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An online intrinsic stabilization strategy for the reduced basis approximation of parametrized advection-dominated problems

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Abstract

We propose a new, black-box online stabilization strategy for reduced basis (RB) approximations of parameter-dependent advection-diffusion problems in the advection-dominated case. Our goal is to stabilize the RB problem irrespectively of the stabilization (if any) operated on the high-fidelity (e.g. finite element) approximation, provided a set of stable RB functions have been computed. Inspired by the *spectral vanishing viscosity* method, our approach relies on the transformation of the basis functions into of modal basis, then on the addition of a vanishing viscosity term over the high RB modes, and on a *rectification* stage – prompted by the *spectral filtering technique* – to further enhance the accuracy of the RB approximation. Numerical results dealing with an advection-dominated problem parametrized with respect to the diffusion coefficient show the accuracy of the RB solution on the whole parametric range.

Résumé

Une strategie intrinsèque de stabilisation en ligne pour l'approximation bases réduites de problèmes parametrés avec transport dominant.

Nous proposons une nouvelle strategie pour stabiliser l'approximation d'un problème de diffusion-transport avec transport dominant par une méthode de bases réduites. Cette strategie, opérée en ligne, est indépendante de la technique "haute fidélité" utilisée "hors ligne", elle trouve son inspiration dans la méthode de la *viscosité spectrale évanescente*. Par une diagonalisation sur l'espace de base réduite, on introduit une nouvelle base modale qui permet d'ajouter, au problème réduit, un terme de viscosité évanescent sur les modes suffisante pour stabiliser l'approximation. Une méthode de réctification de la solution (semblable aux techniques de filtrage spectral) de ce problème est enfin opérée afin d'améliorer la précision de cette approximation. Les résultats numériques obtenus pour un problème avec transport dominant dont l'intensité est parametrisée montrent que l'approximation réduite résulte stable et précise sur tout l'intervalle des paramètres.

Version française abrégée

Nous proposons dans cette note une nouvelle approche pour traiter le problème des instabilités apparaissant lors de l'approximation de problèmes de diffusion-transport avec transport dominant par des méthodes de bases réduites (RB). Ces instabilités sont un problème classique, quelle que soit la discrétisation utilisée qui peut être traité en raffinant le maillage de façon extrême ou en ajoutant des termes de stabilisation appropriés. Dans le cas RB, nous pouvons nous appuyer sur un ensemble de $N \ll N_h$ fonctions de base où chacune est obtenue grâce à une technique classique

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d'éléments finis *stable*, comme la méthode SUPG. Néanmoins la combinaison de ces fonctions à travers une méthode de Galerkin pure, ne suffit pas à assurer la stabilité du problème RB quand N augmente, d'où la nécessité d'introduire des termes de stabilisation appropriés au niveau RB (Sect. 1).

Naturellement, l'espace réduit pour le problème de diffusion-transport doit être indépendant de la méthode classique utilisée pour générer les fonctions de base réduite. Il en est de même pour le schéma, d'autant plus qu'il n'est pas souhaitable — voire impossible — de disposer de la traduction en terme d'algèbre matricielle. Nous proposons ici une technique de stabilisation "en ligne", indépendant de la stabilisation utilisée lors de la création de l'espace réduit. Ceci nous permet d'introduire une stabilisation plus faible que celle utilisée dans la technique SUPG. Elle s'inspire de la méthode de la *viscosité spectrale évanescente* [7], et repose sur trois étapes, voir la Sect. 2. On construit tout d'abord "hors ligne" la matrice $\mathbb{Z}$ dont les colonnes sont les vecteurs de la base réduite (préalablement orthonormalisées pour assurer un bon conditionnement du système réduit):

1. nous obtenons une nouvelle base modale $\tilde{\mathbb{Z}}$ par rotation de $\mathbb{Z}$, selon les vecteurs propres du Laplacien réduit;
2. dans l'étape "en ligne", nous ajoutons un terme de *viscosité* diagonal (dans cette base) dont l'amplitude est nulle sur les modes les plus bas et augmente sur les modes plus hauts; ensuite, on calcule la solution du problème réduit, notée par $\tilde{u}_N^{vv}(\mu)$;
3. nous déterminons enfin l'approximation RB $u_N^{vv}(\mu)$ en opérant une *rectification* de la solution $\tilde{u}_N^{vv}(\mu)$ afin d'augmenter la précision de cette dernière, obtenue au point 2; voir la Sect. 3.

Cette strategie permet de stabiliser de façon intrinsèque l'approximation en base réduite lors de l'étape en ligne, reposant des structures algébriques qui peuvent être construites indépendamment de l'approximation classique (ici SUPG) utilisée. Les résultats numériques présentés dans la Sect. 4 montrent la faisabilité de la technique proposée.

1 Existing stabilization approaches for Galerkin-RB approximations

Let us consider for the sake of exposition the following parametrized advection-diffusion problem:

$$\begin{cases} L(\mu)u := -\mu\Delta u + \mathbf{b}\cdot\nabla u = f & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mu \in \mathcal{P} = [10^{-6}, 1]$ denotes the diffusion coefficient and $\mathbf{b} = (1,1)^T$ the (constant) transport field. The weak form of problem (1) is: find $u(\mu) \in V$ such that

$$a(u(\mu), v; \mu) = F(v) \qquad \forall v \in V, \tag{2}$$

being $V = H_0^1(\Omega)$,

$$a(u, v; \mu) = \mu \int_\Omega (\nabla u \cdot \nabla v + \mathbf{b}\cdot\nabla v)\, d\mathbf{x}, \qquad F(v) = \int_\Omega f v d\mathbf{x}$$

a continuous and coercive bilinear form, for any $\mu \in \mathcal{P}$, and a linear and continuous functional, respectively; note that only a is μ-dependent. The so-called *high-fidelity* solution to problem (2) is obtained by a Galerkin-finite element (FE) method introducing a high-fidelity space $V_h \subset V$ of dimension $dim(V_h) = N_h < \infty$ as: find $u_h(\mu) \in V_h$ such that

$$a(u_h(\mu), v_h; \mu) = F(v_h) \qquad \forall v_h \in V_h. \tag{3}$$

Although problem (2) is well-posed for any choice of $\mu > 0$, when dealing with advection-dominated problems, i.e. when $|\mathbf{b}|/\mu \ll 1$, it is well recognized that (3) yields spurious numerical oscillations unless *(i)* a sufficiently fine mesh is considered, or *(ii)* suitable stabilization techniques are introduced. In this paper we consider this latter option, being interested in finding a way to stabilize a reduced-order model irrespectively of the stabilization operated on the high-fidelity problem.

Hence, we seek $u_h^S(\mu) \in V_h$ such that

$$a_S(u_h^S(\mu), v_h; \mu) = F_S(v_h; \mu) \qquad \forall v_h \in V_h \tag{4}$$

being $a_S(\cdot, \cdot; \mu)$ and $F_S(\cdot; \mu)$ suitable bilinear and linear forms including the stabilization terms. Note that, because of these latter, also the right-hand side might now be μ-dependent. Usual choices are given by the Streamline Upwind Petrov-Galerkin (SUPG) method, the Galerkin-Least Squares method, but also Variational Multi-Scale or Subgrid stabilization methods can be recast under the form (4), see e.g. [2, 6, 8]. For the sake of simplicity, hereon we rely on the SUPG method, even if also other stabilizations can fit the proposed framework. In this case, our high-fidelity FE problem is given by (4), where

$$a_S(u_h^S(\mu), v_h; \mu) = a(u_h^S(\mu), v_h; \mu) + b_h(u_h^S(\mu), v_h), \qquad F_S(v_h) = F(v_h) + G_h(v_h),$$

being

$$b_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} (L(\mu)u_h, \delta_K \mathcal{S}_K(v_h))_{L^2(K)}, \qquad G_h(v_h) = \sum_{K \in \mathcal{T}_h} (f, \delta_K \mathcal{S}_K(v_h))_{L^2(K)}$$

where $\mathcal{S}_K(v_h) = L_{SS}v_h$; here we denote by $L_S(\mu)u = -\mu\Delta u$ the symmetric part of L, by $L_{SS}u = \mathbf{b} \cdot \nabla u$ its skew-symmetric part, and by $\delta_K > 0$ a suitable scaling function. Note that when dealing with linear ($\mathbb{P}_1$) finite elements, i.e. when $V_h = X_h^1 \cap V$, the stabilization term is μ-independent, since $L(\mu)u_h = L_{SS}u_h \ \forall u_h \in V_h = X_h^1 \cap V$. It is straightforward to show that both problems (3) and (4) are coercive, with coercivity factors given by $\alpha_h(\mu)$ and $\alpha_h^S(\mu)$, respectively; nevertheless, the stabilized problem (4) features a larger coercivity factor, since

$$\alpha_h^S(\mu) = \inf_{v_h \in V_h} \frac{a_S(v_h, v_h; \mu)}{\|v_h\|_V^2} \geq \inf_{v_h \in V_h} \frac{a(v_h, v_h; \mu)}{\|v_h\|_V^2} = \alpha_h(\mu). \tag{5}$$

From an algebraic standpoint, in the SUPG case problem (4) yields the following linear system,

$$(\mathbb{A}_h^G(\mu) + \mathbb{A}_h^{SUPG}(\mu))\mathbf{u}_h(\mu) = \mathbf{f}_h^G + \mathbf{f}_h^S \tag{6}$$

being $\mathbf{u}_h^S(\mu) \in \mathbb{R}^{N_h}$ the vector whose components are the degrees of freedom of $u_h^S(\mu)$ and, for $i, j = 1, \ldots, N_h$, $(\mathbb{A}_h^G(\mu))_{ij} = a(\varphi_j, \varphi_i; \mu)$, $(\mathbb{A}_h^{SUPG}(\mu))_{ij} = b_h(\varphi_j, \varphi_i)$, whereas $(\mathbf{f}_h^G)_i = F(\varphi_i)$, $(\mathbf{f}_h^S)_i = G_h(\varphi_i)$; $\{\varphi_i\}_{i=1}^{N_h}$ denote the set of (Lagrangian) basis functions on V_h.

The reduced basis (RB) method allows to speedup the solution of a parametrized PDE under the form (2) by seeking for its approximation in a low-dimensional subspace $V_N \subset V_h$, of dimension $N = dim(V_N) \ll dim(V_h) = N_h$ built from a set of snapshots, that is, high-fidelity solutions computed for properly selected parameter values [4, 9]. The RB problem is then obtained by employing a Galerkin method over V_N and reads as follows: find $u_N(\mu) \in V_N$ such that

$$a(u_N(\mu), v_N; \mu) = F(v_N) \qquad \forall v_N \in V_N. \tag{7}$$

The RB space V_N can be obtained *offline* e.g. through the greedy algorithm (see, e.g., [9]), thus yielding for the case at hand $V_N = \text{span}\{u_h^S(\mu^n), n = 1, \ldots, N\} = \text{span}\{\zeta_1, \ldots, \zeta_N\}$; the RB approximation is thus a linear combination of *stable* FE approximations, obtained for $\mu \in S_N = \{\mu^1, \ldots, \mu^N\}$; a Gram-Schmidt orthonormalization is then performed to get the orthonormal basis $\{\zeta_n\}_{n=1}^N$.

Nevertheless, the RB approximation computed by solving *online* the reduced problem (7) is not stable: similarly to the standard FE approximation, this pure Galerkin-RB (G-RB) approximation show spurious oscillations already for $\mu \approx 10^{-2}$, even if the space V_N is built starting from a set of stabilized snapshots. This shortcoming can be explained by using e.g. the Strang lemma; see, e.g. [10] for numerical results.

On the other hand, performing a Galerkin projection of the stabilized problem (4) onto V_N yields stable RB approximations on the whole parameter range. In this case, the RB approximation of problem (4) reads as: find $u_N^S(\mu) \in V_N$ such that

$$a_S(u_N^S(\mu), v_N; \mu) = F_S(v_N; \mu) \qquad \forall v_N \in V_N. \tag{8}$$

Note that problem (8) – as the Galerkin-RB approximation of any strongly coercive problem – is automatically stable, thanks to the Cea's lemma and the Galerkin orthogonality.

Here we have

$$\|u_h^S(\mu) - u_N^S(\mu)\|_V \leq \left(\frac{M_h^S(\mu)}{\alpha_h^S(\mu)}\right)^{1/2} \inf_{w_N \in V_N} \|u_h(\mu) - w_N\|_V \tag{9}$$

being $M_h^S(\mu)$ and $\alpha_h^S(\mu)$ the discrete continuity and the coercivity factors, respectively, of $a_S(\cdot,\cdot;\mu)$. Thanks to the improved stability expressed by (5), (9) yields a good control of the error $\|u_h^S(\mu) - u_N^S(\mu)\|_V$ in terms of the best approximation error; the same estimate holds for the solution of problem (7) as well, but including a much larger factor $(M_h(\mu)/\alpha_h(\mu))^{1/2}$. We also point out that, as soon as a pure Galerkin method is used for the high-fidelity approximation (on a sufficiently fine mesh) like in (3), a pure Galerkin-RB approximation then yields automatically stable solutions, but entailing a very expensive generation of the RB space.

Algebraically, in the SUPG case the RB approximation (8) (which we can refer to as the SUPG-RB method) turns into the solution of the following RB system:

$$(\mathbb{A}_N^G(\mu) + \mathbb{A}_{N,h}^{SUPG}(\mu))\mathbf{u}_N^S(\mu) = \mathbf{f}_N^G + \mathbf{f}_{N,h}^{SUPG} \tag{10}$$

where $(\mathbb{A}_N^G(\mu))_{mn} = a(\zeta_n, \zeta_m; \mu)$, $(\mathbf{f}_N^G)_m = F(\zeta_m)$, and $(\mathbb{A}_{N,h}^{SUPG}(\mu))_{mn} = b_h(\zeta_n, \zeta_m)$, $(\mathbf{f}_{N,h}^{SUPG})_m = G_h(\zeta_m)$, $m, n = 1, \ldots, N$. Hence, also the stabilization term appearing on the SUPG-RB problem is μ-independent; however, its assembling requires to access the matrices of the stabilization terms in the SUPG-FE problem. In fact, the RB arrays appearing in (10) are expressed in terms of the FE arrays of (6) as

$$\mathbb{A}_N^G(\mu) = \mathbb{Z}^T \mathbb{A}_h^G(\mu)\mathbb{Z}, \quad \mathbb{A}_{N,h}^{SUPG}(\mu) = \mathbb{Z}^T \mathbb{A}_h^{SUPG}(\mu)\mathbb{Z}, \qquad \mathbf{f}_N^G = \mathbb{Z}^T \mathbf{f}_h^G, \quad \mathbf{f}_{N,h}^{SUPG} = \mathbb{Z}^T \mathbf{f}_h^{SUPG}, \tag{11}$$

being $\mathbb{Z} \in \mathbb{R}^{N_h \times N}$ the basis matrix, such that $\mathbb{Z}_{im} = \zeta_m^{(i)}$; the columns of $\mathbb{Z}$ are nothing but the vectors of degrees of freedom corresponding to the basis functions $\zeta_1, \ldots, \zeta_N$, see, e.g., [9] for further details.

2 An *online vanishing viscosity* stabilization method

Here we propose a new way to stabilize the RB problem independently of the stabilization procedure operated on the FE approximation, provided a set of stable RB functions have been computed offline, no matter how. As a matter of fact, the stabilization of the RB problem is built directly at the online stage, without relying on the stabilization terms possibly employed offline, that at this level have no meaning, since not related to the formulation (2). Moreover, we aim at introducing a *weaker* stabilization than the one appearing in the RB approximation of a stabilized FE problem like (4), by adding a suitable *diffusion term* on the RB problem, depending on N and vanishing on the lower modes – that is, the higher the mode, the stronger is the added stabilization.

To do this, we rely on a revisitation of the *spectral vanishing viscosity* technique [7]. First, we rewrite the diffusion operator over the orthonormal reduced basis $\mathbb{Z}$ built offline, that is, we *rotate* the reduced basis Z by the matrix $\mathbb{W}$ of the eigenvectors of the (reduced) diffusion operator, in order to deal with a *diagonalized* (reduced) diffusion operator. For ease of notation, we consider an algebraic formulation of this new scheme. Let us denote by $\mathbb{K}_N \in \mathbb{R}^{N \times N}$ and $\mathbb{M}_N \in \mathbb{R}^{N \times N}$ the reduced stiffness and mass operators, respectively, i.e.,

$$(\mathbb{K}_N)_{mn} = \int_\Omega \nabla\zeta_n \cdot \nabla\zeta_m \, d\mathbf{x}, \qquad (\mathbb{M}_N)_{mn} = \int_\Omega \zeta_n \zeta_m \, d\mathbf{x}, \qquad m, n = 1, \ldots, N,$$

obtained from the full-order matrices as $\mathbb{K}_N = \mathbb{Z}^T \mathbb{K}_h \mathbb{Z}$, $\mathbb{M}_N = \mathbb{Z}^T \mathbb{M}_h \mathbb{Z}$, being

$$(\mathbb{K}_h)_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, d\mathbf{x}, \qquad (\mathbb{M}_h)_{mn} = \int_\Omega \varphi_i \varphi_j \, d\mathbf{x}, \qquad i, j = 1, \ldots, N_h.$$

Then, we solve the following *generalized eigenvalue problem*:

$$\mathbb{K}_N \mathbf{w}_j = \lambda_j \mathbb{M}_N \mathbf{w}_j, \qquad j = 1, \ldots, N$$

and denote by $\mathbb{W} = [\mathbf{w}_1 \mid \ldots \mid \mathbf{w}_N] \in \mathbb{R}^{N \times N}$ the matrix of the eigevectors; note that $\mathbb{W}^T \mathbb{K}_N \mathbb{W} = \mathbb{D}_N$, that is, $\mathbf{w}_i \mathbb{K}_N \mathbf{w}_j = \lambda_j \mathbf{w}_i \mathbb{M}_N \mathbf{w}_j = \lambda_j \delta_{ij}$, being $\mathbb{D}_N = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.

We then rotate the columns of the basis matrix $\mathbb{Z} = [\boldsymbol{\zeta}_1 \mid \ldots \mid \boldsymbol{\zeta}_N]$ by the matrix $\mathbb{W}$ in order to get the *transformed* basis matrix $\tilde{\mathbb{Z}} = \mathbb{Z}\mathbb{W}$. With respect to this new basis, the stiffness term appearing in the RB problem is diagonal, that is, $\tilde{\mathbb{Z}}^T \mathbb{K}_h \tilde{\mathbb{Z}} = \mathbb{W}^T \mathbb{Z}^T \mathbb{K}_h \mathbb{Z}\mathbb{W} = \mathbb{W}^T \mathbb{K}_N \mathbb{W} = \mathbb{D}_N$. We denote the RB space obtained as the span of the new basis functions by $\tilde{V}_N = \text{span}\{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_N\}$, being $\tilde{\boldsymbol{\zeta}}_n$ the algebraic representation (as vector of degrees of freedom) of $\tilde{\zeta}_n$, $n = 1, \ldots, N$.

We now consider the following *vanishing viscosity* (VV) RB approximation: find $\tilde{u}_N^{vv}(\mu) \in \tilde{V}_N$ such that

$$a(\tilde{u}_N^{vv}(\mu), v_N; \mu) + d_N(\tilde{u}_N^{vv}(\mu), v_N) = F(v_N) \qquad \forall v_N \in \tilde{V}_N, \tag{12}$$

being $d_N(\cdot, \cdot)$ an additional viscosity term, whose action on each couple of basis functions $(\tilde{\zeta}_m, \tilde{\zeta}_n)$, $m, n = 1, \ldots, N$ is such that

$$d_N(\tilde{\zeta}_m, \tilde{\zeta}_n) = f(\lambda_n) \int_\Omega \nabla \tilde{\zeta}_n \cdot \nabla \tilde{\zeta}_m \, d\mathbf{x},$$

with $f(\lambda_n)$ to be properly defined. Algebraically, we turn to solve the following reduced system:

$$(\tilde{\mathbb{Z}}^T \mathbb{A}_h(\mu) \tilde{\mathbb{Z}} + \mathbb{S}_N) \mathbf{u}_N(\mu) = \tilde{\mathbb{Z}}^T \mathbf{f}_h$$

where $\mathbb{S}_N \in \mathbb{R}^{N \times N}$ is a diagonal matrix, whose generic element is given by

$$(S_N)_{mn} = d_N(\tilde{\zeta}_m, \tilde{\zeta}_n) = f(\lambda_n) \int_\Omega \nabla \tilde{\zeta}_n \cdot \nabla \tilde{\zeta}_m \, d\mathbf{x} = \begin{cases} 0, & n \neq m \\ f(\lambda_n)(\mathbb{D}_N)_{nn} = f(\lambda_n)\lambda_n, & n = m. \end{cases}$$

In the simpler case $f(\lambda_n) = c$, $c \in \mathbb{R}$, $c > 0$, we would add a viscosity contribution on each mode proportional to λ_n for any $n = 1, \ldots, N$. This is, in fact, what the RB-SUPG method (8) does in practice (upon rotating the basis functions of V_N), and what we want to avoid; we rather seek for a *less intrusive* technique yielding a non-negligible effect only on those (energetic) modes which effectively need to be stabilized.

By choosing a nonconstant function $f(\lambda_n)$, we add an artificial viscosity only to those (energetic) modes which effectively need to be stabilized. To determine $f(\lambda_n)$ we exploit a strategy similar to the *spectral vanishing viscosity* method by Maday and Tadmor [7], i.e., we consider a *non-uniform artificial viscosity* term under the form $(S_N)_{nn} = f(\lambda_n)\lambda_n$, where

$$f(\lambda_n) = \begin{cases} 0 & n < \bar{N}_1 \\ c \dfrac{(\lambda_n - \lambda_{\bar{N}_1})^2}{(\lambda_{\bar{N}_2}^2 - \lambda_{\bar{N}_1})^3} \left(2\lambda_{\bar{N}_2} - (\lambda_{\bar{N}_1} + \lambda_{\bar{N}_2})\lambda_n\right) & \bar{N}_1 \leq n \leq \bar{N}_2 \\ c\lambda_n & n > \bar{N}_2 \end{cases} \tag{13}$$

and $\bar{N}_1 \geq 0$, $\bar{N}_2 \leq N$ are prescribed indices. The expression of f in the range $N_1 \leq n \leq N_2$ is determined by ensuring a $\mathcal{C}^1$ regularity on the weighting coefficient. In other words, we consider no stabilization over the first N_1 modes; on the *intermediate* modes ($\bar{N}_1 \leq n \leq \bar{N}_2$) a stabilization term of the form

$$(S_N)_{nn} = f(\lambda_n)\lambda_n = (\lambda_n - \lambda_{\bar{N}_1})^2 \left(\frac{2c\lambda_{\bar{N}_2}^2}{(\lambda_{\bar{N}_2} - \lambda_{\bar{N}_1})^3} - \frac{\lambda_{\bar{N}_1} + \lambda_{\bar{N}_2}}{(\lambda_{\bar{N}_2} - \lambda_{\bar{N}_1})^3} c\lambda_n \right) \lambda_n,$$

and, finally, a stabilization term proportional to λ_n^2 on the higher modes, that is, $(S_N)_{nn} = f(\lambda_n)\lambda_n = c\lambda_n^2$ for those modes $n \geq \bar{N}_2$ which require a stronger stabilization. The constant $c = \nu/\lambda_N > 0$ is the ratio between the desired added viscosity ν on the highest mode (to be selected, depending on the problem at hand) and the largest eigenvalue λ_N, that is, $f(\lambda_N) = \nu$ on the highest mode.

3 A further post-processing based on a rectification method

Once the problem (12) has been solved, we perform a further *rectification* (introduced in [1]) to improve its accuracy, inspired by the *spectral filtering technique* [3], generally following the use of the SVD. Essentially, we exploit the *consistency* of the RB approximation $\tilde{u}_N^{vv}(\mu) = \sum_{k=1}^N \beta_k(\mu)\tilde{\zeta}_k$, i.e. the fact that

$$\tilde{u}_N^{vv}(\mu^i) = u_h(\mu^i) \qquad \forall \mu^i \in S_N = \{\mu^1, \dots, \mu^N\}. \tag{14}$$

Hence, we express the N snapshots over the reduced basis as $u_h(\mu^i) = \sum_{n=1}^N \alpha_n^i \tilde{\zeta}_n$, for any $i = 1, \dots, N$, and define the matrix $\mathbb{B}$ of components

$$(\mathbb{B})_{in} = \alpha_n^i = (u_h(\mu^i), \zeta_n)_V.$$

Then, we solve the RB problem (12) for each $\mu = \mu^i \in S_N$ and get $\tilde{u}_N^{vv}(\mu^i) = \sum_{k=1}^N \beta_k(\mu^i)\tilde{\zeta}_k$, from which we can define the matrix

$$(\mathbb{B}_R)_{ik} := \beta_k(\mu^i).$$

All these computations can be performed *offline*, since they do not depend on the actual parameter value μ.

Finally, the *rectified* solution $u_N^{vv}(\mu) = \sum_{j=1}^N \beta_j^R(\mu)\tilde{\zeta}_j$ is computed online from $\tilde{u}_N^{vv}(\mu) = \sum_{j=1}^N \beta_j(\mu)\tilde{\zeta}_j$, the solution to problem (12), by computing the new coordinates as $\boldsymbol{\beta}^R(\mu) = \mathbb{B}\mathbb{B}_R^{-1}\boldsymbol{\beta}(\mu)$. Also the matrix $\mathbb{B}\mathbb{B}_R^{-1}$ can be computed once and for all, the rectification requiring a simple multiplication by $\boldsymbol{\beta}(\mu)$ to be performed online for any new $\mu \in \mathcal{D}$. A similar rectification strategy has been also exploited in [5].

Heuristically, the rectification relies on an interpolation process, since $\tilde{u}_N^{vv}(\mu)$ coincides with (the spectral approximation over the reduced basis of) $u_h(\mu)$ for any $\mu = \mu^i$, $i = 1, \dots, N$; this could explain, at some extent, why this post-processing helps to improve (substantially) the results, as shown in Sect. 4.

4 Numerical results

We now compare the approximations obtained through *(i)* the high-fidelity SUPG method, solving (4); *(ii)* the SUPG-RB method, solving (8); *(iii)* the RB method, solving the pure Galerkin problem (7) without any *online* stabilization term; *(iv)* the RB-VV method (12), relying on the online spectral vanishing viscosity and, finally, *(v)* the RB-VV method with the rectification post-processing. These five approximations are denoted, for any $\mu \in \mathcal{P}$, by $u_h^S(\mu)$, $u_N^S(\mu)$, $u_N(\mu)$, $\tilde{u}_N^{vv}(\mu)$, $u_N^{vv}(\mu)$, respectively. We set $f = 1$ and we build the RB space V_N by relying on the greedy algorithm: at each step n, this algorithm retains the snapshot $u_h^S(\mu^n)$ predicted to be worst approximated by the current RB space V_{n-1}; the a posteriori error bound is used as error indicator in this respect. For the case at hand, by imposing a tolerance of 10^{-5} on the relative error bound, the greedy algorithm selects $N_{\max} = 18$ basis functions; the space $\tilde{V}_N$ is then obtained by rotating the basis function of V_N. A latin hypercube sampling on $\eta \in [0, 6]$ allows to define the training sample for $\mu = 10^{-\eta}$ required to run the (weak) greedy algorithm; see, e.g. [9] for further details.

Then, the accuracy of the RB approximations is tested for $N = 1, \dots, N_{\max}$ and different values of $\mu \in \mathcal{P}$; we report in Fig. 1 the errors (in the V-norm) between $u_h^S(\bar{\mu})$ and the four RB approximations introduced above, for $\bar{\mu} = 10^{-6}$, corresponding to the smallest viscosity coefficient in $\mathcal{P}$; then, we report in Fig. 2 the behavior, with respect to N, of the errors (in the V-norm) in three different cases, $\mu = 1, 10^{-3}, 10^{-6}$.

Clearly, the pure Galerkin RB approximation (without SUPG stabilization terms) is not stable, whereas the SUPG-RB method is stable and accurate over the whole parametric range, showing an exponential error decay – this indeed confirms theoretical results dealing with elliptic parametrized PDEs in case of parametric analytic regularity (see e.g. [9]). Applying the rectification method to the G-RB solution does not cure the instability, and for this reason the results have not been reported: indeed, the rectification method needs as many constraints as possible to be effective – i.e., N should be as large as possible – but this usually entails worse and worse G-RB approximations. The spectral vanishing viscosity method, for which here we take $c = 10^{-2}$, $\bar{N}_1 = \lfloor N/3 \rfloor$ and $\bar{N}_2 = \lfloor 2N/3 \rfloor$, yields a stable solution, but inaccurate. A further post-processing relying on the proposed rectification method allows to recover the accuracy in this case, too, yielding an error decay indeed very close to the one provided by the SUPG-RB method.

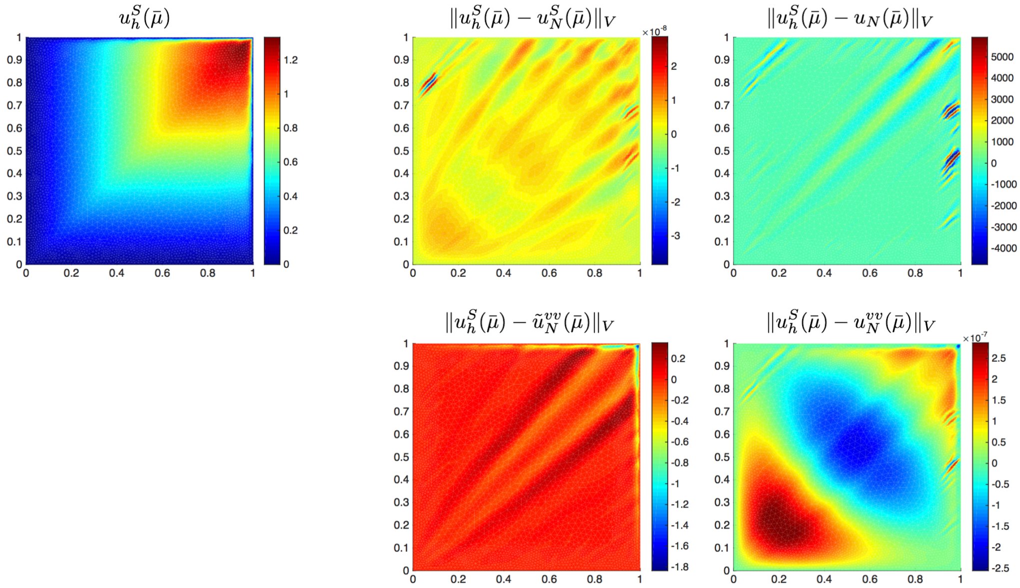

Figure 1: Top: SUPG solution $u_h^S(\bar\mu)$, errors $\|u_h^S(\bar\mu) - u_N^S(\bar\mu)\|_V$, $\|u_h^S(\bar\mu) - u_N(\bar\mu)\|_V$ for the approximations G-RB with stabilization and G-RB without stabilization. Bottom: errors $\|u_h^S(\bar\mu) - \tilde u_N^{vv}(\bar\mu)\|_V$, $\|u_h^S(\bar\mu) - u_N^{vv}(\bar\mu)\|_V$ for the approximations SVV-RB with spectral vanishing viscosity, without and with rectification, respectively. Solutions are reported for the case $\bar\mu = 10^{-6}$; errors are of order 10^{-8}, 10^3, 10^{-1} and 10^{-7}, respectively.

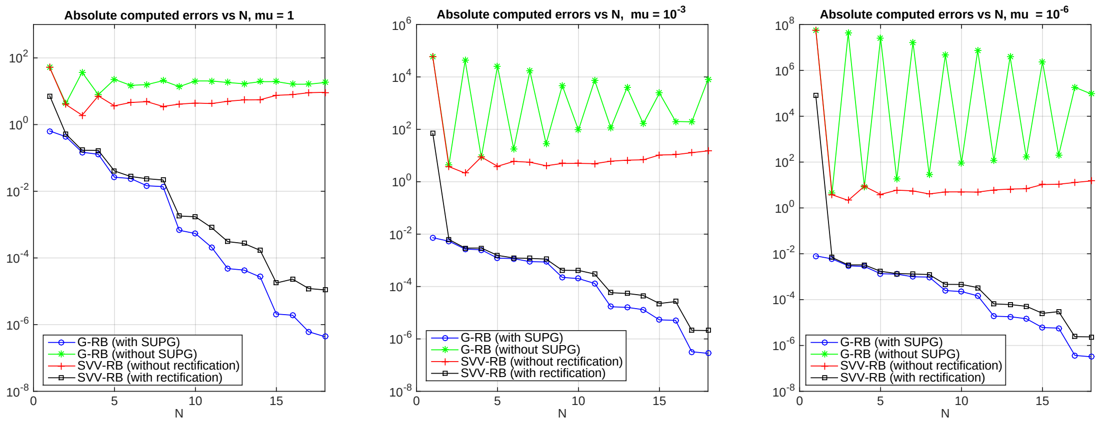

Figure 2: Errors $\|u_h^S(\bar\mu) - u_N^S(\bar\mu)\|_V$ (in blue), $\|u_h^S(\bar\mu) - u_N(\bar\mu)\|_V$ (in green), $\|u_h^S(\bar\mu) - \tilde u_N^{vv}(\bar\mu)\|_V$ (in red), $\|u_h^S(\bar\mu) - u_N^{vv}(\bar\mu)\|_V$ (in black) for different $N = 1, \dots, N_{\max} = 18$, for $\bar\mu = 1$ (left), $\bar\mu = 10^{-3}$ (center), $\bar\mu = 10^{-6}$ (right).

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