[v]$, $v_n, n = 1, 2, \ldots,$ are the orthonormal eigenfunctions, and $y_n, n = 1, 2, \ldots,$ are uncorrelated random variables with zero mean and unit variance. Moreover, the truncated Karhunen-Loève expansion defined as

$$v_N(x, \omega) := \mathbb{E}[v](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} v_n(x) y_n(\omega), \quad \forall x \in D, \tag{2.6}$$

represents the best N-term approximation of the random field v in $\mathcal{H}^s(D)$. The associated approximation/truncation error reads

$$||v - v_N||_{\mathcal{H}^s(D)} = \left(\sum_{n=N+1}^{\infty} \lambda_n ||v_n||^2_{H^s(D)}\right)^{1/2}. \tag{2.7}$$

When $s = 0$, we have $||v_n||_{H^0(D)} = 1, n = 1, 2, \ldots,$ and the approximation error becomes the sum of all the eigenvalues except the first N terms. Provided that the covariance function is smooth, the eigenvalues decay exponentially fast to zero [30]. In general, we can make the following assumption for a random field of interest.

Assumption 1. *The random field* $v : D \times \Gamma \to \mathbb{R}$ *depends on finite dimensional independent random variables* $y := (y_1, \dots, y_N) : \Omega \to \mathbb{R}^N$ *with bounded image* $\Gamma := \prod_{n=1}^N \Gamma_n \subset \mathbb{R}^N$ *and joint probability density function* $\rho := \prod_{n=1}^N \rho_n : \Gamma \to \mathbb{R}$. *Moreover, for the sake of simplicity, we assume*

$$v(x, y) = v_0(x) + \sum_{n=1}^N v_n(x) y_n(\omega) = \sum_{n=0}^N v_n(x) y_n, \tag{2.8}$$

where we can identify $v_0 = \mathbb{E}[v], y_0 = 1$ *and* $v_n = \sqrt{\lambda_n} v_n$ *or* $y_n = \sqrt{\lambda_n} y_n, n = 1, \dots, N$ *for a random field with the truncated Karhunen-Loève expansion (2.6).*

Under Assumption 1, the product Hilbert space $\mathcal{H}^s(D)$ can be rewritten as $\mathcal{H}^s(D) := L^2_\rho(\Gamma) \otimes H^s(D)$ with associated norm

$$||v||_{\mathcal{H}^s(D)} := \left(\int_\Gamma ||v(\cdot, y)||^2_{H^s(D)} \rho(y) dy \right)^{1/2} < \infty. \tag{2.9}$$

2.2. Stochastic elliptic PDEs. We consider the following elliptic homogeneous Dirichlet boundary value problem

$$\begin{cases} -\nabla \cdot (a(x, y) \nabla u(x, y)) & = f(x, y) + g(x, y) \qquad (x, y) \in D \times \Gamma, \\ u(x, y) & = 0 \qquad (x, y) \in \partial D \times \Gamma, \end{cases} \tag{2.10}$$

where a is a random coefficient field, f is a random force field and g is a random field representing a distributed control. We make the following assumptions for the random data

Assumption 2. *The random coefficient* a *is uniformly bounded from above and below, i.e. there exist positive constants* $0 < r < R < \infty$ *such that*

$$P\left(\omega \in \Omega : r ||v||^2_{H^1(D)} \le (a(\cdot, y(\omega)) v, v) \le R ||v||^2_{H^1(D)} \right) = 1, \quad \forall v \in H^1(D). \tag{2.11}$$

The random force term f *and random control function* g *have bounded second moment*

$$\int_\Gamma \int_D f^2(x, y) \rho(y) dx dy < \infty \text{ and } \int_\Gamma \int_D g^2(x, y) \rho(y) dx dy < \infty. \tag{2.12}$$

Moreover, the random fields a *and* f *admit the linear expansion (2.8) as*

$$a(x, y(\omega)) = a_0(x) + \sum_{n=1}^N a_n(x) y_n(\omega) \text{ and } f(x, y(\omega)) = f_0(x) + \sum_{n=1}^N f_n(x) y_n(\omega). \tag{2.13}$$

Let us denote $y_0 = 1$ for ease of notation. Under Assumption 2, we have the following weak formulation for problem (2.10): find $u \in \mathcal{H}^1_0(D)$ such that

$$\mathcal{B}(u, v) = \mathcal{F}(v) + \mathcal{G}(v) \quad \forall v \in \mathcal{H}^1_0(D), \tag{2.14}$$

where $\mathcal{H}^1_0(D) := \{v \in \mathcal{H}^1(D), v = 0 \text{ on } \partial D\}$, $\mathcal{G}(v) = (g, v)$ and the bilinear form $\mathcal{B}$ and the linear functional $\mathcal{F}$ are defined as

$$\mathcal{B}(u, v) = \sum_{n=0}^N \int_\Gamma (B_n(u, v) y_n) \rho(y) dy \text{ and } \mathcal{F}(v) = \sum_{n=0}^N \int_\Gamma (F_n(v) y_n) \rho(y) dy, \tag{2.15}$$

with $B_n(u,v) = (a_n \nabla u, \nabla v); F_n(v) = (f_n, v), n = 0, \ldots, N$. We denote $y_0 = 1$.

THEOREM 2.1. *Provided that the data satisfy Assumption 2, there exists a unique solution $u \in \mathcal{H}_0^1(D)$ of problem (2.14) such that*

$$||u||_{\mathcal{H}_0^1(D)} \le (1/r) \left(||f||_{\mathcal{L}^2(D)} + ||g||_{\mathcal{L}^2(D)} \right). \tag{2.16}$$

Proof. The proof follows directly from that of the deterministic case [24, 22]. ∎

2.3. Constrained optimal control problems. Optimal control problems constrained by stochastic elliptic PDEs consist in finding a stochastic optimal control function $g^* \in \mathcal{L}^2(D)$ that minimizes a cost functional $\mathcal{J}(u,g)$ under an elliptic PDE constraint: find $(u^*, g^*) \in \mathcal{U}$ such that

$$\mathcal{J}(u^*, g^*) = \min_{(u,g)\in\mathcal{U}} \mathcal{J}(u,g) \text{ subject to problem (2.14)}, \tag{2.17}$$

where $\mathcal{U}$ is an admissible solution space defined without loss of generality as $\mathcal{U} = \mathcal{H}_0^1(D) \otimes \mathcal{L}^2(D)$, and the quadratic cost functional is defined as [13, 7]

$$\mathcal{J}(u,g) = \mathbb{E}\left[\frac{1}{2} \int_D |u - u_d|^2 dx + \frac{\alpha}{2} \int_D |g|^2 dx \right], \tag{2.18}$$

in which $u_d \in L^2(D)$ is provided as an observation function, e.g. the mean of a sequence of experimental measures, α is a positive regularization parameter.

THEOREM 2.2. *There exists an optimal solution $(u^*, g^*) \in \mathcal{U}$ to problem (2.17).*

Proof. The proof is straightforward by following Lions' argument for deterministic optimal control problems [17], see also similar proof in [13] for stochastic cases. ∎

Remark 2.1. *When higher moments, e.g. variance, of the observational data u_d or the control function g, or the probability distribution of u_d are incorporated into the cost functional in more general settings as considered in [31], we face essentially nonlinear and fully coupled stochastic problems, which will be addressed in [8].*

3. Saddle point formulation. In this section, we introduce the stochastic optimality system and derive a saddle point formulation of the optimal control problem (2.17). By establishing the equivalence between the optimality system and saddle point problem, we shall prove that there exists a unique solution to both the optimality system and the optimal control problem.

3.1. Stochastic optimality system. Let us first derive the stochastic optimality system to the optimal control problem (2.17) by Lagrangian approach [32]. Define the following stochastic Lagrangian functional associated to problem (2.17) as

$$\mathcal{L}(u, g, p; y) = \mathcal{J}(u,g) + \mathcal{B}(u,p) - \mathcal{F}(p) - \mathcal{G}(p), \tag{3.1}$$

where $p \in \mathcal{H}_0^1(D)$ is named the adjoint variable or Lagrangian parameter [32]. By taking Gâteaux derivative of the Lagrangian functional (3.1) with respect to the variables p, g, u evaluated at q, h, v, we obtain the first order necessary optimality conditions of the stochastic optimal control problem (2.17) - the stochastic optimality system:

$$\left\{ \begin{array}{rll} \mathcal{B}(u,q) - \mathcal{G}(q) & = \mathcal{F}(q) & \forall q \in \mathcal{H}_0^1(D), \\ (\alpha g - p, h) & = 0 & \forall h \in \mathcal{L}^2(D), \\ \mathcal{B}'(p,v) + (u,v) & = (u_d, v) & \forall v \in \mathcal{H}_0^1(D), \end{array} \right. \tag{3.2}$$

where $\mathcal{B}'$ is the adjoint bilinear form of $\mathcal{B}$, $\mathcal{B}'(p,v) = \mathcal{B}(v,p)$. As a consequence of Theorem 2.2, it has been proven in [14, 13] that there exists an adjoint variable $p^* \in \mathcal{H}_0^1(D)$ associated to the optimal solution (u^*, g^*) such that (u^*, g^*, p^*) is a solution of the stochastic optimal system (3.2). In the following, we will show that (u^*, g^*, p^*) is the unique solution to system (3.2); moreover, (u^*, g^*) is also the unique solution of the stochastic optimal control problem (2.17).

3.2. Saddle point formulation. Define the following bilinear form

$$\mathcal{A}(\underline{u}, \underline{v}) := (u, v) + \alpha(g, h) \quad \forall \underline{u}, \underline{v} \in \mathcal{U}, \tag{3.3}$$

where the new variables $\underline{u}, \underline{v}$ are given by $\underline{u} = (u, g) \in \mathcal{U}$ and $\underline{v} = (v, h) \in \mathcal{U}$; $\mathcal{A}$ and the cost functional $\mathcal{J}$ are related as

$$\mathcal{J}(u, g) = \frac{1}{2}\mathcal{A}(\underline{u}, \underline{u}) - (u_d, u) + \frac{1}{2}(u_d, u_d). \tag{3.4}$$

By defining the new observation function $\underline{u}_d = (u_d, 0) \in \mathcal{U}$, we introduce the following cost functional (for ease of notation, we still use $\mathcal{J}$)

$$\mathcal{J}(\underline{u}) := \frac{1}{2}\mathcal{A}(\underline{u}, \underline{u}) - (\underline{u}_d, \underline{u}), \tag{3.5}$$

which is different from the original cost functional (3.4) because of the constant $(\underline{u}_d, \underline{u}_d)/2$. Similarly, let the bilinear form $\mathcal{B}$ in the optimality system (3.2) be redefined as

$$\mathcal{B}(\underline{u}, q) := \mathcal{B}(u, q) - \mathcal{G}(q) \quad \forall \underline{u} \in \mathcal{U}, q \in \mathcal{H}_0^1(D). \tag{3.6}$$

In the following, the definition of $\mathcal{B}$ will be identified by its first argument. To this end, we can rewrite the optimal control problem (2.17) equivalently as

$$\min_{\underline{u} \in \mathcal{U}} \mathcal{J}(\underline{u}) \text{ subject to } \mathcal{B}(\underline{u}, q) = \mathcal{F}(q), \quad \forall q \in \mathcal{H}_0^1(D). \tag{3.7}$$

The following proposition establishes the equivalence of the stochastic optimal control problem (3.7) and the stochastic saddle point problem (3.12) [4, 3, 19]. Its proof follows directly from its deterministic counterpart, see [4].

Proposition 3.1. *Suppose that the bilinear form $\mathcal{A}$ is symmetric, non-negative and continuous, i.e. there exists a constant $\gamma > 0$ such that*

$$\mathcal{A}(\underline{u}, \underline{v}) = \mathcal{A}(\underline{v}, \underline{u}), \mathcal{A}(\underline{u}, \underline{u}) \geq 0 \text{ and } \mathcal{A}(\underline{u}, \underline{v}) \leq \gamma ||\underline{u}||_{\mathcal{U}} ||\underline{v}||_{\mathcal{U}}, \quad \forall \underline{u}, \underline{v} \in \mathcal{U}, \tag{3.8}$$

Moreover, suppose that $\mathcal{A}$ is strongly coercive on the kernel space $\mathcal{U}_0 := \{\underline{u} \in \mathcal{U} : \mathcal{B}(\underline{u}, q) = 0, \forall q \in \mathcal{H}_0^1(D)\}$, i.e. there exists constant $\varsigma > 0$ such that

$$\mathcal{A}(\underline{u}, \underline{u}) \geq \varsigma ||\underline{u}||_{\mathcal{U}}^2, \quad \forall \underline{u} \in \mathcal{U}_0. \tag{3.9}$$

Suppose also that the bilinear form $\mathcal{B}$ is continuous, i.e. there exists constant $\delta > 0$ such that

$$\mathcal{B}(\underline{u}, q) \leq \delta ||\underline{u}||_{\mathcal{U}} ||q||_{\mathcal{H}_0^1(D)}, \quad \forall \underline{u} \in \mathcal{U}, q \in \mathcal{H}_0^1(D), \tag{3.10}$$

and satisfies the compatibility (inf-sup) condition, i.e. there exists constant $\beta > 0$ such that

$$\inf_{q \in \mathcal{H}_0^1(D)} \sup_{\underline{v} \in \mathcal{U}} \frac{\mathcal{B}(\underline{v}, q)}{||\underline{v}||_{\mathcal{U}} ||q||_{\mathcal{H}_0^1(D)}} \geq \beta. \tag{3.11}$$

Then the optimal control problem (3.7) is equivalent to the following saddle point problem: find $(\underline{u}, p) \in \mathcal{U} \otimes \mathcal{H}_0^1(D)$ *such that*

$$\left\{ \begin{array}{rll} \mathcal{A}(\underline{u}, \underline{v}) + \mathcal{B}(\underline{v}, p) & = (\underline{u}_d, \underline{v}) & \forall \underline{v} \in \mathcal{U}, \\ \mathcal{B}(\underline{u}, q) & = \mathcal{F}(q) & \forall q \in \mathcal{H}_0^1(D). \end{array} \right. \tag{3.12}$$

3.3. Equivalence, uniqueness and existence. We provide our main results in the section. The first result is about equivalence, stated in the following lemmas.

LEMMA 3.2. *Under the definition (3.3) for* $\mathcal{A}$ *and (3.6) for* $\mathcal{B}$*, the stochastic optimal control problem (2.17) admits an equivalent saddle point formulation (3.12).*

Proof. By Proposition 3.1, we only need to verify all the properties for the bilinear forms $\mathcal{A}$ and $\mathcal{B}$ specified in the assumption of Proposition 3.1. Let us begin with $\mathcal{A}$.

Properties of $\mathcal{A}$: From the definition (3.3), we have that $\mathcal{A}$ is symmetric and non-negative. As for the continuity property, we have

$$\begin{aligned} \mathcal{A}(\underline{u}, \underline{v}) &\leq ||u||_{\mathcal{L}^2(D)} ||v||_{\mathcal{L}^2(D)} + \alpha ||g||_{\mathcal{L}^2(D)} ||h||_{\mathcal{L}^2(D)} \\ &\leq ||\underline{u}||_{\mathcal{U}} ||\underline{v}||_{\mathcal{U}}, \quad \forall \underline{u}, \underline{v} \in \mathcal{U}, \end{aligned} \tag{3.13}$$

where the first inequality is due to Cauchy-Schwarz inequality [24] and the second one is a result of $||v||_{\mathcal{L}^2(D)} \leq ||v||_{\mathcal{H}_0^1(D)} := ||v||_{\mathcal{L}^2(D)} + ||\nabla v||_{\mathcal{L}^2(D)}, \forall v \in \mathcal{H}_0^1(D)$ and the definition of the norm $||\underline{v}||_{\mathcal{U}} = ||v||_{\mathcal{H}_0^1(D)} + \sqrt{\alpha}||h||_{\mathcal{L}^2(D)}$. To show that $\mathcal{A}$ is strongly coercive in $\mathcal{U}_0$, we have $\mathcal{B}(\underline{u}, q) = 0, \forall q \in \mathcal{H}_0^1(D)$ when $\underline{u} \in \mathcal{U}_0$, thus $\mathcal{B}(u, q) = (g, q), \forall q \in \mathcal{H}_0^1(D)$, and Theorem 2.1 holds with $f = 0$. Consequently, we obtain

$$\begin{aligned} \mathcal{A}(\underline{u}, \underline{u}) &= ||u||^2_{\mathcal{L}^2(D)} + \alpha ||g||^2_{\mathcal{L}^2(D)} \\ &\geq \frac{\alpha r^2}{2} ||u||^2_{\mathcal{H}_0^1(D)} + \frac{\alpha}{2} ||g||^2_{\mathcal{L}^2(D)} \\ &\geq \min\left\{ \frac{\alpha r^2}{4}, \frac{1}{4} \right\} \left(||u||_{\mathcal{H}_0^1(D)} + \sqrt{\alpha} ||g||_{\mathcal{L}^2(D)} \right)^2 \\ &= \min\left\{ \frac{\alpha r^2}{4}, \frac{1}{4} \right\} ||\underline{u}||^2_{\mathcal{U}}, \quad \forall \underline{u} \in \mathcal{U}_0, \end{aligned} \tag{3.14}$$

where we split the second term into two equal parts and used the estimate (2.16) for the first inequality, the second inequality is a result of Cauchy-Schwarz inequality. Therefore, the strong coercivity of the bilinear form $\mathcal{A}$ holds by the estimate (3.14).

Properties of $\mathcal{B}$: The continuity of the bilinear form $\mathcal{B} : \mathcal{U} \otimes \mathcal{H}_0^1(D) \to \mathbb{R}$ defined in (3.6) is shown by

$$\begin{aligned} \mathcal{B}(\underline{u}, q) &\leq R ||u||_{\mathcal{H}_0^1(D)} ||q||_{\mathcal{H}_0^1(D)} + ||g||_{\mathcal{L}^2(D)} ||q||_{\mathcal{L}^2(D)} \\ &\leq \max\left\{ R, \frac{1}{\sqrt{\alpha}} \right\} ||\underline{u}||_{\mathcal{U}} ||q||_{\mathcal{H}_0^1(D)}, \quad \forall \underline{u} \in \mathcal{U}, q \in \mathcal{H}_0^1(D), \end{aligned} \tag{3.15}$$

where the first inequality comes from Assumption 2 with R defined in (2.11) and Cauchy-Schwarz inequality, and the second one is a result of the definition of the norms $||\cdot||_{\mathcal{U}}$ and $||\cdot||_{\mathcal{H}^1_0(D)}$. As for the compatibility (inf-sup) condition of $\mathcal{B}$, we have

$$\begin{aligned}\inf_{q\in\mathcal{H}^1_0(D)}\sup_{\underline{v}\in\mathcal{U}}\frac{\mathcal{B}(\underline{v},q)}{||\underline{v}||_{\mathcal{U}}||q||_{\mathcal{H}^1_0(D)}} &= \inf_{q\in\mathcal{H}^1_0(D)}\sup_{\underline{v}\in\mathcal{U}}\frac{\mathcal{B}(v,q)-(h,q)}{(||v||_{\mathcal{H}^1_0(D)}+\sqrt{\alpha}||h||_{\mathcal{L}^2(D)})||q||_{\mathcal{H}^1_0(D)}}\\ &\geq \inf_{q\in\mathcal{H}^1_0(D)}\sup_{(v,0)\in\mathcal{U}}\frac{\mathcal{B}(v,q)}{||v||_{\mathcal{H}^1_0(D)}||q||_{\mathcal{H}^1_0(D)}}\\ &\geq r,\end{aligned} \tag{3.16}$$

where we set $h=0$ in the first inequality and the second one holds by Assumption 2 with coercivity constant $r>0$. Hence, the inf-sup condition is verified for $\mathcal{B}$. □

LEMMA 3.3. *The stochastic saddle point problem (3.12) is equivalent to the stochastic optimality system (3.2).*

Proof. Let us first rewrite the saddle point formulation (3.12) more explicitly as: find $(u,g,p)\in\mathcal{H}^1_0(D)\otimes\mathcal{L}^2(D)\otimes\mathcal{H}^1_0(D)$ such that

$$\begin{cases} (u,v)+\alpha(g,h)+\mathcal{B}(v,p)-(h,p) &= (u_d,v) \quad \forall v\in\mathcal{H}^1_0(D),\forall h\in\mathcal{L}^2(D),\\ \mathcal{B}(u,q)-(g,q) &= \mathcal{F}(q) \quad\ \ \forall q\in\mathcal{H}^1_0(D).\end{cases} \tag{3.17}$$

We can see that $(3.2)_1$ is the same as $(3.17)_2$. By setting $h=0$ (respectively, $v=0$) in $(3.17)_1$, we can retrieve $(3.2)_3$ (respectively, $(3.2)_2$). On the other hand, by adding $3.2)_2$ to $3.2)_3$ and noting that $\mathcal{B}'(p,v)=\mathcal{B}(v,p)$, we obtain $(3.17)_1$. □

Remark 3.1. *Lemma 3.2 and Lemma 3.3 imply that to solve the stochastic optimal control problem (2.17) is equivalent to solve the stochastic optimality system (3.2), being not only the first order necessary condition but also a sufficient condition.*

The main result for existence and uniqueness are stated in the following theorems.

THEOREM 3.4. *There exists a unique solution $(\underline{u},p)\in\mathcal{U}\otimes\mathcal{H}^1_0(D)$ to the stochastic saddle point problem (3.12). Moreover, we have the following a-priori estimates*

$$||\underline{u}||_{\mathcal{U}} \leq \alpha_1||u_d||_{L^2(D)}+\beta_1||f||_{\mathcal{L}^2(D)}, \tag{3.18}$$

and

$$||p||_{\mathcal{H}^1_0(D)} \leq \alpha_2||u_d||_{L^2(D)}+\beta_2||f||_{\mathcal{L}^2(D)}, \tag{3.19}$$

where the positive constants $\alpha_1,\alpha_2,\beta_1,\beta_2$ are given by

$$\alpha_1=\frac{1}{r'},\quad \alpha_2=\frac{1}{r}(1+R\alpha_1),\quad \beta_1=\frac{\alpha_1}{r}\left(r'+R\right),\quad \beta_2=\frac{R}{r}\beta_1, \tag{3.20}$$

being r and R defined in (2.11) and $r'=\min\{\alpha r^2/4,1/4)\}$ is set according to (3.14) with regularization parameter α introduced in (3.4).

Proof. The result is due to Brezzi's theorem [4] in the stochastic setting, whose proof inherits directly from its deterministic counterpart, see [4] or [24]. □

THEOREM 3.5. *There exists a unique solution to the stochastic optimal control problem (2.17), which is the unique solution of the stochastic optimality system (3.2).*

Proof. The result is a consequence of the results in Lemma 3.2 and Lemma 3.3 for the equivalence of the stochastic optimal control problem (2.17), the stochastic saddle point problem (3.12) and the stochastic optimality system (3.2) as well as the result in Theorem 3.4 for existence and uniqueness of a solution to (3.12). □

4. Stochastic regularity. In order to derive error estimates for the numerical approximation of the stochastic optimal control problem (2.17), we first study the regularity of the optimal solution in the stochastic space. In particular, we consider the stochastic regularity of the saddle point problem (3.12) in semi-weak formulation, i.e. weak formulation only in deterministic space: $\forall y \in \Gamma$, find $(\underline{u}(y), p(y)) \in U \otimes V$ (where $U := H_0^1(D) \otimes L^2(D)$ and $V := H_0^1(D)$) such that

$$\begin{cases} A(\underline{u}(y), \underline{v}) + B(\underline{v}, p(y); y) &= (\underline{u}_d, \underline{v}) \qquad \forall \underline{v} \in U, \\ B(\underline{u}(y), q; y) &= F(q; y) \qquad \forall q \in V, \end{cases} \tag{4.1}$$

where the semi-weak bilinear forms A and B and linear functional F are the deterministic counterparts (without taking stochastic integral $\int_\Gamma \cdot \rho(y) dy$) of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ defined in (3.3), (3.6) and (2.15), respectively. Note that B depends on y also through the random coefficient $a(y)$ and F through the random force $f(y)$.

THEOREM 4.1. *For every multi-index $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$, with $|\nu| = \nu_1 + \dots + \nu_N$, the $|\nu|$-th order partial derivative of the solution of (4.1) with respect to y, defined as $\partial_y^\nu(\cdot) = \frac{\partial^{|\nu|}(\cdot)}{\partial_{y_1}^{\nu_1} \cdots \partial_{y_N}^{\nu_N}}$, are bounded by*

$$\begin{aligned} ||\partial_y^\nu \underline{u}(y)||_U + ||\partial_y^\nu p(y)||_V \leq & C^{|\nu|+1} |\nu|! ||a||_\infty^\nu (||u_d||_2 + ||f(y)||_2) \\ & + C^{|\nu|} \sum_{n:\nu_n \neq 0} |\nu - e_n|! ||a||_\infty^{\nu - e_n} ||f_n||_2 \quad \forall y \in \Gamma, \end{aligned} \tag{4.2}$$

where $||\cdot||_\infty$ and $||\cdot||_2$ are shorthand notations standing for $||\cdot||_{L^\infty(D)}$ and $||\cdot||_{L^2(D)}$, $||a||_\infty^\nu := \prod_{n=1}^N ||a_n||_\infty^{\nu_n}$ where $a_n, f_n, 1 \leq n \leq N$ are defined in (2.13), e_n is a N-dimensional vector with the n-th element equal to 1 and other elements zeros, the constant $C = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$.

Proof. The proof is based on a recursive procedure and the use of Brezzi's theorem for stability estimate. When $|\nu| = 0$, by Theorem 3.4 in the deterministic setting we have

$$||\underline{u}(y)||_U + ||p(y)||_V \leq (\alpha_1 + \alpha_2)||u_d||_2 + (\beta_1 + \beta_2)||f(y)||_2 \leq C(||u_d||_2 + ||f(y)||_2), \tag{4.3}$$

which satisfies (4.2). For $|\nu| \geq 1$, by taking $|\nu|$-th order derivative of (4.1) with respect to y, we obtain

$$\begin{cases} A(\partial_y^\nu \underline{u}(y), \underline{v}) + B(\underline{v}, \partial_y^\nu p(y); y) = - \sum_{n:\nu_n \neq 0} \nu_n (a_n \nabla v, \nabla \partial_y^{\nu - e_n} p(y)) \quad \forall \underline{v} \in U, \\ B(\partial_y^\nu \underline{u}(y), q; y) = (\partial_y^\nu f(y), q) - \sum_{n:\nu_n \neq 0} \nu_n (a_n \nabla \partial_y^{\nu - e_n} u(y), \nabla q) \quad \forall q \in V, \end{cases} \tag{4.4}$$

where $\partial_y^\nu f(y)$ vanishes for all $|\nu| > 1$ due to assumption (2.13). By applying Brezzi's theorem 3.4 in the deterministic setting, we obtain the stability estimate for the solution $(\partial_y^\nu \underline{u}(y), \partial_y^\nu p(y)) \in U \otimes V$ to the problem (4.4) as

$$\begin{aligned} & ||\partial_y^\nu \underline{u}(y)||_U + ||\partial_y^\nu p(y)||_V \leq (\beta_1 + \beta_2)||\partial_y^\nu f(y)||_2 \\ & + \sum_{n:\nu_n \neq 0} \nu_n ||a_n||_\infty \Big((\beta_1 + \beta_2)||\partial_y^{\nu - e_n} \underline{u}(y)||_U + (\alpha_1 + \alpha_2)||\partial_y^{\nu - e_n} p(y)||_V \Big) \\ & \leq C||\partial_y^\nu f(y)||_2 + C \sum_{n:\nu_n \neq 0} \nu_n ||a_n||_\infty \Big(||\partial_y^{\nu - e_n} \underline{u}(y)||_U + ||\partial_y^{\nu - e_n} p(y)||_V \Big), \end{aligned} \tag{4.5}$$

for which we have summed up the two stability estimates in Theorem 3.4 and used the inequalities $||\nabla \partial_y^{\nu-e_n} u(y)||_2 \leq ||\partial_y^{\nu-e_n} \underline{u}(y)||_U$ and $||\nabla \partial_y^{\nu-e_n} p(y)||_2 \leq ||\partial_y^{\nu-e_n} p(y)||_V$. For $|\nu| = 1$, i.e. $\nu = e_n$, for some $1 \leq n \leq N$, the result (4.2) can be obtained by substituting (4.3) into (4.5). For $\nu \geq 2$, suppose that the result (4.2) holds for $\tilde{\nu}$ with $|\tilde{\nu}| = |\nu| - 1$, i.e. $\tilde{\nu} = \nu - e_n$ for some $1 \leq n \leq N$, we are about to verify that it also holds for ν. From (4.5), where the first term vanishes as $|\nu| \geq 2$, we have by induction

$$
\begin{aligned}
&||\partial_y^\nu \underline{u}(y)||_U + ||\partial_y^\nu p(y)||_V \\
&\leq C \sum_{n:\nu_n \neq 0} \nu_n ||a_n||_\infty \Big(||\partial_y^{\nu-e_n} \underline{u}(y)||_U + ||\partial_y^{\nu-e_n} p(y)||_V \Big), \\
&\leq C \sum_{n:\nu_n \neq 0} \nu_n ||a_n||_\infty \left(C^{|\nu-e_n|+1} |\nu - e_n|! ||a||_\infty^{\nu-e_n} \Big(||u_d||_2 + ||f(y)||_2 \Big) \right) \\
&+ C \sum_{n:\nu_n \neq 0} \nu_n ||a_n||_\infty \left(C^{|\nu-e_n|} \sum_{m:(\nu-e_n)_m \neq 0} |\nu - e_n - e_m|! ||a||_\infty^{\nu-e_n-e_m} ||f_m||_2 \right) \\
&= C^{|\nu|+1} \left(\sum_{n:\nu_n \neq 0} \nu_n \right) (|\nu| - 1)! ||a||_\infty^\nu \left(||u_d||_2 + ||f(y)||_2 \right) \\
&\quad + C^{|\nu|} \sum_{m:\nu_m \neq 0} \left(\sum_{n:(\nu-e_m)_n \neq 0} \nu_n \right) (|\nu - e_m| - 1)! ||a||_\infty^{\nu-e_m} ||f_m||_2 \\
&= C^{|\nu|+1} |\nu|! ||a||_\infty^\nu \left(||u_d||_2 + ||f(y)||_2 \right) + C^{|\nu|} \sum_{m:\nu_m \neq 0} |\nu - e_m|! ||a||_\infty^{\nu-e_m} ||f_m||_2,
\end{aligned}
\tag{4.6}
$$

where we have used $|\nu - e_n| = |\nu| - 1$ for the first equality. □

Based on this regularity result, we can prove the following result about the analytic extension of the solution of problem (4.1) to a complex domain.

THEOREM 4.2. *The solution of problem (4.1) can be analytically extended to the complex region*

$$
\Sigma := \left\{ z \in \mathbb{C}^N : \exists\, y \in \Gamma \text{ such that } \sum_{n=1}^{N} C ||a_n||_\infty |z_n - y_n| < 1 \text{ holds} \right\}. \tag{4.7}
$$

In particular, we consider the following subregion for error estimates

$$
\Sigma(\Gamma; \tau) := \{ z \in \Sigma : \exists\, y \in \Gamma \text{ such that } |z_n - y_n| \leq \tau_n, 1 \leq n \leq N \}, \tag{4.8}
$$

where $\tau = (\tau_1, \ldots, \tau_N)$ is the largest possible vector with positive elements.

Proof. For every $y \in \Gamma$, the Taylor expansion of the solution $(\underline{u}, p)$ of the problem (4.1) about y is given by

$$
\underline{u}(z) = \sum_\nu \frac{\partial_y^\nu \underline{u}(y)}{\nu!} (z - y)^\nu \text{ and } p(z) = \sum_\nu \frac{\partial_y^\nu p(y)}{\nu!} (z - y)^\nu, \tag{4.9}
$$

where $(z - y)^\nu = \prod_{n=1}^{N} (z_n - y_n)^{\nu_n}$. From the regularity result stated in Theorem

4.1, we have

$$
\begin{aligned}
&\left|\left|\sum_{\nu}\frac{\partial_y^\nu \underline{u}(y)}{\nu!}(z-y)^\nu\right|\right|_U + \left|\left|\sum_{\nu}\frac{\partial_y^\nu p(y)}{\nu!}(z-y)^\nu\right|\right|_V \\
&\leq \sum_{\nu}\frac{|z-y|^\nu}{\nu!}\left(||\partial_y^\nu \underline{u}(y)||_U + ||\partial_y^\nu p(y)||_V\right) \\
&\leq \sum_{\nu}\frac{|z-y|^\nu}{\nu!}C^{|\nu|+1}|\nu|!||a||_\infty^\nu\left(||u_d||_2 + ||f(y)||_2\right) \\
&\quad + \sum_{\nu}\frac{|z-y|^\nu}{\nu!}C^{|\nu|}\sum_{n:\nu_n\neq 0}|\nu-e_n|!||a||_\infty^{\nu-e_n}||f_n||_2,
\end{aligned}
\tag{4.10}
$$

where $|z-y| = (|z_1-y_1|,\dots,|z_N-y_N|)$. To proceed, we need the generalized Newton binomial formula: for any N-dimensional vector with complex element $\eta = (\eta_1,\dots,\eta_N) \in \mathbb{C}^N$, and for every $k = 0,1,2,\dots$, we have

$$
\sum_{|\nu|=k}\frac{k!}{\nu!}\eta^\nu = \left(\sum_{n=1}^N \eta_n\right)^k, \tag{4.11}
$$

by which we can evaluate the first term of (4.10) as

$$
\begin{aligned}
&\sum_{\nu}\frac{|z-y|^\nu}{\nu!}C^{|\nu|+1}|\nu|!||a||_\infty^\nu\left(||u_d||_2 + ||f(y)||_2\right) \\
&= C\left(||u_d||_2 + ||f(y)||_2\right)\sum_{\nu}\frac{|\nu|!}{\nu!}(C||a||_\infty|z-y|)^\nu \\
&= C\left(||u_d||_2 + ||f(y)||_2\right)\sum_{k=0}^\infty\sum_{|\nu|=k}\frac{k!}{\nu!}(C||a||_\infty|z-y|)^\nu \\
&= C\left(||u_d||_2 + ||f(y)||_2\right)\sum_{k=0}^\infty\left(\sum_{n=1}^N C||a_n||_\infty|z_n-y_n|\right)^k < \infty,
\end{aligned}
\tag{4.12}
$$

where $||a||_\infty|z-y| := (||a_1||_\infty|z_1-y_1|,\dots,||a_N||_\infty|z_N-y_N|)$. The first term converges thanks to the condition satisfied in the complex region (4.7). As for the second term of (4.10), we have

$$
\begin{aligned}
&\sum_{\nu}\frac{|z-y|^\nu}{\nu!}C^{|\nu|}\sum_{n:\nu_n\neq 0}|\nu-e_n|!||a||_\infty^{\nu-e_n}||f_n||_2 \\
&\leq \sum_{n=1}^N\frac{||f_n||_2}{||a_n||_\infty}\sum_{\nu}\frac{|\nu|!}{\nu!}(C||a||_\infty|z-y|)^\nu < \infty,
\end{aligned}
\tag{4.13}
$$

where we have used $|\nu-e_n| < |\nu|, 1 \leq n \leq N, \nu_n \neq 0$ for the first inequality. Assumption 2 for the data a and f, which implies $||a_n||_\infty > 0$ and $||f_n||_2 < \infty, 1 \leq n \leq N$, together with the estimate (4.12) guarantee the convergence in (4.13). Therefore, the solution of the problem (4.1) can be analytically extended in Σ defined in (4.7). $\square$

5. Numerical approximation. Thanks to the equivalence of the stochastic optimal control problem (2.17) and its saddle point formulation (3.12), it is sufficient to consider numerical approximation of (3.12) to solve (2.17), which involves both deterministic approximation of the optimal solution in the physical domain D and stochastic approximation in the probability domain Γ. In this section, we employ a finite element method with suitable preconditioning techniques [29, 22] for deterministic approximation and a stochastic collocation method [33, 1, 21] for stochastic approximation. In order to alleviate the global computational cost, we propose the model-order reduction strategy empowered by a weighted reduced basis method [10].

5.1. Finite element method. Given a regular triangulation $\mathcal{T}_h$ of the physical domain $\bar{D} \subset \mathbb{R}^d$ with mesh size h [24, 22], we define the finite element space

$$X_h = X_h^k := \{v_h \in C^0(\bar{D}) : v_h|_K \in \mathbb{P}_k \quad \forall K \in \mathcal{T}_h\}, \quad k \geq 1, \tag{5.1}$$

where $C^0(\bar{D})$ is the space of continuous functions in $\bar{D}$, $\mathbb{P}_k, k \geq 1$, is the space of polynomials of degree less than or equal to k in the variables $x_1, \dots, x_d$. Given any $y \in \Gamma$, by applying Galerkin projection of the solution $(\underline{u}(y), p(y))$ in the finite element space $U_h \otimes X_h \subset U \otimes H_0^1(D)$, where $U_h := X_h \otimes X_h$, we obtain the semi-weak saddle point problem (4.1) in finite element formulation as: find $(\underline{u}_h(y), p_h(y)) \in U_h \otimes X_h$

$$\begin{cases} A(\underline{u}_h(y), \underline{v}_h) + B(\underline{v}_h, p_h(y); y) &= (\underline{u}_d, \underline{v}_h) \qquad \forall \underline{v}_h \in U_h, \\ \qquad\qquad B(\underline{u}_h(y), q_h; y) &= F(q_h; y) \qquad \forall q_h \in X_h. \end{cases} \tag{5.2}$$

All the conditions in Proposition 3.1 are satisfied in the deterministic settings for the bilinear forms A and B in (5.2) with the choice of finite element space $U_h \otimes X_h$. In particular, following the same argument in the proof of the compatibility condition in (3.16), we have that B satisfies the compatibility condition in $U_h \otimes X_h$. Therefore, there exists a unique solution of problem (5.2), expanded on finite element bases as

$$u_h(x,y) = \sum_{i=1}^{N_h} u_i(y)\phi_i(x), g_h(x,y) = \sum_{i=1}^{N_h} g_i(y)\phi_i(x), p_h(x,y) = \sum_{i=1}^{N_h} p_i(y)\phi_i(x), \tag{5.3}$$

where $\phi_i, 1 \leq i \leq N_h$ are the finite element bases in X_h, N_h is the number of degrees-of-freedom (d.o.f). The algebraic formulation of (5.2) reads

$$\begin{pmatrix} A_h & B_h^T(y) \\ B_h(y) & 0 \end{pmatrix} \begin{pmatrix} \underline{\mathbf{u}}_h(y) \\ \mathbf{p}_h(y) \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{u}}_{dh} \\ \mathbf{f}_h(y) \end{pmatrix}, \tag{5.4}$$

which can be written in a more explicit formulation corresponding to the optimality system (3.2) in the deterministic setting as

$$\begin{pmatrix} M_h & 0 & C_h^T(y) \\ 0 & \alpha M_h & -M_h^T \\ C_h(y) & -M_h & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_h(y) \\ \mathbf{g}_h(y) \\ \mathbf{p}_h(y) \end{pmatrix} = \begin{pmatrix} \mathbf{u}_{dh} \\ 0 \\ \mathbf{f}_h(y) \end{pmatrix}. \tag{5.5}$$

Notations have the following meanings: $A_h = (M_h, 0_{N_h \times N_h}; 0_{N_h \times N_h}, \alpha M_h)$ with

$$(M_h)_{i,j} = (\phi_j, \phi_i), \quad (0_{N_h \times N_h})_{i,j} = 0, \quad 1 \leq i,j \leq N_h, \tag{5.6}$$

$B_h(y) = (C_h(y); -M_h)$ with

$$(C_h(y))_{i,j} = (a(y)\nabla\phi_j, \nabla\phi_i), \quad 1 \leq i,j \leq N_h, \tag{5.7}$$

the finite element optimal solution is $\underline{\mathbf{u}}_h(y) = (\mathbf{u}_h(y); \mathbf{g}_h(y))$, with

$$\mathbf{u}_h(y) = (u_1(y), \dots, u_{N_h}(y))^T, \quad \mathbf{g}_h(y) = (g_1(y), \dots, g_{N_h}(y))^T, \tag{5.8}$$

the adjoint variable $\mathbf{p}_h(y)$ is defined as

$$\mathbf{p}_h(y) = (p_1(y), \dots, p_{N_h}(y))^T, \tag{5.9}$$

the right hand side $\underline{\mathbf{u}}_{dh} = (\mathbf{u}_{dh}; \mathbf{0}_{N_h}), \mathbf{u}_{dh}, \mathbf{f}_h(y)$ as

$$(\mathbf{u}_{dh})_i = (u_d, \phi_i), \quad (\mathbf{0}_{N_h})_i = 0, \quad \mathbf{f}_h(y) = (f(y), \phi_i), \quad 1 \le i \le N_h. \tag{5.10}$$

When $\alpha \ll 1$ and $N_h \gg 1$, the matrix of the linear system (5.5) is ill-conditioned with very large condition number, leading to computational challenge for direct solve of (5.5). We prefer using GMRES iterations with the following (optimal) preconditioner [29, 23]

$$P = \begin{pmatrix} \hat{M}_h & 0 & 0 \\ 0 & \alpha \hat{M}_h & 0 \\ 0 & 0 & \hat{C}_h(\bar{y}) M_h^{-1} \hat{C}_h^T(\bar{y}) \end{pmatrix}, \tag{5.11}$$

where $\hat{M}_h$ is approximated by symmetric Gauss-Seidel method and $\hat{C}_h$ represents an algebraic multigrid V-cycles approximation for C_h at a reference value $\bar{y} \in \Gamma$ [25].

5.2. Stochastic collocation method. Stochastic collocation method adopts a non-intrusive approach for approximating the random solution of stochastic problems [33, 1, 21]. It features the advantages of fast convergence of the intrusive (usually hard to implement) stochastic Galerkin method and easy implementation of the slow convergent Monte-Carlo method. In this section, we present the construction of stochastic collocation method with both tensor-product [1] and sparse-grid structures [33, 21].

5.2.1. Tensor-product structure. Let us define

$$C(\Gamma; X) := \{v : \Gamma \to X | v \text{ is continuously measurable and } \max_{y \in \Gamma} ||v(y)||_X < \infty\}. \tag{5.12}$$

Let $\mathcal{P}_m(\Gamma)$ be a space of polynomials with degree less than or equal to m in every coordinate $y_1, \dots, y_N$, we define the Lagrangian interpolation operator $\mathcal{I}_{\mathbf{i}} : C(\Gamma; X) \to \mathcal{P}_{m(\mathbf{i})-1}(\Gamma) \otimes X$ as

$$\mathcal{I}_{\mathbf{i}} v(y) = (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_N}) v(y) = \sum_{j_1=1}^{m(i_1)} \cdots \sum_{j_N=1}^{m(i_N)} v(y_1^{j_1}, \dots, y_N^{j_N}) \bigotimes_{n=1}^{N} l_n^{j_n}(y_n), \tag{5.13}$$

where $\mathcal{U}^{i_n} : C(\Gamma_n; X) \to \mathcal{P}_{m(i_n)-1}(\Gamma_n) \otimes X$ is a one-dimensional Lagrangian interpolation operator based on the collocation nodes $y_n^1, \dots, y_n^{m(i_n)}$, $1 \le n \le N$, reads

$$\mathcal{U}^{i_n} v(y_n) = \sum_{j_n=1}^{m(i_n)} v(y_n^{j_n}) l_n^{j_n}(y_n) \text{ with } l_n^{j_n}(y_n) = \prod_{1 \le k \le m(i_n): k \ne j_n} \frac{y_n - y_n^k}{y_n^{j_n} - y_n^k}, \tag{5.14}$$

$\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{N}^N$, and $m(k)$ is a function of k depending on the choice of collocation nodes, e.g. $m(k) = 1$ when $k = 1$ and $m(k) = 2^k + 1, 1 \le n \le N$ when $k > 1$ [21].

With the definition of Lagrangian interpolation operator (5.13) in a tensor product structure, we can approximate statistics of interest, e.g. expectation, by

$$\mathbb{E}[v] \approx \mathbb{E}[\mathcal{I}_{\mathbf{i}} v] = \sum_{j_1=1}^{m(i_1)} \cdots \sum_{j_N=1}^{m(i_N)} v(y_1^{j_1}, \dots, y_N^{j_N}) \prod_{n=1}^{N} w_n^{j_n}, \tag{5.15}$$

where $w_n^{j_n} = \int_{\Gamma_n} l_n^{j_n}(y_n)\rho(y_n)dy_n, 0 \leq j_n \leq m(i_n), 1 \leq n \leq N$ represent quadrature weights. Accuracy of stochastic collocation approximation depends on the choice of the collocation nodes, the most popular of which are Clenshaw-Curtis abscissas, Gauss abscissas of certain orthogonal polynomials corresponding to the joint probability density function ρ, e.g. Gauss-Hermite abscissas for normal density function, [21, 5].

5.2.2. Sparse-grid structure. The tensor-product structure of the stochastic collocation method results in an exponential growth of collocation nodes with respect to dimensions, which prevent its application in high dimensional problems. In order to alleviate the heavy computational burden, we take advantage of sparse-grid structure for the stochastic collocation approximation with Smolyak interpolation $\mathcal{S}_q : C(\Gamma; X) \to \mathcal{P}_{m(q-N+1)-1}(\Gamma) \otimes X$ [21]

$$\mathcal{S}_q v(y) = \sum_{q-N+1 \leq |\mathbf{i}| \leq q} (-1)^{q-|\mathbf{i}|} \begin{pmatrix} N-1 \\ q-|\mathbf{i}| \end{pmatrix} \left(\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_N}\right) v(y), \tag{5.16}$$

where $|\mathbf{i}| = i_1 + \cdots + i_N$ with the multivariate index $\mathbf{i} \geq \mathbf{1}$ and $q \geq N$. Thanks to the difference operator $\triangle^{i_n} = \mathcal{U}^{i_n} - \mathcal{U}^{i_n - 1}$ with $\mathcal{U}^0 = 0$, the Smolyak interpolant admits the alternative formulation

$$\begin{aligned} \mathcal{S}_q v(y) &= \sum_{\mathbf{i} \in X(q,N)} (\triangle^{i_1} \otimes \cdots \otimes \triangle^{i_N}) v(y) \\ &= \mathcal{S}_{q-1} v(y) + \sum_{|\mathbf{i}|=q} (\triangle^{i_1} \otimes \cdots \otimes \triangle^{i_N}) v(y), \end{aligned} \tag{5.17}$$

with the multivariate index set defined as

$$X(q,N) := \left\{ \mathbf{i} \in \mathbb{N}_+^N, \forall i_n \geq 1 : \sum_{n=1}^{N} i_n \leq q \right\}, \tag{5.18}$$

which enables hierarchical sparse-grid construction with nested collocation nodes, e.g. nested Clenshaw-Curtis nodes and Gauss-Patterson nodes [15]. The number of collocation nodes of the sparse grid interpolation (5.17) is much less than that of the interpolation on tensor-product grid but still grows exponentially with respect to the dimension of the problem. In order to tackle high dimensional problem (in the order of $O(100)$), we have to take different importance of each dimension into account by applying anisotropic sparse grid interpolation formula [20], reading as

$$\mathcal{S}_q^{\alpha} v(y) = \sum_{\mathbf{i} \in X_\alpha(q,N)} (\triangle^{i_1} \otimes \cdots \otimes \triangle^{i_N}) v(y), \tag{5.19}$$

where the anisotropic multivariate index set $X_\alpha(q,N)$ is defined as

$$X_\alpha(q,N) := \left\{ \mathbf{i} \in \mathbb{N}_+^N, \forall i_n \geq 1 : \sum_{n=1}^{N} \alpha_n i_n \leq \min_{1 \leq n \leq N} \alpha_n q \right\} \tag{5.20}$$

with a positive multivariate weight $\alpha = (\alpha_1, \dots, \alpha_N)$, which can be obtained by a priori or a posteriori estimate [20], or dimension adaptive algorithm [11]. Note that the isotropic sparse grid interpolation (5.17) is a particular anisotropic case as $\alpha = \mathbf{1}$. Evaluation of statistics based on the isotropic/anisotropic sparse-grid stochastic collocation method, e.g. expectation, is straightforward by the following approximation

$$\mathbb{E}[\mathcal{S}_q^\alpha v] = \sum_{\mathbf{i} \in X_\alpha(q,N)} \mathbb{E}\left[\left((\mathcal{U}^{i_1} - \mathcal{U}^{i_1 - 1}) \otimes \cdots \otimes (\mathcal{U}^{i_N} - \mathcal{U}^{i_N - 1})\right) v(y)\right]. \tag{5.21}$$

5.2.3. Stochastic approximation. In applying stochastic collocation method to solve the stochastic optimal control problem (2.17), we first solve the algebraic optimality system (5.5) at a series of prescribed collocation nodes, or realizations of the random vector $y \in \Gamma$, and then evaluate statistics of interest, e.g. the cost functional (3.4) or the expectation of the solution, by formula (5.15) for tensor-product quadrature or (5.21) for sparse-grid quadrature.

5.3. Weighted reduced basis method. Solving the optimization problem (5.5) is rather expensive when N_h becomes very large and only a few tens or hundreds of complete solves of the system (5.5) may become affordable in practice. Then the stochastic collocation method (even with (anisotropic) sparse-grid structure [21, 20]) can hardly be employed because the number of collocation nodes easily overpasses this computational constraint, especially for high dimensional problems. The approach that we propose relies on a weighted reduced basis method [10].

5.3.1. Reduced basis method. For any given choice of $y \in \Gamma$, e.g. the collocation points used by stochastic collocation method, we seek a reduced basis solution $(\underline{u}_r(y), p_r(y)) \in U_{N_r} \otimes X^p_{N_r}$ such that

$$\begin{cases} A(\underline{u}_r(y), \underline{v}_r) + B(\underline{v}_r, p_r(y); y) &= (\underline{u}_d, \underline{v}_r) \qquad \forall \underline{v}_r \in U_{N_r}, \\ B(\underline{u}_r(y), q_r; y) &= F(q_r; y) \qquad \forall q_r \in X^p_{N_r}, \end{cases} \tag{5.22}$$

where the reduced basis space $U_{N_r} = X^e_{N_r} \otimes X^g_{N_r}$ and $X^p_{N_r}$ are constructed from "snapshots" - solutions of (5.5) at some selected samples $y^n, 1 \le n \le N_r$, i.e.

$$\begin{aligned} X^u_{N_r} &= \text{span}\{u_h(y^n), 1 \le n \le N_r\}, \\ X^g_{N_r} &= \text{span}\{g_h(y^n), 1 \le n \le N_r\}, \\ X^p_{N_r} &= \text{span}\{p_h(y^n), 1 \le n \le N_r\}. \end{aligned} \tag{5.23}$$

Note that in order to guarantee the assumptions in Proposition 3.1 in reduced basis space, in particular the inf-sup condition for system (5.22), we use an enriched reduced basis space $X^e_{N_r}$ as union of $X^u_{N_r}$ and $X^p_{N_r}$ [19], i.e.

$$X^e_{N_r} = X^u_{N_r} \cup X^p_{N_r} = \text{span}\{u_h(y^n), p_h(y^n), 1 \le n \le N_r\}. \tag{5.24}$$

For the sake of algebraic stability in assembling the reduced basis matrices and performing Galerking projection [28], we orthonormalize the snapshots in the reduced basis space $X^e_{N_r}$ and $X^g_{N_r}$ by Gram-Schmidt process with respect to the inner-products $(a(\bar{y})\nabla\cdot, \nabla\cdot)$ ($\bar{y}$ being a reference value, e.g. the center of Γ) and $(\cdot,\cdot)$, yielding

$$X^e_{N_r} = \{\zeta^e_n, 1 \le n \le 2N_r\} \text{ and } X^g_{N_r} = \{\zeta^g_n, 1 \le n \le N_r\}. \tag{5.25}$$

Let the reduced basis solution at $y \in \Gamma$ be written as

$$u_r(y) = \sum_{n=1}^{2N_r} u_n(y)\zeta_n^e, \quad g_r(y) = \sum_{n=1}^{N_r} g_n(y)\zeta_n^g, \quad p_r(y) = \sum_{n=1}^{2N_r} p_n(y)\zeta_n^e, \tag{5.26}$$

and the solution coefficient vector at $y \in \Gamma$ as $\mathbf{u}_r(y) = (u_1(y), \dots, u_{2N_r}(y))^T$, $\mathbf{g}_r(y) = (g_1(y), \dots, g_{N_r}(y))^T$, $\mathbf{p}_r(y) = (p_1(y), \dots, p_{2N_r}(y))^T$, we obtain the reduced algebraic optimality system corresponding to the full algebraic optimality system (5.5) as

$$\begin{pmatrix} M_r & 0 & C_r^T(y) \\ 0 & \alpha D_r & -E_r^T \\ C_r(y) & -E_r & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_r(y) \\ \mathbf{g}_r(y) \\ \mathbf{p}_r(y) \end{pmatrix} = \begin{pmatrix} \mathbf{u}_{dr} \\ 0 \\ \mathbf{f}_r(y) \end{pmatrix}, \tag{5.27}$$

being a $5N_r \times 5N_r$ dense system, where the reduced optimality matrix is defined as

$$\begin{pmatrix} M_r & 0 & C_r^T(y) \\ 0 & \alpha D_r & -E_r^T \\ C_r(y) & -E_r & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_e^T M_h \mathcal{Z}_e & 0 & \mathcal{Z}_e^T C_h^T(y) \mathcal{Z}_e \\ 0 & \alpha \mathcal{Z}_g^T M_h \mathcal{Z}_g & -\mathcal{Z}_g^T M_h^T \mathcal{Z}_e \\ \mathcal{Z}_e^T C_h(y) \mathcal{Z}_e & -\mathcal{Z}_e^T M_h \mathcal{Z}_g & 0 \end{pmatrix}, \tag{5.28}$$

and the reduced optimal solution and the right hand side are given by

$$\begin{pmatrix} \mathbf{u}_r(y) \\ \mathbf{g}_r(y) \\ \mathbf{p}_r(y) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_e^T \mathbf{u}_h(y) \\ \mathcal{Z}_g^T \mathbf{g}_h(y) \\ \mathcal{Z}_e^T \mathbf{p}_h(y) \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{u}_{dr} \\ 0 \\ \mathbf{f}_r(y) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_e^T \mathbf{u}_{dh} \\ 0 \\ \mathcal{Z}_e^T \mathbf{f}_h(y) \end{pmatrix}, \tag{5.29}$$

where $\mathcal{Z}_e = (\zeta_1^e, \dots, \zeta_{2N_r}^e)$ and $\mathcal{Z}_g = (\zeta_1^g, \dots, \zeta_{N_r}^g)$ are column vector matrices.

5.3.2. A weighted greedy algorithm. The efficiency of reduced basis method depends critically on the choice of the samples $y^n, 1 \le n \le N_r$, for which we turn to a weighted greedy algorithm [10]. To start, we randomly choose a realization $y^1 \in \Gamma$ (or use the reference value $\bar{y}$), and solve the full optimality system (5.5) to get the solution $(u_h(y^1), g_h(y^1), p_h(y^1))$. By Gram-Schmidt process, we construct the first reduced basis space X_1^e and X_1^g. For $N_r = 2, \dots, N_{max}$ (where N_{max} is a prescribed maximum number of reduced bases), we solve the following weighted $L^\infty(\Gamma; X_\rho)$ optimization problem

$$y^{N_r} = \arg\sup_{y \in \Gamma} ||(u_h(y), g_h(y), p_h(y)) - (u_r(y), g_r(y), p_r(y))||_{X_\rho}, \tag{5.30}$$

where X_ρ is a weighted Hilbert space (with weight ρ) equipped with the norm

$$||(v(y), h(y), q(y))||_{X_\rho} = \left\{ \left(||v(y)||_{\bar{X}}^2 + \alpha ||h(y)||_{L^2(D)}^2 + ||q(y)||_{\bar{X}}^2 \right) \rho(y) \right\}^{1/2}, \tag{5.31}$$

where α is the regularization parameter given in the cost functional (2.18), $\rho(y)$ is taken as the joint probability density function evaluated at $y \in \Gamma$ and $||v(y)||_{\bar{X}}^2 = (a(\bar{y})\nabla v(y), \nabla v(y))$ at a reference value $\bar{y} \in \Gamma$. However, solving accurately the infinite-dimensional optimization problem (5.30) is computationally impossible. Instead, we replace Γ by a training set Ξ_{train}, e.g. the collocation nodes used in the stochastic collocation method. Moreover, instead of using the "truth" error defined in (5.30), we consider a cheap, sharp and reliable error bound $\triangle_{N_r}^\rho$ such that

$$||(u_h(y), g_h(y), p_h(y)) - (u_r(y), g_r(y), p_r(y))||_{X_\rho} \le \triangle_{N_r}^\rho (u_r(y), g_r(y), p_r(y)). \tag{5.32}$$

Upon replacement of Γ and the truth error, we have the weighted greedy algorithm

$$y^{N_r} = \arg \sup_{y \in \Xi_{train}} \triangle^{\rho}_{N_r}(u_r(y), g_r(y), p_r(y)), \quad N_r = 2, \dots, N_{max}, \tag{5.33}$$

based on which, we can hierarchically build the reduced basis spaces $X^e_{N_r}$ and $X^g_{N_r}$.

Remark 5.1. *The weighted norm $||\cdot||_{X_\rho}$ is defined with the joint probability density function ρ in order to assign associated weight or importance in choosing the samples for construction of the reduced basis space. In this way the a posteriori error bound and the truth error are kept small, resulting in a more accurate solution, when the probability at the sample is large. This numerical scheme aims to balance the accuracy and importance of the stochastic solution, achieving higher accuracy of statistical moments of interest [10], see illustration in Section 7.1.*

5.3.3. A weighted a posteriori error bound. For the purpose of computing a weighted a posteriori error bound $\triangle^{\rho}_{N_r}$, we reformulate the saddle point problem (4.1) as a weakly coercive problem at first [3, 34].

For every $y \in \Gamma$, let $\mathrm{u}(y) := (u(y), g(y), p(y)) \in \mathrm{U} := \bar{X} \otimes L^2(D) \otimes \bar{X} \simeq X_\rho$ and $\mathrm{v} := (v, h, q) \in \mathrm{U}$, we define the bilinear form $\mathrm{B} : \mathrm{U} \otimes \mathrm{U} \to \mathbb{R}$ as

$$\mathrm{B}(\mathrm{u}(y), \mathrm{v}; y) := A(\underline{u}(y), \underline{v}) + B(\underline{v}, p(y); y) + B(\underline{u}(y), q; y), \tag{5.34}$$

and the linear functional $\mathrm{F} : \mathrm{U} \to \mathbb{R}$ as

$$\mathrm{F}(\mathrm{v}; y) := (\underline{u}_d, \underline{v}) + F(q; y). \tag{5.35}$$

Then the saddle point problem (4.1) is equivalent to the following problem: given $y \in \Gamma$, find $\mathrm{u} \in \mathrm{U}$ such that

$$\mathrm{B}(\mathrm{u}(y), \mathrm{v}; y) = \mathrm{F}(\mathrm{v}; y) \quad \mathrm{v} \in \mathrm{U}. \tag{5.36}$$

It can be shown [34] that the bilinear form B is continuous and weakly coercive, i.e.

$$\gamma(y) := \sup_{\mathrm{v} \in \mathrm{U}} \sup_{\mathrm{u}(y) \in \mathrm{U}} \frac{\mathrm{B}(\mathrm{u}(y), \mathrm{v}; y)}{||\mathrm{u(y)}||_{\mathrm{U}} ||\mathrm{v}||_{\mathrm{U}}} < \infty \text{ and } \beta(y) := \inf_{\mathrm{v} \in \mathrm{U}} \sup_{\mathrm{u}(y) \in \mathrm{U}} \frac{\mathrm{B}(\mathrm{u}(y), \mathrm{v}; y)}{||\mathrm{u(y)}||_{\mathrm{U}} ||\mathrm{v}||_{\mathrm{U}}} > 0, \tag{5.37}$$

where $||\mathrm{v}||_{\mathrm{U}} := ||v||_{\bar{X}} + \sqrt{\alpha}||h||_{L^2(D)} + ||q||_{\bar{X}}$ corresponding to (5.31). Moreover, there exists a unique solution $\mathrm{u}(y) \in \mathrm{U}$ of problem (5.36) satisfying the stability estimate

$$||\mathrm{u}(y)||_{\mathrm{U}} \le \frac{1}{\beta(y)} ||\mathrm{F}(y)||_{\mathrm{U}'}. \tag{5.38}$$

Consequently, we have similar results (5.37) and (5.38) for the finite element solution $\mathrm{u}_h(y)$ of problem (5.5) with constants $\gamma_h(y), \beta_h(y)$. Let the residual be defined as

$$\mathrm{R}(\mathrm{v}_h; y) = \mathrm{F}(\mathrm{v}_h; y) - \mathrm{B}(\mathrm{u}_r(y), \mathrm{v}_h; y) \quad \forall \mathrm{v}_h \in \mathrm{U}_h := X_h \otimes X_h \otimes X_h, \tag{5.39}$$

then we have that the error between the finite element solution and the reduced basis solution $\mathrm{e}(y) = (u_h(y), g_h(y), p_h(y)) - (u_r(y), g_r(y), p_r(y))$ satisfies

$$\mathrm{B}(\mathrm{e}(y), \mathrm{v}_h; y) = \mathrm{R}(\mathrm{v}_h; y) \quad \mathrm{v}_h \in \mathrm{U}_h. \tag{5.40}$$

which yields, by the stability estimate (5.38), that

$$||\mathrm{e}(y)||_{\mathrm{U}_h} \le \frac{1}{\beta_h(y)} ||\mathrm{R}(\mathrm{v}_h; y)||_{\mathrm{U}'_h}. \tag{5.41}$$

Therefore, we can define the a weighted posteriori error bound as: for $\forall y \in \Xi_{train}$

$$\triangle^{\rho}_{N_r}(\mathrm{u}_r(y)) := \frac{\sqrt{\rho(y)}}{\beta_{LB}(y)}||\mathrm{R}(y)||_{\mathrm{U}'} \geq \sqrt{\rho(y)}||\mathrm{e}(y)||_{\mathrm{U}} = ||\mathrm{e}(y)||_{X_\rho}, \tag{5.42}$$

where a lower bound $\beta_{LB}(y) \leq \beta_h(y), \forall y \in \Xi_{train}$ can be evaluated by a cheap successive constraint method [28]. As for evaluation of the weighted residual norm $||\mathrm{R}(y)||_{X_\rho}$, we turn to an efficient offline-online decomposition procedure.

Remark 5.2. *In the definition of the compound Hilbert space* U*, we use the Hilbert space* $\bar{X}$ *equipped with norm* $||\cdot||_{\bar{X}} = (a(\bar{y})\nabla\cdot, \nabla\cdot)$ *for both the state variable* $u(y)$ *and the adjoint variable* $p(y)$ *in order to obtain good stability of the inf-sup constant* $\beta_h(y), \forall y \in \Gamma$. *In fact, when* $a(y)$ *is not far from the reference value* $a(\bar{y})$, *the inf-sup constant* $\beta_h(y)$ *is also close to* $\beta_h(\bar{y})$, *which enables us to use a uniformly lower bound* $\beta_{LB} \leq \beta_h(y), \forall y \in \Gamma$, *for the sake of computational efficiency [10].*

5.3.4. Offline-online decomposition. The offline-online decomposition procedure decomposes the reduced basis method into the expensive offline construction stage and cheap online evaluation stage. More explicitly, we build the reduced basis space $X^e_{N_r}$ and $X^g_{N_r}$, assemble and store all matrices in (5.28) and the right hand side vector (5.29) in an offline stage. In particular, the quantities in (5.28) and (5.29) that depend on the random variable $y \in \Gamma$ are assembled as

$$\mathcal{Z}_e^T C_h(y)\mathcal{Z}_e = \sum_{n=0}^{N} y_n \mathcal{Z}_e^T C_h^n \mathcal{Z}_e \text{ and } \mathcal{Z}_e^T \mathbf{f}_h(y) = \sum_{n=0}^{N} y_n \mathcal{Z}_e^T \mathbf{f}_h^n, \tag{5.43}$$

where $\mathcal{Z}_e^T C_h^n \mathcal{Z}_e$ and $\mathcal{Z}_e^T \mathbf{f}_h^n, 0 \leq n \leq N$ are assembled offline with the matrices $(C_h^n)_{i,j} = (a_n \nabla\phi_j, \nabla\phi_i), 0 \leq n \leq N, 1 \leq i,j \leq N_h$ and the vectors $\mathbf{f}_h^n = (f_n, \phi_i), 0 \leq n \leq N, 1 \leq i \leq N_h$. Recall that $y_0 = 1$ and $a_n, f_n, 0 \leq n \leq N$ are defined in (2.13). For a more compact notation, we define

$$\mathrm{B}_r^0 = \begin{pmatrix} \mathcal{Z}_e^T M_h \mathcal{Z}_e & 0 & \mathcal{Z}_e^T (C_h^0)^T \mathcal{Z}_e \\ 0 & \alpha \mathcal{Z}_g^T M_h \mathcal{Z}_g & -\mathcal{Z}_g^T M_h^T \mathcal{Z}_e \\ \mathcal{Z}_e^T C_h^0 \mathcal{Z}_e & -\mathcal{Z}_e^T M_h \mathcal{Z}_g & 0 \end{pmatrix}, \quad \mathrm{F}_r^0 = \begin{pmatrix} \mathcal{Z}_e^T \mathbf{u}_{dh} \\ 0 \\ \mathcal{Z}_e^T \mathbf{f}_h^0 \end{pmatrix}, \tag{5.44}$$

and

$$\mathrm{B}_r^n = \begin{pmatrix} 0 & 0 & \mathcal{Z}_e^T (C_h^n)^T \mathcal{Z}_e \\ 0 & 0 & 0 \\ \mathcal{Z}_e^T C_h^n \mathcal{Z}_e & 0 & 0 \end{pmatrix}, \quad \mathrm{F}_r^n = \begin{pmatrix} 0 \\ 0 \\ \mathcal{Z}_e^T \mathbf{f}_h^n \end{pmatrix}, \quad 1 \leq n \leq N. \tag{5.45}$$

Then the reduced algebraic optimality system (5.27) can be written as

$$\sum_{n=0}^{N} y_n \mathrm{B}_r^n \mathbf{u}_r^c(y) = \sum_{n=0}^{N} y_n \mathrm{F}_r^n, \tag{5.46}$$

where $\mathbf{u}_r^c(y) = (\mathbf{u}_r(y); \mathbf{g}_r(y); \mathbf{p}_r(y))$ is the coefficient of reduced basis solution at $y \in \Gamma$. A direct solver, e.g. by Gauss elimination, can be applied to solve the reduced basis optimality system (5.46) with complexity $O((5N_r)^3)$, since $N_r \ll N_h$ in practice.

From the definition of the residual (5.39), we have by Riesz representation theorem [24] that there exists a unique element $\hat{\mathrm{e}}(y) \in \mathrm{U}_h$ such that

$$(\hat{\mathrm{e}}(y), \mathrm{v}_h)_{\mathrm{U}_h} = \mathrm{R}(\mathrm{v}_h; y) \quad \forall \mathrm{v}_h \in \mathrm{U}_h. \tag{5.47}$$

Therefore, we have $||\mathrm{R}(y)||_{\mathrm{U}'_h} = ||\hat{\mathrm{e}}(y)||_{\mathrm{U}_h}$, to evaluate which we make the following definition of bilinear form and linear function corresponding to (5.34) and (5.35):

$$\mathrm{B}^0(\mathrm{u}(y),\mathrm{v}) = A(\underline{u}(y),\underline{v}) + B^0(\underline{v},p(y)) + B^0(\underline{u}(y),q), \mathrm{F}^0(\mathrm{v}) = (\underline{u}_d,\underline{v}) + F^0(q), \quad (5.48)$$

and

$$\mathrm{B}^n(\mathrm{u}(y),\mathrm{v}) = B^n(\underline{v},p(y)) + B^n(\underline{u}(y),q),\ \mathrm{F}^n(\mathrm{v}) = F^n(q), \quad 1 \le n \le N, \qquad (5.49)$$

where $\forall(\underline{v},q) \in \mathrm{U}$, we have $B^0(\underline{v},q) = (a_0\nabla v, \nabla q) - (h,q)$, $F^0(q) = (f_0,q)$, and $B^n(\underline{v},q) = (a_n\nabla v,\nabla q), F^n(q) = (f_n,q), 1 \le n \le N$. By the above definition, we obtain by Riesz representation theorem that there exist $\mathrm{f}_n, \mathrm{b}_n^k$ such that $(\mathrm{f}_n,\mathrm{v}_r) = \mathrm{F}^n(\mathrm{v}_r)$ and $(\mathrm{b}_n^k,\mathrm{v}_r) = -\mathrm{B}^n(\zeta_k^c,\mathrm{v}_r)$ for $\forall \mathrm{v}_r \in \mathrm{U}_r, 0 \le n \le N, 1 \le k \le 5N_r$, where $\zeta_k^c = (\zeta_k^e,0,0), 1 \le k \le 2N_r$, $\zeta_k^c = (0,\zeta_{k-2N_r}^g,0), 2N_r+1 \le k \le 3N_r$, $\zeta_k^c = (0,0,\zeta_{k-3N_r}^e), 3N_r+1 \le k \le 5N_r$ and the compound reduced basis space $\mathrm{U}_r = X_{N_r}^e \otimes X_{N_r}^g \otimes X_{N_r}^e$. To this end, we have by the definition of the residual (5.39)

$$\begin{aligned} ||\hat{\mathrm{e}}(y)||^2_{\mathrm{U}_h} = &\sum_{n=0}^{N}\sum_{n'=0}^{N} y_n(\mathrm{f}_n,\mathrm{f}_{n'})y_{n'} + 2\sum_{n=0}^{N}\sum_{n'=0}^{N}\sum_{k=1}^{5N_r} y_n(\mathrm{f}_n,\mathrm{b}_{n'}^k)(\mathbf{u}_r)_k y_{n'} \\ &+ \sum_{n=0}^{N}\sum_{n'=0}^{N}\sum_{k=1}^{5N_r}\sum_{k'=1}^{5N_r} y_n(\mathbf{u}_r)_k(\mathrm{b}_n^k,\mathrm{b}_{n'}^{k'})(\mathbf{u}_r)_{k'}y_{n'}, \end{aligned} \qquad (5.50)$$

where all the quantities of inner-product are computed and stored in the offline stage, and only $O((N+1)^2 \times (5N_r)^2)$ operations, being N and $N_r \ll N_h$ very small, are needed for online evaluation of the a posterior error bound $\triangle_{N_r}^{\rho}$ defined in (5.42).

6. Error estimates. In this section, we carry out a global error analysis of our numerical approximation. We first consider the error contribution from different sources, including finite element approximation, stochastic collocation approximation as well as weighted reduced basis approximation separately and then provide a global error estimate for several quantities of interest, including the stochastic optimal solution $(\underline{u}(y),p(y)), \forall y \in \Gamma$, and its statistical moments, e.g. expectation $\mathbb{E}[(\underline{u},p)]$.

6.1. Finite element error. Given the finite element solution of the semi-weak saddle point problem (5.2) $(\underline{u}_h(y),p_h(y)) \in U_h \otimes X_h, \forall y \in \Gamma$, we aim to estimate the finite element error defined as

$$\mathcal{E}_h(y) = ||\underline{u}(y) - \underline{u}_h(y)||_U + ||p(y) - p_h(y)||_{\bar{X}}, \qquad (6.1)$$

where the norm for the first term is given by

$$||\underline{u}(y) - \underline{u}_h(y)||_U = ||u(y) - u_h(y)||_{\bar{X}} + \sqrt{\alpha}||g(y) - g_h(y)||_{L^2(D)}. \qquad (6.2)$$

We remark that the two norms are equivalent $||v||_{\bar{X}} \simeq ||v||_{H_0^1(D)}, \forall v \in H_0^1(D)$, thanks to the assumption (2.11) and Poincaré inequality [24]. Therefore, all the assumptions in Proposition 3.1 hold true in the deterministic setting for the bilinear form A and B with $H_0^1(D)$ replaced by $\bar{X}$. For ease of notation, we still use the same symbols for the continuity , coercivity and inf-sup constants of A and B with $\bar{X}$, i.e.

$$A(\underline{u},\underline{v}) \le \gamma||\underline{u}||_U||\underline{v}||_U,\ B(\underline{v},q;y) \le \delta||\underline{v}||_U||q||_{\bar{X}}, \forall \underline{u},\underline{v} \in U, \forall q \in \bar{X}, \forall y \in \Gamma, \quad (6.3)$$

where γ is corresponding to the continuity constant 1 of $\mathcal{A}$ in (3.13) and $\delta := \max\{R, 1/\sqrt{\alpha}\}$ is the uniform continuity constant corresponding to that of $\mathcal{B}$ in (3.15).

Moreover, in the kernel of B, $U_h^0 := \{\underline{v}_h \in U_h, B(\underline{v}_h, q_h) = 0, \forall q_h \in X_h\}$, we have the finite element coercivity constant $\varsigma := \min\{\alpha r^2/4, 1/4\}$ of A corresponding to that of $\mathcal{A}$ in (3.14), i.e.

$$A(\underline{v}_h, \underline{v}_h) \geq \varsigma ||\underline{v}_h||_U^2, \quad \forall \underline{v}_h \in U_h^0. \tag{6.4}$$

Lastly, the uniform finite element inf-sup constant $\bar{\beta}_h$ of B corresponding to that of $\mathcal{B}$ in (3.16) is given by $\bar{\beta}_h := r$, i.e.

$$\forall q_h \in X_h, \exists \underline{v}_h \in U_h, \underline{v}_h \neq 0, \text{ s.t. } B(\underline{v}_h, q_h; y) \geq \bar{\beta}_h ||\underline{v}||_U ||q_h||_{\bar{X}}, \forall y \in \Gamma. \tag{6.5}$$

Under Assumption 2 (Sec 2.2), the conditions (6.3), (6.4) and (6.5) are satisfied in the finite element space $U_h \otimes X_h$, then $\forall y \in \Gamma$ we have the following error estimate for the stochastic finite element solution $(\underline{u}_h(y), p_h(y)) \in U_h \otimes X_h$, which can be directly derived from the proof in the deterministic case, see [24]

$$\begin{aligned} \mathcal{E}_h(y) &\leq C_1^h \inf_{\underline{v}_h \in U_h} ||\underline{u}(y) - \underline{v}_h||_U + C_2^h \inf_{q_h \in X_h} ||p(y) - q_h||_{\bar{X}} \\ &= O(h^l) \left(C_1^h ||u(y)||_{H^{l+1}(D)} + \alpha C_1^h h ||g(y)||_{H^{l+1}(D)} + C_2^h ||p(y)||_{H^{l+1}(D)} \right), \end{aligned} \tag{6.6}$$

where $l = \min\{k, s-1\}$, being k the polynomial degree and s such that $u(y), g(y), p(y) \in H^s(D), s \geq 2, \forall y \in \Gamma$, and the constants C_1^h, C_2^h are given by

$$C_1^h = \left(1 + \frac{\gamma}{\varsigma}\right)\left(1 + \frac{\gamma}{\bar{\beta}_h}\right)\left(1 + \frac{\delta}{\bar{\beta}_h}\right) \text{ and } C_2^h = 1 + \frac{\delta}{\varsigma} + \frac{\delta}{\bar{\beta}_h} + \frac{\gamma\delta}{\varsigma\bar{\beta}_h}. \tag{6.7}$$

Remark 6.1. *The above result is obtained based on the finite element formulation of the saddle point problem (5.2). We remark that a similar result can be achieved based on a finite element approximation of the weakly coercive problem (5.36).*

6.2. Stochastic collocation error. The error arising from the stochastic collocation approximation depends mainly on the regularity of the stochastic optimal solution. Thanks to the analytic regularity stated in Theorem 4.2, the stochastic optimal solution $(\underline{u}, p)$ can be analytically extended to the complex region $\Sigma(\Gamma; \tau)$. Then the error of the tensor-product stochastic collocation approximation (5.13), in the case that Γ is bounded, satisfies [1]

$$\mathcal{E}_s = ||\underline{u} - \mathcal{I}_{\mathbf{i}}\underline{u}||_{C(\Gamma;U)} + ||p - \mathcal{I}_{\mathbf{i}}p||_{C(\Gamma;\bar{X})} \leq \sum_{n=1}^N C_n^{\mathbf{i}} \exp(-(m(i_n) - 1)r_n), \tag{6.8}$$

where the constants $C_n^{\mathbf{i}}, 1 \leq n \leq N$ are bounded by [1, 6]

$$C_n^{\mathbf{i}} \leq (1 + \Lambda(m(i_n))) \frac{2}{e^{r_n} - 1} \left(\max_{z \in \Sigma(\Gamma;\tau)} ||\underline{u}(z)||_U + \max_{z \in \Sigma(\Gamma;\tau)} ||p(z)||_{\bar{X}} \right), \tag{6.9}$$

with Lebesgue constant $\Lambda(m) \leq 1 + (2/\pi)\log(m+1)$, and convergence rate

$$r_n = \log\left(\frac{2\tau_n}{|\Gamma_n|} + \sqrt{1 + \frac{4\tau_n^2}{|\Gamma_n|^2}} \right) > 1, \quad 1 \leq n \leq N. \tag{6.10}$$

Moreover, the error of the expectation of the stochastic optimal solution $\mathbb{E}[(\underline{u}, p)]$ evaluated by the tensor-product quadrature in (5.15) satisfies [1]

$$\begin{aligned} \mathcal{E}_s^e &= ||\mathbb{E}[\underline{u}] - \mathbb{E}[\mathcal{I}_{\mathbf{i}}\underline{u}]||_U + ||\mathbb{E}[p] - \mathbb{E}[\mathcal{I}_{\mathbf{i}}p]||_{\bar{X}} \\ &\leq ||\underline{u} - \mathcal{I}_{\mathbf{i}}\underline{u}||_{L^2_\rho(\Gamma;U)} + ||p - \mathcal{I}_{\mathbf{i}}p||_{L^2_\rho(\Gamma;\bar{X})} \leq \sum_{n=1}^N C_n \exp(-(m(i_n) - 1)r_n), \end{aligned} \tag{6.11}$$

where C_n is a constant that can be upper bounded by the product of the last two factors in (6.9) (thus independent of **i**). Similar results have also been obtained for unbounded Γ in [1].

As for the isotropic sparse-grid Smolyak interpolation (5.16) with Gauss-abscissas, we have the following estimate for the worst case scenario error

$$\mathcal{E}_s = ||\underline{u} - \mathcal{S}_q \underline{u}||_{C(\Gamma;U)} + ||p - \mathcal{S}_q p||_{C(\Gamma;\bar{X})} \leq C_s N_q^{-r}, \tag{6.12}$$

where N_q is the number of collocation nodes, C_s is a constant independent of N_q (see [21, 6] for more explicit expression), r is the algebraic convergence rate given by [6]

$$r \geq \frac{e \log(2) \min\{r_n, 1 \leq n \leq N\}}{3 + \log(N)}, \tag{6.13}$$

being $r_n, 1 \leq n \leq N$ defined in (6.10). When it comes to anisotropic sparse-grid stochastic collocation method with Clenshaw-Curtis nodes, we have the following error estimate

$$\mathcal{E}_s^\alpha = ||\underline{u} - \mathcal{S}_q^\alpha \underline{u}||_{C(\Gamma;U)} + ||p - \mathcal{S}_q^\alpha p||_{C(\Gamma;\bar{X})} \leq C_s^\alpha N_q^{-r(\alpha)}, \tag{6.14}$$

where C_s^α is independent of N_q and the algebraic convergence rate $r(\alpha)$ is defined as

$$r(\alpha) = \left(\ln(2)e - \frac{1}{2}\right) \frac{\alpha_{min}}{\ln(2) + \sum_{n=1}^N \frac{\alpha_{min}}{\alpha_n}}, \tag{6.15}$$

being $\alpha_{min} = \min_{1 \leq n \leq N} \alpha_n$ with the choice $\alpha_n = r_n/2, 1 \leq n \leq N$, with r_n defined in (6.10). Moreover, the error of the expectation of the stochastic optimal solution evaluated by isotropic or anisotropic sparse grid Smolyak formula is bounded by [21, 6]

$$\begin{aligned} \mathcal{E}_s^e &= ||\mathbb{E}[\underline{u}] - \mathbb{E}[\mathcal{S}_q^\alpha \underline{u}]||_U + ||\mathbb{E}[p] - \mathbb{E}[\mathcal{S}_q^\alpha p]||_{\bar{X}} \\ &\leq ||\underline{u} - \mathcal{S}_q^\alpha \underline{u}||_{L^2_\rho(\Gamma;U)} + ||p - \mathcal{S}_q^\alpha p||_{L^2_\rho(\Gamma;\bar{X})} \leq C_s^e N_q^{-r(\alpha)}, \end{aligned} \tag{6.16}$$

where C_s^e is a constant independent of N_q, see [6] for its explicit expression.

Remark 6.2. *We remark that the error estimate for tensor-product stochastic collocation approximation is given with respect to the polynomial degree while we use the number of collocation nodes for the error estimate of the sparse-grid type.*

6.3. Reduced basis error. In addition to the a posteriori error estimate for a weighted reduced basis approximation that has been proved in section 5.3.3, now we consider the a priori error estimate for the reduced basis solution. In particular, we provide a direct convergence result for the reduced basis error when $\Gamma \subset \mathbb{R}$, an indirect result via Kolmogorov N-width [2] regardless of dimension and its comparison with the stochastic collocation error.

Holding the analytic regularity in Theorem 4.2, we have the following error estimate for reduced basis solution of (5.22) when $\Gamma \subset \mathbb{R}$ [10]

$$\mathcal{E}_r = ||\underline{u}_h - \underline{u}_r||_{C(\Gamma;U)} + ||p_h - p_r||_{C(\Gamma;U)} \leq C_r \exp(-rN_r) \tag{6.17}$$

where r is defined as in (6.10) for a single dimension, the constant C_r is bounded by

$$C_r \leq C \left(\max_{z \in \Sigma(\Gamma;\tau)} ||\underline{u}_h(z)||_U + \max_{z \in \Sigma(\Gamma;\tau)} ||p_h(z)||_{\bar{X}} \right), \tag{6.18}$$

being C a constant independent of the number of reduced bases N_r.

Let d_N be the Kolmogorov N-width defined in an abstract Hilbert space X as

$$d_N(\Gamma; X) := \inf_{X_N \subset X} \sup_{y \in \Gamma} \inf_{w_N \in X_N} ||v(y) - w_N||_X, \tag{6.19}$$

where X_N is a N-dimensional subspace of X. We have the following result for $\mathcal{E}_r$ [2]: suppose that there exists $M > 0$ such that $d_0(\Gamma) \leq M$; moreover, suppose that there exist two positive constants $c_1 > 0, c_2 > 0$, such that

$$\text{if } d_{N_r}(\Gamma; U_h \otimes X_h) \leq M \exp(-c_1 N_r^{c_2}) \text{ then } \mathcal{E}_r \leq c_5 M \exp(-c_3 N_r^{c_4}), \tag{6.20}$$

where $c_4 = c_2/(c_2+1)$, $c_3 > 0, c_5 > 0$ depend only on c_1, c_2 and $c_6 > 0$, which measures the sharpness of the reduced basis error bound in (5.42), i.e.

$$c_6 \triangle^{\rho}_{N_r}(\mathrm{u}_r(y)) \leq ||\mathrm{u}_h(y) - \mathrm{u}_r(y)||_{X_\rho}. \tag{6.21}$$

Furthermore, we obtained its direct comparison with that of the stochastic collocation approximation in [6]: provided that the training set Ξ_{train} for the weighed reduced basis approximation is taken the same as or including the collocation set Ξ_{sc} for the stochastic collocation approximation, we have the following error comparison

$$||\underline{u}_h - \underline{u}_r||_{C(\Gamma;U)} + ||p_h - p_r||_{C(\Gamma;U)} \leq C \left(||\underline{u}_h - \underline{u}_s||_{C(\Gamma;U)} + ||p_h - p_s||_{C(\Gamma;\bar{X})} \right), \tag{6.22}$$

where the constant C only depends on the constants $\gamma, \delta, \varsigma, \bar{\beta}_h$ defined in (6.3), (6.4) and (6.5), and $\underline{u}_s, p_s$ are the optimal solution by stochastic collocation method with either tensor-product or sparse-grid technique, see [6] for a similar proof.

Remark 6.3. *The result (6.20) implies that whenever the error of the best possible approximation decays exponentially, the reduced basis error also enjoys an exponential decay with rate depending on the sharpness of the greedy algorithm (6.21). The result (6.22) guarantees that the weighted reduced basis approximation converges no slower than the stochastic collocation approximation, in fact much faster in practice, see [6] for detailed comparison based on examples in various test settings.*

6.4. Global error estimate. In this section, we provide a global error estimate by assembling error contribution from the different sources analyzed above.

To start, we consider a pointwise (at certain $y \in \Gamma$) error estimate for finite element approximation and weighted reduced basis approximation.

THEOREM 6.1. *Given any $y \in \Gamma$, the reduced basis solution $(\underline{u}_r(y), p_r(y))$ of problem (5.22) satisfies the following mixed error estimate (a priori error for finite element approximation and a posteriori error for reduced basis approximation)*

$$||\underline{u}(y) - \underline{u}_r(y)||_U + ||p(y) - p_r(y)||_{\bar{X}} \leq \mathcal{E}_h(y) + \triangle_{N_r}(y), \quad \forall y \in \Gamma, \tag{6.23}$$

where the finite element error $\mathcal{E}_h(y)$ is defined in (6.6), and the reduced basis error $\triangle_{N_r}(y)$ is defined similar to (5.42) without multiplying by the weight $\rho(y)$.

Proof. The proof is straightforward by separating the global error into two parts

$$\begin{aligned} ||\underline{u}(y) - \underline{u}_r(y)||_U + ||p(y) - p_r(y)||_{\bar{X}} \leq\ & ||\underline{u}(y) - \underline{u}_h(y)||_U + ||\underline{u}_h(y) - \underline{u}_r(y)||_U \\ & + ||p(y) - p_h(y)||_{\bar{X}} + ||p_h(y) - p_r(y)||_{\bar{X}}, \end{aligned} \tag{6.24}$$

which yields (6.23) by gathering the first and third terms for $\mathcal{E}_h(y)$ by finite error (6.6) and the left terms for $\triangle_{N_r}(y)$ by reduced basis error (5.42). □

The $C(\Gamma)$-error (in worst case scenario) for both stochastic collocation approximation and weighted reduced basis approximation is stated as follows.

THEOREM 6.2. *The global $C(\Gamma)$-error by the finite element and the stochastic collocation approximations can be estimated by*

$$||\underline{u}(y) - \underline{u}_s(y)||_{C(\Gamma;U)} + ||p(y) - p_s(y)||_{C(\Gamma;\bar{X})} \leq \mathcal{E}_h + \mathcal{E}_s, \tag{6.25}$$

where $\mathcal{E}_h = \sup_{y\in\Gamma} \mathcal{E}_h(y)$ and $\mathcal{E}_s$ is defined in (6.8) for tensor product stochastic collocation approximation and in (6.12) for sparse-grid type. As for reduced basis and finite element approximations, we have

$$||\underline{u}(y) - \underline{u}_r(y)||_{C(\Gamma;U)} + ||p(y) - p_r(y)||_{C(\Gamma;\bar{X})} \leq \mathcal{E}_h + \mathcal{E}_r, \tag{6.26}$$

where $\mathcal{E}_r$ is given in (6.17) or (6.20).

Finally, we present the probability-averaged error (error of expectation) for a combination of stochastic collocation and weighted reduced basis approximations.

THEOREM 6.3. *For the approximation of the expectation of the stochastic optimal solution, we use reduced basis solution at all the collocation nodes Ξ_{sc} and quadrature formula (5.15) or (5.21), with the global error bounded by*

$$||\mathbb{E}[\underline{u}] - \mathbb{E}[\underline{u}_r]||_U + ||\mathbb{E}[p] - \mathbb{E}[p_r]||_{\bar{X}} \leq \mathcal{E}_h + \mathcal{E}_s^e + \sqrt{|\Gamma|}\triangle^{\rho}_{N_r}(y^{N_r}). \tag{6.27}$$

where $|\Gamma|$ is the Lebesgue measure of the probability domain Γ, $\mathcal{E}_h$ is defined as in Theorem 6.2, $\mathcal{E}_s^e$ is defined in (6.11) or (6.16), and $\triangle^{\rho}_{N_r}(y^{N_r})$ is defined by (5.42) evaluated at the N_r-th sample $y^{N_r} \in \Gamma$ picked by the weighted greedy algorithm.

Proof. The global probability-averaged error can be bounded by

$$\begin{aligned}
||\mathbb{E}[\underline{u}] - \mathbb{E}[\underline{u}_r]||_U + ||\mathbb{E}[p] - \mathbb{E}[p_r]||_{\bar{X}} &\leq ||\underline{u} - \underline{u}_r||_{L^2_\rho(\Gamma;U)} + ||p - p_r||_{L^2_\rho(\Gamma;\bar{X})} \\
&\leq ||\underline{u} - \underline{u}_h||_{L^2_\rho(\Gamma;U)} + ||\underline{u}_h - \underline{u}_s||_{L^2_\rho(\Gamma;U)} + ||\underline{u}_s - \underline{u}_r||_{L^2_\rho(\Gamma;U)} \\
&+ ||p - p_h||_{L^2_\rho(\Gamma;\bar{X})} + ||p_h - p_s||_{L^2_\rho(\Gamma;\bar{X})} + ||p_s - p_r||_{L^2_\rho(\Gamma;\bar{X})} \\
&\leq \mathcal{E}_h + \mathcal{E}_s^e + \left(\int_\Gamma \left(\triangle^{\rho}_{N_r}(y)\right)^2 dy\right)^{1/2} \leq \mathcal{E}_h + \mathcal{E}_s^e + \sqrt{|\Gamma|}\triangle^{\rho}_{N_r}(y^{N_r}).
\end{aligned} \tag{6.28}$$

We remark that $(\underline{u}_s(y), p_s(y)) = (\underline{u}_h(y), p_h(y))$ at the collocation nodes $y \in \Xi_{sc}$ and $\sqrt{|\Gamma|}\triangle^{\rho}_{N_r}(y^{N_r})$ can be replaced by $\sup_{y\in\Xi_{sc}} \triangle_{N_r}(y)$ using the fact that $\triangle^{\rho}_{N_r}(y) = \triangle_{N_r}(y)\sqrt{\rho(y)}$ in the last inequality. ∎

7. Numerical tests. In this section, we carry out several numerical tests to illustrate the computational efficiency and numerical accuracy of the weighted reduced basis method compared to the non-weighted reduced basis method and stochastic collocation method with tensor product grid, isotropic and anisotropic sparse grid. Theoretical error estimates obtained in the last section are verified by three examples with different dimensions, ranging from one dimension to moderate dimension $(1-10)$ and to high dimension $(10-100)$, and with different probability distributions.

7.1. One dimensional problems. The first example focuses on the demonstration of the convergence property of the weighted reduced basis method compared to other methods with probability density functions of distinct shape. The physical domain is specified as $D = (0,1)^2$ with a uniform triangulation mesh of 712 vertices, over which we construct finite element space for spatial discretization by continuous piecewise linear polynomials. We set $f = 1$ and the coefficient a of problem (2.10) as

$$a(x, y) = \frac{1}{10}(1.1 + \sin(2\pi x_1)y), \tag{7.1}$$

where $x = (x_1, x_2) \in D$ and the random variable $y \sim Beta(\mu_1, \mu_2)$ obeys beta distribution supported on $\Gamma = [-1, 1]$ with two shape parameters $\mu_1, \mu_2 \in \mathbb{N}_+$. The probability density function of y is displayed in Fig. 7.1 when (μ_1, μ_2) take values of $(1, 1)$, $(10, 10)$ and $(100, 100)$, featuring very different shapes with distinct weight. The observation data u_d is set as the solution of (2.10) at the reference value $\bar{y} = 0$ and control $g = sin(\pi x_1)sin(\pi x_2)$. We define the worst case scenario error as

$$\max_{1\leq m\leq M_{test}} (||\underline{u}(y^m) - \underline{u}_N(y^m)||_U + ||p(y^m) - p_N(y^m)||_{\bar{X}}), \tag{7.2}$$

where $y^m, 1 \leq m \leq M_{test}$ are testing samples randomly drawn according to its probability density function, $(\underline{u}, p)$ is the finite element solution and $(\underline{u}_N, p_N)$ is the solution by (weighted) reduced basis method or stochastic collocation method with N bases or collocation nodes. The expectation error is defined in a posteriori way as

$$\left| ||E_l[\underline{u}]||^2_U - ||E_L[\underline{u}]||^2_U \right| + \left| ||E_l[p]||^2_{\bar{X}} - E_L[p]||^2_{\bar{X}} \right| \tag{7.3}$$

for ease of computation, where $l, 1 \leq l \leq L-1$ represents the level of approximation by quadrature formula. We apply the weighted reduced basis method and reduced basis method with M_{train} training samples drawn according to the probability distribution, and also stochastic collocation method based on Gauss-Jacobi quadrature nodes to solve the stochastic optimal control problem (2.17) with regularization parameter $\alpha = 1$. The convergence results are shown in the following few figures Fig. 7.1 - 7.4, for which we have used $M_{train} = 100$ training samples and $M_{test} = 100$ test samples.

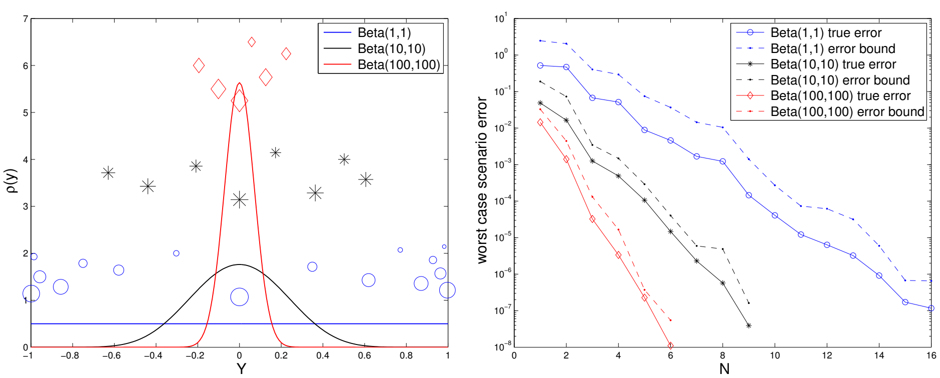

FIG. 7.1. *Left: Probability density function of $Beta(\mu_1, \mu_2)$ distribution with different (μ_1, μ_2) and samples selected by weighted reduced basis approximation in order, the bigger the size the earlier it has been selected; Right: convergence result of the true error and error bound by wRBM.*

On the left of Fig. 7.1, the samples selected by weighted reduced basis method are plotted in sequential order, where the larger the markers are, the earlier the samples have been selected. The right of Fig. 7.1 shows the convergence of the true error (error between approximation and true value) and the error bound $\triangle_N$ defined in (5.42) in three different settings. From Fig. 7.1 we can see that the most important samples (or samples with large probability) can be efficiently selected by the weighted reduced basis method, leading to less samples (thus less bases in the reduced basis space) for the more concentrated probability distribution.

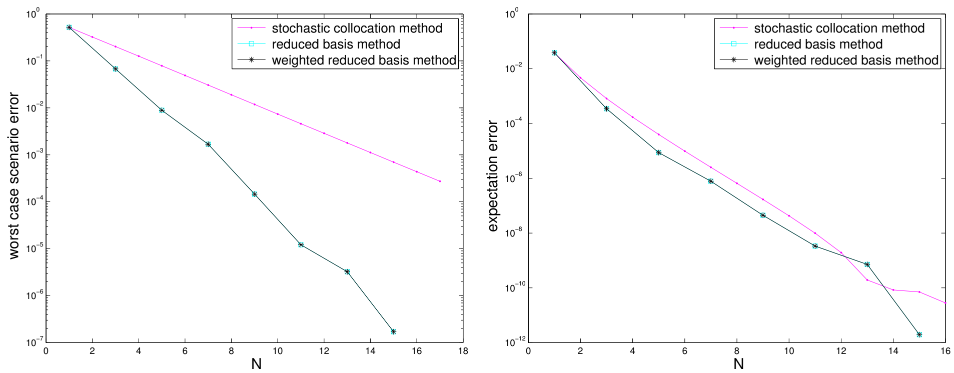

FIG. 7.2. *Comparison of worst case scenario error (left) and expectation error (right) by (weighted) reduced basis method and stochastic collocation method with* $(\mu_1, \mu_2) = (1, 1)$.

When $(\mu_1, \mu_2) = (1, 1)$, the beta distribution becomes a uniform distribution with probability density function $\rho = 1/2$, in which case the weighted reduced basis method is the same as reduced basis method, as we can see from their convergence results in Fig. 7.3, from which we can also observe that the reduced basis method converges faster than stochastic collocation method for worst case scenario error, while for the expectation error they display quite close convergence rates.

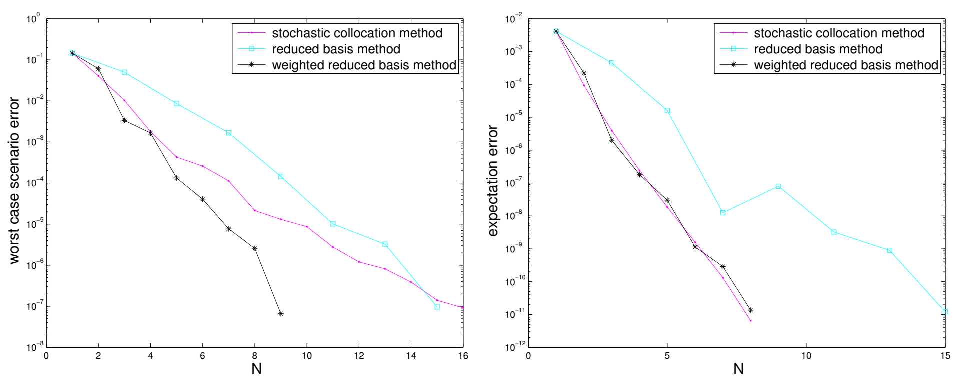

FIG. 7.3. *Comparison of worst case scenario error (left) and expectation error (right) by (weighted) reduced basis method and stochastic collocation method with* $(\mu_1, \mu_2) = (10, 10)$.

For $(\mu_1, \mu_2) = (10, 10)$, the weighted reduced basis method performs evidently better than the reduced basis method measured in both errors, and converges faster than stochastic collocation method as for worst case scenario error and comparable in expectation error (note that here the Gauss-Jacobi quadrature formula is optimal for evaluation of expectation), which demonstrates that the weighted reduced basis method works efficiently for evaluation of statistical moments of the solution. This conclusion has been further illustrated by the convergence results displayed in Fig. 7.4 for the test with $(\mu_1, \mu_2) = (100, 100)$. However, we remark that the computation for both offline construction and online evaluation by the (weighted) reduced basis method is more expensive than that by stochastic collocation method in one dimensional problems, see [6] for more detailed comparison of computational cost between reduced basis method and stochastic collocation method.

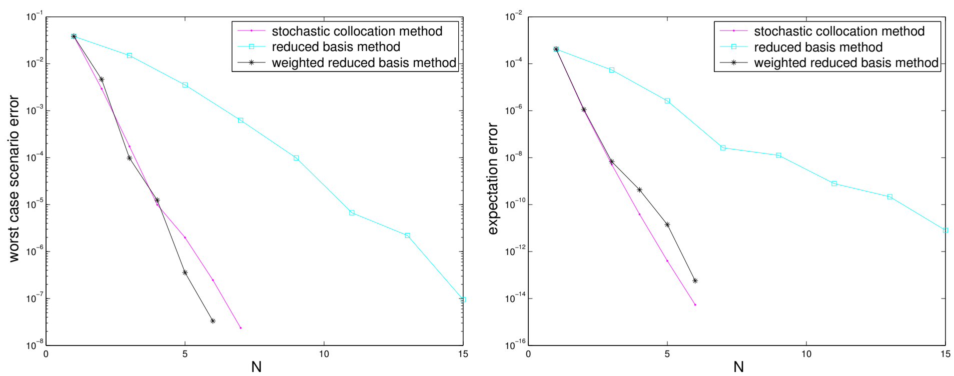

Fig. 7.4. *Comparison of worst case scenario error (left) and expectation error (right) by (weighted) reduced basis method and stochastic collocation method with* $(\mu_1, \mu_2) = (100, 100)$.

7.2. Moderate dimensional problems. The example presented in this section is devoted to verify the error estimates obtained in Section 6 and demonstrate the computational efficiency and numerical accuracy of the weighted reduced basis method. We define a general random field g as a truncation of Karhunen-Loève expansion of a Gaussian random field with correction length L [21]

$$g(x_i, y) = \mathbb{E}[g] + \left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} y_1 + \sum_{n=1}^{N} \sqrt{\lambda_n} \left(\sin(n\pi x_i) y_{2n} + \cos(n\pi x_i) y_{2n+1}\right), \quad (7.4)$$

where the random variables $y_n, 1 \le n \le 2N+1$ follow standard normal distribution, the eigenvalues $\lambda_1 = 0.4782, \lambda_2 = 0.0752, \lambda_3 = 0.0034$, accounting for around 99.5% uncertainties of the random field truncated with 7 random variables. In order to guarantee assumption (2.11), we cut off [1] the random variables $|y_n| \le 3, 1 \le n \le N$ (with tail probability less than 0.5%) and set $\mathbb{E}[g] = 8$. For simplicity, we do not consider the cut-off error and the truncation error. We set $a = g(x_1, y)/10$, $f = g(x_2, y)$, $\alpha = 1$ and the observation data u_d as the solution of (2.10) at the reference value $\bar{y}_n = 0, 1 \le n \le 2N+1$ and control function $g(x, y) = sin(\pi x_1) sin(\pi x_2)$.

To test the finite element error, we set $y = \bar{y}$, $h = 1/4, 1/8, 1/16, 1/32, 1/64$ and use the optimal solution at $h = 1/64$ as the "true" value. Fig. 7.5 (left) displays the linear and quadratic decay of finite element error $\mathcal{E}_h(\bar{y})$ with $\mathbb{P}_1$ and $\mathbb{P}_2$ elements, which verified the results in (6.6). The right of Fig. 7.5 depicts the reduced basis error $\mathcal{E}_r$ and the error bound $\triangle_{N_r}$ of the optimal solution (in fact, we take the worse case scenario error at 100 test samples), from which we can see that the cheap error bound is rather sharp and accurate, decaying exponentially fast with respect to the number of reduced bases. We remark that the error bound depends on the lower bound of the inf-sup constant $\beta_{LB}(y), y \in \Gamma$ in (5.42), which falls inside $[0.5, 1]$ in the training set

[1] Alternative to the necessity of cut-off, we may assume log-normal structure [21] of the random field and apply weighted empirical interpolation method [9] to obtain an affine decomposition (2.8).

$y \in \Xi_{train}$ with 1000 samples. For the sake of computational efficiency, we can take a uniform lower bound $\beta_{LB} = 0.5$ for any new $y \in \Gamma$.

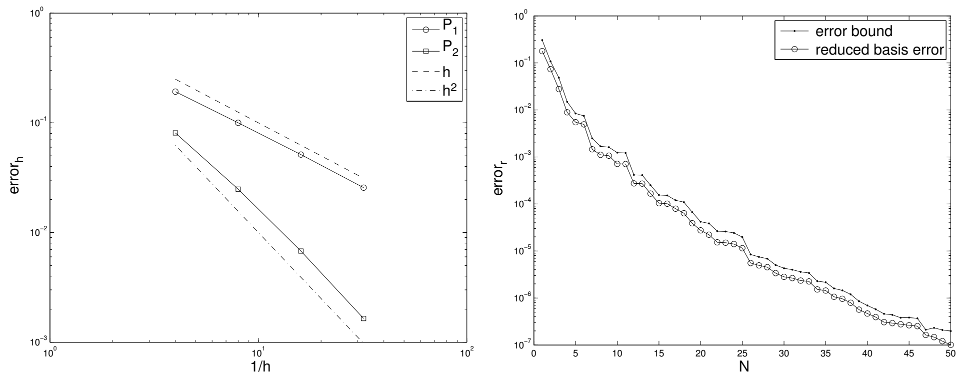

FIG. 7.5. *Left: finite element error of $\mathbb{P}_1$ and $\mathbb{P}_2$; right: reduced basis error and error bound.*

Fig. 7.6 reports the comparison of stochastic approximation errors between the weighted reduced basis method (RBM) and the stochastic collocation method (SCM) with Gauss-Hermite collocation nodes in both tensor-product and sparse-grid settings. The convergence comparison measured by worst case scenario error is depicted on the left of Fig. 7.6, which shows that the reduced basis approximation converges much faster than the stochastic collocation approximation, with error reaching 10^{-7} with only 50 bases (thus 50 solve of the full optimality system (5.5)), while it requires $78079 \approx 1562 \times 50$ collocation nodes (thus 78079 solve) for sparse-grid setting to attain the same error although it converges faster than the tensor-product setting.

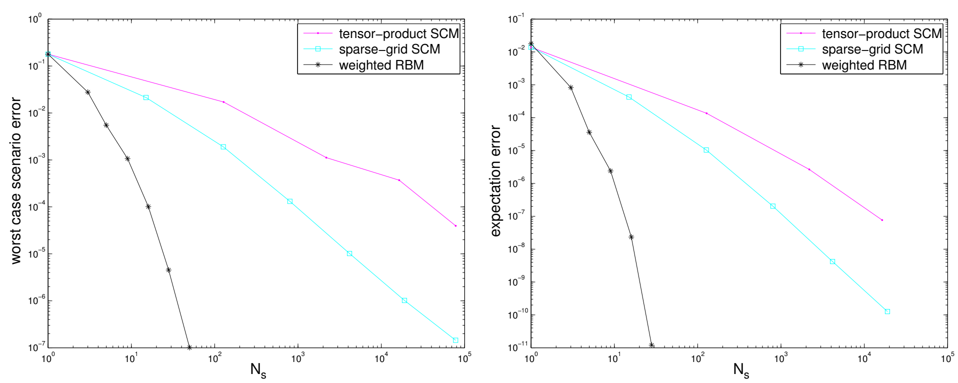

FIG. 7.6. *Comparison between weighted reduced basis method and stochastic collocation method (tensor-product and sparse-grid) for worst case scenario error (left) and error of expectation (right).*

For evaluation of the expectation error, we use the expectation of the optimal solution by the sparse-grid stochastic collocation method at the deepest level ($q - N = 6$) as a "true" value. The weighted reduced basis expectation is evaluated via formula (5.21) at the deepest level of sparse-grid with the optimal solution computed by online reduced basis procedure at all the collocation nodes. We can see from the right of Fig. 7.6 that only 28 bases or solve are needed for weighted reduced basis method to obtain a more accurate expectation than the stochastic collocation method (with 18943 solve in sparse-grid setting and 16384 by tensor-product setting). Thanks to the cheap online evaluation, the weighted reduced basis method is much more efficient than the stochastic collocation method to evaluate the statistics of the solution, especially when a solve of the full optimality system is very expensive.

7.3. High dimensional problems. In this section, we show that the weighted reduced basis method (wRBM) can be effectively applied to solve high dimensional problems and its combination with the anisotropic sparse grid stochastic collocation method (aSCM) are efficient to evaluate statistical moments of the solution.

We assume that the random coefficient $a = g(x_1, y)/10$ with g defined in (7.4) where $L = 1/128$, which features a slow decay of the eigenvalues ($\lambda_1 = 0.0138, \lambda_{50} = 0.0095$). Moreover, we assume that the random variables $y_n, 1 \leq n \leq 2N+1$ follow uniform distribution with zero mean and unit variance, supported on $[-\sqrt{3}, \sqrt{3}]$. We set $f = 10$ and $\mathbb{E}[a] = 20$ that satisfy Assumption 2 and u_d as for moderate dimensional problems in the last section with $g(x,y) = sin(\pi x_1) sin(\pi x_2)$. We apply a dimensional-adaptive algorithm (see [11] for details) with maximum number of collocation nodes specified as $10^1, 10^2, 10^3, 10^4, 10^5$ to construct the anisotropic sparse-grid stochastic collocation approximation. The weighted reduced basis approximation is constructed with 1000 training samples [2] and tested with 100 test samples. The convergence results are depicted in Fig. 7.7 for 11, 31 and 101 dimensional problems. In the reduced basis construction, only 30 bases have been used to achieve more accurate approximation (measured in worse case scenario error) than the anisotropic sparse grid stochastic collocation method with 10^5 collocation nodes, requiring 10^5 full solve of the optimality systems.

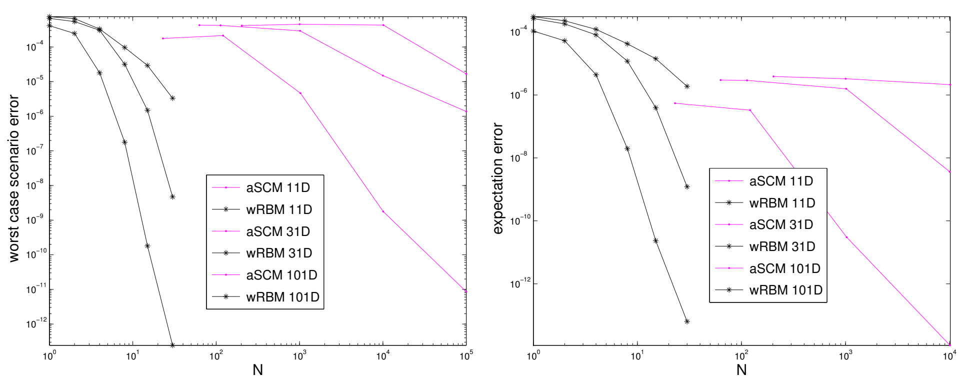

FIG. 7.7. *Comparison between weighted reduced basis method and stochastic collocation method (tensor-product and sparse-grid) for worst case scenario error (left) and error of expectation (right).*

As for evaluation of the expectation by weighted reduced basis method, we first compute the reduced basis solution at the collocation nodes in the anisotropic sparse-grid in the deepest level (with 10^5 collocation nodes) and then compute the expectation by Clenshaw-Curtis quadrature formula [21] on the sparse grid. From the right of Fig. 7.7, we can see that 30 reduced bases are sufficient for the weighted reduced basis method to achieve comparable accuracy as the stochastic collocation method.

[2]Instead of using a fixed number of training samples, we can choose adaptively the collocation nodes on the sparse grid as the training samples or use an adaptive greedy algorithm [35].

8. Concluding remarks. In this paper we studied stochastic optimal control problems with elliptic PDE constraint and developed and analyzed an efficient computational method to solve them. An analysis of existence, uniqueness and stochastic regularity of the optimal solution was carried out by virtue of a saddle point formulation of the optimal control problems. In numerical approximation of the stochastic optimality system, we applied finite element method with proper preconditioning techniques in the deterministic space and stochastic collocation method in the stochastic space. In order to alleviate the computational effort, we proposed a model order reduction approach based on a weighted reduced basis method. A global error analysis of our computation method was conducted thanks to the stochastic regularity result of the optimal solution. Numerical tests have well verified and illustrated the efficiency and accuracy of the computational method proposed in this paper. Generalization and application of the method in stochastic optimal control problems with more complex constraints, e.g. time-dependent and nonlinear problems, are ongoing.

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