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***Abstract*—Optimal control and shape optimization problems governed by partial differential equations (PDEs) arise in many applications involving computational fluid dynamics; they can be seen as *many-query problems* since they involve repetitive evaluations of outputs expressed as functionals of field variables. Since they usually require big computational efforts, looking for computational efficiency in numerical methods and algorithms becomes mandatory. We aim at reformulating optimal control and shape optimization problems as parametric optimization problems, where parameters can be either physical or geometrical quantities related to shape. In particular, we rely on suitable parametrization paradigms (such as free-form deformation techniques or radial basis functions) in order to describe shapes and deformations in a very low-dimensional but versatile way. We thus exploit the reduced basis method for parametrized problems (built upon a high-fidelity "truth" finite element approximation) in order to contain computational efforts. We develop these techniques in view of haemodynamics applications, to control blood flows through the shape optimization of cardiovascular geometries.**

***Keywords*—Optimal Control, Shape Optimization, Model Order Reduction, Reduced Basis Methods, Finite Elements, Stokes Equations.**

I. INTRODUCTION

In a broad variety of applications dealing with fluid dynamics the design of devices able to reduce drag forces or dissipations greatly enhances the efficiency of a system. The reduction of drag in transportation vehicles and of vorticity/stresses in biomedical devices [1] represent further instances in which optimization techniques are called into play.

Optimal flow control problems [2] can thus be formulated as the minimization of a given cost functional (or *output*) controlling some *input* parameters which can be physical quantities (e.g. source terms or boundary values) or, alternatively, geometrical quantities; we refer to the latter case as shape optimization problems [3], [4]. Such problems involve the study of a system of PDEs and the evaluation of an output depending on the field variables, combining flow simulation, optimization and – in case of shape optimization problems – shape variation [2]. Since *(i)* optimization procedures require repetitive evaluations of outputs, *(ii)* PDEs can be hard to solve and *(iii)* discretization is expensive when geometry keeps changing, computational costs are usually very high; we thus want to address suitable strategies to reduce numerical efforts in *many-query* problems.

Our proposed approach to solve optimal control or shape optimization is as follows. First we express the *control* as a *parametric input*; this stage is straightforward in the former case, while in the latter a suitable parametrization of the geometry is required. Then, an equivalent parametrized formulation of the flow problem – now embedding the input/control as a parametric quantity – can be derived. If a suitable condition of *affine parametric dependence* holds (otherwise it could be recovered through an intermediate *empirical interpolation* process), we can exploit the *reduced basis (RB)* method [5] for parametrized PDEs, based on a powerful Offline-Online computational strategy. This method offers the possibility to evaluate the *output* very rapidly; hence, at the outer level, a suitable iterative procedure for the optimization is performed, now involving a very reduced version of the original problem. In this paper we present a new *recipe* for the solution of shape optimization problems, addressing a quick presentation of the required *ingredients* (along with their *quantities*), of the most important *preparation* steps and a texture assessment of some results concerning two problems arising in haemodynamics.

II. SHAPE OPTIMIZATION FOR STOKES FLOWS

From an abstract point of view, optimal control and shape optimization problems can be written as the minimization, over an admissible controls set $U = (u, \Omega_o) \in \mathcal{U}_{ad} = U_{ad} \times \mathcal{O}_{ad}$,

$$\text{find} \quad \hat{U} = \arg\min_{U \in \mathcal{U}_{ad}} \mathcal{J}_o(Y(U)) \tag{1}$$

of a cost functional $\mathcal{J}(Y(U))$ depending on the solution $Y = Y(U)$ of a state problem (e.g. under weak form)

$$Y \in \mathcal{Y}(\Omega_o): \quad \mathcal{A}_o(Y, W; U) = \mathcal{F}_o(W; U), \qquad \forall\, W \in \mathcal{Y}(\Omega_o); \tag{2}$$

$\mathcal{A}_o(\cdot,\cdot;U)$ is a bilinear form and $\mathcal{F}_o(\cdot;U)$ is a linear form, both depending on the control variable $U = (u, \Omega_o)$, being $u \in U_{ad}$ a control function and $\Omega_o \in \mathcal{O}_{ad}$ the *original* domain where the problem is defined; $\mathcal{Y}(\Omega_o)$ denotes a suitable functional space defined over Ω_o. Let us assume that the control variable U depends on a set of *input* parameters $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p) \in \mathcal{D} \subset \mathbb{R}^p$; in this way, problem (1)-(2) can be reduced to the following *parametric optimization* problem:

$$\text{find} \quad \hat{\boldsymbol{\mu}} = \arg\min_{\boldsymbol{\mu} \in \mathcal{D}_{ad}} \mathcal{J}_o(Y(\boldsymbol{\mu})) \tag{3}$$

where $\mathcal{D}_{ad} \subseteq \mathcal{D}$ and $Y(\boldsymbol{\mu})$ solves

$$Y(\boldsymbol{\mu}) \in \mathcal{Y}(\Omega_o(\boldsymbol{\mu})): \quad \mathcal{A}_o(Y(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = \mathcal{F}_o(W; \boldsymbol{\mu}), \qquad \forall\, W \in \mathcal{Y}(\Omega_o(\boldsymbol{\mu})). \tag{4}$$

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Let us now focus on shape optimization problems for Stokes flows [6], and characterize the abstract parametric framework (3)-(4). Let us denote $Y(\boldsymbol{\mu}) = (\mathbf{u}(\boldsymbol{\mu}), p(\boldsymbol{\mu}))$ the velocity and the pressure of the fluid, $W = (\mathbf{w}, q)$ and

$$\mathcal{A}_o(Y(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = a_o(\mathbf{u}(\boldsymbol{\mu}), \mathbf{w}; \boldsymbol{\mu}) + b_o(p(\boldsymbol{\mu}), \mathbf{w}; \boldsymbol{\mu}) + b_o(q, \mathbf{u}(\boldsymbol{\mu}); \boldsymbol{\mu}),$$

where

$$a_o(\mathbf{u}, \mathbf{w}; \boldsymbol{\mu}) = \int_{\Omega_o(\boldsymbol{\mu})} \frac{\partial \mathbf{u}}{\partial x_{oi}} \nu^o_{ij} \frac{\partial \mathbf{w}}{\partial x_{oj}} d\Omega_o, \qquad \nu^o_{ij} = \nu \delta_{ij},$$

$$b_o(p, \mathbf{w}; \boldsymbol{\mu}) = -\int_{\Omega_o(\boldsymbol{\mu})} p \chi^o_{ij} \frac{\partial w_j}{\partial x_{oi}} d\Omega_o, \qquad \chi^o_{ij} = \delta_{ij},$$

being $\nu > 0$ the kynematic viscosity of the fluid and δ_{ij} the Kronecker symbol. In the same way,

$$\mathcal{F}_o(W; \boldsymbol{\mu}) = \langle F^s_o, \mathbf{w} \rangle + \langle F^l_o, \mathbf{w} \rangle + \langle G^l_o, q \rangle$$

where

$$\langle F^s_o, \mathbf{w} \rangle = \int_{\Omega_o(\boldsymbol{\mu})} \mathbf{f} \cdot \mathbf{w} d\Omega_o$$

is a source term and

$$\langle F^l_o, \mathbf{w} \rangle = -a_o(R_o, \mathbf{w}; \boldsymbol{\mu}), \quad \langle G^l_o, q \rangle = -b_o(q, R_o; \boldsymbol{\mu})$$

are the terms related to the lifting of (possibly) nonhomogeneous Dirichlet boundary conditions. Here $\mathcal{Y}(\Omega_o(\boldsymbol{\mu})) = X(\boldsymbol{\mu}) \times Q(\boldsymbol{\mu})$, being $(H^1_0(\Omega_o(\boldsymbol{\mu}))^2 \subseteq X(\boldsymbol{\mu}) \subseteq (H^1(\Omega_o(\boldsymbol{\mu}))^2$ the velocity space (taking into account the homogeneous Dirichlet boundary conditions) and $Q(\Omega_o) \subseteq L^2(\Omega_o)$ the pressure space, respectively; $R_o \in X(\boldsymbol{\mu})$ is a proper lifting function. Concerning the cost functional to be minimized, the case of a quadratic *output* depending on the velocity field, i.e.

$$s_o(\boldsymbol{\mu}) = \mathcal{J}_o(\mathbf{u}(\boldsymbol{\mu})) = \int_{\Omega_o(\boldsymbol{\mu})} Q(\mathbf{u}(\boldsymbol{\mu})) d\Omega_o, \tag{5}$$

is considered, being $Q = Q(\cdot)$ a physical quantity of interest.

Our approach to shape optimization takes advantage of *reduced basis* (RB) methods for rapid and reliable prediction of engineering outputs associated with parametric PDEs [5]. This method is premised upon a classical finite element (FE) method "truth" approximation space $\mathcal{Y}^{\mathcal{N}}$ of (typically very large) dimension $\mathcal{N}$ and is based on the use of "snapshot" FE solutions of the PDEs, corresponding to certain parameter values, as global approximation basis functions previously computed and stored [5]. The RB framework requires a reference (parameter independent) domain Ω in order to compare, and combine, FE solutions that would be otherwise computed on different domains and grids; moreover, this procedure enables to avoid shape deformation and remeshing that normally occur at each step of an iterative optimization procedure [7].

The reference domain Ω is related to the original domain $\Omega_o(\boldsymbol{\mu})$ through a parametric mapping $T(\cdot; \boldsymbol{\mu})$, such that $\Omega_o(\boldsymbol{\mu}) = T(\Omega; \boldsymbol{\mu})$; a possible shape parametrization used for the construction of the map will be discussed in Sec. III-B. By tracing the problem (3)-(4) back on the reference domain Ω, we obtain the following parametrized formulation:

$$\begin{aligned} &\text{find } \hat{\boldsymbol{\mu}} = \arg \min_{\boldsymbol{\mu} \in \mathcal{D}_{ad}} s(\boldsymbol{\mu}) = \mathcal{J}(Y(\boldsymbol{\mu})) \quad \text{s.t.} \\ &Y(\boldsymbol{\mu}) \in \mathcal{Y}: \ \mathcal{A}(Y(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = \mathcal{F}(W; \boldsymbol{\mu}), \ \forall W \in \mathcal{Y}. \end{aligned} \tag{6}$$

For the parametrized Stokes problem [8], [9],

$$\mathcal{A}(Y(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = a(\mathbf{u}(\boldsymbol{\mu}), \mathbf{w}; \boldsymbol{\mu}) + b(p(\boldsymbol{\mu}), \mathbf{w}; \boldsymbol{\mu}) + b(q, \mathbf{u}(\boldsymbol{\mu}); \boldsymbol{\mu}),$$

where

$$a(\mathbf{u}, \mathbf{w}; \boldsymbol{\mu}) = \int_\Omega \frac{\partial \mathbf{u}}{\partial x_i} \nu_{ij}(\boldsymbol{\mu}) \frac{\partial \mathbf{w}}{\partial x_j} d\Omega, \quad \boldsymbol{\nu}(\boldsymbol{\mu}) = \boldsymbol{J}_T^{-1} \boldsymbol{\nu}^o \boldsymbol{J}_T^{-T} |\boldsymbol{J}_T|,$$

$$b(p, \mathbf{w}; \boldsymbol{\mu}) = -\int_\Omega p \chi_{ij}(\boldsymbol{\mu}) \frac{\partial w_j}{\partial x_i} d\Omega, \quad \boldsymbol{\chi}(\boldsymbol{\mu}) = \boldsymbol{J}_T^{-1} \boldsymbol{\chi}^o |\boldsymbol{J}_T|,$$

being $\boldsymbol{J}_T = \boldsymbol{J}_T(\boldsymbol{\mu})$ the Jacobian of $T(\cdot; \boldsymbol{\mu})$ and $|\boldsymbol{J}_T|$ its determinant;

$$\mathcal{F}(W; \boldsymbol{\mu}) = \langle F^s, \mathbf{w} \rangle + \langle F^l, \mathbf{w} \rangle + \langle G^l, q \rangle$$

where

$$\langle F^s, \mathbf{w} \rangle = \int_\Omega \mathbf{f} \cdot \mathbf{w} |\boldsymbol{J}_T| d\Omega,$$

$$\langle F^l, \mathbf{w} \rangle = -a(R, \mathbf{w}; \boldsymbol{\mu}), \quad \langle G^l, q \rangle = -b(q, R; \boldsymbol{\mu}).$$

As before, $\mathcal{Y} = X \times Q$, being $(H^1_0(\Omega))^2 \subseteq X \subseteq (H^1(\Omega))^2$ and $Q \subseteq L^2(\Omega)$; $R \in X$ is a proper lifting function. Moreover, the *output* to be minimized is given by

$$s(\boldsymbol{\mu}) = \mathcal{J}(\mathbf{u}(\boldsymbol{\mu})) = \int_\Omega Q \circ T(\mathbf{u}(\boldsymbol{\mu})) |\boldsymbol{J}_T| d\Omega. \tag{7}$$

Hence, the original problem has been reformulated on a reference configuration, resulting in a parametrized problem where the effect of geometry variations is traced back onto its parametrized transformation tensors.

III. Reduction Ingredients

Our model reduction framework is based on the coupling between the RB method and suitable shape parametrization techniques. The combination of these tools allows a considerable reduction in the number of design parameters as well as a computational saving (thanks to the reduced dimension of the linear system associated to the resulting discretized problems).

A. Reduced Basis Method

Following the so-called *discretize than optimize* approach, the standard Galerkin FE approximation of (6) is as follows:

$$\begin{aligned} &\text{find } \hat{\boldsymbol{\mu}} = \arg \min_{\boldsymbol{\mu} \in \mathcal{D}_{ad}} s^{\mathcal{N}}(\boldsymbol{\mu}) = \mathcal{J}(Y^{\mathcal{N}}(\boldsymbol{\mu})) \quad \text{s.t.} \\ &Y^{\mathcal{N}}(\boldsymbol{\mu}) \in \mathcal{Y}^{\mathcal{N}}: \ \mathcal{A}(Y^{\mathcal{N}}(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = \mathcal{F}(W; \boldsymbol{\mu}), \ \forall W \in \mathcal{Y}^{\mathcal{N}}. \end{aligned}$$

The reduced basis method provides an efficient way to compute an approximation $Y_N(\boldsymbol{\mu})$ of $Y^{\mathcal{N}}(\boldsymbol{\mu})$ (and related output) by using a Galerkin projection on a reduced subspace made up of well-chosen FE solutions, corresponding to a specific choice $S_N = \{\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^N\}$ of parameter values. Indicating by

$$\mathcal{Y}^{\mathcal{N}}_N = \text{span}\{Y^{\mathcal{N}}(\boldsymbol{\mu}^n), n = 1, \ldots, N\}, \tag{8}$$

the RB space, the RB formulation of (6) is as follows:

$$\begin{aligned} &\text{find } \hat{\boldsymbol{\mu}} = \arg \min_{\boldsymbol{\mu} \in \mathcal{D}_{ad}} s_N(\boldsymbol{\mu}) = \mathcal{J}(Y_N(\boldsymbol{\mu})) \quad \text{s.t.} \\ &Y_N(\boldsymbol{\mu}) \in \mathcal{Y}^{\mathcal{N}}_N: \ \mathcal{A}(Y_N(\boldsymbol{\mu}), W; \boldsymbol{\mu}) = \mathcal{F}(W; \boldsymbol{\mu}), \ \forall W \in \mathcal{Y}^{\mathcal{N}}_N. \end{aligned}$$

Thanks to the (considerably) reduced dimension $O(N) \ll O(\mathcal{N})$ of the linear systems obtained from RB approximation, we can provide both reliable results and rapid response in the real-time (e.g. parameter estimation) and multi-query contexts (e.g. optimal design/control). In particular:

- *Reliability* is ensured by rigorous a posteriori estimations for the error in the RB approximation w.r.t. truth FE discretization; using the general Babuška framework, we can derive *a posteriori* error bounds under the form [9]:

$$\|Y^{\mathcal{N}}(\boldsymbol{\mu})-Y_N(\boldsymbol{\mu})\|_{\mathcal{Y}} \leq \Delta_N(\boldsymbol{\mu}) := \frac{\|r(\cdot;\boldsymbol{\mu})\|_{(\mathcal{Y}^{\mathcal{N}})'}}{\beta_{LB}^{\mathcal{N}}(\boldsymbol{\mu})}; \quad (9)$$

  $r(W;\boldsymbol{\mu}) = \mathcal{F}(W;\boldsymbol{\mu}) - \mathcal{A}(Y_N(\boldsymbol{\mu}), W;\boldsymbol{\mu})$ is the residual and $\beta_{LB}^{\mathcal{N}}(\boldsymbol{\mu})$ is a lower bound of the inf-sup constant [9]

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{V\in\mathcal{Y}^{\mathcal{N}}} \sup_{W\in\mathcal{Y}^{\mathcal{N}}} \frac{\mathcal{A}(V,W;\boldsymbol{\mu})}{\|V\|_{\mathcal{Y}}\|W\|_{\mathcal{Y}}};$$

  similar error bounds are available also for the output.
- *Rapid response* is ensured by an Offline–Online computational strategy which minimizes marginal cost and a rapidly convergent RB space assembling, based on a *greedy algorithm* [5]. To achieve the rapid response goal, RB methods rely on the assumption of affine parametric dependence in $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{F}(\cdot;\boldsymbol{\mu})$, given by:

$$\begin{aligned} \mathcal{A}(Y,W;\boldsymbol{\mu}) &= \sum_{q=1}^{Q_{\mathcal{A}}} \Theta^q_{\mathcal{A}}(\boldsymbol{\mu})\mathcal{A}^q(Y,W), \\ \mathcal{F}(W;\boldsymbol{\mu}) &= \sum_{q=1}^{Q_{\mathcal{F}}} \Theta^q_{\mathcal{F}}(\boldsymbol{\mu})\mathcal{F}^q(\boldsymbol{\mu}). \end{aligned} \quad (10)$$

Hence, in an expensive Offline stage we prepare a very small RB "database", while in the Online stage, for each new $\boldsymbol{\mu} \in \mathcal{D}$, we rapidly evaluate both the field and the output (with error bounds) whose computational complexity is independent of FE dimension $\mathcal{N}$. This is essential in a parametrized optimization problem, where a great number of such evaluations are required. In case of nonaffine parametric dependence, an affine approximation is recovered through an empirical interpolation method (EIM) [10]. In order to guarantee an inf-sup stability condition for the Stokes problem the following RB spaces are built:

$$\begin{aligned} Q_N^{\mathcal{N}} &= \text{span}\{\zeta_n := p^{\mathcal{N}}(\boldsymbol{\mu}^n),\ n=1,\dots,N\}, \\ X_N^{\mathcal{N}} &= \text{span}\{\sigma_n := \mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}^n),\ T_p^{\boldsymbol{\mu}}\zeta_n,\ n=1,\dots,N\}, \end{aligned}$$

where $T_p^{\boldsymbol{\mu}} : Q_N^{\mathcal{N}} \rightarrow X_N^{\mathcal{N}}$ is the *supremizer operator*, $(T_p^{\boldsymbol{\mu}} q, \mathbf{w})_X = b(q,\mathbf{w};\boldsymbol{\mu}), \forall \mathbf{w} \in X_N^{\mathcal{N}}$, used to enrich the velocity space. In the nonlinear case (e.g. Navier-Stokes problem) Newton or fixed-point method is used for solving the nonlinear RB system [11].

B. Shape Parametrization Techniques

Since shape representation is highly problem-dependent, various methods have been proposed; common strategies for shape deformation involve the use of *(i)* the coordinates of the boundary points as design variables (*local boundary variation*) or *(ii)* some families of basis shapes combined by means of a set of control point (*polynomial boundary parametrizations*). These techniques are not well suited within the RB framework, since a global mapping $T(\cdot;\boldsymbol{\mu})$ is needed, rather than a boundary representation [7]. A more versatile parametrization

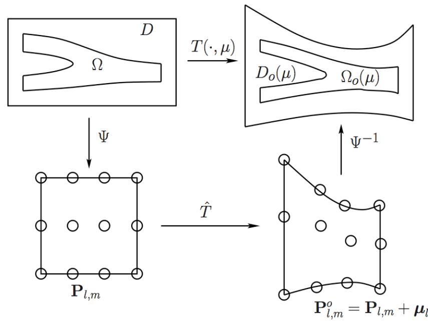

Fig. 1. Schematic diagram of the FFD technique

can be introduced by exploiting the *free-form deformation* (FFD) technique, in which the deformations of an initial design, rather than the geometry itself, are parametrized [12], [6]. We build a fixed rectangular domain D s.t. $\Omega \subset D$, a regular map $\hat{\mathbf{x}} = \Psi(\mathbf{x})$ s.t. $\Psi(D) = (0,1)^2$, and select an ordered lattice of control points $\mathbf{P}_{l,m} = [l/L, m/M]^T$, $l = 0,\dots,L$, $m = 0,\dots,M$ (see Fig. 1).

Then, we deform the object by the moving control points to a new position $\mathbf{P}^o_{l,m}(\boldsymbol{\mu}_{l,m}) = \mathbf{P}_{l,m} + \boldsymbol{\mu}_{l,m}$, by specifying a set of $(L+1)(M+1)$ parameter vectors $\boldsymbol{\mu}_{l,m} \in \mathbb{R}^2$. Usually, only small subsets of these *deformations* are selected (by means of problem-dependent criteria) as input parameters if we want a geometrical reduction; in this way, each parameter $\mu_1,\dots,\mu_p$ identifies a selected directional displacement of a corresponding control point. We can define the mapping $T(\cdot;\boldsymbol{\mu}) : D \rightarrow D_o(\boldsymbol{\mu})$ as $T(\mathbf{x};\boldsymbol{\mu}) = \Psi^{-1} \circ \hat{T} \circ \Psi(\mathbf{x};\boldsymbol{\mu})$, where

$$\hat{T}(\hat{\mathbf{x}};\boldsymbol{\mu}) = \left(\sum_{l=0}^{L}\sum_{m=0}^{M} b^{L,M}_{l,m}(\hat{\mathbf{x}}) \mathbf{P}^o_{l,m}(\boldsymbol{\mu}_{l,m}) \right),$$

and $b^{L,M}_{l,m}(\hat{\mathbf{x}}) = \binom{L}{l}\binom{M}{m}(1-\hat{x}_1)^{L-l}\hat{x}_1^l(1-\hat{x}_2)^{M-m}\hat{x}_2^m$ are tensor products of 1D *Bernstein basis polynomials*. Other shape parametrization techniques (e.g. based on radial basis functions) are currently under investigation.

IV. Shape Optimization Recipe

Here are the general directions for a reduced order shape optimization scheme, based on the ingredients presented above.

1) *Shape parametrization*. Select a shape parametrization technique, input parameters $\mu_1,\dots,\mu_p$ and the parameter set $\mathcal{D}$ on the basis of problem-dependent criteria.
2) *Parametrized formulation*. Compute the parametric map $T(\cdot;\boldsymbol{\mu})$ and the parametrized tensors $\boldsymbol{J}_T(\boldsymbol{\mu})$, $\boldsymbol{\nu}(\boldsymbol{\mu})$ and $\boldsymbol{\chi}(\boldsymbol{\mu})$ in order to get the parametrized formulation (6).
3) *Empirical interpolation*. Compute proper affine expansions in case of nonaffinely parametrized tensors.
4) *FE structures*. Assemble finite element structures corresponding to the $\boldsymbol{\mu}$-independent terms of (10), using the affine expansions of the tensors computed at point 3.
5) $\beta^{\mathcal{N}}_{LB}(\boldsymbol{\mu})$ *Lower bounds*. By means of the *successive constraint method* (SCM) compute the structures for the lower bound of the inf-sup constant $\beta^{\mathcal{N}}_{LB}(\boldsymbol{\mu})$.
6) *RB space construction*. Compute $\beta^{\mathcal{N}}_{LB}(\boldsymbol{\mu})$ lower bounds and dual norms of residuals in order to obtain lower bound (9). Use a *greedy algorithm* to adaptively select parameter values in (8) as follows. Let $\Xi_{train} \subset \mathcal{D}$ be

a (sufficiently rich) finite training sample; given a first value $\boldsymbol{\mu}^1$, the remaining parameter values are chosen as

$$\boldsymbol{\mu}^n := \arg \max_{\boldsymbol{\mu} \in \Xi_{train}} \Delta_{n-1}(\boldsymbol{\mu}), \quad \text{for } n = 2, \dots, N$$

until an error tolerance ε_{tol}^{RB} a priori fixed is achieved:

$$\Delta_N(\boldsymbol{\mu}) \leq \varepsilon_{tol}^{RB} \quad \text{for all } \boldsymbol{\mu} \in \Xi_{train}.$$

For each selected $\boldsymbol{\mu}^n$, compute the corresponding *snapshot* solution $Y^{\mathcal{N}}(\boldsymbol{\mu}^n)$ and perform a Gram-Schmidt orthonormalization of $\{Y^{\mathcal{N}}(\boldsymbol{\mu}^1), \dots, Y^{\mathcal{N}}(\boldsymbol{\mu}^n)\}$.

To date, the *Offline* preparation[1] has been performed. Then,

7) *Numerical Optimization*. Solve the optimization problem by a suitable optimization subroutine (e.g. sequential quadratic programming); at each step, an *Online* evaluation of RB solution and output is requested.

This combined "RB+FFD" approach features several advantages: besides being very flexible, it involves the solution of low dimensional problems, yielding substantial computational savings without sacrificing numerical accuracy, even when addressing complex shape optimization problems.

V. Study Cases in Haemodynamics

The proposed approach is rather general and can be used in a broad variety of application contexts. To provide a proof of its versatility, we apply it to the design of cardiovascular prostheses [6] and to the study of blood flows in vessels of variable shape[2]. More specifically, we want to find the optimal shape of an aorto-coronaric bypass graft, which represents the standard treatment of advanced coronary arteries diseases.

Some correlations between high vorticity areas and post-surgical complications (e.g. restenosis, long term graft failure) have been established. Moreover, computational studies have highlighted the correlation between shape and vorticity as effect of recirculation and flow separation. We use the Stokes equations for modelling low Reynolds number blood flow in mid size arteries. We are interested in the minimization of the blood vorticity in the down-field region $\Omega_o^{df}(\boldsymbol{\mu}) \subset \Omega_o(\boldsymbol{\mu})$, i.e. $Q(\mathbf{w}) = |\nabla \times \mathbf{w}|^2 \mathbf{1}_{\Omega_o^{df}(\boldsymbol{\mu})}$. With the framework described in Fig. 2, a vorticity reduction of about 60% has been obtained after 45 optimization steps. RB approximation have been built upon a FE approximation on $\mathbb{P}^2/\mathbb{P}^1$ spaces of total dimension $\mathcal{N} = 35,997$; the parametric complexity of the problem, i.e. the number of affine terms in (10) obtained through EIM, is quite high, since $Q_{\mathcal{A}} = 219$ and $Q_{\mathcal{F}} = 5$ with a tolerance $\varepsilon_{tol}^{EIM} = 10^{-4}$. By means of the *greedy algorithm*, with $\varepsilon_{tol}^{RB} = 10^{-3}$, $N = 24$ basis functions have been selected, giving an RB space of total dimension 72. The reduction in linear systems dimension leads to a significant (about 100 times) computational speedup (average times are about $2s$ vs. $200s$ for RB and FE approximation, respectively). Indeed, geometrical reduction in term of the number of parameters is of about 100 with respect to traditional shape parametrization based on local boundary variation.

[1] Efficient and reliable methods of computing both the dual norm of the residual and lower bounds $\beta_{LB}^{\mathcal{N}}(\boldsymbol{\mu})$ are used in order to exploit the *Offline-Online* computational stratagem also for the error estimation.

[2] Needless to say, these techniques prove to be useful also in facing other optimization problems arising in CFD.

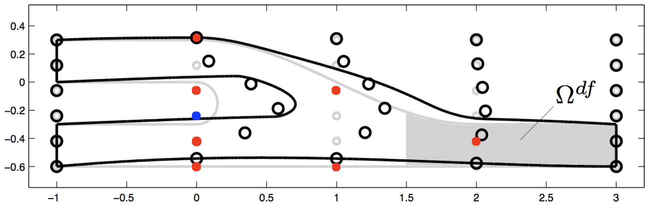

Fig. 2. Initial reference (in grey) and optimal (in black) shape, down-field region Ω^{df} for the reference shape. Control points depicted in red and blue can be freely moved in vertical or horizontal direction, respectively. Here $\mathcal{D} = \{\boldsymbol{\mu} = (\mu_1, \dots, \mu_8) \in \mathbb{R}^8 : \mu_i \in [-0.2, 0.2] \ \forall \ i \neq 5, \mu_5 \in [0, 1]\}$.

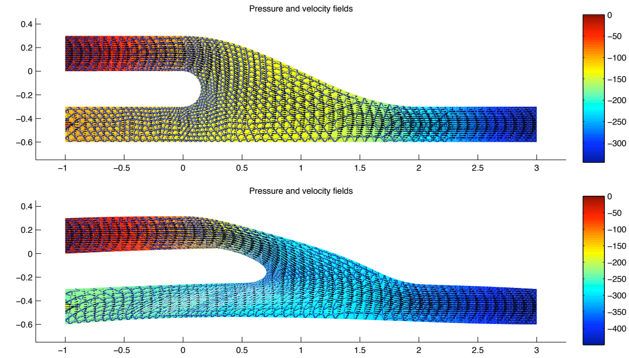

Fig. 3. Pressure and velocity fields for initial and optimal bypass shapes.

Following this path, other blood flows in variable geometries (e.g. carotid artery bifurcation) are under investigation. Moreover, in order to deal with more complex fluid dynamics, the extension of this framework to Navier-Stokes equations is in order and represents our current research activity.

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