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## Abstract

We analyze the stability and accuracy of discrete least squares on multivariate polynomial spaces to approximate a given function depending on a multivariate random variable uniformly distributed on a hypercube. The polynomial approximation is calculated starting from pointwise noise-free evaluations of the target function at low-discrepancy point sets. We prove that the discrete least-squares approximation, in a multivariate anisotropic tensor product polynomial space and with evaluations at low-discrepancy point sets, is stable and accurate under the condition that the number of evaluations is proportional to the square of the dimension of the polynomial space, up to logarithmic factors. This result is analogous to those obtained in [7, 22, 19, 6] for discrete least squares with random point sets, however it holds with certainty instead of just with high probability. The result is further generalized to arbitrary polynomial spaces associated with downward closed multi-index sets, but with a more demanding (and probably nonoptimal) proportionality between the number of evaluation points and the dimension of the polynomial space.

*Keywords:* approximation theory, discrete least squares, error analysis, multivariate polynomial approximation, low-discrepancy point set,  $(t, m, s)$ -net,  $(t, s)$ -sequence, nonparametric regression.

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## 1. Introduction

In recent years, an increasing interest has been dedicated to the various fields of applied mathematics gravitating around the issue of uncertain knowledge of data in computational models. The uncertainty can be treated by means of random variables

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distributed according to a given or unknown probability distribution. In the applications, the presence of multiple sources of uncertainties demands that a large number of random variables be employed. Therefore, the underlying challenge is the approximation of target quantities of interest which functionally depend on a large number of random variables. Starting from the classical Monte Carlo method, *i.e.* with random sampling points, several approaches have been proposed. When the functional dependencies on the random variables are smooth, polynomial approximation techniques [18] such as stochastic Galerkin [2], stochastic collocation on sparse grids [5] and discrete least squares with random evaluations [7, 22, 19, 6] have been proposed as an efficient approximation tool. Another approach is the quasi-Monte Carlo method [24, 29, 10], which relies on the careful development of specific sets of deterministic quadrature points, so-called low-discrepancy points, to approximate multidimensional integrals. The combination of random and deterministic points has proven advantageous as well.

In recent works, it has been proven that univariate discrete least squares on polynomial spaces with *random* evaluations uniformly distributed on an interval are stable and optimally convergent in expectation [7] and in probability [22], under the condition that the number of evaluations is proportional to the square of the dimension of the polynomial space. The analysis has been extended to the multivariate case in [6], for any dimension of the random variable, for polynomial spaces associated with any arbitrary downward closed multi-index set, for the uniform and Chebyshev densities. The same analysis can be extended to any tensorized densities on a hypercube in the beta family using the results proven in [20].

In the present work we focus only on the case of uniform density, and we analyze discrete least squares on multivariate polynomial spaces with evaluations at *low-discrepancy* point sets. We prove in Theorem 9 that, in multivariate anisotropic tensor product polynomial spaces and using low-discrepancy point sets, the discrete least-squares approximation of any uniformly continuous function is stable and accurate, when the number of evaluation points is proportional to the square of the dimension of the polynomial space (up to logarithmic factors). As in [6], accurate means that the error of the discrete least-squares projection in the  $L^2$  norm is comparable with the best approximation error in the  $L^\infty$  norm. Therefore, with anisotropic tensor product spaces, the use of low-discrepancy point sets leads to analogous theoretical results as those with random points proven in [6]. The results with low-discrepancy points hold with certainty, whereas the results with random points only hold with high probability or in expectation. A closer look to the logarithmic factors reveals that in the low-discrepancy case the stability condition contains a logarithmic dependence which worsens as the dimension increases, whereas the same logarithmic dependence is dimension-free in the random case.

In the multivariate case, when the polynomial space differs from the anisotropic tensor product the quadratic growth worsens: in any case we have proven the stability and accuracy of discrete least squares in any polynomial space associated with

arbitrary downward closed multi-index sets, if the number of evaluation points is proportional to the quartic power of the dimension of the polynomial space. Notice that this is a sufficient but not necessary condition. An analogous quartic proportionality can be proven using probabilistic estimates for the star discrepancy of random points independent and uniformly distributed.

A relevant quantity in our analysis is the superposition of star discrepancies of low-order projections of point sets, which has proven to be related to the convergence of quasi-Monte Carlo and to tractability issues, see [28, 32] and references therein.

Recently, in [34] an analysis of discrete least squares with deterministic points has been presented in the case of the Chebyshev density, however, using techniques quite different than those used here. The authors prove stability and accuracy under the condition that the number of points scales as the square of the dimension of the polynomial space associated with any downward closed multi-index set, with the proportionality constant depending on the number of components of the multivariate random variable.

We point out that, the use of quasi-Monte Carlo and low-discrepancy point sets for integration usually requires strong smoothness assumptions on the integrand, *e.g.* existence of mixed derivatives, see [10]. However, in our case the discrete least-squares approximation does not require any assumption of mixed regularity on the function to approximate. The quasi-Monte Carlo estimates involving mixed derivatives are applied here only on polynomial functions (which of course have enough regularity) to prove the stability of the discrete least-squares approximation.

The outline of the paper is the following: in §2 we recall the approximation methodology based on discrete least squares on multivariate polynomial spaces. In §3 we introduce the notion of star discrepancy of a point set, the latest developments of its upper bounds for nets and sequences, and some estimates for the superposition of star discrepancies of low-order projections of a point set. In §4 we prove a norm equivalence on multivariate polynomial spaces using the star discrepancy. In §5 we prove stability and accuracy of discrete least squares on multivariate polynomial spaces with evaluations at low-discrepancy point sets. Finally in §6 we draw some conclusions.

## 2. Discrete least-squares approximation

Let  $I_s \subset \mathbb{R}^s$  be the  $s$ -dimensional hypercube  $I_s := [0, 1]^s$  in the Euclidean  $s$ -dimensional space, with  $s \in \mathbb{N}$  denoting the dimension. Consider a random variable  $Y \in I_s$  distributed according to the probability density  $\rho : I_s \rightarrow \mathbb{R}_0^+$ , and a target function  $\phi : I_s \rightarrow \mathbb{R}$  that depends on the random variable. Throughout this article we consider only the tensorized  $s$ -dimensional uniform density  $\rho = \rho(y) := \otimes_{q=1}^s \mathbb{I}_{[0,1]}(y_q) dy_q$ , where  $\mathbb{I}_{[0,1]}$  denotes the characteristic function on the interval  $[0, 1]$ . We would like to approximate the function  $\phi = \phi(Y)$  in the  $L^2$  probability sense, using pointwise noise-free evaluations. The dependence of the function  $\phi$  on the random variable  $Y$  is assumed to be smooth, and this justifies the use of an approximation approach based on poly-

nomial expansions. Given  $n$  distinct points  $y^1, \dots, y^n \in I_s$ , we introduce the  $L^2$  scalar product and its discrete counterpart,

$$\langle f_1, f_2 \rangle_{L^2(I_s)} := \int_{I_s} f_1(y) f_2(y) dy, \quad \langle f_1, f_2 \rangle_n := \frac{1}{n} \sum_{i=1}^n f_1(y^i) f_2(y^i),$$

and the associated norm  $\|\cdot\|_{L^2(I_s)} := \langle \cdot, \cdot \rangle_{L^2(I_s)}^{1/2}$  and seminorm  $\|\cdot\|_n := \langle \cdot, \cdot \rangle_n^{1/2}$ .

We denote by  $\{\varphi_q\}_{q \geq 0}$  the family of univariate Legendre polynomials orthonormal w.r.t. the standard  $L^2$  scalar product on  $[0, 1]$ , i.e.  $\langle \varphi_q, \varphi_t \rangle_{L^2(0,1)} = \delta_{qt}$ , see [30]. Denote by  $\Lambda \subset \mathbb{N}_0^s$  a finite multi-index set, and for any  $\boldsymbol{\nu} \in \Lambda$  define the multivariate Legendre polynomials  $\psi_{\boldsymbol{\nu}}$  as

$$\psi_{\boldsymbol{\nu}}(y) := \prod_{q=1}^s \varphi_{\nu_q}(y_q), \quad y \in I_s, \quad (1)$$

by tensorization of the univariate  $L^2$ -orthonormal Legendre polynomials  $\{\varphi_q\}_{q \geq 0}$ . The space of polynomials  $\mathbb{P}_{\Lambda} = \mathbb{P}_{\Lambda}(I_s)$  associated with the multi-index set  $\Lambda$  is defined as

$$\mathbb{P}_{\Lambda} := \text{span}\{\psi_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \Lambda\},$$

and of course it holds  $\dim(\mathbb{P}_{\Lambda}) = \#\Lambda$ . Notice that the seminorm  $\|\cdot\|_n$  becomes a norm over any polynomial space  $\mathbb{P}_{\Lambda}$ , provided  $n$  is sufficiently large ( $n \geq \#\Lambda$ ) and the  $n$  points  $\{y^i\}_{i=1}^n$  are distinct. A particular class of multi-index sets, that we consider in our analysis in §4–§5, is characterized by the following property.

**Definition 1** (Downward closedness of the multi-index set  $\Lambda$ ). *The finite multi-index set  $\Lambda \subset \mathbb{N}_0^s$  is downward closed (or it is a lower set) if*

$$(\boldsymbol{\nu} \in \Lambda \quad \text{and} \quad \boldsymbol{\mu} \leq \boldsymbol{\nu}) \Rightarrow \boldsymbol{\mu} \in \Lambda,$$

where  $\boldsymbol{\mu} \leq \boldsymbol{\nu}$  means that  $\mu_q \leq \nu_q$  for all  $q = 1, \dots, s$ .

According to this definition, the multi-index set  $\Lambda = \{\mathbf{0}\}$ , which contains only the null multi-index, is downward closed.

Denoting by  $w$  a nonnegative integer, common isotropic polynomial spaces  $\mathbb{P}_{\Lambda_w}$  are

$$\begin{aligned} \text{Tensor Product (TP)} : \quad & \Lambda_w = \{\boldsymbol{\nu} \in \mathbb{N}_0^s : \|\boldsymbol{\nu}\|_{\ell_{\infty}(\mathbb{N}_0^s)} \leq w\}, \\ \text{Total Degree (TD)} : \quad & \Lambda_w = \{\boldsymbol{\nu} \in \mathbb{N}_0^s : \|\boldsymbol{\nu}\|_{\ell_1(\mathbb{N}_0^s)} \leq w\}, \\ \text{Hyperbolic Cross (HC)} : \quad & \Lambda_w = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^s : \prod_{q=1}^s (\nu_q + 1) \leq w + 1 \right\}. \end{aligned}$$

An anisotropic polynomial space, that will be used in the present paper, is the anisotropic tensor product space with maximum degrees  $w_1, \dots, w_s$  in each coordinate:

$$\text{anisotropic Tensor Product (aTP)} : \quad \Lambda_{w_1, \dots, w_s} = \{\boldsymbol{\nu} \in \mathbb{N}_0^s : \nu_q \leq w_q, \forall q = 1, \dots, s\}.$$

In the remaining part of this section, the multi-index set  $\Lambda$  need not be downward closed, but can be any finite multi-index set  $\Lambda \subset \mathbb{N}_0^s$ . We consider a discrete least-squares approximation of  $\phi$  over the polynomial space  $\mathbb{P}_\Lambda$ . Given  $n$  points  $y^1, \dots, y^n$ , we compute the noise-free evaluations of the target function  $\phi$  in these points. The discrete  $L^2$  projection  $\Pi_\Lambda^n \phi$  of the function  $\phi$  over the polynomial space  $\mathbb{P}_\Lambda$  is defined as

$$\Pi_\Lambda^n \phi := \operatorname{argmin}_{u \in \mathbb{P}_\Lambda} \|\phi - u\|_n, \quad (2)$$

and corresponds to a minimization problem whose unknown is the coefficient vector  $\beta$  in the expansion

$$(\Pi_\Lambda^n \phi)(y) = \sum_{\nu \in \Lambda} \beta_\nu \psi_\nu(y), \quad y \in I_s.$$

We introduce the design matrix  $A$  and the right-hand side  $b$  defined element-wise as  $[A]_{ij} = \psi_j(y^i)$  and  $b(y^i) = \phi(y^i)$ , respectively, for all  $i = 1, \dots, n$  and  $j = 1, \dots, \#\Lambda$ . From a linear algebra point of view, solving problem (2) is equivalent to finding the solution  $\beta$  to the normal equations

$$n^{-1} A^\top A \beta = n^{-1} A^\top b.$$

Problem (2) approximates the continuous  $L^2$  projection

$$\Pi_\Lambda \phi := \operatorname{argmin}_{u \in \mathbb{P}_\Lambda} \|\phi - u\|_{L^2(I_s)},$$

which usually cannot be directly computed.

In any dimension  $s$ , for any multi-index set  $\Lambda$  and any set of  $n$  distinct points  $y^1, \dots, y^n \in I_s$ , we define the following nonnegative quantities as in [22]:

$$\mathcal{Q}(n, \Lambda) := \sup_{u \in \mathbb{P}_\Lambda \setminus \{u \equiv 0\}} \frac{\|u\|_n^2}{\|u\|_{L^2(I_s)}^2} \quad \text{and} \quad \mathcal{S}(n, \Lambda) := \sup_{u \in \mathbb{P}_\Lambda \setminus \{u \equiv 0\}} \frac{\|u\|_{L^2(I_s)}^2}{\|u\|_n^2}. \quad (3)$$

In §4 we analyze these quantities using low-discrepancy point sets. In the case that the  $n$  points  $y^1, \dots, y^n$  are realizations of the random variables  $Y^1, \dots, Y^n \stackrel{\text{iid}}{\sim} \rho$ , the quantities  $\mathcal{Q} = \mathcal{Q}(n, \Lambda)$  and  $\mathcal{S} = \mathcal{S}(n, \Lambda)$  defined in (3) are two random variables themselves. This framework has been analyzed in [7, 22, 6, 21, 19], and we report in §5.2 the main results achieved.

In the remaining part of this section, the points  $y^1, \dots, y^n$  can be either deterministic or random. In the following we report two results from [22], that give an insight into the importance of the quantities (3) in the stability and convergence properties of the discrete  $L^2$  projection (2).

**Proposition 1.** *For any multi-index set  $\Lambda$  in any dimension  $s$ , with  $\mathcal{S}(n, \Lambda)$  defined as in (3) and  $n \geq \#\Lambda$ , it holds that*

$$\|\phi - \Pi_\Lambda^n \phi\|_{L^2(I_s)} \leq \left(1 + \sqrt{\mathcal{S}(n, \Lambda)}\right) \inf_{u \in \mathbb{P}_\Lambda} \|\phi - u\|_{L^\infty(I_s)}, \quad \forall \phi \in C^0(I_s). \quad (4)$$

*Proof.* See [22, Proposition 1]. □

To quantify the stability of the least-squares problem (2), we define the spectral condition number of its associated matrix  $A^\top A$  as

$$\text{cond}(A^\top A) := \frac{\sigma_{\max}(A^\top A)}{\sigma_{\min}(A^\top A)}, \quad (5)$$

with  $\sigma_{\max}(\cdot)$  and  $\sigma_{\min}(\cdot)$  being the maximum and minimum singular values.

**Proposition 2.** *For any multi-index set  $\Lambda$  and any dimension  $s$ , the spectral condition number (2-norm) of the matrix  $A^\top A$ , as defined in (5), is equal to*

$$\text{cond}(A^\top A) = \mathcal{Q}(n, \Lambda) \mathcal{S}(n, \Lambda), \quad (6)$$

since  $\sigma_{\max}(A^\top A) = \mathcal{Q}(n, \Lambda)$  and  $\sigma_{\min}(A^\top A) = (\mathcal{S}(n, \Lambda))^{-1}$ .

*Proof.* See [22, Proposition 4]. □

**Remark 1.** *In any dimension  $s$  and for any multi-index set  $\Lambda$  it holds that*

$$\mathcal{S}(n, \Lambda) = \sup_{\substack{u \in \mathbb{P}_\Lambda \\ \|u\|_{L^2(I_s)} = 1}} \frac{1}{\|u\|_n^2} = \left( \inf_{\substack{u \in \mathbb{P}_\Lambda \\ \|u\|_{L^2(I_s)} = 1}} \|u\|_n^2 \right)^{-1}, \quad \mathcal{Q}(n, \Lambda) = \sup_{\substack{u \in \mathbb{P}_\Lambda \\ \|u\|_{L^2(I_s)} = 1}} \|u\|_n^2.$$

### 3. Low-discrepancy point sets

In this section we introduce the notions of local discrepancy and star discrepancy of a given set of points, which aim at quantifying how well the points are uniformly distributed in the domain  $I_s$ . The topic is extensively introduced and covered in [24, 26, 11, 10], with complete lists of references.

Let  $S := \{1, \dots, s\}$  be the set containing all the  $s$  directions, and let  $R$  and  $T$  be subsets of  $S$  satisfying  $T \subseteq R \subseteq S$ . Unless explicitly mentioned otherwise, the empty sets  $R = \emptyset$  and  $T = \emptyset$  are allowed as well. The ordering of the directions is not taken into account, and will not play any role in this paper. We denote the cardinalities of the sets  $S, R, T$  by  $|S|, |R|, |T|$  rather than by the hash symbol used for the cardinalities of multi-index sets. Of course  $s = |S|$ . Following the notation  $I_s$  to denote the  $s$ -dimensional hypercube, for any  $\emptyset \neq R \subseteq S$  we denote by  $I_{|R|} := [0, 1]^{|R|}$  the  $|R|$ -dimensional hypercube. In the one-dimensional case we simplify the notation  $I_1$  to  $I := [0, 1]$ .

Given a point  $y \in I_s$ , we denote by  $(y_R, 1) \in I_s$  the point with the same values as  $y$  in the coordinates corresponding to the elements of  $R$ , and with values equal to 1 in the remaining coordinates in the set  $S \setminus R$ . In the following discussion, the value 1 could be replaced by any other arbitrary (but fixed) value in  $I$ . Also, we often use the



notation  $y = (y_R, y_{S \setminus R})$  to denote the point  $y \in I_s$ , to emphasize its components in  $R$  and  $S \setminus R$ , respectively.

We introduce the anchored Sobolev space  $H_{\text{mix}}^s(I_s)$  with inner product

$$\langle f_1, f_2 \rangle_{H_{\text{mix}}^s(I_s)} := \sum_{R \subseteq S} \int_{I_{|R|}} \frac{\partial^{|R|}}{\partial y_R} f_1(y_R, 1) \frac{\partial^{|R|}}{\partial y_R} f_2(y_R, 1) dy_R, \quad (7)$$

where  $\partial^{|R|} f(y_R, 1)/\partial y_R$  denotes the mixed first derivative of  $f$  in the directions specified by the elements of the set  $R$ , and evaluated in the point 1 in all the remaining directions contained in the set  $S \setminus R$ . The inner product (7) induces the norm  $\|f\|_{H_{\text{mix}}^s} := \langle f, f \rangle_{H_{\text{mix}}^s}^{1/2}$  over the space  $H_{\text{mix}}^s$ , which contains all the functions with square-integrable mixed first derivatives and with finite  $H_{\text{mix}}^s$  norm. These spaces can be characterized by means of reproducing kernels, see *e.g.* [10].

Given a set of  $n$  points  $y^1, \dots, y^n \in I_s$  and any subset  $\emptyset \neq R \subseteq S$ , we introduce the anchored local discrepancy

$$\Delta^{n,R}(t_R, 1) := \frac{1}{n} \sum_{i=1}^n \prod_{q \in R} \mathbb{I}_{[0, t_q]}(y_q^i) - \prod_{q \in R} t_q, \quad t_R \in I_{|R|}, \quad (8)$$

and the anchored star discrepancy

$$D_R^n := \sup_{t_R \in I_{|R|}} |\Delta^{n,R}(t_R, 1)|, \quad (9)$$

that quantifies how much the empirical distribution of the components in  $R$  of the  $n$  points differs from the uniform distribution, while the remaining  $|S \setminus R|$  components are frozen to 1. On the one hand, a well uniformly distributed set of points has a small star discrepancy. On the other hand, large values of the star discrepancy imply a poor uniformity of the empirical distribution. The quantifiers “small” and “large” will be made more precise in the next section.

The star discrepancy  $D_R^n$  corresponds to the  $L^\infty$  norm of the local discrepancy  $\Delta^{n,R}$ . Similarly, the  $L^p$  discrepancy can be defined by means of the  $L^p$  norm with any  $p \geq 1$ . Notice that, when  $R = S$ ,  $\Delta^{n,S}$  and  $D_S^n$  correspond to the usual anchored local discrepancy and anchored star discrepancy. These notions of (unweighted) discrepancies can be extended to their weighted counterparts [29], by introducing suitable weights that specify the mutual importance of combinations of coordinates. Several types of weights have been proposed, *e.g.* product weights, finite-order weights, order-dependent weights and general weights, see [11]. However, in this work, we restrict ourselves to “unweighted” discrepancies.

### 3.1. Upper bounds for the star discrepancy

In this section we recall upper bounds for the star discrepancy. We make use of the same distinction introduced in [10], where “closed” set of points refers to a finite set

of say  $n$  points, and “open” set of points refers to the first, say,  $n$  points of an infinite sequence.

Concerning the upper bound for the star discrepancy, there exist sequences of points such that

$$D_S^n \leq \mathcal{B}_s \frac{(\ln n)^s}{n}, \quad s \geq 1, \quad \text{for all } n \geq 1, \quad (10)$$

where the constant  $\mathcal{B}_s$  depends only on the sequence and on the dimension  $s$  but not on  $n$ . Notice that, for fixed  $s$ , the function  $n \mapsto n^{-1}(\ln n)^s$  increases w.r.t.  $n$  unless  $n \geq \exp(s)$ . Therefore one has to take at least  $n \geq \exp(s)$  points to make the right-hand side in (10) lower than  $\mathcal{B}_s$ . Common low-discrepancy sequences are by Sobol’, Niederreiter, Faure, van der Corput, Halton, see *e.g.* [11].

A “closed” point set with  $n$  points sometimes allows a further decrease of the exponent of the logarithm: *e.g.* in the case of  $(t, m, s)$ -nets the star discrepancy satisfies

$$D_S^n \leq \mathcal{B}_s \frac{(\ln n)^{s-1}}{n}, \quad s \geq 1. \quad (11)$$

As remarked in [10, Example 2.5], typically the upper bounds for “closed” point sets are better than those for “open” point sets. On the other hand, “open” point sets allow to arbitrarily increase the number of points  $n$  keeping all the previously chosen points in the set. In general this does not hold for “closed” point sets, and a different number of points  $n$  corresponds to a completely different set of points.

In the next section, we introduce two classes of low-discrepancy point sets: nets and sequences. A net is a “closed” point set, and the first say  $n$  points of a sequence is an “open” point set.

### 3.1.1. Nets and sequences

In this article, we focus on so-called  $(t, m, s)$ -nets and  $(t, s)$ -sequences, see [23, 24, 11] and references therein. We start by introducing the notion of  $(t, m, s)$ -net, following [11].

**Definition 2** ( $(t, m, s)$ -net in base  $b$ ). *Let  $s \geq 1$ ,  $b \geq 2$ ,  $t \geq 0$  and  $m \geq 1$  be integers with  $t \leq m$ . A  $(t, m, s)$ -net in base  $b$  is a point set consisting of  $b^m$  points in  $[0, 1]^s$  such that every elementary interval of the form*

$$\prod_{i=1}^s \left[ \frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}} \right)$$

*with integers  $d_i \geq 0$ ,  $0 \leq a_i < b^{d_i}$ , and  $d_1 + \dots + d_s = m - t$ , contains exactly  $b^t$  points of the net.*

Here  $b$  is an integer denoting the base of the net,  $t$  is the quality parameter, and  $m$  specifies the total number of points in the net given by  $n = b^m$ . It is known that for every prime base  $b$  there exist  $(0, m, s)$ -nets in base  $b$  for all  $s \leq b + 1$ , see [11, page 198]. Moreover, we have the following theorem.

**Theorem 1** ([11, Theorem 5.28]). *For every dimension  $s$  there is a  $(t, m, s)$ -net in base  $b = 2$  consisting of  $2^{11s}$  points in  $[0, 1]^s$  whose star discrepancy is less than  $s/2^{1.09s}$ .*

In [12, Theorem 1] an upper bound for the star discrepancy of a  $(t, m, s)$ -net in base  $b$  is proven. For any positive integer  $k$  and any integer  $v$ , we denote by  $\binom{k}{v}$  the usual binomial coefficient with  $\binom{k}{v} = 0$  whenever  $k < v$  or  $v < 0$ .

**Theorem 2** ([12, Theorem 1]). *Let  $s \geq 2$ ,  $m \geq t \geq 0$ , and let  $b \geq 2$ . The star discrepancy of a  $(t, m, s)$ -net in base  $b$  with  $n = b^m$  points satisfies*

$$D_S^n \leq \frac{b^t}{n} \sum_{v=0}^{s-1} a_{v,b}^{(s)} m^v, \quad (12)$$

with

$$a_{v,b}^{(s)} := \binom{s-2}{v} \left(\frac{b+2}{2}\right)^{s-2-v} \frac{(b-1)^v}{2^v v!} (a_{0,b}^{(2)} + s^2 - 4) + \binom{s-2}{v-1} \left(\frac{b+2}{2}\right)^{s-1-v} \frac{(b-1)^{v-1}}{2^{v-1} v!} a_{1,b}^{(2)},$$

for any  $0 \leq v \leq s-1$ , with

$$a_{0,b}^{(2)} = \begin{cases} \frac{b+8}{4}, & \text{if } b \text{ is even,} \\ \frac{b+4}{2}, & \text{if } b \text{ is odd,} \end{cases} \quad \text{and} \quad a_{1,b}^{(2)} = \begin{cases} \frac{b^2}{4(b+1)}, & \text{if } b \text{ is even,} \\ \frac{b-1}{4}, & \text{if } b \text{ is odd.} \end{cases}$$

We report in the following a corollary of this result. Throughout the article  $\tau_b$  is defined for any integer  $b \geq 2$  as

$$\tau_b := \begin{cases} \frac{b^2}{b^2-1}, & \text{if } b \text{ is even,} \\ 1, & \text{if } b \text{ is odd.} \end{cases}$$

Notice that, for any  $b \geq 2$ , it holds that  $1 \leq \tau_b \leq 4/3$ .

**Corollary 1** ([12, Corollary 1]). *Given a  $(t, m, s)$ -net in base  $b$  with  $n = b^m$  points, then it holds that*

$$D_S^n \leq \frac{b^t \tau_b}{2(s-1)!} \left(\frac{b-1}{2 \ln b}\right)^{s-1} \frac{(\ln n)^{s-1}}{n} + \mathcal{O}\left(\frac{(\ln n)^{s-2}}{n}\right). \quad (13)$$

The term  $\mathcal{O}((\ln n)^{s-2}/n)$  in (13) can be precisely quantified taking into account all the terms in the right-hand side of (12).

A  $(t, s)$ -sequence, according to [11], is defined as in the following.

**Definition 3** ( $(t, s)$ -sequence in base  $b$ ). *Let  $t \geq 0$  and  $s \geq 1$  be integers. A  $(t, s)$ -sequence in base  $b$  is a sequence of points  $(y^1, y^2, \dots)$  in  $[0, 1]^s$  such that for all integers  $m > t$  and  $l \geq 0$ , every block of  $b^m$  points*

$$y^{lb^m+1}, \dots, y^{(l+1)b^m}$$

*in the sequence  $(y^1, y^2, \dots)$  forms a  $(t, m, s)$ -net in base  $b$ .*

As an example, the Sobol' sequence is a special kind of a  $(t, s)$ -sequence in base  $b = 2$ . An analogous result of [12, Theorem 1] has been proposed in [12, Theorem 2] for a  $(t, s)$ -sequence in base  $b$ , which we report here, as well as its corollary.

**Theorem 3** ([12, Theorem 2]). *Let  $s \geq 2$ ,  $m \geq t \geq 0$ , and let  $b \geq 2$ . The star discrepancy of a  $(t, s)$ -sequence in base  $b$  satisfies*

$$D_S^n \leq \frac{b^t}{n} \sum_{v=0}^s A_{v,b}^{(s)} (\log_b n)^v, \quad (14)$$

for any  $n \geq \max\{b, b^t\}$ , and with

$$\begin{aligned} A_{0,b}^s &:= \frac{b+2}{2} a_{0,b}^{(s)}, \\ A_{v,b}^s &:= \left(2^v + \frac{b-1}{2}\right) a_{v,b}^{(s)} + \frac{b-1}{2^v} a_{v-1,b}^{(s)}, \quad \text{for } 1 \leq v \leq s-1, \\ A_{s,b}^s &:= \frac{b-1}{2s} a_{s-1,b}^{(s)}, \end{aligned}$$

where the coefficients  $a_{v,b}^{(s)}$  are the same as in Theorem 2.

**Corollary 2** ([12, Corollary 2]). *Given the first  $n$  points of a  $(t, s)$ -sequence in base  $b$ , then, for any  $n \geq 1$ , it holds that*

$$D_S^n \leq \frac{b^t \tau_b}{2s!} \left(\frac{b-1}{2 \ln b}\right)^s \frac{(\ln n)^s}{n} + \mathcal{O}\left(\frac{(\ln n)^{s-1}}{n}\right). \quad (15)$$

Again, the term  $\mathcal{O}((\ln n)^{s-1}/n)$  in (15) can be precisely quantified taking into account all the terms in the right-hand side of (14).

Notice that the main difference between the bound (15) for sequences and the bound (13) for nets is in the exponent of the logarithmic terms as well as in the factorial  $s!$  in the denominator compared with  $(s-1)!$ . Recently, the upper bound on the star discrepancy for  $(t, s)$ -sequences has been further improved in the non-asymptotic regime in [14, Theorem 1].

The digital construction of  $(t, m, s)$ -nets and  $(t, s)$ -sequences has been introduced in [23]. Afterwards, these point sets have been named digital  $(t, m, s)$ -nets and digital  $(t, s)$ -sequences, see also [11]. We do not introduce them in the present paper, and just mention that our results proven for nets and sequences hold true for digital nets and digital sequences as well. Recently  $(t, m, \mathbf{e}, s)$ -nets have also been introduced, see [31, 16], but we will address their application in forthcoming analyses. Neither will we consider generalized nets, *i.e.*  $(t, \alpha, \beta, n \times m, s)$ -nets as introduced in [8, 3, 4].

### 3.1.2. The one-dimensional set of deterministic equispaced points

In one dimension, the following result provides an explicit formula for the star discrepancy of any point set.

**Theorem 4** ([24, Theorem 2.6]). *If  $0 \leq y^1 \leq \dots \leq y^n \leq 1$ , then*

$$D_S^n = \frac{1}{2n} + \max_{1 \leq i \leq n} \left| y^i - \frac{2i-1}{2n} \right|. \quad (16)$$

In this case, the point set of “closed” type with minimal star discrepancy (16) is

$$y^i = \frac{2i-1}{2n} \in I, \quad i = 1, \dots, n, \quad (17)$$

and the value of its star discrepancy, with  $S = \{1\}$ , is

$$D_S^n = (2n)^{-1}, \quad n \geq 1. \quad (18)$$

The point set (17) is a  $(0, 1, 1)$ -net in base  $b = n$ , and it is optimal in the sense that it has the best star discrepancy w.r.t. all point sets with  $n$  points in  $[0, 1]$ .

### 3.2. Low-order projections of a low-discrepancy point set

The overall accuracy of the quasi-Monte Carlo method relies on the low-discrepancy properties of the set of quadrature points, and on the properties of the integrand, *e.g.* its smoothness. A well known explicit formula for the integration error is (26), which involves indeed the local discrepancy of all the low-order projections. Therefore, the discrepancy quality of low-order projections is as much important as the discrepancy quality of the point set itself. However, the number of low-order projections of an  $s$ -dimensional point set is  $2^s - 1$ , and a quantitative analysis of their role corresponds to taking into account the different importance that the interplay of any subset of coordinates might have. The discrepancies of low-order projections of low-discrepancy point sets and their influence in the convergence of quasi-Monte Carlo have already been studied in the literature, see [28, 32], and will play a main role in our analysis as well.

In the following we estimate the superposition of star discrepancies of low-order projections

$$\mathbb{D}_s^n(\theta) := \sum_{\emptyset \neq R \subseteq S} D_R^n \theta^{|R|} \quad (19)$$

for point sets like  $(t, m, s)$ -nets and  $(t, s)$ -sequences with any  $s \geq 1$  and with  $\theta$  being a nonnegative real parameter. In the case  $S = \{1\}$  and  $\theta = 1$ ,  $\mathbb{D}_s^n(\theta)$  coincides with the usual star discrepancy. We recall a useful propagation rule for the low-order projections of a  $(t, m, s)$ -net.

**Lemma 1** ([11, Lemma 4.16]). *Given a  $(t, m, s)$ -net in base  $b$ , its projection onto any combination of  $1 \leq s' \leq s$  dimensions is a  $(t, m, s')$ -net in base  $b$ .*

An analogous propagation rule holds also for a  $(t, \alpha, \beta, n \times m, s)$ -net in base  $b$ , see [3, Theorem 1.2], and for a digital  $(t, \alpha, \beta, n \times m, s)$ -net, see [9, Theorem 2, propagation rule V].

In the next lemma we explicitly calculate an upper bound for the superposition of star discrepancies of low-order projections of  $(t, m, s)$ -nets, starting from the upper bound in Theorem 2.

**Lemma 2.** *In any dimension  $s \geq 2$ , given a  $(t, m, s)$ -net in base  $b$  with  $n = b^m$  points: for any real  $\theta \geq 0$  the superposition of star discrepancies of low-order projections satisfies*

$$\mathbb{D}_s^n(\theta) \leq \frac{\theta b^t}{n} \left( \left( 1 + \theta \frac{b+2}{2} \right)^{s-2} \left( \left( 1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) + s \right), \quad (20)$$

with

$$g(s, b, \theta) := \frac{s}{2} \left( \theta(s-1)a_{0,b}^{(2)} + a_{1,b}^{(2)} \right), \quad (21)$$

$$h(s, b) := s \frac{a_{1,b}^{(2)}}{2}. \quad (22)$$

*Proof.* See [Appendix A](#). □

From Lemma 1, the quality parameter of any low-order projection of a  $(t, s)$ -sequence cannot be worse than  $t$ . Therefore, an upper bound for the superposition of star discrepancies of low-order projections of  $(t, s)$ -sequences can be explicitly calculated starting from the upper bound in Theorem 3.

**Lemma 3.** *In any dimension  $s \geq 2$ , given the first  $n \geq \max\{b, b^t\}$  points in a  $(t, s)$ -sequence in base  $b$ : for any real  $\theta \geq 0$  the superposition of star discrepancies of low-order projections satisfies*

$$\begin{aligned} \mathbb{D}_s^n(\theta) \leq & \frac{\theta b^t}{n} \left( b \left( 1 + \theta \frac{b+2}{2} \right)^{s-2} \left( \left( 1 + \frac{2\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) \right. \\ & \left. + s \left( a_{1,b}^{(2)} \left( 1 + \frac{\ln n}{\ln b} \right) + 2 \right) + \frac{1}{2} \left( \frac{b+2}{2} \right)^{-2} \left( r(b) + w(s, b) \left( 1 + \theta \frac{b+2}{2} \right)^s \right) \right), \end{aligned} \quad (23)$$

with the functions  $g$  and  $h$  being defined in (21)–(22), and the functions  $r = r(b)$  and  $w = w(s, b)$  defined as

$$r(b) := 4 - a_{0,b}^{(2)} + \left( 3 - a_{0,b}^{(2)} \right) \left( \frac{b+2}{2} \right), \quad (24)$$

$$w(s, b) := \left( a_{0,b}^{(2)} + s^2 - 4 \right). \quad (25)$$

*Proof.* See [Appendix A](#). □

For some types of  $(t, s)$ -sequences, *e.g.* for digital sequences of Sobol' and Niederreiter type, it is possible to quantify how much the quality parameter of low-order projections is smaller than the quality parameter of the starting sequence. More specifically, for any  $\emptyset \neq R \subseteq S$  an explicit expression depending on  $|R|$  can be obtained for the quality parameter of the  $(t_R, |R|)$ -sequence obtained by projecting the  $(t, s)$ -sequence onto the coordinates in  $R$ , see [27, Section 2]. Thanks to this explicit expression, for these particular  $(t, s)$ -sequences it is possible to improve the term  $b^t$  in (23).

### 3.3. Koksma-Hlawka inequalities

In this section we recall in Lemma 4 the Hlawka/Zaremba's identity (see [15, 33] for the proof), and then prove in Lemma 5 a Koksma-Hlawka-type inequality, which is the starting point of the analysis developed in §4.

**Lemma 4** (Hlawka's identity or Zaremba's identity). *For any function  $f \in H_{mix}^s$  and any set of  $n$  points  $y^1, \dots, y^n \in I_s$  it holds that*

$$\frac{1}{n} \sum_{i=1}^n f(y^i) - \int_{I_s} f(y) dy = \sum_{R \subseteq S} (-1)^{|R|} \int_{I_{|R|}} \Delta^{n,R} \frac{\partial^{|R|}}{\partial y_R} f(y_R, 1) dy_R. \quad (26)$$

**Lemma 5.** *For any function  $f \in H_{mix}^s$  and any set of  $n$  points  $y^1, \dots, y^n \in I_s$  it holds that*

$$\left| \|f\|_{L^2(I_s)}^2 - \|f\|_n^2 \right| \leq \sum_{\emptyset \neq R \subseteq S} D_R^n \sum_{T \subseteq R} \left\| \frac{\partial^{|T|}}{\partial y_T} f(y_R, 1) \right\|_{L^2(I_{|R|})} \left\| \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} f(y_R, 1) \right\|_{L^2(I_{|R|})}. \quad (27)$$

*Proof.* Using the Zaremba's identity (26) we have

$$\|f\|_{L^2(I_s)}^2 - \|f\|_n^2 = \int_{I_s} (f(y))^2 dy - \frac{1}{n} \sum_{i=1}^n (f(y^i))^2 = \sum_{\emptyset \neq R \subseteq S} (-1)^{|R|} \int_{I_{|R|}} \Delta^{n,R} \frac{\partial^{|R|}}{\partial y_R} (f(y_R, 1))^2 dy_R. \quad (28)$$

Then,

$$\begin{aligned} \left| \|f\|_{L^2(I_s)}^2 - \|f\|_n^2 \right| &\leq \sum_{\emptyset \neq R \subseteq S} \left| \int_{I_{|R|}} \Delta^{n,R} \frac{\partial^{|R|}}{\partial y_R} (f(y_R, 1))^2 dy_R \right| \\ &= \sum_{\emptyset \neq R \subseteq S} \left| \int_{I_{|R|}} \Delta^{n,R} \sum_{T \subseteq R} \frac{\partial^{|T|}}{\partial y_T} f(y_R, 1) \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} f(y_R, 1) dy_R \right| \\ &\leq \sum_{\emptyset \neq R \subseteq S} D_R^n \int_{I_{|R|}} \left| \sum_{T \subseteq R} \frac{\partial^{|T|}}{\partial y_T} f(y_R, 1) \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} f(y_R, 1) \right| dy_R \\ &\leq \sum_{\emptyset \neq R \subseteq S} D_R^n \sum_{T \subseteq R} \left( \int_{I_{|R|}} \left( \frac{\partial^{|T|}}{\partial y_T} f(y_R, 1) \right)^2 dy_R \right)^{\frac{1}{2}} \left( \int_{I_{|R|}} \left( \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} f(y_R, 1) \right)^2 dy_R \right)^{\frac{1}{2}} \\ &= \sum_{\emptyset \neq R \subseteq S} D_R^n \sum_{T \subseteq R} \left\| \frac{\partial^{|T|}}{\partial y_T} f(y_R, 1) \right\|_{L^2(I_{|R|})} \left\| \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} f(y_R, 1) \right\|_{L^2(I_{|R|})}. \end{aligned}$$

□

There are mainly two differences between (27) and the classical Koksma-Hlawka inequality, see [17, Theorem 5.6]: first we directly bound the difference of the norms instead of the integration error, and second we keep the combinatorial summation with the low-order star discrepancies out of the norm on the right-hand side, rather than including it in the Hardy–Krause variation.

#### 4. Norm equivalence on polynomial spaces

In this section we prove a norm equivalence between the discrete and continuous  $L^2$  norms over  $\mathbb{P}_\Lambda$ , *i.e.* we derive conditions which ensure the existence of  $\delta \in (0, 1)$ , such that

$$(1 - \delta)\|u\|_{L^2(I_s)}^2 \leq \|u\|_n^2 \leq (1 + \delta)\|u\|_{L^2(I_s)}^2, \quad \forall u \in \mathbb{P}_\Lambda, \quad (29)$$

where  $\Lambda$  is an arbitrary downward closed set. The norm equivalence (29) corresponds to

$$\frac{1}{1 + \delta} \leq \frac{\|u\|_{L^2(I_s)}^2}{\|u\|_n^2} \leq \frac{1}{1 - \delta}, \quad \forall u \in \mathbb{P}_\Lambda,$$

and therefore, taking the supremum and infimum over  $u \in \mathbb{P}_\Lambda \setminus \{u \equiv 0\}$ , it allows us to obtain lower and upper bounds of the quantities introduced in (3), namely:

$$\frac{1}{1 + \delta} \leq \mathcal{S}(n, \Lambda) \leq \frac{1}{1 - \delta}, \quad \text{and} \quad 1 - \delta \leq \mathcal{Q}(n, \Lambda) \leq 1 + \delta.$$

It also provides, using Proposition 2, a bound on the condition number of  $A^\top A$ , namely

$$\text{cond}(A^\top A) \leq \frac{1 + \delta}{1 - \delta}.$$

To begin with, in §4.1 we derive from [20] some useful multivariate Markov-type and Nikolskii-type inequalities for polynomials associated with downward closed multi-index sets. Afterwards, in §4.2 we prove the norm equivalence (29) with the equivalence constant  $\delta$  being dependent on the particular polynomial space  $\mathbb{P}_\Lambda$  characterized by the multi-index set  $\Lambda$ , on the star discrepancy of all the low-order projections of the points  $y^1, \dots, y^n$ , and on the dimension  $s$ .

##### 4.1. Multidimensional inequalities for polynomials associated with downward closed multi-index sets

We first recall two standard results on univariate Legendre polynomials. Given an interval  $[a, b] \subset \mathbb{R}$ , for any  $q \in \mathbb{N}_0$  the  $L^2$ -orthonormal Legendre polynomial  $\varphi_q$  with degree  $q$  satisfies

$$\|\varphi'_q\|_{L^2(a,b)} = \frac{2}{b-a} \sqrt{q \left( q + \frac{1}{2} \right) (q+1)}, \quad (30)$$

$$\|\varphi_q\|_{L^\infty(a,b)} = \sqrt{\frac{2q+1}{b-a}}. \quad (31)$$



The proof of (30) follows from [20, Lemma 4], taking into account the scaling factor due to the change of the interval.

To keep the present paper self-contained, we now recall from [20] some results that will be used a number of times in the following. Given  $\eta \in \mathbb{N}_0$  and  $\eta+1$  real nonnegative coefficients  $\alpha_0, \dots, \alpha_\eta$ , we define the univariate polynomial  $p \in \mathbb{P}_\eta(\mathbb{N}_0)$  of degree  $\eta$  as

$$p : \mathbb{N}_0 \rightarrow \mathbb{R} : n \mapsto p(n) := \sum_{l=0}^{\eta} \alpha_l n^l, \quad (32)$$

with the convention that  $0^0 = 1$  to avoid the splitting of the summation. In any dimension  $s$  and given an arbitrary downward closed multi-index set  $\Lambda$ , we define the quantity  $K_p(\Lambda)$  as

$$K_p(\Lambda) := \sum_{\nu \in \Lambda} \prod_{q=1}^s p(\nu_q) = \sum_{\nu \in \Lambda} \prod_{q=1}^s (\alpha_0 + \alpha_1 \nu_q + \dots + \alpha_\eta \nu_q^\eta), \quad (33)$$

which depends only on  $\Lambda$  when  $p$  is fixed. We introduce the following condition concerning the coefficients of the polynomial  $p$ .

**Definition 4** (Binomial condition). *The polynomial  $p$  defined in (32) satisfies the binomial condition if its coefficients  $\alpha_0, \dots, \alpha_\eta$  satisfy*

$$\alpha_l \leq \binom{\eta+1}{l}, \quad \text{for any } l = 0, \dots, \eta. \quad (34)$$

**Theorem 5** ([20, Theorem 1]). *In any dimension  $s$ , for any downward closed multi-index set  $\Lambda$  and for any  $\eta \in \mathbb{N}_0$ , if the coefficients  $\alpha_0, \dots, \alpha_\eta$  of the polynomial  $p$  satisfy the binomial condition (34) then the quantity  $K_p(\Lambda)$  defined in (33) satisfies*

$$K_p(\Lambda) \leq (\#\Lambda)^{\eta+1}. \quad (35)$$

In our analysis in the present paper we need also Markov and Nikolskii inequalities for multivariate polynomials that have been proven in [20], and we report them in the following adapted to the domain  $I_s$  instead of  $[-1, 1]^s$ .

**Theorem 6** ([20, Theorem 3]). *For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda(I_s)$  with  $\Lambda$  downward closed it holds that*

$$\left\| \frac{\partial^s}{\partial y_1 \cdots \partial y_s} u \right\|_{L^2(I_s)} \leq (\#\Lambda)^2 \|u\|_{L^2(I_s)}.$$

**Theorem 7** ([20, Theorem 6]). *For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda(I_s)$  with  $\Lambda$  downward closed it holds that*

$$\|u\|_{L^\infty(I_s)}^2 \leq (\#\Lambda)^2 \|u\|_{L^2(I_s)}^2.$$

The proofs of these inequalities rely on the use of Theorem 5 combined with the one-dimensional equalities (30) and (31) for Legendre polynomials. These results have been proven also for weighted  $L^2$  norms, with the orthonormalization weight of Chebyshev, Jacobi and Gegenbauer orthogonal polynomials, see [20].

Given any set  $\emptyset \neq R \subseteq S$ , we define the multi-index set

$$\Lambda_R := \text{proj}_R \Lambda,$$

which is obtained by projecting the multi-index set  $\Lambda$  onto the coordinates in the set  $R$ . This corresponds to building a multi-set with all the elements in  $\Lambda$  truncated to the components in the set  $R$ , and then take out possible multiple occurrences of the same element to obtain a properly-said set. Unless mentioned otherwise, we allow also the empty set  $R = \emptyset$ , in which case we define  $\#\Lambda_R := 1$ . This is a natural choice to ensure that

$$\#\Lambda_R \leq (\#\Lambda_T)(\#\Lambda_{R \setminus T}), \quad \forall T \subseteq R, \quad \forall R \subseteq S,$$

and allows us, for example, to make sense of the case  $S \setminus R = \emptyset$  in the following equation (36), where equality is attained. Notice that, if  $\Lambda$  is downward closed, then the set  $\Lambda_R$  is downward closed for any  $R \subseteq S$ .

In the following two lemmas, we prove Nikolskii-type and Markov-type inequalities for multivariate polynomials associated with downward closed multi-index sets.

**Lemma 6.** *For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda$  with  $\Lambda$  downward closed and for any set  $\emptyset \neq R \subseteq S$  it holds that*

$$\max_{y_{S \setminus R} \in I_{|S \setminus R|}} \|u(y_R, y_{S \setminus R})\|_{L^2(I_{|R|})}^2 \leq (\#\Lambda_{S \setminus R})^2 \|u\|_{L^2(I_s)}^2. \quad (36)$$

*Proof.* For any  $u \in \mathbb{P}_\Lambda(I_s)$  it holds that

$$u(y_R, \cdot) \in \mathbb{P}_{\Lambda_{S \setminus R}}, \quad \forall y_R \in I_{|R|}.$$

Then, using Theorem 7 we have

$$\max_{y_{S \setminus R} \in I_{|S \setminus R|}} |u(y_R, y_{S \setminus R})| \leq (\#\Lambda_{S \setminus R}) \sqrt{\int_{I_{S \setminus R}} u^2(y_R, y_{S \setminus R}) dy_{S \setminus R}}, \quad \forall y_R \in I_{|R|}.$$

Moreover,

$$\begin{aligned} \|u(y_R, y_{S \setminus R})\|_{L^2(I_{|R|})}^2 &\leq \max_{y_{S \setminus R} \in I_{|S \setminus R|}} \int_{I_{|R|}} u^2(y_R, y_{S \setminus R}) dy_R \\ &\leq \int_{I_{|R|}} \max_{y_{S \setminus R} \in I_{|S \setminus R|}} u^2(y_R, y_{S \setminus R}) dy_R \\ &\leq (\#\Lambda_{S \setminus R})^2 \int_{I_{|R|}} \int_{I_{|S \setminus R|}} u^2(y_R, y_{S \setminus R}) dy_{S \setminus R} dy_R \\ &= (\#\Lambda_{S \setminus R})^2 \|u\|_{L^2(I_s)}^2. \end{aligned}$$

□

**Lemma 7.** For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda$  with  $\Lambda$  downward closed, and for any set  $\emptyset \neq R \subseteq S$  and any subset  $T \subseteq R$ , it holds that

$$\left\| \frac{\partial^{|T|}}{\partial y_T} u(y_R, y_{S \setminus R}) \right\|_{L^2(I_{|R|})} \leq (\#\Lambda_T)^2 \|u(y_R, y_{S \setminus R})\|_{L^2(I_{|R|})}, \quad \forall y_{S \setminus R} \in I_{|S \setminus R|}. \quad (37)$$

*Proof.* For any  $u \in \mathbb{P}_\Lambda(I_S)$  it holds that

$$u(\cdot, y_{S \setminus R}) \in \mathbb{P}_{\Lambda_R}, \quad \forall y_{S \setminus R} \in I_{|S \setminus R|}.$$

Given any arbitrary  $y_{S \setminus R} \in I_{|S \setminus R|}$ , we define  $u_R := u(\cdot, y_{S \setminus R}) \in \mathbb{P}_{\Lambda_R}$ . Using Theorem 6 adapted to the domain  $I_{|T|}$ , for any arbitrary  $y_{S \setminus R} \in I_{|S \setminus R|}$  we have

$$\begin{aligned} \left\| \frac{\partial^{|T|}}{\partial y_T} u(y_R, y_{S \setminus R}) \right\|_{L^2(I_{|R|})}^2 &= \int_{I_{|R|}} \left( \frac{\partial^{|T|}}{\partial y_T} u(y_R, y_{S \setminus R}) \right)^2 dy_R \\ &= \int_{I_{|R|}} \left( \frac{\partial^{|T|}}{\partial y_T} u_R(y_R) \right)^2 dy_R \\ &= \int_{I_{|R \setminus T|}} \int_{I_{|T|}} \left( \frac{\partial^{|T|}}{\partial y_T} u_R(y_R) \right)^2 dy_T dy_{R \setminus T} \\ &\leq \int_{I_{|R \setminus T|}} (\#\Lambda_T)^4 \|u_R(y_R)\|_{L^2(I_{|T|})}^2 dy_{R \setminus T} \\ &= (\#\Lambda_T)^4 \int_{I_{|R \setminus T|}} \int_{I_{|T|}} u^2(y_T, y_{R \setminus T}) dy_T dy_{R \setminus T} \\ &= (\#\Lambda_T)^4 \|u(y_R, y_{S \setminus R})\|_{L^2(I_{|R|})}^2. \end{aligned}$$

□

#### 4.2. Norm equivalence on polynomial spaces using the star discrepancy

This section contains several results where a norm equivalence between the  $L^2$  continuous and discrete norms is proven, with the equivalence constant depending on the star discrepancy of the low-order projections.

**Lemma 8.** For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda$  with  $\Lambda$  downward closed, using any point set with  $n$  points it holds that

$$\left| \|u\|_{L^2(I_S)}^2 - \|u\|_n^2 \right| \leq \|u\|_{L^2(I_S)}^2 (\#\Lambda)^4 \mathbb{D}_s^n(1). \quad (38)$$

*Proof.* First we prove the following intermediate result. For any set  $\emptyset \neq R \subseteq S$  and

any  $\nu, \mu \in \Lambda$ , the  $L^2$ -orthonormal Legendre polynomials  $\varphi_\nu$  and  $\varphi_\mu$  satisfy

$$\begin{aligned}
\int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} \left( \prod_{q \in R} \varphi_{\nu_q}(y_q) \varphi_{\mu_q}(y_q) \right) \right| dy_R &= \int_{I_{|R|}} \left| \prod_{q \in R} \frac{\partial}{\partial y_q} (\varphi_{\nu_q}(y_q) \varphi_{\mu_q}(y_q)) \right| dy_R \\
&= \prod_{q \in R} \left( \int_I \left| \frac{\partial}{\partial y_q} (\varphi_{\nu_q}(y_q) \varphi_{\mu_q}(y_q)) \right| dy_q \right) \\
&= \prod_{q \in R} \left( \int_I \left| \left( \varphi_{\mu_q}(y_q) \frac{\partial}{\partial y_q} \varphi_{\nu_q}(y_q) + \varphi_{\nu_q}(y_q) \frac{\partial}{\partial y_q} \varphi_{\mu_q}(y_q) \right) \right| dy_q \right) \\
&\leq \prod_{q \in R} \left( \int_I \left| \left( \varphi_{\mu_q}(y_q) \frac{\partial}{\partial y_q} \varphi_{\nu_q}(y_q) \right) \right| dy_q + \int_I \left| \left( \varphi_{\nu_q}(y_q) \frac{\partial}{\partial y_q} \varphi_{\mu_q}(y_q) \right) \right| dy_q \right) \\
&\leq \prod_{q \in R} \left( \left\| \frac{\partial}{\partial y_q} \varphi_{\nu_q}(y_q) \right\|_{L^2(I)} + \left\| \frac{\partial}{\partial y_q} \varphi_{\mu_q}(y_q) \right\|_{L^2(I)} \right) \\
&= \prod_{q \in R} \left( \sqrt{4\nu_q^3 + 6\nu_q^2 + 2\nu_q} + \sqrt{4\mu_q^3 + 6\mu_q^2 + 2\mu_q} \right).
\end{aligned} \tag{39}$$

In the last but one step we have used the Cauchy-Schwarz inequality. In the last step we have used (30) for each one of the two derivatives. We can now expand any  $u \in \mathbb{P}_\Lambda$  in Legendre series  $u = \sum_{\nu \in \Lambda} \beta_\nu \psi_\nu$  with coefficients  $\beta = (\beta_\nu)_{\nu \in \Lambda}$ . Then, using in sequence (39), (31),  $(\sqrt{a} + \sqrt{b})^2 \leq (a+1)(b+1)$  for any reals  $a, b \geq 0$  and Theorem 5,

we obtain the following:

$$\begin{aligned}
\int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} u^2(y_R, 1) \right| dy_R &= \int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} \left( \left( \sum_{\nu \in \Lambda} \beta_\nu \psi_\nu(y_R, 1) \right) \left( \sum_{\mu \in \Lambda} \beta_\mu \psi_\mu(y_R, 1) \right) \right) \right| dy_R \\
&= \int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} \left( \sum_{\nu \in \Lambda} \sum_{\mu \in \Lambda} \beta_\nu \beta_\mu \psi_\nu(y_R, 1) \psi_\mu(y_R, 1) \right) \right| dy_R \\
&\leq \sum_{\nu \in \Lambda} \sum_{\mu \in \Lambda} |\beta_\nu| |\beta_\mu| \prod_{q \notin R} |\varphi_{\nu_q}(1)| |\varphi_{\mu_q}(1)| \int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} \left( \prod_{q \in R} \varphi_{\nu_q}(y_q) \varphi_{\mu_q}(y_q) \right) \right| dy_R \\
&\leq \sum_{\nu \in \Lambda} \sum_{\mu \in \Lambda} |\beta_\nu| |\beta_\mu| \prod_{q \notin R} \sqrt{(2\nu_q + 1)(2\mu_q + 1)} \prod_{q \in R} \left( \sqrt{4\nu_q^3 + 6\nu_q^2 + 2\nu_q} + \sqrt{4\mu_q^3 + 6\mu_q^2 + 2\mu_q} \right) \\
&\leq \|\beta\|_{\ell_2}^2 \sqrt{\sum_{\nu \in \Lambda} \sum_{\mu \in \Lambda} \prod_{q \notin R} (2\nu_q + 1)(2\mu_q + 1) \prod_{q \in R} (4\nu_q^3 + 6\nu_q^2 + 2\nu_q + 1)(4\mu_q^3 + 6\mu_q^2 + 2\mu_q + 1)} \\
&\leq \|\beta\|_{\ell_2}^2 \sqrt{\sum_{\nu \in \Lambda} \sum_{\mu \in \Lambda} \prod_{q=1}^s (4\nu_q^3 + 6\nu_q^2 + 2\nu_q + 1)(4\mu_q^3 + 6\mu_q^2 + 2\mu_q + 1)} \\
&= \|\beta\|_{\ell_2}^2 \sqrt{\sum_{\nu \in \Lambda} \prod_{q=1}^s (4\nu_q^3 + 6\nu_q^2 + 2\nu_q + 1) \sum_{\mu \in \Lambda} \prod_{q=1}^s (4\mu_q^3 + 6\mu_q^2 + 2\mu_q + 1)} \\
&\leq \|u\|_{L^2(I_s)}^2 (\#\Lambda)^4. \tag{40}
\end{aligned}$$

Finally, from (28) and using (40) we obtain the thesis for any  $u \in \mathbb{P}_\Lambda$ :

$$\begin{aligned}
\left| \|u\|_{L^2(I_s)}^2 - \|u\|_n^2 \right| &\leq \sum_{\emptyset \neq R \subseteq S} \left| \int_{I_{|R|}} \Delta^{n,R} \frac{\partial^{|R|}}{\partial y_R} u^2(y_R, 1) dy_R \right| \\
&\leq \sum_{\emptyset \neq R \subseteq S} D_R^n \int_{I_{|R|}} \left| \frac{\partial^{|R|}}{\partial y_R} u^2(y_R, 1) \right| dy_R \\
&\leq \|u\|_{L^2(I_s)}^2 (\#\Lambda)^4 \sum_{\emptyset \neq R \subseteq S} D_R^n.
\end{aligned}$$

□

An alternative and sometimes better estimate can be obtained starting from (27) instead of (28).

**Lemma 9.** *For any  $s$ -variate polynomial  $u \in \mathbb{P}_\Lambda$  with  $\Lambda$  downward closed, using any point set with  $n$  points it holds that*

$$\left| \|u\|_{L^2(I_s)}^2 - \|u\|_n^2 \right| \leq \|u\|_{L^2(I_s)}^2 \sum_{\emptyset \neq R \subseteq S} D_R^n (\#\Lambda_{S \setminus R})^2 \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \tag{41}$$

$$\leq \|u\|_{L^2(I_s)}^2 \max_{\substack{\emptyset \neq R \subseteq S \\ T \subseteq R}} \left\{ (\#\Lambda_{S \setminus R})^2 (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \mathbb{D}_s^n(2). \tag{42}$$

*Proof.* From (27), using Lemma 7 and Lemma 6 we obtain

$$\begin{aligned}
\left| \|u\|_{L^2(I_s)}^2 - \|u\|_n^2 \right| &\leq \sum_{\emptyset \neq R \subseteq S} D_R^n \sum_{T \subseteq R} \left\| \frac{\partial^{|T|}}{\partial y_T} u(y_R, 1) \right\|_{L^2(I_{|R|})} \left\| \frac{\partial^{|R \setminus T|}}{\partial y_{R \setminus T}} u(y_R, 1) \right\|_{L^2(I_{|R|})} \\
&\leq \sum_{\emptyset \neq R \subseteq S} D_R^n \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \|u(y_R, 1)\|_{L^2(I_{|R|})}^2 \\
&= \sum_{\emptyset \neq R \subseteq S} D_R^n \|u(y_R, 1)\|_{L^2(I_{|R|})}^2 \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \\
&\leq \|u\|_{L^2(I_s)}^2 \sum_{\emptyset \neq R \subseteq S} D_R^n (\#\Lambda_{S \setminus R})^2 \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2,
\end{aligned}$$

and (41) is proven. Starting from the right-hand side of (41) we obtain (42) as in the following:

$$\begin{aligned}
\sum_{\emptyset \neq R \subseteq S} D_R^n (\#\Lambda_{S \setminus R})^2 \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 &\leq \sum_{\emptyset \neq R \subseteq S} D_R^n (\#\Lambda_{S \setminus R})^2 \max_{T \subseteq R} \left\{ (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \sum_{T \subseteq R} 1 \\
&\leq \max_{\substack{\emptyset \neq R \subseteq S \\ T \subseteq R}} \left\{ (\#\Lambda_{S \setminus R})^2 (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \sum_{\emptyset \neq R \subseteq S} D_R^n 2^{|R|}.
\end{aligned}$$

□

In the case of an isotropic polynomial space, the previous result particularizes as follows.

**Corollary 3.** *Let  $\mathbb{P}_\Lambda$  be an isotropic polynomial space, i.e.  $\Lambda$  is invariant under any permutation of the directions. Then, for any polynomial  $u \in \mathbb{P}_\Lambda$  with  $\Lambda$  downward closed and using any point set with  $n$  points it holds that*

$$\left| \|u\|_{L^2(I_s)}^2 - \|u\|_n^2 \right| \leq \max_{0 \leq t \leq q \leq s} \left\{ (\#\Lambda_{\{1, \dots, t\}})^2 (\#\Lambda_{\{1, \dots, q-t\}})^2 (\#\Lambda_{\{1, \dots, s-q\}})^2 \right\} \mathbb{D}_s^n(2) \|u\|_{L^2(I_s)}^2.$$

**Corollary 4** (Anisotropic TP spaces). *In any dimension  $s$ , when  $\Lambda$  is an anisotropic tensor product space with degrees  $w_1, \dots, w_s$ , the following quantity appearing in (42) satisfies*

$$\max_{\substack{\emptyset \neq R \subseteq S \\ T \subseteq R}} \left\{ (\#\Lambda_{S \setminus R})^2 (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} = \prod_{q=1}^s (w_q + 1)^2 = (\#\Lambda)^2. \quad (43)$$

**Remark 2.** *For any polynomial space  $\mathbb{P}_\Lambda$  with the downward closed multi-index set  $\Lambda$  contained in the anisotropic tensor product with degrees  $w_1, \dots, w_s$  it holds that*

$$\max_{\substack{\emptyset \neq R \subseteq S \\ T \subseteq R}} \left\{ (\#\Lambda_{S \setminus R})^2 (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \leq \prod_{q=1}^s (w_q + 1)^2. \quad (44)$$

*In the cases with  $s = 1$ ,  $s = 2$  or  $s = 3$ , if  $\Lambda$  is such that the maximal degrees in each direction are equal to  $w_1, \dots, w_s$  then the equality holds in (44), and therefore the set*

$\Lambda$  always behaves like the anisotropic tensor product despite it could be more sparse. When  $s \geq 4$  this is not the case and the sparsity of  $\Lambda$  might pay off: if

$$\exists \emptyset \neq R \subseteq S : 2 \leq |R| \leq s-2 \text{ and } \Lambda_R < \prod_{q \in R} (w_q + 1),$$

then the strict inequality holds in (44). In other words, in high-dimension ( $s \geq 4$ ) the largest three-term product of the square of the cardinalities of the low-order projections of  $\Lambda$  can be effectively smaller than  $\prod_{q=1}^s (w_q + 1)^2$ , if  $\Lambda$  is sufficiently more sparse than an anisotropic tensor product. This cannot happen in dimension  $s = 1, 2, 3$ .

## 5. Stability and accuracy of discrete least squares

In this section we present the main result on the stability and accuracy of discrete least squares with deterministic evaluations. First, in §5.1 we prove that the same conditions that ensure the norm equivalence (29) between the continuous and the discrete  $L^2$  norms are sufficient conditions for the stability and accuracy of discrete least squares on polynomial spaces. Afterwards, in §5.2 we recall the main results achieved in [7, 6, 19, 21, 22, 20] concerning the analysis of discrete least squares with uniformly distributed random points. Finally in §5.3 we compare the cases of low-discrepancy - and random points.

### 5.1. Evaluations at low-discrepancy point sets

Using the results in Lemmas 8 and 9, we introduce the following positive quantity, which depends on  $\Lambda$ ,  $n$  and  $s$ :

$$\mathcal{Z}_{s,n}(\Lambda) := \min \left\{ (\#\Lambda)^4 \mathbb{D}_s^n(1), \sum_{\emptyset \neq R \subseteq S} D_R^n (\#\Lambda_{S \setminus R})^2 \sum_{T \subseteq R} (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \quad (45)$$

$$\leq \min \left\{ (\#\Lambda)^4 \mathbb{D}_s^n(1), \max_{\substack{\emptyset \neq R \subseteq S \\ T \subseteq R}} \left\{ (\#\Lambda_{S \setminus R})^2 (\#\Lambda_T)^2 (\#\Lambda_{R \setminus T})^2 \right\} \mathbb{D}_s^n(2) \right\}. \quad (46)$$

The quantity  $\mathcal{Z}_{s,n}(\Lambda)$  can be made arbitrarily small by choosing an open or closed low-discrepancy point set with a sufficiently large number of points. In any dimension  $s$  and for any  $\Lambda$  downward closed, it holds that

$$\lim_{n \rightarrow +\infty} \mathcal{Z}_{s,n}(\Lambda) = 0,$$

because from Lemmas 2–3 the upper bounds of  $\mathbb{D}_s^n(1)$  and  $\mathbb{D}_s^n(2)$  in (46) converge to zero as  $n$  goes to infinity. Notice that, when using a point set of closed type, different values of  $n$  in  $\mathcal{Z}_{s,n}(\Lambda)$  might correspond to completely different sets of points. Using specific types of low-discrepancy point sets, thanks to the same upper bounds from Lemmas 2–3, it can be shown that the quantity  $\mathcal{Z}_{s,n}(\Lambda)$  is monotonically decreasing w.r.t.  $n$  for all  $n$  large enough, in any dimension  $s$  and for any  $\Lambda$  downward closed.

The following theorem gives an upper bound on  $\mathcal{Z}_{s,n}(\Lambda)$  for  $(t, m, s)$ -nets and  $(t, s)$ -sequences.

**Theorem 8.** *For any  $s \geq 2$ , when  $\Lambda$  is of anisotropic tensor product type, using the  $n = b^m$  points of a  $(t, m, s)$ -net in base  $b$  it holds that*

$$\mathcal{Z}_{s,n}(\Lambda) \leq (\#\Lambda)^2 \frac{b^t}{n} \left( (b+3)^{s-2} \left( \left( 1 + \frac{b-1 \ln n}{b+3 \ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) + s \right), \quad (47)$$

and using the first  $n \geq \max\{b, b^t\}$  points of a  $(t, s)$ -sequence in base  $b$  it holds that

$$\begin{aligned} \mathcal{Z}_{s,n}(\Lambda) &\leq (\#\Lambda)^2 \frac{b^t}{n} \left( b(b+3)^{s-2} \left( \left( 1 + \frac{2(b-1) \ln n}{b+3 \ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) \right. \\ &\quad \left. + s \left( a_{1,b}^{(2)} \left( 1 + \frac{\ln n}{\ln b} \right) + 2 \right) + \frac{1}{2} \left( \frac{b+2}{2} \right)^{-2} (r(b) + w(s, b) (b+3)^s) \right). \end{aligned} \quad (48)$$

*Proof.* Starting from (46), using Corollary 4 and Lemma 2 for  $(t, m, s)$ -nets or Lemma 3 for  $(t, s)$ -sequences, we obtain (47) and (48), respectively.  $\square$

**Theorem 9.** *In any dimension  $s \geq 1$  and for any downward closed multi-index set  $\Lambda$ , fix  $\delta \in (0, 1)$  and choose  $n$  such that the following condition holds*

$$\delta \geq \mathcal{Z}_{s,n}(\Lambda). \quad (49)$$

Then it holds that

$$1 \leq \text{cond}(A^\top A) \leq \frac{1+\delta}{1-\delta}, \quad (50)$$

and for any  $\phi \in C^0(I_s)$

$$\|\phi - \Pi_\Lambda^n \phi\|_{L^2(I_s)} \leq \left( 1 + \frac{1}{\sqrt{1-\delta}} \right) \inf_{u \in \mathbb{P}_\Lambda} \|u - \phi\|_{L^\infty(I_s)}. \quad (51)$$

*Proof.* When using a low-discrepancy point set of “open” or “closed” type with  $n$  points, combining Lemmas 8 and 9 we obtain

$$1 - \mathcal{Z}_{s,n}(\Lambda) \leq \frac{\|u\|_n^2}{\|u\|_{L^2(I_s)}^2} \leq 1 + \mathcal{Z}_{s,n}(\Lambda), \quad \forall u \in \mathbb{P}_\Lambda \setminus \{u \equiv 0\}. \quad (52)$$

The right-hand side of (52) does not depend on the polynomial  $u$ , and therefore we can take the supremum over the space  $\mathbb{P}_\Lambda \setminus \{u \equiv 0\}$  and substitute the definition (3) of the quantity  $\mathcal{Q}$ :

$$1 - \delta \leq 1 - \mathcal{Z}_{s,n}(\Lambda) \leq \mathcal{Q}(n, \Lambda) \leq 1 + \mathcal{Z}_{s,n}(\Lambda) \leq 1 + \delta. \quad (53)$$



Taking the inverse of each term in (52), with the same argument and using the definition (3) of the quantity  $\mathcal{S}$  gives

$$\frac{1}{1+\delta} \leq \frac{1}{1+\mathcal{Z}_{s,n}(\Lambda)} \leq \mathcal{S}(n, \Lambda) \leq \frac{1}{1-\mathcal{Z}_{s,n}(\Lambda)} \leq \frac{1}{1-\delta}. \quad (54)$$

Using the result in Proposition 2 and thanks to (53) and (54) we obtain the thesis (50). To prove (51) it suffices to substitute the bound (54) into (4).  $\square$

The following corollary highlights the case of anisotropic tensor product polynomial spaces with low-discrepancy point sets of “open” and “closed” type.

**Corollary 5.** *Fix any  $\delta \in (0, 1)$ . In one dimension  $s = 1$ , if the number of sampling points  $n$  satisfies*

$$n \geq \delta^{-1}(\#\Lambda)^2, \quad \text{with the } (0, 1, 1)\text{-net in base } b = n \text{ given by (17),} \quad (55)$$

$$\frac{n}{\ln n} \geq 2\delta^{-1}(\#\Lambda)^2 \mathcal{B}_1, \quad \text{with any point set of “open” type,} \quad (56)$$

then (50) and (51) hold true. In any dimension  $s \geq 2$  with  $\Lambda$  being of anisotropic tensor product type: if the number of sampling points  $n$  satisfies

$$\frac{n}{(b+3)^{s-2} \left(1 + \frac{b-1 \ln n}{b+3 \ln b}\right)^{s-1} \mathcal{O}(s^2)} \geq \delta^{-1}(\#\Lambda)^2 b^t, \quad (57)$$

with any  $(t, m, s)$ -net in base  $b$  with  $n = b^m$  points, or

$$\frac{n}{b(b+3)^{s-2} \left(1 + \frac{2(b-1) \ln n}{b+3 \ln b}\right)^{s-1} \mathcal{O}(s^2) + \left(1 + \frac{\ln n}{\ln b}\right) \mathcal{O}(s)} \geq \delta^{-1}(\#\Lambda)^2 b^t, \quad (58)$$

with any  $(t, s)$ -sequence in base  $b$ , then (50) and (51) hold true.

*Proof.* In the one-dimensional case ( $s = 1$ ), from (45) we have  $\mathcal{Z}_{s,n}(\Lambda) = 2D_S^n(\#\Lambda)^2$ , and combining this with (18) and (10) we can rewrite condition (49) as (55) and (56), respectively. In the multidimensional case ( $s \geq 2$ ), in the case of anisotropic tensor product polynomial spaces using Theorem 8 we can rewrite condition (49) as (57) and (58). Thanks to Theorem 9, conditions (55), (56), (57) and (58) ensure that (50) and (51) hold true in each one of the cases.  $\square$

Notice that, in conditions (57) and (58), for any  $b \geq 2$  it holds

$$\frac{b-1}{(b+3) \ln b} \leq 0.311,$$

and the terms  $\mathcal{O}(s^2)$  and  $\mathcal{O}(s)$  are precisely quantified in Theorem 8.

### 5.2. Evaluations at random point sets

Discrete least squares with evaluations at random points have been analyzed in [7, 6, 19, 21, 22, 20]. In [22, Theorem 3] it is proven that the univariate discrete least-squares approximation with random evaluations is stable and accurate with high probability, when the number of evaluations is proportional to the square of the dimension of the polynomial space, and for any “quasi-uniform” density  $\rho$ , *i.e.* densities which are bounded and bounded away from zero. An analogous univariate result has been proven in [7] but in expectation rather than in probability. The case of beta and gaussian densities have been analyzed in [19, Chap. 3]. In the following we report the multivariate result which has been proven in [6], in the particular case of the uniform density. Extensions to the Chebyshev density can also be found in [6], and further generalizations to the *beta* family can be obtained using the results proven in [20].

For a given  $M > 0$ , we assume that the target function satisfies a uniform bound  $|\phi(y)| \leq M$  for any  $y \in I_s$ . In addition, we introduce the truncation operator  $T_M(t) := \text{sign}(t) \min\{M, |t|\}$  and define the truncated discrete least-squares projector  $\tilde{\Pi}_\Lambda^n := T_M \circ \Pi_\Lambda^n$ . For any  $\delta \in (0, 1)$ , we define  $\zeta(\delta) := \delta + (1 - \delta) \ln(1 - \delta) > 0$ .

**Theorem 10** (from [6]). *For any  $\gamma > 0$ , any  $\delta \in (0, 1)$  and any downward closed multi-index set  $\Lambda \subset \mathbb{N}_0^s$ , if  $n$  satisfies*

$$\frac{n}{\ln n} \geq \frac{1 + \gamma}{\zeta(\delta)} (\#\Lambda)^2 \quad (59)$$

*then for any  $\phi \in C^0(I_s)$  with  $\|\phi\|_{L^\infty(I_s)} \leq M$ , the following hold*

$$\begin{aligned} \mathbb{E} \left( \|\phi - \tilde{\Pi}_\Lambda^n \phi\|_{L^2(I_s)}^2 \right) &\leq \left( 1 + \frac{4\zeta(\delta)}{(1 + \gamma) \ln n} \right) \|\phi - \Pi_\Lambda \phi\|_{L^2(I_s)}^2 + 8M^2 n^{-\gamma}, \quad (60) \\ \Pr \left( \|\phi - \Pi_\Lambda^n \phi\|_{L^2(I_s)} \leq \left( 1 + \sqrt{\frac{1}{1 - \delta}} \right) \inf_{u \in \mathbb{P}_\Lambda} \|\phi - u\|_{L^\infty(I_s)} \right) &\geq 1 - 2n^{-\gamma}, \\ \Pr \left( \text{cond}(A^\top A) \leq \frac{1 + \delta}{1 - \delta} \right) &\geq 1 - 2n^{-\gamma}, \end{aligned}$$

*where the expectation in (60) is taken over all possible random point sets.*

Theorem 10 asserts that the discrete least-squares approximation is stable and optimally convergent in any dimension and for any downward closed multi-index set  $\Lambda$ , if the number of sampling points is proportional to the square of the dimension of the polynomial space (up to logarithmic factors). We aim now at comparing this result with the one obtained for low-discrepancy point sets derived in §5.1.

### 5.3. Low-discrepancy point sets versus random point sets

With both deterministic or random points, in any dimension  $s$  and for any downward closed multi-index set  $\Lambda$ , the discrete least-squares approximation is stable and accurate under condition (49) or (59), respectively.

The first notable difference is that in Theorem 10 the stability and accuracy of discrete least squares on polynomial spaces are proven with high probability, whereas in Corollary 5 the stability and accuracy are proven with certainty.

In the one-dimensional case we summarize the following situation. The condition to ensure stability and accuracy, with evaluations in random uniformly distributed points, requires the number of evaluations  $n$  to scale like  $n \propto w^2$  up to a logarithmic factor in  $n$ , with respect to the highest degree  $w$  retained in the polynomial space. The choice of evaluations at low-discrepancy point sets of “open” type requires  $n \propto w^2$ , again up to a logarithmic factor in  $n$ , whereas the choice of evaluations at the low-discrepancy point set (17) of “closed” type requires  $n \propto w^2$ , without any logarithmic factor. Therefore, in one dimension, the same proportionality relation  $n \propto w^2$  ensures stability and accuracy, no matter which type of points is being used.

In the multidimensional case in any dimension, the condition  $n \propto (\#\Lambda)^2$  ensures stability and accuracy of discrete least squares on polynomial spaces of anisotropic tensor product type, up to a dimension-free logarithmic factor in the case of random points, and up to a dimension-dependent logarithmic factor in the case of low-discrepancy points. With more general polynomial spaces  $\mathbb{P}_\Lambda$ , associated with arbitrary downward closed multi-index sets  $\Lambda$ , the condition  $n \propto (\#\Lambda)^2$  with random points might worsen to  $n \propto (\#\Lambda)^4$ , again up to a dimension-dependent logarithmic factor, in the case of low-discrepancy points, according to our estimates.

The number of points  $n$  required by condition (59) with random points can be lower or larger than the number of points required by condition (49) with deterministic points, depending on the dimension  $s$ , on the multi-index set  $\Lambda$ , on the parameter  $\gamma$ , and on the parameters  $b$  and  $t$  which determine the low-discrepancy point set of “open” or “closed” type. In particular, the parameters  $t$  and  $b$  still depend on the dimension  $s$  and on the number of points  $n$ , see *e.g.* <http://mint.sbg.ac.at>, complicating the comparison between condition (59) and condition (49), *i.e.* (57)–(58). The two conditions (59) and (49) have different consequences: with random points, stability and accuracy are achieved with a confidence level which still depends on  $\gamma$ ; with low-discrepancy points, stability and accuracy are achieved with certainty. The one-dimensional case  $s = 1$  is aside: with the  $(0, 1, 1)$ -net in base  $b = n$  given by (17), condition (49) is always less demanding than (59). In higher dimension  $s \geq 2$ , on the one hand, for any admissible choice of the parameters  $b$ ,  $m$  and  $t$  there might be a choice of  $\gamma > 0$  such that (59) is less demanding than (49). Here “admissible” means that the choice of the parameters  $b$ ,  $m$  and  $t$  is not arbitrary but obeys to specific constraints. On the other hand, it is always possible to choose a sufficiently large  $\gamma$  such that (59) becomes more demanding than (49), but still cannot reach a confidence level equal to one, which can be achieved only in the limit  $n$  going to infinity. The precise comparison between (59) and (49) should also take into account all the constants arising from the upper bounds of the star discrepancy and the interplay among the parameters  $n$ ,  $s$ ,  $b$  and  $t$  outlined in §3.

**Remark 3.** *In the case of independent and uniformly distributed points, probabilistic*

bounds for the star discrepancy have been derived in [1] showing that

$$\Pr\left(D_s^n \leq c(s, \xi) \frac{\sqrt{s}}{\sqrt{n}}\right) \geq \xi, \quad \text{with } c(s, \xi) := 5.7 + \sqrt{4.9 + \frac{\ln((1 - \xi)^{-1})}{s}},$$

with  $\xi \in (0, 1)$ . The use of this bound allows us to prove the stability and accuracy with high probability of discrete least squares with evaluations at random points, following the lines of the proof of Theorem 9. Unfortunately in this case the condition requires  $n \propto (\#\Lambda)^4$ , which is nonoptimal w.r.t. condition (59) in Theorem 10.

## 6. Conclusions

We have proven that, in anisotropic tensor product polynomial spaces in any dimension, discrete least squares with evaluations at low-discrepancy point sets are stable and accurate if the number of evaluations is proportional to the square of the dimension of the polynomial space, up to a dimension-dependent logarithmic factor. Here, accuracy is evaluated in terms of the best approximation error in the  $L^\infty$  norm. With any polynomial space associated with an arbitrary downward closed multi-index set, stability and accuracy have been proven under a more demanding sufficient condition, with at most a quartic power rather than a quadratic power. The conditions derived in our analysis will automatically take advantage of any future improvement in the upper bounds for the star discrepancy of “open” and “closed” point sets.

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## Appendix A. Proofs of Lemmas 2 and 3

*Proof of Lemma 2.* From Definition 2, in the one-dimensional case  $S = \{1\}$  the star discrepancy of any  $(t, m, s)$ -net in base  $b$  with  $n = b^m$  points satisfies  $D_s^n \leq b^t n^{-1}$ . Using the propagation rule of Lemma 1 and the upper bound stated in Theorem 2 for  $s \geq 2$  we obtain

$$\mathbb{D}_s^n(\theta) = \sum_{\emptyset \neq R \subseteq S} D_R^n \theta^{|R|} = \sum_{\substack{R \subseteq S \\ |R|=1}} D_R^n \theta^{|R|} + \sum_{\substack{R \subseteq S \\ |R| \geq 2}} D_R^n \theta^{|R|} \leq \frac{b^t}{n} \left( s\theta + \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=0}^{q-1} a_{v,b}^{(q)} m^v}_{T_1} \right).$$

Now we estimate the term  $T_1$  as:

$$\begin{aligned}
T_1 &: \sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=0}^{q-1} a_{v,b}^{(q)} m^v = \sum_{v=0}^{s-1} m^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
&= \sum_{v=0}^{s-1} m^v \binom{s-1}{v} \frac{1}{v!} \sum_{q=0}^{s-v-2} \theta^q \binom{s-v-2}{q} \frac{s(s-v-1)}{(q+v+1)(q+v+2)} \\
&\quad \times \left(\frac{b+2}{2}\right)^q \left(\frac{b-1}{2}\right)^v \theta^{v+2} \left(a_{0,b}^{(2)} + (q+v+2)^2 - 4\right) + \sum_{v=1}^{s-1} m^v \binom{s-1}{v} \frac{1}{(v-1)!} \\
&\quad \times \sum_{q=0}^{s-v-2} \theta^q \binom{s-1-v}{q} \frac{s}{(q+v)(q+v+1)} \left(\frac{b+2}{2}\right)^q \left(\frac{b-1}{2}\right)^v \theta^{v+1} a_{1,b}^{(2)} \\
&\leq \sum_{v=0}^{s-1} m^v \binom{s-1}{v} \frac{1}{v!} \left(\frac{b-1}{2}\right)^v \theta^{v+2} s(s-v-1) \\
&\quad \times \left(\frac{a_{0,b}^{(2)} - 4}{(v+1)(v+2)} + 1 + \frac{1}{(v+1)}\right) \sum_{q=0}^{s-v-2} \theta^q \binom{s-v-2}{q} \left(\frac{b+2}{2}\right)^q \\
&\quad + \sum_{v=1}^{s-1} m^v \binom{s-1}{v} \frac{1}{(v-1)!} \left(\frac{b-1}{2}\right)^v \theta^{v+1} a_{1,b}^{(2)} \frac{s}{v(v+1)} \sum_{q=0}^{s-v-2} \theta^q \binom{s-2-v}{q} \left(\frac{b+2}{2}\right)^q \\
&= \underbrace{\sum_{v=0}^{s-1} m^v \binom{s-1}{v} \frac{1}{v!} \left(\frac{b-1}{2}\right)^v \theta^{v+2} s(s-v-1) \left(\frac{a_{0,b}^{(2)} - 4}{(v+1)(v+2)} + 1 + \frac{1}{(v+1)}\right) \left(1 + \theta \frac{b+2}{2}\right)^{s-v-2}}_{T_2} \\
&\quad + \underbrace{\sum_{v=1}^{s-1} m^v \binom{s-1}{v} \frac{1}{v!} \left(\frac{b-1}{2}\right)^v \theta^{v+1} a_{1,b}^{(2)} \frac{s}{v+1} \left(1 + \theta \frac{b+2}{2}\right)^{s-v-2}}_{T_3}.
\end{aligned}$$

We introduce the functions

$$\begin{aligned}
f_1(v, s, b, \theta) &:= \theta s(s-v-1) \frac{1}{v!} \left(\frac{a_{0,b}^{(2)} - 4}{(v+1)(v+2)} + 1 + \frac{1}{(v+1)}\right), \\
f_2(v, s, b, \theta) &:= \frac{1}{v!} \left(a_{1,b}^{(2)} \frac{s}{v+1}\right),
\end{aligned}$$

that are products of decreasing functions in  $v$ . Hence, for any choice of the parameters  $s$ ,  $b$  and  $\theta$ , the two points  $\widehat{v}_1 = 0$  and  $\widehat{v}_2 = 1$  satisfy

$$\begin{aligned}
f_1(v, s, b, \theta) &\leq f_1(\widehat{v}_1, s, b, \theta), \quad \forall v = 0, \dots, s-1, \\
f_2(v, s, b, \theta) &\leq f_2(\widehat{v}_2, s, b, \theta), \quad \forall v = 1, \dots, s-1.
\end{aligned}$$

Afterwards, we estimate the terms  $T_2$  and  $T_3$  as

$$T_2 \leq f_1(0, s, b, \theta) \sum_{v=0}^{s-1} \binom{s-1}{v} \left(\frac{b-1}{2}\right)^v \theta^{v+1} \left(1 + \theta \frac{b+2}{2}\right)^{s-v-2} m^v,$$

$$T_3 \leq f_2(1, s, b, \theta) \sum_{v=1}^{s-1} \binom{s-1}{v} \left(\frac{b-1}{2}\right)^v \theta^{v+1} \left(1 + \theta \frac{b+2}{2}\right)^{s-v-2} m^v,$$

and summing up the series we finally obtain

$$T_1 \leq T_2 + T_3$$

$$\leq \theta \left(1 + \theta \frac{b+2}{2}\right)^{s-2} \left( \left(1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b}\right)^{s-1} (f_1(0, s, b, \theta) + f_2(1, s, b, \theta)) - f_2(1, s, b, \theta) \right).$$

To shorten formulas, we introduce the functions  $g = g(s, b, \theta)$  and  $h = h(s, b)$  defined in (21)–(22) such that  $g(s, b, \theta) = f_1(0, s, b, \theta) + f_2(1, s, b, \theta)$  and  $h(s, b) = f_2(1, s, b, \theta)$ .  $\square$

**Remark 4.** For given values of  $\theta$ ,  $b$ ,  $s$  and  $n$ , the upper estimate (20) can be optimized. We introduce two real constants  $\mathcal{C}_1$  and  $\mathcal{C}_2$  such that

$$\mathcal{C}_1, \mathcal{C}_2 \geq 1,$$

and the parametric upper estimate

$$\mathbb{D}_s^n(\theta) \leq \frac{\theta b^t}{n} \left( \left(1 + \theta \frac{b+2}{2}\right)^{s-2} \left( \left(1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\mathcal{C}_1 \ln b}\right)^{s-1} \tilde{f}_1(\hat{v}_1, s, b, \theta, \mathcal{C}_1) \right. \right.$$

$$\left. \left. + \left(1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\mathcal{C}_2 \ln b}\right)^{s-1} \tilde{f}_2(\hat{v}_2, s, b, \theta, \mathcal{C}_2) - \tilde{f}_2(\hat{v}_2, s, b, \theta, \mathcal{C}_2) \right) + s \right),$$

(A.1)

with the functions  $\tilde{f}_1$  and  $\tilde{f}_2$  being defined as

$$\tilde{f}_1(v, s, b, \theta, \mathcal{C}_1) := \theta s(s-v-1) \frac{\mathcal{C}_1^v}{v!} \left( \frac{\binom{a_{0,b}^{(2)} - 4}{(v+1)(v+2)} + 1 + \frac{1}{(v+1)}} \right),$$

$$\tilde{f}_2(v, s, b, \theta, \mathcal{C}_2) := \frac{\mathcal{C}_2^v}{v!} \left( a_{1,b}^{(2)} \frac{s}{v+1} \left(1 + \theta \frac{b+2}{2}\right) \right),$$

and with the points  $\hat{v}_1 \in [0, \dots, s-1]$  and  $\hat{v}_2 \in [1, \dots, s-1]$  such that

$$\tilde{f}_1(v, s, b, \theta) \leq \tilde{f}_1(\hat{v}_1, s, b, \theta), \quad \forall v = 0, \dots, s-1,$$

$$\tilde{f}_2(v, s, b, \theta) \leq \tilde{f}_2(\hat{v}_2, s, b, \theta), \quad \forall v = 1, \dots, s-1.$$

The upper bound (A.1) differs from (20) due to the smaller multiplicative term in front of  $\log n$  and to the presence of the additional terms  $\mathcal{C}_1^v$  and  $\mathcal{C}_2^v$  in the functions  $\tilde{f}_1$  and  $\tilde{f}_2$ , competing with the factorials  $v!$ . The upper bound (20) is a particular instance of (A.1) with  $\mathcal{C}_1 = \mathcal{C}_2 = 1$ , and therefore can only be improved by a constrained optimization over the parameter set  $\{(\mathcal{C}_1, \mathcal{C}_2) \in \mathbb{R}^2 : \mathcal{C}_1, \mathcal{C}_2 \geq 1\}$ .

*Proof of Lemma 3.* It holds that

$$\mathbb{D}_s^n(\theta) = \sum_{\emptyset \neq R \subseteq S} D_R^n \theta^{|R|} = \left( \underbrace{\sum_{\substack{R \subseteq S \\ |R|=1}} D_R^n \theta^{|R|}}_{Q_1} + \underbrace{\sum_{\substack{R \subseteq S \\ |R| \geq 2}} D_R^n \theta^{|R|}}_{Q_2} \right).$$

A result from [13, Corollary 1] states that for any  $(t, 1)$ -sequence in base  $b \geq 2$  the star discrepancy satisfies

$$D_S^n \leq \frac{b^t}{n} \left( a_{1,b}^{(2)} (1 + \log_b n) + 2 \right),$$

where  $a_{1,b}^{(2)}$  is the same coefficient introduced in Theorem 2 and  $S = \{1\}$ . We use this bound to estimate the term  $Q_1$ , and obtain

$$Q_1 \leq \frac{s\theta b^t}{n} \left( a_{1,b}^{(2)} (1 + \log_b n) + 2 \right).$$

To estimate the term  $Q_2$  we use the upper bound stated in Theorem 3 for any  $s \geq 2$ , and then split the innermost summation:

$$\begin{aligned} Q_2 &\leq \frac{b^t}{n} \left( \sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=0}^q A_{v,b}^{(q)} (\log_b n)^v \right) \\ &= \frac{b^t}{n} \left( \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q \frac{b+2}{2} a_{0,b}^{(q)}}_I + \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=1}^{q-1} a_{v,b}^{(q)} 2^v (\log_b n)^v}_{II} \right. \\ &\quad \left. + \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=1}^{q-1} a_{v,b}^{(q)} \frac{b-1}{2} (\log_b n)^v}_{III} + \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=0}^{q-1} a_{v,b}^{(q)} \frac{b-1}{2(v+1)} (\log_b n)^{v+1}}_{IV} \right), \end{aligned}$$

where in the term IV we have merged the terms

$$\begin{aligned} \sum_{v=1}^{q-1} a_{v-1,b}^{(q)} \frac{b-1}{2v} (\log_b n)^v + a_{q-1,b}^{(q)} \frac{b-1}{2q} (\log_b n)^q &= \sum_{v=1}^q a_{v-1,b}^{(q)} \frac{b-1}{2v} (\log_b n)^v \\ &= \sum_{v=0}^{q-1} a_{v,b}^{(q)} \frac{b-1}{2(v+1)} (\log_b n)^{v+1}. \end{aligned}$$

For the terms II, III, IV, swapping the two summations we get

$$\begin{aligned}
II : \quad \sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=1}^{q-1} a_{v,b}^{(q)} 2^v (\log_b n)^v &= \sum_{v=1}^{s-1} 2^v (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
&= \underbrace{\sum_{v=0}^{s-1} 2^v (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)}}_{IIa} - \underbrace{\sum_{q=2}^s \binom{s}{q} \theta^q a_{0,b}^{(q)}}_{IIb},
\end{aligned}$$

$$\begin{aligned}
III : \quad \sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=1}^{q-1} a_{v,b}^{(q)} \frac{b-1}{2} (\log_b n)^v &= \frac{b-1}{2} \sum_{v=1}^{s-1} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
&= \underbrace{\frac{b-1}{2} \sum_{v=0}^{s-1} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)}}_{IIIa} - \underbrace{\frac{b-1}{2} \sum_{q=2}^s \binom{s}{q} \theta^q a_{0,b}^{(q)}}_{IIIb},
\end{aligned}$$

$$IV : \quad \sum_{q=2}^s \binom{s}{q} \theta^q \sum_{v=0}^{q-1} a_{v,b}^{(q)} \frac{b-1}{2(v+1)} (\log_b n)^{v+1} = \sum_{v=0}^{s-1} \frac{b-1}{2(v+1)} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)}.$$

For the term IIa, using the result obtained with nets in the proof of Lemma 2, we obtain

$$\begin{aligned}
IIa : \quad \sum_{v=0}^{s-1} 2^v (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
\leq \theta \left( \left( 1 + \theta \frac{b+2}{2} \right)^{s-2} \left( \left( 1 + \frac{2\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) + s \right),
\end{aligned}$$

with a doubled multiplicative factor in front of  $\ln n$ . For the terms IIIa and IV, again proceeding as in the proof of Lemma 2 with nets, we get

$$\begin{aligned}
IIIa : \quad \frac{b-1}{2} \sum_{v=0}^{s-1} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
\leq \theta \frac{b-1}{2} \left( \left( 1 + \theta \frac{b+2}{2} \right)^{s-2} \left( \left( 1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) + s \right), \\
IV : \quad \sum_{v=0}^{s-1} \frac{b-1}{2(v+1)} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
\leq \frac{b-1}{2} \sum_{v=0}^{s-1} (\log_b n)^v \sum_{q=v+2}^s \binom{s}{q} \theta^q a_{v,b}^{(q)} \\
\leq \theta \frac{b-1}{2} \left( \left( 1 + \theta \frac{b+2}{2} \right)^{s-2} \left( \left( 1 + \frac{\theta(b-1)}{2 + \theta(b+2)} \frac{\ln n}{\ln b} \right)^{s-1} g(s, b, \theta) - h(s, b) \right) + s \right).
\end{aligned}$$



For the term I we have

$$I : \sum_{q=2}^s \binom{s}{q} \theta^q \frac{b+2}{2} a_{0,b}^{(q)} = \sum_{q=2}^s \binom{s}{q} \theta^q \left(\frac{b+2}{2}\right)^{q-2} \left(a_{0,b}^{(2)} + q^2 - 4\right),$$

and putting together the terms I, IIb and IIIb, we arrive at

$$\begin{aligned} I - IIb - IIIb &= \sum_{q=2}^s \binom{s}{q} \theta^q \frac{b+2}{2} a_{0,b}^{(q)} - \frac{b+1}{2} \sum_{q=2}^s \binom{s}{q} \theta^q a_{0,b}^{(q)} \\ &= \frac{1}{2} \sum_{q=2}^s \binom{s}{q} \theta^q \left(\frac{b}{2} + 1\right)^{q-2} \left(a_{0,b}^{(2)} + q^2 - 4\right) \\ &= \frac{1}{2} \left( -(a_{0,b}^{(2)} - 4) \left(\frac{b+2}{2}\right)^{-2} - (a_{0,b}^{(2)} - 3) \left(\frac{b+2}{2}\right)^{-1} \right. \\ &\quad \left. + \sum_{q=0}^s \binom{s}{q} \theta^q \left(\frac{b}{2} + 1\right)^{q-2} \left(a_{0,b}^{(2)} + q^2 - 4\right) \right) \\ &\leq \frac{1}{2} \left(\frac{b+2}{2}\right)^{-2} \left(4 - a_{0,b}^{(2)} + \left(3 - a_{0,b}^{(2)}\right) \left(\frac{b+2}{2}\right)\right) \\ &\quad + \left(a_{0,b}^{(2)} + s^2 - 4\right) \left(1 + \theta \frac{b+2}{2}\right)^s. \end{aligned} \tag{A.2}$$

Now collecting the estimates for the terms IIa, IIIa, IV and (A.2) (that is an upper bound for the summation of the terms I, IIb and IIIb) we obtain the upper bound of  $Q_2$ , and then summing the upper bounds of  $Q_1$  and  $Q_2$  we obtain the thesis. To shorten formulas we write the thesis using the auxiliary functions  $r = r(b)$  and  $w = w(s, b)$  defined in (24)–(25).  $\square$

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