A posteriori error estimation for the steady Navier-Stokes equations in random domains

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Abstract

We consider finite element error approximations of the steady incompressible Navier-Stokes equations defined on a randomly perturbed domain, the perturbation being small. Introducing a random mapping, these equations are transformed into PDEs on a fixed reference domain with random coefficients. Under suitable assumptions on the random mapping and the input data, in particular the so-called *small data* assumption, we prove the well-posedness of the problem. We assume then that the mapping depends affinely on L independent random variables and adopt a perturbation approach expanding the solution with respect to a small parameter ε that controls the amount of randomness in the problem. We perform an *a posteriori* error analysis for the first order approximation error, namely the error between the exact (random) solution and the finite element approximation of the first term in the expansion with respect to ε. Numerical results are given to illustrate the theoretical results and the effectiveness of the error indicator.

1 Introduction

It is nowadays common to include uncertainty in the modelling of complex phenomena arising for instance in physics, biology or engineering to reflect an intrinsic variability of the system or our inability to adequately characterize the input data, due for instance to experimental measurements, such as the coefficients, forcing term, boundary conditions or geometry. The goal is then to determine the effect of the uncertainty on the solution or a specific quantity of interest. In a probability setting, the uncertainty is characterized via random variables or more generally random fields, and yields to stochastic partial differential equations (SPDEs).

In this paper, we focus our study on the steady-state Navier-Stokes equations defined on a randomly perturbed domain, considering small perturbations on the domain. For simplicity, we assume that the uncertainty in the system is only due to the random domain, but the analysis can be straightforwardly extended to include other sources of randomness. Moreover, we stress that all the analysis, namely the well-posedness of the problem (in particular the uniqueness) and the error analysis, is performed under the assumption of small data.

Several approaches have been developed to perform analysis and numerical approximation of PDEs in random domains, such as the fictitious domain method [12], the perturbation method based on shape calculus [30] and the domain mapping method initially proposed by [45] and also used for instance in [13, 29, 11]. In the first approach, the PDEs are extended to a fixed reference domain, the so-called fictitious domain, which contains all the random domains. The original boundary condition is then imposed through a Lagrange multiplier yielding a saddle-point problem to be solved in the fictitious domain. In the perturbation method, which is suitable for small perturbations only, the solution is represented using a shape Taylor expansion with respect to the (random) perturbation field of the boundary of the domain. Finally, the domain mapping approach, which is the one considered in this work, transforms the deterministic PDEs defined on a random domain into PDEs on a fixed reference domain with random coefficients via a random mapping. We give on Figure 1 an illustration of the mapping for a given ω between the physical domain and the reference one, supplemented with some

notation. For this method, contrary to the method based on shape derivatives, the random mapping needs to be known not only at the boundary but in the whole domain. If the random mapping is not given analytically, it can be obtained by solving appropriate equations, e.g. Laplace equation as it is done in [45]. The domain mapping method prevents the need of remeshing and can make use of the well-developed theory for stochastic PDEs on deterministic domains. Numerical approximation of the solution on the fixed reference domain can indeed be obtained through any of the well-known techniques, such as Monte-Carlo methods [21] and their generalizations as quasi-Monte Carlo [10, 25, 18] and multi-level Monte-Carlo [32, 17, 23], or the stochastic spectral methods comprising the Stochastic Galerkin [4, 22] and the Stochastic Collocation [3, 37, 44] methods.
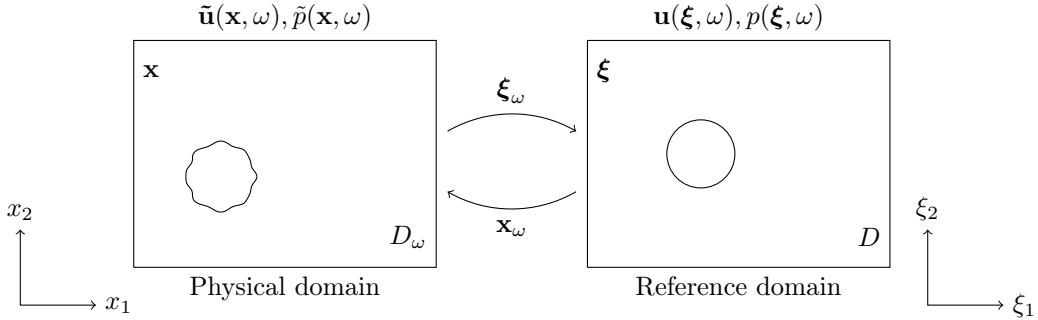

Figure 1: Illustration and notation for the domain mapping approach.

In this work, once the PDEs are transformed on the reference domain, we proceed as in [28] and use a perturbation approach [33] expanding the exact random solution with respect to a parameter ε that controls the level of uncertainty in the problem. This approach yields uncoupled deterministic problems for each term in this expansion, which can be solved using for instance the finite element (FE) method. The main goal of this paper is to perform an *a posteriori* error analysis for the error between the exact random solution and the finite element approximation of the first term in the expansion, that is the solution corresponding to the case $\varepsilon = 0$. The two error estimators we obtain are constituted of two parts, namely one part due to the (FE) space discretization and another one due to the uncertainty. Their computation require only the FE approximation of the solution of the problem for $\varepsilon = 0$ and the Jacobian matrix of the mapping between the reference domain and the physical random domain. These estimators can be used for instance to adaptively determine a mesh that yields a numerical accuracy comparable with the model uncertainty. Notice that the error estimates we get here using the domain mapping method combined with a perturbation technique are defined for any fixed ε. The only restriction is that ε is sufficiently small for the problem to be well-posed. The more common perturbation method is to use shape calculus [30], thus avoiding to recast the equations in a reference domain. However, the derivation of *a posteriori* error estimates for a fixed value of ε is, in our opinion, not obvious in this context and, to the best of our knowledge, it is still an open question.

Finally, we mention that the formulation of the Navier-Stokes equations we get on the reference domain is similar to the one obtained for instance in [26] where a fluid-structure interaction problem is considered or in [39, 36] where the Navier-Stokes equations in parametrized domains are solved approximately using the Reduced Basis method.

The paper is organized as follows. We give in Section 2 the statement of the problem, namely the steady-state Navier-Stokes equations defined on a random domain. We introduce in Section 3 the corresponding problem on a fixed reference domain using a random mapping and show its well-posedness in Section 4 under the *small data* assumption and suitable assumptions on the mapping. A specific but rather general form of the random mapping is introduced in Section 5, namely that it depends linearly on a finite number of independent random variables. In Section 6, which is the core of this paper, an *a posteriori* error analysis is performed with the derivation of two *a posteriori* error estimates for the first order approximation. Finally, numerical experiments are presented in Section 7 and agree with the theoretical results.

2 Navier-Stokes equations on a random domain

Let $D_\omega \subseteq \hat{D} \subset \mathbb{R}^d$, $d = 2, 3$, be an open bounded domain with Lipschitz continuous boundary that depends on a random parameter $\omega \in \Omega$, where $\hat{D}$ is a fixed bounded domain that contains D_ω for all $\omega \in \Omega$. Here $(\Omega, \mathcal{F}, P)$ denotes a complete probability space, where Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events and $P : \mathcal{F} \to [0, 1]$ is a probability measure. By a slight abuse of notations, we will denote

$$D_\omega \times \Omega := \{(\mathbf{x}, \omega) : \mathbf{x} \in D_\omega, \omega \in \Omega\}.$$

We consider the steady incompressible Navier-Stokes equations in D_ω:

find a velocity $\tilde{\mathbf{u}} : D_\omega \times \Omega \to \mathbb{R}^d$ and a pressure $\tilde{p} : D_\omega \times \Omega \to \mathbb{R}$ such that P-almost everywhere (a.e.) in Ω, or in other words almost surely (a.s.), the following equations hold

$$\begin{cases} -\nu \Delta_\mathbf{x} \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla_\mathbf{x}) \tilde{\mathbf{u}} + \nabla_\mathbf{x} \tilde{p} &=& \tilde{\mathbf{f}} & \text{for } \mathbf{x} \in D_\omega \\ \nabla_\mathbf{x} \cdot \tilde{\mathbf{u}} &=& 0 & \text{for } \mathbf{x} \in D_\omega \\ \tilde{\mathbf{u}} &=& \mathbf{0} & \text{for } \mathbf{x} \in \partial D_\omega, \end{cases} \tag{1}$$

where ν is the kinematic viscosity, $\tilde{\mathbf{f}} \in \left[L^2(\hat{D}) \right]^d$ is the external force field per unit mass that we assume to be deterministic and well-defined for all $\mathbf{x} \in \hat{D}$. Note that $\tilde{p}$ is the pressure divided by the density of the fluid. We consider homogeneous Dirichlet boundary conditions for the sake of simplicity. In (1), we have used the following notation: if we write $\mathbf{x} = (x_1, ..., x_d)$ and $\tilde{\mathbf{u}} = (\tilde{u}_1, ..., \tilde{u}_d)^T$ then for $i, j = 1, ..., d$

$$\nabla_\mathbf{x} \tilde{p} = (\frac{\partial \tilde{p}}{\partial x_1}, ..., \frac{\partial \tilde{p}}{\partial x_d})^T, \quad (\nabla_\mathbf{x} \tilde{\mathbf{u}})_{ij} = \frac{\partial \tilde{u}_i}{\partial x_j}, \quad \nabla_\mathbf{x} \cdot \tilde{\mathbf{u}} = \sum_{i=1}^d \frac{\partial \tilde{u}_i}{\partial x_i}$$

and

$$(\Delta_\mathbf{x} \tilde{\mathbf{u}})_i = (\nabla_\mathbf{x} \cdot \nabla_\mathbf{x} \tilde{\mathbf{u}})_i = \sum_{j=1}^d \frac{\partial}{\partial x_j} \frac{\partial \tilde{u}_i}{\partial x_j} = \Delta_\mathbf{x} \tilde{u}_i, \quad [(\tilde{\mathbf{u}} \cdot \nabla_\mathbf{x}) \tilde{\mathbf{u}}]_i = \sum_{j=1}^d \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}.$$

Note that we will use the same notation to denote the norm of a scalar, vector or matrix-valued function, with the natural extension $\|\mathbf{v}\|^2 = \sum_{i=1}^d \|v_i\|^2$ (Euclidean norm) and $\|B\|^2 = \sum_{i,j=1}^d \|B_{ij}\|^2$ (Frobenius norm) for any vector $\mathbf{v} = (v_1, ..., v_d) \in \mathbb{R}^d$ and any matrix $B = (B_{ij})_{i,j=1}^d \in \mathbb{R}^{d \times d}$. In order to write the weak formulation of the problem, we need to introduce some functional spaces. For a given Banach space W with norm $\| \cdot \|_W$, we define the Bochner space

$$L_P^2(\Omega; W) := \{v : \Omega \to W, v \text{ is strongly measurable and } \|v\|_{L_P^2(\Omega;W)} < +\infty\},$$

where $\|v\|_{L_P^2(\Omega;W)}^2 := \int_\Omega \|v(\omega)\|_W^2 dP(\omega) = \mathbb{E}[\|v\|_W^2]$ using the shorthand notation $v(\omega) = v(\cdot, \omega)$ for ease of presentation. Notice that if W is a separable Hilbert space, then $L_P^2(\Omega; W)$ is isomorphic [4] to the tensor product space $L_P^2(\Omega) \otimes W$. Finally, we define $\tilde{V}_\omega = \left[H_0^1(D_\omega) \right]^d$ equipped with the gradient norm $\| \cdot \|_{\tilde{V}_\omega} := \|\nabla_\mathbf{x} \cdot \|_{L^2(D_\omega)}$ and $\tilde{Q}_\omega = L^2(D_\omega)$. Note that unless otherwise clearly stated, the Lebesgue measure is used in D_ω. The (pointwise in ω) weak formulation of Problem (1) reads:

find $(\tilde{\mathbf{u}}(\omega), \tilde{p}(\omega)) \in \tilde{V}_\omega \times \tilde{Q}_\omega$ such that

$$\begin{cases} \nu \int_{D_\omega} \nabla_\mathbf{x} \tilde{\mathbf{u}} : \nabla_\mathbf{x} \tilde{\mathbf{v}} d\mathbf{x} + \int_{D_\omega} [(\tilde{\mathbf{u}} \cdot \nabla_\mathbf{x}) \tilde{\mathbf{u}}] \cdot \tilde{\mathbf{v}} d\mathbf{x} - \int_{D_\omega} \tilde{p} \nabla_\mathbf{x} \cdot \tilde{\mathbf{v}} d\mathbf{x} = \int_{D_\omega} \tilde{\mathbf{f}} \cdot \tilde{\mathbf{v}} d\mathbf{x} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \int_{D_\omega} \tilde{q} \nabla_\mathbf{x} \cdot \tilde{\mathbf{u}} d\mathbf{x} = 0 \end{cases} \tag{2}$$

for all $(\tilde{\mathbf{v}}, \tilde{q}) \in \tilde{V}_\omega \times \tilde{Q}_\omega$ and a.s. in Ω. Since we impose Dirichlet conditions on the whole boundary, the pressure is only defined up to an additive constant. We come back to this point in the next section (see Remark 3.1). Under the assumption of *small data*, the well-posedness of the problem on the family of random domains $(D_\omega)_{\omega \in \Omega}$ can be proved using two different approaches. The first one would be to consider the Navier-Stokes equations directly on $D_\omega \times \Omega$. Another approach, adopted here, consists in mapping the random domain to a reference one, yielding PDEs on a (fixed, deterministic) reference domain with random coefficients.

3 Formulation on a reference domain

Let $D \subset \mathbb{R}^d$ be an open bounded reference domain with Lipschitz continuous boundary ∂D. We assume that there exists a mapping $\mathbf{x} : D \times \Omega \to \mathbb{R}^d$ that transforms D into D_ω. For each $\omega \in \Omega$, we denote

$$\mathbf{x}_\omega : \quad D \quad \to \quad D_\omega$$
$$\boldsymbol{\xi} \quad \mapsto \quad \mathbf{x} = \mathbf{x}_\omega(\boldsymbol{\xi}).$$

We assume that for any $\omega \in \Omega$, $\mathbf{x}_\omega$ is invertible and sufficiently regular so that everything that follows makes sense, the precise regularity assumptions on the random mapping $\mathbf{x}$ being given in Section 4. Let $\boldsymbol{\xi}_\omega$ be the inverse of $\mathbf{x}_\omega$ defined by

$$\boldsymbol{\xi}_\omega : \quad D_\omega \quad \to \quad D$$
$$\mathbf{x} \quad \mapsto \quad \boldsymbol{\xi} = \boldsymbol{\xi}_\omega(\mathbf{x}).$$

We also introduce the $d \times d$ Jacobian matrices $A^{-1} = A^{-1}(\boldsymbol{\xi}, \omega)$ and $\tilde{A} = \tilde{A}(\mathbf{x}, \omega)$ corresponding respectively to the random transformations $\mathbf{x}_\omega$ and $\boldsymbol{\xi}_\omega$ and defined by

$$A^{-1} = \left(A_{ij}^{-1} \right)_{1 \leq i,j \leq d} \quad \text{with} \quad A_{ij}^{-1} := \frac{\partial(\mathbf{x}_\omega)_i}{\partial \xi_j}$$

and

$$\tilde{A} = \left(\tilde{A}_{ij} \right)_{1 \leq i,j \leq d} \quad \text{with} \quad \tilde{A}_{ij} := \frac{\partial(\boldsymbol{\xi}_\omega)_i}{\partial x_j}.$$

We mention that the matrix A^{-1} is often denoted F in the continuum mechanics literature. For any function $\tilde{g}$ defined on $D_\omega \times \Omega$, we denote by $g = \tilde{g} \circ \mathbf{x}_\omega$ its corresponding function on $D \times \Omega$, i.e. $g(\boldsymbol{\xi}, \omega) = \tilde{g}(\mathbf{x}, \omega)$ with $\mathbf{x} = \mathbf{x}_\omega(\boldsymbol{\xi})$. Notice that the matrix $A = \tilde{A} \circ \mathbf{x}_\omega$ is the inverse (in the matrix sense) of A^{-1}. From the chain rule, the following relations hold true

$$\nabla_{\mathbf{x}} = \tilde{A}^T \nabla_{\boldsymbol{\xi}} \quad \text{and} \quad \nabla_{\mathbf{x}} \tilde{\mathbf{u}} = (\nabla_{\boldsymbol{\xi}} \mathbf{u} \circ \boldsymbol{\xi}_\omega) \tilde{A},$$

where $\tilde{A}^T \nabla_{\boldsymbol{\xi}}$ is a matrix-vector product. For the sake of notation, we will write ∇ instead of $\nabla_{\boldsymbol{\xi}}$ from now on and use the notation

$$[(B\nabla)p]_i = \sum_{j=1}^d B_{ij} \frac{\partial p}{\partial \xi_j}, \quad (B\nabla) \cdot \mathbf{u} = \sum_{i,j=1}^d B_{ij} \frac{\partial u_i}{\partial \xi_j} = B : \nabla \mathbf{u}$$

and

$$[(B\nabla)\mathbf{u}]_{ij} = \sum_{k=1}^d B_{jk} \frac{\partial u_i}{\partial \xi_k}, \quad [(\mathbf{u} \cdot B\nabla)\mathbf{v}]_i = \sum_{j,k=1}^d u_j B_{jk} \frac{\partial v_i}{\partial \xi_k}$$

for a $d \times d$ matrix $B = (B_{ij})_{1 \leq i,j \leq d}$. Note that $(A\nabla)p = A(\nabla p)$. Moreover, let $J_{\mathbf{x}} = \det(A^{-1})$ denotes the determinant of the Jacobian matrix A^{-1} associated to $\mathbf{x}_\omega$. Finally, we introduce the spaces $V = \left[H_0^1(D) \right]^d$ and $Q = L_0^2(D) = \{ q \in L^2(D) : \int_D q d\xi = 0 \}$.

Remark 3.1. *We choose to fix the constant part of the pressure by imposing zero average on D and not on D_ω, the goal being not to estimate this constant when performing the error analysis. Notice that if we fix $\tilde{p}$ with zero average on D_ω, then the average of the corresponding pressure $p = \tilde{p} \circ \mathbf{x}_\omega$ on D would be small when $\mathbf{x}_\omega$ is a small perturbation of the identity mapping. Indeed, we have then $\int_D p d\boldsymbol{\xi} = \int_D p d\boldsymbol{\xi} - \int_{D_\omega} \tilde{p} d\mathbf{x} = \int_D p(1 - |J_{\mathbf{x}}|) d\boldsymbol{\xi}$.*

We are now able to write the weak formulation of Problem (1) on the reference domain, using the change of variable $\mathbf{x} = \mathbf{x}_\omega(\boldsymbol{\xi})$:

find $(\mathbf{u}(\omega), p(\omega)) \in V \times Q$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}; \omega) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \omega) + b(\mathbf{v}, p; \omega) & = & F(\mathbf{v}; \omega) \\ b(\mathbf{u}, q; \omega) & = & 0 \end{cases} \tag{3}$$

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for all $(\mathbf{v}, q) \in V \times Q$ and a.s. in Ω, where

$$a(\mathbf{u}, \mathbf{v}; \omega) := \nu \int_D (\nabla \mathbf{u} A(\omega)) : (\nabla \mathbf{v} A(\omega)) J_{\mathbf{x}}(\omega) d\boldsymbol{\xi}, \quad b(\mathbf{v}, q; \omega) := -\int_D q J_{\mathbf{x}}(\omega) (A(\omega)^T \nabla) \cdot \mathbf{v} d\boldsymbol{\xi}$$

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \omega) := \int_D [(\mathbf{u} \cdot A(\omega)^T \nabla) \mathbf{v}] \cdot \mathbf{w} J_{\mathbf{x}}(\omega) d\boldsymbol{\xi}, \quad F(\mathbf{v}; \omega) := \int_D \mathbf{f}(\omega) \cdot \mathbf{v} J_{\mathbf{x}}(\omega) d\boldsymbol{\xi}. \tag{4}$$

Using the relations (see Appendix D for proofs)

$$(\nabla \mathbf{u} A) : (\nabla \mathbf{u} A) = (\nabla \mathbf{u} A A^T) : (\nabla \mathbf{u}), \quad \nabla \mathbf{u} A = (A^T \nabla) \mathbf{u} \tag{5}$$

and

$$-\int_D q J_{\mathbf{x}} (A^T \nabla) \cdot \mathbf{v} d\boldsymbol{\xi} = \int_D J_{\mathbf{x}} (A^T \nabla q) \cdot \mathbf{v} d\boldsymbol{\xi}, \tag{6}$$

the strong form of (3) can be written

find $\mathbf{u} : D \times \Omega \to \mathbb{R}^d$ and $p : D \times \Omega \to \mathbb{R}$ such that P-almost everywhere in Ω there holds:

$$\begin{cases} -\nu \nabla \cdot [(J_{\mathbf{x}} A A^T \nabla) \mathbf{u}] + (\mathbf{u} \cdot J_{\mathbf{x}} A^T \nabla) \mathbf{u} + (J_{\mathbf{x}} A^T \nabla) p & = & \mathbf{f} J_{\mathbf{x}}, & \boldsymbol{\xi} \in D \\ (J_{\mathbf{x}} A^T \nabla) \cdot \mathbf{u} & = & 0, & \boldsymbol{\xi} \in D \\ \mathbf{u} & = & \mathbf{0}, & \boldsymbol{\xi} \in \partial D. \end{cases} \tag{7}$$

Notice that similarly to the formulation in [26], the continuity equation can be equivalently written $\nabla \cdot (J_{\mathbf{x}} A \mathbf{u})$ thanks to Piola's identity (see Appendix D).

Remark 3.2. *If homogeneous Neumann boundary conditions $\nu \frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{\mathbf{n}}} - \tilde{p} \tilde{\mathbf{n}} = \mathbf{0}$ are prescribed for Problem (1) on a part of the boundary ∂D_ω, typically at the outflow part of the boundary, the corresponding boundary conditions for the problem on the reference domain D read $\nu J_{\mathbf{x}} \nabla \mathbf{u} A A^T \mathbf{n} - p J_{\mathbf{x}} A^T \mathbf{n} = \mathbf{0}$. However, the problem might no longer be well-posed due to the loss of (uniform) coercivity of $a(\cdot, \cdot; \omega) + c(\cdot, \cdot, \cdot; \omega)$ or its counter part on D_ω. Indeed, we are not able to control the negative part of the boundary integral. Braack and al. proved in [8] the existence and uniqueness of a solution to the Navier-Stokes equations with small data and homogeneous Neumann conditions on a part of the boundary after introducing what they called a directed-do-nothing condition, adding a (boundary integral) term in the weak formulation of the problem.*

4 Well-posedness

The goal is now to show the well-posedness of Problem (1), under suitable conditions on the family of random mapping $(\mathbf{x}_\omega)_{\omega \in \Omega}$ and restriction on the input data. We will show that there exists a unique solution $(\mathbf{u}, p)$ to Problem (3), the weak solution of Problem (1) being then given by $(\tilde{\mathbf{u}}, \tilde{p}) = (\mathbf{u} \circ \boldsymbol{\xi}_\omega, p \circ \boldsymbol{\xi}_\omega)$.

For any $\omega \in \Omega$, we assume that $\mathbf{x}_\omega : D \to D_\omega$, with $D_\omega = \mathbf{x}_\omega(D)$, is a one-to-one mapping such that $\mathbf{x}_\omega \in [W^{1,\infty}(D)]^d$, $\boldsymbol{\xi}_\omega \in [W^{1,\infty}(D_\omega)]^d$ and D_ω is bounded with Lipschitz continuous boundary ∂D_ω. Since $\mathbf{x}_\omega$ is invertible, the determinant $J_{\mathbf{x}}$ of its Jacobian matrix A^{-1} does not vanish. Without loss of generality, we can assume that $J_{\mathbf{x}} > 0$, namely that the mapping is orientation-preserving. Moreover, we make the following assumption [13, 29] on the singular values σ_i of A^{-1}: there exist two constants $\sigma_{min}, \sigma_{max}$ such that for $i = 1, ..., d$

$$0 < \sigma_{min} \leq \sigma_i(A^{-1}(\boldsymbol{\xi}, \omega)) \leq \sigma_{max} < \infty \quad \text{a.e. in } D \text{ and a.s. in } \Omega. \tag{8}$$

Notice that the singular values of A are then bounded uniformly from below and above by σ_{max}^{-1} and σ_{min}^{-1}, respectively. Therefore, the random mapping $\mathbf{x}$ have finite moment of any order and with the above regularity assumption we have $\mathbf{x} \in L_P^\infty(\Omega; [W^{1,\infty}(D)]^d)$. Moreover, the following properties are immediate consequences of assumption (8).

Proposition 4.1. *Under assumption (8), we have a.e. in D and a.s. in Ω*

- $\sigma_{min}^d \leq \det(A^{-1}) \leq \sigma_{max}^d$,

- $\sigma_{max}^{-2} \leq \lambda_i(AA^T) \leq \sigma_{min}^{-2}$ for $i = 1, ..., d$.

Proof. Since the eigenvalues of $A^{-1}A^{-T}$ (and thus of the so-called (right) Cauchy-Green strain tensor $A^{-T}A^{-1}$) are the square of the singular values of A^{-1}, the first relation follows directly from (8) and the fact that

$$\det(A^{-1}) = \sqrt{\det(A^{-1}A^{-T})} = \sqrt{\Pi_{i=1}^d \lambda_i(A^{-1}A^{-T})} = \Pi_{i=1}^d \sigma_i(A^{-1}).$$

The second relation is just a consequence of $\lambda_i(AA^T) = \sigma_i(A)^2$. $\qquad\square$

The following proposition ensures that the spaces $L^2(D_w)$ and $L^2(D)$, respectively $\left[H_0^1(D_w)\right]^d$ and $\left[H_0^1(D)\right]^d$, are isomorphic.

Proposition 4.2. *Under assumption (8), for any $\tilde{g} \in L^2(D_\omega)$ and $\tilde{\mathbf{v}} \in \left[H^1(D_\omega)\right]^d$ we have a.s. in Ω*

$$\sigma_{min}^{\frac{d}{2}} \|g\|_{L^2(D)} \leq \|\tilde{g}\|_{L^2(D_\omega)} \leq \sigma_{max}^{\frac{d}{2}} \|g\|_{L^2(D)} \tag{9}$$

and

$$\frac{\sigma_{min}^{\frac{d}{2}}}{\sigma_{max}} \|\nabla \mathbf{v}\|_{L^2(D)} \leq \|\nabla_{\mathbf{x}} \tilde{\mathbf{v}}\|_{L^2(D_\omega)} \leq \frac{\sigma_{max}^{\frac{d}{2}}}{\sigma_{min}} \|\nabla \mathbf{v}\|_{L^2(D)} \tag{10}$$

with $g = \tilde{g} \circ \mathbf{x}_\omega$ and $\mathbf{v} = \tilde{\mathbf{v}} \circ \mathbf{x}_\omega$. The same relations hold true for any $g \in L^2(D)$ and $\mathbf{v} \in \left[H^1(D)\right]^d$ with $\tilde{g} = g \circ \boldsymbol{\xi}_\omega$ and $\tilde{\mathbf{v}} = \mathbf{v} \circ \boldsymbol{\xi}_\omega$.

Proof. Let $\tilde{g} \in L^2(D_\omega)$ and $\tilde{\mathbf{v}} \in \left[H^1(D_\omega)\right]^d$. The proof of (9) is immediate using the uniform bounds on $\det(A^{-1})$ given by Proposition 4.1. For (10), we use the fact that $\sigma_{min}^d \sigma_{max}^{-2}$ and $\sigma_{max}^d \sigma_{min}^{-2}$ are uniform bounds for the eigenvalues (or equivalently singular values) of the symmetric positive definite matrix $\det(A^{-1})AA^T$ and the relation

$$\|\nabla_{\mathbf{x}} \tilde{\mathbf{u}}\|_{L^2(D_\omega)}^2 = \int_D (\nabla \mathbf{u} A) : (\nabla \mathbf{u} A) \det(A^{-1}) d\boldsymbol{\xi} = \int_D \sum_{i=1}^d (\det(A^{-1})AA^T \nabla u_i) \cdot \nabla u_i d\boldsymbol{\xi}.$$

The proof of (9) and (10) for the case $g \in L^2(D)$ and $\mathbf{v} \in \left[H^1(D)\right]^d$ is similar using the relations $\sigma_{max}^{-d} \leq \det(A) \leq \sigma_{min}^{-d}$ and $\sigma_{max}^{-2}\sigma_{min}^2 \leq \lambda_i(\det(A)A^{-1}A^{-T}) \leq \sigma_{min}^{-d}\sigma_{max}^2$ a.e. in D and a.s. in Ω. $\qquad\square$

To show the well-posedness of Problem (3), the forms a, b and c defined in (4) have to satisfy (uniformly) some properties, which we verify in the following proposition.

Proposition 4.3. *For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and any $q \in L^2(D)$ we have a.s. in Ω*

- *a is continuous: $|a(\mathbf{u}, \mathbf{v}; \omega)| \leq \nu M \|\nabla \mathbf{u}\|_{L^2(D)} \|\nabla \mathbf{v}\|_{L^2(D)}$ with $M = \sigma_{min}^{-2}\sigma_{max}^d$,*

- *a is coercive: $a(\mathbf{v}, \mathbf{v}; \omega) \geq \nu\alpha \|\nabla \mathbf{v}\|_{L^2(D)}^2$ with $\alpha = \sigma_{max}^{-2}\sigma_{min}^d$,*

- *b is continuous: $|b(\mathbf{v}, q; \omega)| \leq \sigma_{max}^d \sigma_{min}^{-1} \|q\|_{L^2(D)} \|\nabla \mathbf{v}\|_{L^2(D)}$,*

- *c is continuous: $|c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \omega)| \leq \hat{C} \|\nabla \mathbf{u}\|_{L^2(D)} \|\nabla \mathbf{v}\|_{L^2(D)} \|\nabla \mathbf{w}\|_{L^2(D)}$ with $\hat{C} = C_I^2 \sigma_{max}^d \sigma_{min}^{-1}$,*

where $C_I = C_I(D)$ is the constant in $\|\mathbf{v}\|_{L^4(D)} \leq C_I \|\nabla \mathbf{v}\|_{L^2(D)}$, resulting from Sobolev embedding's theorem and Poincaré's inequality on D.

Proof. The proof is immediate from Proposition 4.1, Hölder's inequality and the Sobolev embedding theorem.

$\qquad\square$

Notice that we do not include the parameter ν in the constants α and M linked to the coercivity and continuity of a, respectively, because we will track its occurrence in the derivation of our *a posteriori* error estimates, the goal being to minimize the sensitivity of the effectivity index with respect to ν. We mention that b is also continuous on $[H^1(D)]^d$ with the same constant as in Proposition 4.3 up to a multiplication by a factor $\sqrt{d}$. Moreover, we assume that $b(\cdot,\cdot;\omega)$ satisfies uniformly the so-called (Brezzi [9]) inf-sup condition: there exists a constant $\beta > 0$ such that

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, q; \omega)}{\|q\|_{L^2(D)} \|\nabla \mathbf{v}\|_{L^2(D)}} \geq \beta \quad \text{a.s. in } \Omega. \tag{11}$$

Remark 4.4. *The inf-sup condition (11) can be easily shown under the assumption that the mapping* $\mathbf{x}_\omega$ *is in* $[W^{2,\infty}(D)]^d$, *proceeding similarly to [26]. Indeed, for any* $q \in Q$ *there exists* $\mathbf{z} \in V$ *such that* $\nabla \cdot \mathbf{z} = q$ *and* $\|\nabla \mathbf{z}\|_{L^2(D)} \leq C_1 \|q\|_{L^2(D)}$ *with a constant* C_1 *depending only on the reference domain* D, *see for instance [24]. Setting* $\mathbf{v} = -(J_{\mathbf{x}} A)^{-1} \mathbf{z}$ *we get*

$$b(\mathbf{v}, q; \omega) = \|q\|_{L^2(D)}^2 \geq \frac{1}{C_1} \|q\|_{L^2(D)} \|\nabla \mathbf{z}\|_{L^2(D)} \text{ and } \|\nabla \mathbf{v}\|_{L^2(D)} \leq C_2 \|(J_{\mathbf{x}} A)^{-1}\|_{W^{1,\infty}(D)} \|\nabla \mathbf{z}\|_{L^2(D)},$$

where C_2 *depends only on the Poincaré constant on* D. *From these two inequalities, we deduce that* $\frac{b(\mathbf{v}, q; \omega)}{\|\nabla \mathbf{v}\|_{L^2(D)}} \geq \beta \|q\|_{L^2(D)}$ *with* $\beta^{-1} = C_1 C_2 \|(J_{\mathbf{x}} A)^{-1}\|_{W^{1,\infty}(D)}$.

Let us introduce the subspace $\tilde{V}_{\mathrm{div},\omega} \subset \tilde{V}_\omega$ constituted of all (weakly) divergence-free functions of $\tilde{V}_\omega$, and its counterpart on D given by

$$V_{\mathrm{div},\omega} := \{\mathbf{v} \in V : \ b(\mathbf{v}, q; \omega) = 0 \quad \forall q \in Q, \text{ a.s. in } \Omega\}.$$

We can then formulate the (reduced, pointwise in ω) weak formulation of Problem (3): find $\mathbf{u}(\omega) \in V_{\mathrm{div},\omega}$ such that

$$a(\mathbf{u}, \mathbf{v}; \omega) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \omega) = F(\mathbf{v}; \omega) \quad \forall \mathbf{v} \in V_{\mathrm{div},\omega}, \text{ a.s. in } \Omega. \tag{12}$$

Proposition 4.5. *For* $\mathbf{u}(\omega) \in V_{\mathrm{div},\omega}$ *solution of (12), there exists a unique pressure* $p(\omega) \in Q$ *so that* $(\mathbf{u}, p)$ *is a solution of (3), a.s in* Ω.

Proof. Follows from the inf-sup condition (see [24, p.283]). $\qquad\square$

Therefore, to show the well-posedness of Problem (3), and thus of the original problem (2), it only remains to prove that the nonlinear problem (12) admits a unique solution. Recalling that F is defined in (4) with $\mathbf{f} = \tilde{\mathbf{f}} \circ \mathbf{x}_\omega$, the following proposition give a sufficient condition on the input data so that Problem (12) is well-posed.

Proposition 4.6. *If*

$$\frac{C_P C_I^2 \sigma_{max}^{\frac{3d}{2}+4}}{\nu^2 \sigma_{min}^{2d+1}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \leq \theta < 1 \quad \text{a.s. in } \Omega \tag{13}$$

for some $\theta \in [0, 1[$, *where* $C_P = C_P(D)$ *denotes the Poincaré constant on* D, *then Problem (12) has a unique solution. Moreover, its solution satisfies*

$$\|\nabla \mathbf{u}(\omega)\|_{L^2(D)} \leq \theta \frac{\nu \sigma_{min}^{d+1}}{C_I^2 \sigma_{max}^{d+2}} = \theta \frac{\nu \alpha}{\hat{C}} \quad \text{a.s. in } \Omega, \tag{14}$$

with α *and* $\hat{C}$ *defined in Proposition 4.3.*

Remark 4.7. *Notice that if condition (13) holds, then* $\frac{\hat{C}}{(\nu\alpha)^2} \|F(\cdot; \omega)\|_{V'_{\mathrm{div},\omega}} < 1$ *a.s. in* Ω, *where the norm on the dual space is defined in the usual way, which is nothing else but the standard small data assumption for uniqueness (see e.g. [24, 42, 20]). Indeed, we have*

$$\frac{\hat{C}}{(\nu\alpha)^2} \|F(\cdot; \omega)\|_{V'_{\mathrm{div},\omega}} = \frac{\hat{C}}{(\nu\alpha)^2} \sup_{\mathbf{v} \in V_{\mathrm{div},\omega}} \frac{|F(\mathbf{v}; \omega)|}{\|\nabla \mathbf{v}\|_{L^2(D)}} \leq \frac{C_P C_I^2 \sigma_{max}^{\frac{3d}{2}+4}}{\nu^2 \sigma_{min}^{2d+1}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \quad \text{a.s. in } \Omega,$$

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where for the last inequality we used the relation

$$|F(\mathbf{v};\omega)| \leq \sigma_{max}^{\frac{d}{2}} \|\mathbf{f} J_{\mathbf{x}}^{\frac{1}{2}}\|_{L^2(D)} \|\mathbf{v}\|_{L^2(D)} \leq C_P \sigma_{max}^{\frac{d}{2}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \|\nabla \mathbf{v}\|_{L^2(D)} \quad a.s. \ in \ \Omega.$$

Moreover, instead of (13), we could impose that

$$\frac{C_P C_I^2 \sigma_{max}^{2(d+2)}}{\nu^2 \sigma_{min}^{2d+1}} \|\mathbf{f}(\omega)\|_{L^2(D)} \leq \theta < 1 \quad a.s. \ in \ \Omega \tag{15}$$

since $\|\mathbf{f}(\omega)\|_{L^2(D)} \geq \sigma_{max}^{-\frac{d}{2}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)}$ *by Proposition 4.2, and thus (15) implies (13).*

The proof of Proposition 4.6, given in Appendix A for completeness, follows the same procedure as the one proposed in [40] for deterministic steady Navier-Stokes equations in a given domain and is based on a fixed point argument.

5   Specific form of the random mapping

We assume from now on that the random mapping $\mathbf{x}(\boldsymbol{\xi}, \omega)$ is parametrized by L mutually independent random variables and write $\mathbf{x}(\boldsymbol{\xi}, \omega) = \mathbf{x}(\boldsymbol{\xi}, Y_1(\omega), ..., Y_L(\omega))$ with a slight abuse of notation. This assumption with L finite, usually referred to as *finite dimensional noise assumption*, is necessary to make the problem feasible for numerical simulation. Such approximation of a random field can be achieved by several techniques, for instance using truncated Karhunen-Loève (see Remark 5.1) or Fourier expansions. More precisely, we assume that the mapping $\mathbf{x}_\omega$ from D to D_ω writes

$$\mathbf{x}_\omega(\boldsymbol{\xi}) = \boldsymbol{\varphi}_0(\boldsymbol{\xi}) + \varepsilon \sum_{j=1}^{L} \boldsymbol{\varphi}_j(\boldsymbol{\xi}) Y_j(\omega), \tag{16}$$

where the Y_j, $j = 1, ..., L$, are independent random variables with zero mean and unit variance, the deterministic functions $\boldsymbol{\varphi}_j : D \to \mathbb{R}^d$ are assumed to be smooth so that $\nabla\boldsymbol{\varphi}_0 \in \left[W^{1,\infty}(D)\right]^{d \times d}$ and $\nabla\boldsymbol{\varphi}_j \in [L^\infty(D)]^{d \times d}$ for $j = 1, ..., L$, and $\varepsilon \in [0, \varepsilon_{max}]$ is a parameter that controls the amount of randomness. We assume that the random variables Y_j, $j = 1, ..., L$, and the functions $\boldsymbol{\varphi}_j$, $j = 0, 1, ..., L$, are independent of ε. Without loss of generality, we can assume that $\boldsymbol{\varphi}_0$ is the identity mapping (see [29]), i.e.

$$\mathbf{x}_\omega(\boldsymbol{\xi}) = \boldsymbol{\xi} + \varepsilon \sum_{j=1}^{L} \boldsymbol{\varphi}_j(\boldsymbol{\xi}) Y_j(\omega). \tag{17}$$

The Jacobian matrix A^{-1} associated to $\mathbf{x}_\omega$ therefore reads

$$A^{-1}(\boldsymbol{\xi}, \omega) = I + \varepsilon A_1(\boldsymbol{\xi}, \omega) \quad \text{with} \quad A_1(\boldsymbol{\xi}, \omega) = \sum_{j=1}^{L} \nabla\boldsymbol{\varphi}_j(\boldsymbol{\xi}) Y_j(\omega),$$

where I denotes the $d \times d$ identity matrix and $\nabla\boldsymbol{\varphi}_j(\boldsymbol{\xi})$ is the Jacobian matrix of $\boldsymbol{\varphi}_j$ for $j = 1, ..., L$. Finally, we make the following additional assumptions to ensure that (8) is satisfied:

$$Y_j(\Omega) = [-\gamma_j, \gamma_j] =: \Gamma_j \text{ with } \gamma_j > 0, \ j = 1, ..., L, \tag{18}$$

and

$$\varepsilon_{max} < \frac{1}{\delta} \text{ with } \delta \text{ such that } \sum_{j=1}^{L} \gamma_j \|\nabla\boldsymbol{\varphi}_j(\boldsymbol{\xi})\|_2 \leq \delta \quad a.e. \ in \ D, \tag{19}$$

where $\|\cdot\|_2$ is the spectral norm. It is straightforward to show that under assumptions (18) and (19), then (8) is fullfield for any $\varepsilon \in [0, \varepsilon_{max}]$ with $\sigma_{min} = 1 - \varepsilon_{max}\delta$ and $\sigma_{max} = 1 + \varepsilon_{max}\delta$.

Remark 5.1. *A (truncated) Karhunen-Loève expansion of the random vector field $\mathbf{x}_\omega$ (see [29, 34, 35]) yields a characterization of $\mathbf{x}_\omega$ that can be recast into the form (16). In this case, the functions $\boldsymbol{\varphi}_j$, $j = 1, ..., L$, write $\boldsymbol{\varphi}_j = \sqrt{\lambda_j}\boldsymbol{\psi}_j$ with $\{\lambda_j, \boldsymbol{\psi}_j\}$ the eigenpairs of the (compact, self-adjoint) integral operator associated with the covariance kernel $V : D \times D \to \mathbb{R}^{d \times d}$ given by*

$$V(\boldsymbol{\xi}, \boldsymbol{\xi}') := \frac{1}{\varepsilon^2} \mathbb{E}\left[(\mathbf{x}_\omega(\boldsymbol{\xi}) - \boldsymbol{\varphi}_0(\boldsymbol{\xi}))(\mathbf{x}_\omega(\boldsymbol{\xi}') - \boldsymbol{\varphi}_0(\boldsymbol{\xi}'))^T\right].$$

We underline that in this work, we do not take into account the error made when the random mapping is approximated via a finite number of random variables. Therefore, we assume here that (16) is an exact representation of the random mapping introduced in Section 3.

Due to the Doob-Dynkin Lemma, the solutions $\mathbf{u}$ and p of (7) depend on the same random variables as $\mathbf{x}_\omega$. Defining the random vector $\mathbf{Y} = (Y_1, ..., Y_L)$, we can thus write $\mathbf{u}(\boldsymbol{\xi}, \omega) = \mathbf{u}(\boldsymbol{\xi}, \mathbf{Y}(\omega))$ and $p(\boldsymbol{\xi}, \omega) = p(\boldsymbol{\xi}, \mathbf{Y}(\omega))$. The complete probability space $(\Omega, \mathcal{F}, P)$ can be replaced by $(\Gamma, B(\Gamma), \rho(\mathbf{y})d\mathbf{y})$, where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_L$, $B(\Gamma)$ is the Borel σ-algebra on Γ and $\rho(\mathbf{y})d\mathbf{y}$ is the probability measure of the random vector $\mathbf{Y}$. Notice that since the random variables Y_j, $j = 1, ..., L$, are assumed independent, the joint density function ρ factorizes as $\rho(\mathbf{y}) = \Pi_{j=1}^{L}\rho_j(y_j)$ for all $\mathbf{y} = (y_1, ..., y_L) \in \Gamma$. Therefore, for any integrable function $\hat{g} : \Gamma \to \mathbb{R}$ on $(\Gamma, B(\Gamma), \rho(\mathbf{y})d\mathbf{y})$, the expectation of the random variable $g = g(\omega) = \hat{g}(\mathbf{Y}(\omega))$ is by definition given by

$$\mathbb{E}\left[g\right] = \int_\Omega g(\omega)dP(\omega) = \int_\Omega \hat{g}(\mathbf{Y}(\omega))dP(\omega) = \int_\Gamma \hat{g}(\mathbf{y})\rho(\mathbf{y})d\mathbf{y}.$$

With a little abuse of notation, we will not distinguish $\hat{g}$ and g in what follows. The problem (3) can then be rewritten into the following parametric form:

find $(\mathbf{u}(\mathbf{y}), p(\mathbf{y})) \in V \times Q$ such that

$$\left\{ \begin{array}{rcl} a(\mathbf{u}, \mathbf{v}; \mathbf{y}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \mathbf{y}) + b(\mathbf{v}, p; \mathbf{y}) & = & F(\mathbf{v}; \mathbf{y}) \\ b(\mathbf{u}, q; \mathbf{y}) & = & 0 \end{array} \right. \tag{20}$$

for all $(\mathbf{v}, q) \in V \times Q$ and ρ-a.e. in Γ, where the various forms are defined as in (4) with $A(\boldsymbol{\xi}, \omega)$, $A^{-1}(\boldsymbol{\xi}, \omega)$, $J_{\mathbf{x}}(\boldsymbol{\xi}, \omega)$ and $\mathbf{f}(\boldsymbol{\xi}, \omega)$ replaced by $A(\boldsymbol{\xi}, \mathbf{y})$, $A^{-1}(\boldsymbol{\xi}, \mathbf{y})$, $J_{\mathbf{x}}(\boldsymbol{\xi}, \mathbf{y})$ and $\mathbf{f}(\boldsymbol{\xi}, \mathbf{y})$, respectively. This problem is well-posed under the so-called *small data* assumption (13) with $\mathbf{f}(\omega)$ and a.s. in Ω replaced by $\mathbf{f}(\mathbf{y})$ and ρ-a.e. in Γ, respectively, the proof being essentially the same as the proof of Proposition 4.6. The random weak solution of Problem (7), i.e. the solution of (3), is then given by $(u(\mathbf{Y}(\omega)), p(\mathbf{Y}(\omega)))$ with $(\mathbf{u}, p)$ the parametric solution of (20).

Remark 5.2. *Notice that for any $\mathbf{y} \in \Gamma$, the partial derivative with respect to y_j of the solutions $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\mathbf{x}, \mathbf{y})$ and $\tilde{p} = \tilde{p}(\mathbf{x}, \mathbf{y})$ of the problem defined on $D_{\mathbf{y}}$ is given for $j = 1, ..., L$ by*

$$\frac{\partial \tilde{\mathbf{u}}}{\partial y_j} = \frac{\partial \mathbf{u}}{\partial y_j} \circ \boldsymbol{\xi}_{\mathbf{y}} + (\frac{\partial \boldsymbol{\xi}_{\mathbf{y}}}{\partial y_j} \cdot \nabla_{\boldsymbol{\xi}})\mathbf{u} \circ \boldsymbol{\xi}_{\mathbf{y}} \quad and \quad \frac{\partial \tilde{p}}{\partial y_j} = \frac{\partial p}{\partial y_j} \circ \boldsymbol{\xi}_{\mathbf{y}} + \frac{\partial \boldsymbol{\xi}_{\mathbf{y}}}{\partial y_j} \cdot (\nabla_{\boldsymbol{\xi}} p \circ \boldsymbol{\xi}_{\mathbf{y}}). \tag{21}$$

In other words, the (Eulerian) partial derivative with respect to y_j of $\tilde{\mathbf{u}}$ (resp. $\tilde{p}$) is equal to the material derivative with respect to y_j of $\mathbf{u} = \tilde{\mathbf{u}} \circ \mathbf{x}_{\mathbf{y}}$ (resp. $p = \tilde{p} \circ \mathbf{x}_{\mathbf{y}}$), transported back to $D_{\mathbf{y}}$. Moreover, we have the relation

$$(\frac{\partial \boldsymbol{\xi}_{\mathbf{y}}}{\partial y_j} \cdot \nabla_{\boldsymbol{\xi}})\mathbf{u} \circ \boldsymbol{\xi}_{\mathbf{y}} = -(\frac{\partial \mathbf{x}_{\mathbf{y}}}{\partial y_j} \circ \boldsymbol{\xi}_{\mathbf{y}} \cdot \nabla_{\mathbf{x}})\tilde{\mathbf{u}} \tag{22}$$

and using it in (21) we recognize an analogy with the Arbitrary Lagrangian Eulerian (ALE) formulation of PDEs on moving domains [19, 7], where the (Eulerian) partial time-derivative is replaced by the partial time-derivative on the ALE frame written in the Eulerian coordinate plus the convective-type term of the right-hand side of (22) in which the so-called domain velocity is involved.

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6 Error estimation

To simplify the presentation, we assume from now on that $d = 2$ and that $\tilde{\mathbf{f}} \in \left[H^2(\hat{D})\right]^2$. Since the forcing term on D is given by $\mathbf{f} = \tilde{\mathbf{f}} \circ \mathbf{x_Y}$ and we assumed $\boldsymbol{\varphi}_0$ to be the identity mapping, the regularity assumption on $\tilde{\mathbf{f}}$ allows us to write $\mathbf{f} = \mathbf{f}(\boldsymbol{\xi}, \omega) = \mathbf{f}(\boldsymbol{\xi}, \mathbf{Y}(\omega))$ as

$$\mathbf{f}(\mathbf{Y}) = \mathbf{f}_0 + \varepsilon \mathbf{f}_1(\mathbf{Y}) + \mathcal{O}(\varepsilon^2) \quad \text{with} \quad \mathbf{f}_0 := \tilde{\mathbf{f}}, \quad \mathbf{f}_1(\mathbf{Y}) := \sum_{j=1}^{L} \mathbf{F}_j Y_j, \quad \mathbf{F}_j := (\nabla_{\mathbf{x}}\tilde{\mathbf{f}})\boldsymbol{\varphi}_j. \tag{23}$$

The constant in the term of order ε^2 in (23) depends on the second derivatives of $\tilde{\mathbf{f}}$ and products $\boldsymbol{\varphi}_i \boldsymbol{\varphi}_j$, $i, j = 1, ..., L$. Moreover, since $d = 2$ we have

$$J_{\mathbf{x}} = \det(A^{-1}) = \det(I + \varepsilon A_1) = 1 + \varepsilon tr(A_1) + \varepsilon^2 \det(A_1) \quad \text{with} \quad \det(A_1) \le \delta^2 \tag{24}$$

using assumption (19) to bound $\det(A_1)$ and

$$A = I - \varepsilon A_1 + \sum_{k=2}^{\infty} (-1)^k \varepsilon^k A_1^k \quad \text{with} \quad \|\sum_{k=2}^{\infty} (-1)^k \varepsilon^k A_1^k\|_2 \le \frac{\varepsilon^2 \delta^2}{1 - \varepsilon \delta} \le \frac{\varepsilon^2 \delta^2}{\sigma_{min}}, \tag{25}$$

where we have used a von Neumann series to expand $A = (I + \varepsilon A_1)^{-1}$. We use a perturbation approach expanding the solution $(\mathbf{u}, p)$ on the reference domain D with respect to ε up to a certain order as

$$(\mathbf{u}(\boldsymbol{\xi}, \mathbf{Y}(\omega)), p(\boldsymbol{\xi}, \mathbf{Y}(\omega))) = (\mathbf{u}_0(\boldsymbol{\xi}), p_0(\boldsymbol{\xi})) + \varepsilon(\mathbf{u}_1(\boldsymbol{\xi}, \mathbf{Y}(\omega)), p_1(\boldsymbol{\xi}, \mathbf{Y}(\omega))) + ... \tag{26}$$

where $(\mathbf{u}_0, p_0)$ is the solution of the *standard* Navier-Stokes equations on D, i.e. it solves

find $\mathbf{u}_0 : D \to \mathbb{R}^d$ and $p_0 : D \to \mathbb{R}$ such that:

$$\begin{cases} -\nu \Delta \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla p_0 &= \mathbf{f}_0, \quad \boldsymbol{\xi} \in D \\ \nabla \cdot \mathbf{u}_0 &= 0, \quad \boldsymbol{\xi} \in D \\ \mathbf{u}_0 &= \mathbf{0}, \quad \boldsymbol{\xi} \in \partial D. \end{cases} \tag{27}$$

Writing $\mathbf{u}_1 = \sum_{j=1}^{L} \mathbf{U}_j Y_j$ and $p_1 = \sum_{j=1}^{L} P_j Y_j$, it can be shown that the couple $(\mathbf{u}_1, p_1)$ is obtained by solving the L (linear) problems

for $j = 1, ..., L$, find $\mathbf{U}_j : D \to \mathbb{R}^d$ and $P_j : D \to \mathbb{R}$ such that:

$$\begin{cases} -\nu \Delta \mathbf{U}_j + (\mathbf{u}_0 \cdot \nabla)\mathbf{U}_j + (\mathbf{U}_j \cdot \nabla)\mathbf{u}_0 + \nabla P_j &= g_j(\mathbf{u}_0, p_0), \quad \boldsymbol{\xi} \in D \\ \nabla \cdot \mathbf{U}_j &= h_j(\mathbf{u}_0), \quad \boldsymbol{\xi} \in D \\ \mathbf{U}_j &= \mathbf{0}, \quad \boldsymbol{\xi} \in \partial D, \end{cases} \tag{28}$$

where

$$\begin{aligned} g_j(\mathbf{u}_0, p_0) &= (tr(\nabla \boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j) + \nu \nabla \cdot [(\hat{B}_j \nabla)\mathbf{u}_0] - (\mathbf{u}_0 \cdot B_j \nabla)\mathbf{u}_0 - (B_j \nabla)p_0, \\ h_j(\mathbf{u}_0) &= -(B_j \nabla) \cdot \mathbf{u}_0 \end{aligned}$$

with

$$B_j := tr(\nabla \boldsymbol{\varphi}_j)I - \nabla \boldsymbol{\varphi}_j^T \quad \text{and} \quad \hat{B}_j := tr(\nabla \boldsymbol{\varphi}_j)I - (\nabla \boldsymbol{\varphi}_j + \nabla \boldsymbol{\varphi}_j^T). \tag{29}$$

Some details about the derivation of Problems (27) and (28) is given in Appendix B. In this paper, we approximate the solution of the deterministic problem (27) using the finite element method to obtain an approximation $(\mathbf{u}_{0,h}, p_{0,h})$ and we provide an *a posteriori* error estimate of $(\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h})$. For any $h > 0$, let $\mathcal{T}_h$ be a family of shape regular partitions (see [14]) of D into d-simplices K of diameter $h_K \le h$. Moreover, let (V_h, Q_h) with $V_h \subset V$ and $Q_h \subset Q$ be a pair of inf-sup stable finite element spaces, such as mini-elements $\mathbb{P}_1 bubble - \mathbb{P}_1$ (see [2] or [24, p.175] for a proof of stability of these spaces) or Taylor-Hood $\mathbb{P}_2 - \mathbb{P}_1$. We denote by $(\mathbf{u}_{0,h}, p_{0,h})$ the FE approximation of the (weak) solution $(\mathbf{u}_0, p_0)$ of Problem (27). Writing $\mathbf{y}_0 = \mathbb{E}[\mathbf{Y}] = \mathbf{0}$, it is obtained by solving

find $(\mathbf{u}_{0,h}, p_{0,h}) \in V_h \times Q_h$ such that

$$\begin{cases} a(\mathbf{u}_{0,h}, \mathbf{v}_h; \mathbf{y}_0) + c(\mathbf{u}_{0,h}; \mathbf{u}_{0,h}, \mathbf{v}_h; \mathbf{y}_0) + b(\mathbf{v}_h, p_{0,h}; \mathbf{y}_0) &= F(\mathbf{v}_h; \mathbf{y}_0) \\ b(\mathbf{u}_{0,h}, q_h; \mathbf{y}_0) &= 0 \end{cases} \tag{30}$$

for all $(\mathbf{v}_h, q_h) \in V_h \times Q_h$. The rest of this section is devoted to an *a posteriori* error analysis for the error $|||(\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h})|||$, where the norm $||| \cdot |||$ is defined for any $(\mathbf{v}, q) \in L_P^2(\Omega; V) \times L_P^2(\Omega; Q)$ by

$$|||\mathbf{v}, q||| := \left(\mathbb{E} \left[\nu \|\nabla \mathbf{v}\|_{L^2(D)}^2 + \frac{1}{\nu} \|q\|_{L^2(D)}^2 \right] \right)^{\frac{1}{2}}.$$

Remark 6.1. *Notice that we obtain the same results if we use the norm $\nu^2 \|\nabla \mathbf{v}\|^2 + \|q\|^2$ or $\|\nabla \mathbf{v}\|^2 + \frac{1}{\nu^2} \|q\|^2$ on $V \times Q$. This choice of scaling is guided by the dimension unit of ν, p and $\nabla \mathbf{u}$. This is moreover the natural scaling that arises when analysing the a priori estimates on the solution or when performing the a posteriori error analysis (see Appendix C for more details).*

As we will see in the following, the error estimate consists of two parts, namely a part due to the finite element approximation (in h) and another one due to the uncertainty (in ε). Let us define for any $\mathbf{y} \in \Gamma$ the residual $R(\cdot; \mathbf{y}) : V \times Q \to \mathbb{R}$, which depends on $(\mathbf{u}_{0,h}, p_{0,h})$, by

$$R((\mathbf{v}, q); \mathbf{y}) := F(\mathbf{v}; \mathbf{y}) - a(\mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) - c(\mathbf{u}_{0,h}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) - b(\mathbf{v}, p_{0,h}; \mathbf{y}) - b(\mathbf{u}_{0,h}, q; \mathbf{y}).$$

We distinguish two parts in the residual $R((\mathbf{v}, q); \mathbf{y}) = R_1(\mathbf{v}; \mathbf{y}) + R_2(q, \mathbf{y})$, depending only on $\mathbf{v}$ and q respectively, with

$$\begin{aligned} R_1(\mathbf{v}; \mathbf{y}) &:= F(\mathbf{v}; \mathbf{y}) - a(\mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) - b(\mathbf{v}, p_{0,h}; \mathbf{y}) - c(\mathbf{u}_{0,h}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) \\ R_2(q; \mathbf{y}) &:= -b(\mathbf{u}_{0,h}, q; \mathbf{y}). \end{aligned}$$

The first step in the residual-based error estimation consists in linking the error to the residual. The norm of the residual is then bounded by a computable quantity (possibly up to a multiplicative constant).

Proposition 6.2. *If (8), (11) and (13) are satisfied and h is small enough, then there exists a constant $C > 0$ depending only on θ, σ_{min}, σ_{max} and β such that a.s. in Ω*

$$\nu \|\nabla(\mathbf{u}(\mathbf{Y}) - \mathbf{u}_{0,h})\|_{L^2(D)}^2 + \frac{1}{\nu} \|p(\mathbf{Y}) - p_{0,h}\|_{L^2(D)}^2 \leq C \left(\frac{1}{\nu} \|R_1(\cdot, \mathbf{Y})\|_{V'}^2 + \nu \|R_2(\cdot, \mathbf{Y})\|_{Q'}^2 \right). \tag{31}$$

We mention that the closer θ to 1, the larger C in Proposition 6.2, see relation (40). Similarly, the closer σ_{min} to 0, the larger C will be. The proof of this proposition is inspired by what is done in [1] for the deterministic steady Navier-Stokes equations. In order to simplify the notation, we will write $\| \cdot \|$ instead of $\| \cdot \|_{L^2(D)}$ in the sequel.

Proof. In what follows, all equations depending on $\mathbf{y}$ hold ρ-a.e. in Γ, without specifically mentioning it. Moreover, the dependence of the functions with respect to $\mathbf{y} \in \Gamma$ will not necessarily be indicated. Let $\mathbf{e}(\mathbf{y}) := \mathbf{u}(\mathbf{y}) - \mathbf{u}_{0,h}$ and $E(\mathbf{y}) := p(\mathbf{y}) - p_{0,h}$. Then (20) yields

$$a(\mathbf{e}, \mathbf{v}; \mathbf{y}) + b(\mathbf{v}, E; \mathbf{y}) + b(\mathbf{e}, q; \mathbf{y}) + D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) = R((\mathbf{v}, q); \mathbf{y}) \tag{32}$$

where

$$D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) := c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \mathbf{y}) - c(\mathbf{u}_{0,h}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}).$$

We can show that

$$D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) \leq (2\theta\nu\alpha + \hat{C}\|\nabla \mathbf{e}_0\|)\|\nabla \mathbf{e}\|\|\nabla \mathbf{v}\| \tag{33}$$

and

$$D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{u} - \mathbf{u}_{0,h}; \mathbf{y}) \leq (\theta\nu\alpha + \hat{C}\|\nabla \mathbf{e}_0\|)\|\nabla \mathbf{e}\|^2 \tag{34}$$

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where $\mathbf{e}_0 := \mathbf{u}_0 - \mathbf{u}_{0,h}$ and M, α and $\hat{C}$ are defined in Proposition 4.3. Indeed, for any $\mathbf{v} \in V$ we have

$$
\begin{aligned}
D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) &= c(\mathbf{u}, \mathbf{u} - \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) + c(\mathbf{u} - \mathbf{u}_{0,h}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) \\
&\leq \hat{C} \left(\|\nabla \mathbf{u}\| + \|\nabla \mathbf{u}_0\| + \|\nabla \mathbf{e}_0\| \right) \|\nabla \mathbf{e}\| \|\nabla \mathbf{v}\| \\
&\leq \hat{C} \left(2\theta \frac{\alpha \nu}{\hat{C}} + \|\nabla \mathbf{e}_0\| \right) \|\nabla \mathbf{e}\| \|\nabla \mathbf{v}\|
\end{aligned}
$$

thanks to (14), which proves relation (33). Relation (34) is proved analogously using the fact that $c(\mathbf{u}, \mathbf{v}, \mathbf{v}; \mathbf{y}) = 0$ for any $\mathbf{v} \in V$. The rest of the proof consists of two steps, first the derivation of a bound on $\|E\|$ and then a bound on $\|\nabla \mathbf{e}\|$.

Using the inf-sup condition (11) for b, the bound (33) on D, the continuity of a and the relation (32) with $q = 0$, we have

$$
\begin{aligned}
\|E\| &\leq \frac{1}{\beta} \sup_{\mathbf{v} \in V} \frac{|b(\mathbf{v}, p - p_{0,h}; \mathbf{y})|}{\|\nabla \mathbf{v}\|} = \frac{1}{\beta} \sup_{\mathbf{v} \in V} \frac{|R_1(\mathbf{v}; \mathbf{y}) - a(\mathbf{u} - \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y}) - D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{v}; \mathbf{y})|}{\|\nabla \mathbf{v}\|} \\
&\leq \frac{1}{\beta} \left[\|R_1(\cdot; \mathbf{y})\|_{V'} + (\nu M + 2\nu \alpha + \hat{C} \|\nabla \mathbf{e}_0\|) \|\nabla \mathbf{e}\| \right] .
\end{aligned} \tag{35}
$$

Therefore, using the relation $(a + b)^2 \leq 2(a^2 + b^2)$ we obtain

$$
\frac{1}{\nu} \|E\|^2 \leq \frac{2}{\beta^2 \nu} \|R_1(\cdot; \mathbf{y})\|_{V'}^2 + \frac{2(M + 2\alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\|)^2}{\beta^2} \nu \|\nabla \mathbf{e}\|^2 . \tag{36}
$$

We now give a bound on the error $\|\nabla \mathbf{e}\|$ for the velocity. Using the inequalities (34) and (35), the coercivity of the bilinear form a, Young's inequality several times and taking $\mathbf{v} = \mathbf{e}$ and $q = -E$ in (32), we get

$$
\begin{aligned}
\nu \alpha \|\nabla \mathbf{e}\|^2 &\leq a(\mathbf{e}, \mathbf{e}; \mathbf{y}) = R_1(\mathbf{e}; \mathbf{y}) - R_2(E; \mathbf{y}) - D(\mathbf{u}, \mathbf{u}_{0,h}, \mathbf{e}) \\
&\leq \|R_1(\cdot; \mathbf{y})\|_{V'} \|\nabla \mathbf{e}\| + \|R_2(\cdot; \mathbf{y})\|_{Q'} \|E\| + (\theta \nu \alpha + \hat{C} \|\nabla \mathbf{e}_0\|) \|\nabla \mathbf{e}\|^2 \\
&\leq \frac{1}{2\gamma_1 \nu} \|R_1(\cdot; \mathbf{y})\|_{V'}^2 + \frac{\nu}{2\beta^2 \gamma_2} \|R_2(\cdot; \mathbf{y})\|_{Q'}^2 + \frac{1}{\beta} \|R_1(\cdot; \mathbf{y})\|_{V'} \|R_2(\cdot; \mathbf{y})\|_{Q'} \\
&\quad + \left(\frac{\gamma_1}{2} + \frac{\gamma_2 (M + 2\alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\|)^2}{2} + \theta \alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\| \right) \nu \|\nabla \mathbf{e}\|^2 \\
&\leq \frac{c_1}{\nu} \|R_1(\cdot; \mathbf{y})\|_{V'}^2 + c_2 \nu \|R_2(\cdot; \mathbf{y})\|_{Q'}^2 \\
&\quad + \left(\frac{\gamma_1}{2} + \frac{\gamma_2 (M + 2\alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\|)^2}{2} + \theta \alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\| \right) \nu \|\nabla \mathbf{e}\|^2 ,
\end{aligned} \tag{37}
$$

with

$$
c_1 = \frac{1}{2\gamma_1} + \frac{1}{2} \quad \text{and} \quad c_2 = \frac{1}{2\gamma_2 \beta^2} + \frac{1}{2\beta^2} .
$$

Recalling that $\theta \in [0, 1[$ and using the convergence of $\mathbf{u}_{0,h}$ to $\mathbf{u}_0$ as h tends to 0, we can choose h, γ_1 and γ_2 small enough so that

$$
\frac{\gamma_1}{2} + \frac{\gamma_2 (M + 2\alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\|)^2}{2} + \theta \alpha + \frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\| \leq \frac{1 + \theta}{2} \alpha . \tag{38}
$$

For instance, we can choose h small enough so that

$$
\frac{\hat{C}}{\nu} \|\nabla \mathbf{e}_0\| \leq \frac{1 - \theta}{6} \alpha \tag{39}
$$

and take

$$
\gamma_1 = \frac{1 - \theta}{3} \alpha \quad \text{and} \quad \gamma_2 = \frac{1 - \theta}{3(M + 2\alpha + \frac{1 - \theta}{6} \alpha)^2} \alpha
$$

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which depends only on θ, σ_{min} and σ_{max}. Therefore, the last term of the right-hand side of inequality (37) can be moved to the left and we get

$$\nu\|\nabla\mathbf{e}\|^2 \leq \frac{2}{(1-\theta)\alpha} \left[\frac{c_1}{\nu}\|R_1(\cdot;\mathbf{y})\|^2_{V'} + c_2\nu\|R_2(\cdot;\mathbf{y})\|^2_{Q'} \right]. \tag{40}$$

Using this bound in (36) together with (39) we get

$$\frac{1}{\nu}\|E\|^2 \leq \left(\frac{2}{\beta^2} + \frac{4c_1}{3\gamma_2\beta^2} \right) \frac{1}{\nu}\|R_1(\cdot;\mathbf{y})\|^2_{V'} + \frac{4c_2}{3\gamma_2\beta^2}\nu\|R_2(\cdot;\mathbf{y})\|^2_{Q'}.$$

Replacing finally $\mathbf{y}$ by $\mathbf{Y}(\omega)$, the combination of last two inequalities allows us to conclude the proof since c_1 and c_2 depend only on β as well as γ_1 and γ_2, which in turn depend only on θ, σ_{min} et σ_{max}. $\qquad\square$

From Proposition 6.2, we deduce the following bound on the error in the $|||\cdot|||$-norm

$$|||\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h}||| \leq \sqrt{C} \left(\frac{1}{\nu}\mathbb{E}\left[\|R_1\|^2_{V'}\right] + \nu\mathbb{E}\left[\|R_2\|^2_{Q'}\right] \right)^{\frac{1}{2}} \tag{41}$$

by simply taking first the expected value and then the square root on both sides of inequality (31). The goal is now to derive a computable (deterministic) error estimator by estimating the residuals that appear in the right-hand side of (41). We use a standard procedure to estimate the part due to the space discretization and proceed in two different ways for the part due to the uncertainty, more precisely the truncation in (26). The first one is straightfoward and does not require the resolution of additional problems. However, it uses the triangle inequality as well as the Poincaré inequality (on the fixed domain D) to bound the terms due to the external forces and the convection. Even though the Poincaré constant is a uniform bound, the loss when using Poincaré's inequality can be different depending of the problem, affecting the sharpness of the error estimate from case to case. The second procedure consists in computing the dual norm of some functional, and therefore requires the resolution of additional (linear) problems. However, it has the advantage of requiring the use of Cauchy-Schwarz's inequality only and thus does not suffer from the drawback mentioned above.

6.1 First error estimate

Let $[\cdot]_{\mathbf{n}_e}$ denotes the jump across an edge $e \in \mathcal{T}_h$ in the direction $\mathbf{n}_e$ defined by

$$[\mathbf{g}]_{\mathbf{n}_e}(\boldsymbol{\xi}) := \lim_{t\to 0} \left[\mathbf{g}(\boldsymbol{\xi} + t\mathbf{n}_e) - \mathbf{g}(\boldsymbol{\xi} - t\mathbf{n}_e) \right],$$

where $\mathbf{n}_e$ is a unit normal vector to e of arbitrary (but fixed) direction for internal edges and the outward unit vector for boundary edges. Since we impose homogeneous Dirichlet conditions at the boundary, we set the jump to zero for boundary edges. We now have all the ingredients necessary to derive our first error estimate.

Proposition 6.3. *Let $(\mathbf{u}, p)$ be the (weak) solution of Problem (7) and let $(\mathbf{u}_{0,h}, p_{0,h})$ be the solution of Problem (30). If the assumptions of Proposition 6.2 are satisfied, then there exist positive constants C_1, C_2 and C_3 independent of h and ε such that*

$$|||\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h}||| \leq \sqrt{2C} \left(C_1\eta_h^2 + C_2\eta_\varepsilon^2 \right)^{\frac{1}{2}} + \sqrt{C}C_3\varepsilon^2 \quad with \quad \eta_h^2 = \sum_{K\in\mathcal{T}_h} \eta_K^2 \ and \ \eta_\varepsilon^2 = \sum_{j=1}^{L} \eta_j^2, \tag{42}$$

where C is the constant in Proposition 6.2 and

$$\eta_K^2 := \frac{1}{\nu}\eta_{K,1}^2 + \nu\eta_{K,2}^2 \quad and \quad \eta_j^2 := \frac{1}{\nu}\eta_{j,1}^2 + \nu\eta_{j,2}^2 \tag{43}$$

with

$$\eta_{K,1}^2 \ := \ h_K^2 \|\mathbf{f}_0 + \nu\Delta\mathbf{u}_{0,h} - (\mathbf{u}_{0,h}\cdot\nabla)\mathbf{u}_{0,h} - \nabla p_{0,h}\|_{L^2(K)}^2 + \sum_{e\subset K} h_e \|\tfrac{1}{2}\left[\nu(\nabla\mathbf{u}_{0,h})\mathbf{n}_e - p_{0,h}\mathbf{n}_e\right]_{\mathbf{n}_e}\|_{L^2(e)}^2$$

$$\eta_{K,2}^2 \ := \ \|\nabla\cdot\mathbf{u}_{0,h}\|_{L^2(K)}^2$$

$$\eta_{j,1}^2 \ := \ \varepsilon^2\left(\|tr(\nabla\boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j\|^2 + \nu^2\|(\hat{B}_j\nabla)\mathbf{u}_{0,h}\|^2 + \|p_{0,h}B_j\|^2 + \|(\mathbf{u}_{0,h}\cdot B_j\nabla)\mathbf{u}_{0,h}\|^2\right)$$

$$\eta_{j,2}^2 \ := \ \varepsilon^2\|(B_j\nabla)\cdot\mathbf{u}_{0,h}\|^2, \tag{44}$$

B_j and $\hat{B}_j$ being defined in (29), $\mathbf{f}_0$ and $\mathbf{F}_j$ given in (23). Moreover, C_1 depends only on the mesh aspect ratio while C_2 depends only on the Poincaré constant on D.

Remark 6.4. *Notice that if $\varepsilon_{\max}\delta$ is close to 1, or in other words σ_{min} is close to 0, then the constant C_3 in Proposition 6.3 might be large, see (25). Therefore, in order for the last term of (42) to be negligible, we need to assume small perturbations of the domain, for instance by imposing $\varepsilon_{max}\leq\frac{1}{2\delta}$.*

Proof. Similarly to the proof of Proposition 6.2, it is understood that all equations depending on $\mathbf{y}$ hold ρ-a.e. in Γ unless explicitly stated. Thanks to (41), we only need to bound the expectation of $\frac{1}{\nu}\|R_1(\cdot;\mathbf{Y})\|_{V'}^2$ and $\nu\|R_2(\cdot;\mathbf{Y})\|_{Q'}^2$, that is

$$\int_\Gamma \frac{1}{\nu}\|R_1(\cdot;\mathbf{y})\|_{V'}^2\,\rho(\mathbf{y})d\mathbf{y} \quad\text{and}\quad \int_\Gamma \nu\|R_2(\cdot;\mathbf{y})\|_{Q'}^2\,\rho(\mathbf{y})d\mathbf{y},$$

by computable quantities. We decompose each term R_1 and R_2 into two parts which control the FE error and the error due to truncation in the expansion (26), respectively. For $\mathbf{y}_0 = \mathbb{E}[\mathbf{Y}] = \mathbf{0}$ and for all $\mathbf{y}\in\Gamma$, $\mathbf{v}\in V$ and $q\in Q$ we write

$$R_1(\mathbf{v};\mathbf{y}) = R_1(\mathbf{v};\mathbf{y}_0) + [R_1(\mathbf{v};\mathbf{y}) - R_1(\mathbf{v};\mathbf{y}_0)]$$

and

$$R_2(q;\mathbf{y}) = R_2(q;\mathbf{y}_0) + [R_2(q;\mathbf{y}) - R_2(q;\mathbf{y}_0)].$$

Using standard procedure (Galerkin orthogonality, Clément interpolation [16]), see for instance [43], and taking the contribution of the constant ν into account, the deterministic quantities can be bounded by

$$\frac{1}{\nu}\|R_1(\cdot;\mathbf{y}_0)\|_{V'}^2 + \nu\|R_2(\cdot;\mathbf{y}_0)\|_{Q'}^2 \leq C_1\sum_{K\in\mathcal{T}_h}\eta_K^2$$

where C_1 depends only on the *Clément interpolation constant* and the regularity of the mesh and the local error estimator η_K is defined in (43). We now bound the terms due to the uncertainty. We have

$$R_1(\mathbf{v};\mathbf{y}) - R_1(\mathbf{v};\mathbf{y}_0) = \text{II}_1 + \text{II}_2 + \text{II}_3 + \text{II}_4 \quad\text{and}\quad R_2(q;\mathbf{y}) - R_2(q;\mathbf{y}_0) = \text{II}_5$$

with

$$\begin{aligned}
\text{II}_1 &= F(\mathbf{v};\mathbf{y}) - F(\mathbf{v};\mathbf{y}_0) \leq C_P\|J_{\mathbf{x}}\mathbf{f} - \mathbf{f}_0\|\|\nabla\mathbf{v}\| \\
\text{II}_2 &= a(\mathbf{u}_{0,h},\mathbf{v};\mathbf{y}_0) - a(\mathbf{u}_{0,h},\mathbf{v};\mathbf{y}) \leq \nu\|[(J_{\mathbf{x}}AA^T - I)\nabla]\mathbf{u}_{0,h}\|\|\nabla\mathbf{v}\| \\
\text{II}_3 &= b(\mathbf{v},p_{0,h};\mathbf{y}_0) - b(\mathbf{v},p_{0,h};\mathbf{y}) \leq \|(J_{\mathbf{x}}A^T - I)p_{0,h}\|\|\nabla\mathbf{v}\| \\
\text{II}_4 &= c(\mathbf{u}_{0,h},\mathbf{u}_{0,h},\mathbf{v};\mathbf{y}_0) - c(\mathbf{u}_{0,h},\mathbf{u}_{0,h},\mathbf{v};\mathbf{y}) \leq C_P\|[\mathbf{u}_{0,h}\cdot(J_{\mathbf{x}}A^T - I)\nabla]\mathbf{u}_{0,h}\|\|\nabla\mathbf{v}\| \\
\text{II}_5 &= b(\mathbf{u}_{0,h},q;\mathbf{y}_0) - b(\mathbf{u}_{0,h},q;\mathbf{y}) \leq \|[(J_{\mathbf{x}}A^T - I)\nabla]\cdot\mathbf{u}_{0,h}\|\|q\|.
\end{aligned}$$

The bound for each term is straightforward, except the one for the term II_3 which can be obtained by writing it in component form. Therefore, we obtain

$$\frac{1}{\nu}\|R_1(\cdot;\mathbf{y})\|_{V'}^2 + \nu\|R_2(\cdot;\mathbf{y})\|_{Q'}^2 \leq C_1\eta_h^2 + C_2\kappa_\varepsilon(\mathbf{y})^2,$$

where C_2 is a (deterministic) constant that depends only on C_P and

$$\kappa_\varepsilon^2 \quad := \quad \frac{1}{\nu}\|J_\mathbf{x}\mathbf{f} - \mathbf{f}_0\|^2 + \nu\|[(J_\mathbf{x}AA^T - I)\nabla]\mathbf{u}_{0,h}\|^2 + \frac{1}{\nu}\|(J_\mathbf{x}A^T - I)p_{0,h}\|^2$$
$$+ \frac{1}{\nu}\|[\mathbf{u}_{0,h}\cdot(J_\mathbf{x}A^T - I)\nabla]\mathbf{u}_{0,h}\|^2 + \nu\|[(J_\mathbf{x}A^T - I)\nabla]\cdot\mathbf{u}_{0,h}\|^2.$$

Since the independent random variables $\{Y_j\}$ are assumed to be of zero mean and unite variance, we have $\mathbb{E}[Y_j] = 0$ and $\mathbb{E}[Y_iY_j] = \delta_{ij}$ for $i, j = 1, ..., L$ and thus, using Young's inequality and the relations (23), (24) and (25), among others, we easily get

$$\mathbb{E}\left[\|J_\mathbf{x}\mathbf{f} - \mathbf{f}_0\|^2\right] \quad = \quad \varepsilon^2\sum_{j=1}^{L}\|tr(\nabla\boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j\|^2 + \mathcal{O}(\varepsilon^3)$$

$$\mathbb{E}\left[\|[(J_\mathbf{x}AA^T - I)\nabla]\mathbf{u}_{0,h}\|^2\right] \quad = \quad \varepsilon^2\sum_{j=1}^{L}\|(\hat{B}_j\nabla)\mathbf{u}_{0,h}\|^2 + \mathcal{O}(\varepsilon^3)$$

$$\mathbb{E}\left[\|(J_\mathbf{x}A^T - I)p_{0,h}\|^2\right] \quad = \quad \varepsilon^2\sum_{j=1}^{L}\|p_{0,h}B_j\|^2 + \mathcal{O}(\varepsilon^3)$$

$$\mathbb{E}\left[\|[\mathbf{u}_{0,h}\cdot(J_\mathbf{x}A^T - I)\nabla]\mathbf{u}_{0,h}\|^2\right] \quad = \quad \varepsilon^2\sum_{j=1}^{L}\|(\mathbf{u}_{0,h}\cdot B_j\nabla)\mathbf{u}_{0,h}\|^2 + \mathcal{O}(\varepsilon^3)$$

$$\mathbb{E}\left[\|(J_\mathbf{x}A^T - I)\nabla\cdot\mathbf{u}_{0,h}\|^2\right] \quad = \quad \varepsilon^2\sum_{j=1}^{L}\|(B_j\nabla)\cdot\mathbf{u}_{0,h}\|^2 + \mathcal{O}(\varepsilon^3)$$

with B_j and $\hat{B}_j$ defined in (29). Therefore, for some constant $c_3 > 0$ independent of ε and h we get

$$\frac{1}{\nu}\mathbb{E}\left[\|R_1\|_{V'}^2\right] + \nu\mathbb{E}\left[\|R_2\|_{Q'}^2\right] \le C_1\sum_{K\in\mathcal{T}_h}\eta_K^2 + C_2\sum_{j=1}^{L}\eta_j^2 + c_3\varepsilon^3, \tag{45}$$

where η_j is defined in (43). To conclude the proof, it only remains to take the square root on both sides of inequality (45). Indeed, using the notation η_h and η_ε introduced in (42), we have

$$\left(\frac{1}{\nu}\mathbb{E}\left[\|R_1\|_{V'}^2\right] + \nu\mathbb{E}\left[\|R_2\|_{Q'}^2\right]\right)^{\frac{1}{2}} \le \left(C_1\eta_h^2 + C_2\eta_\varepsilon^2 + c_3\varepsilon^3\right)^{\frac{1}{2}} \le \sqrt{C_1}\eta_h + \left(C_2\eta_\varepsilon^2 + c_3\varepsilon^3\right)^{\frac{1}{2}}$$

thanks to the inequality $\sqrt{a^2 + b^2} \le a + b$ for any $a, b \ge 0$. Moreover, since $\eta_\varepsilon = \mathcal{O}(\varepsilon)$ we get for some constant $C_3 > 0$ independent of ε and h

$$\left(C_2\eta_\varepsilon^2 + c_3\varepsilon^3\right)^{\frac{1}{2}} = \sqrt{C_2}\eta_\varepsilon\left(1 + \frac{c_3\varepsilon^3}{C_2\eta_\varepsilon^2}\right)^{\frac{1}{2}} = \sqrt{C_2}\eta_\varepsilon\left(1 + \frac{1}{2}\frac{c_3\varepsilon^3}{C_2\eta_\varepsilon^2} - \frac{1}{8}\left(\frac{c_3\varepsilon^3}{C_2\eta_\varepsilon^2}\right)^2 + ...\right) \le \sqrt{C_2}\eta_\varepsilon + C_3\varepsilon^2.$$

Finally, using the inequality $a + b \le \sqrt{2}\left(a^2 + b^2\right)^{\frac{1}{2}}$ we obtain

$$\left(\frac{1}{\nu}\mathbb{E}\left[\|R_1\|_{V'}^2\right] + \nu\mathbb{E}\left[\|R_2\|_{Q'}^2\right]\right)^{\frac{1}{2}} \le \sqrt{C_1}\eta_h + \sqrt{C_2}\eta_\varepsilon + C_3\varepsilon^2 \le \sqrt{2}\left(C_1\eta_h^2 + C_2\eta_\varepsilon^2\right)^{\frac{1}{2}} + C_3\varepsilon^2,$$

which yields (42) thanks to (41).

$\square$

6.2 Second error estimate

As mentioned above, the use of the triangle inequality to bound each term linked to R_1 separately, plus the Poincaré inequality for some of them, in the derivation of the error estimate controlling the

randomness of the problem can affect the sharpness of the error estimate. However, it has the advantage to require the resolution of only one (nonlinear) problem, namely the problem for $(\mathbf{u}_{0,h}, p_{0,h})$. We propose in this section a second error estimate for which the use of these inequalities is not required. It is obtained by computing, approximately, the dual norm of the residual $R_1(\mathbf{v}; \mathbf{y}) - R_1(\mathbf{v}; \mathbf{y}_0)$. Similarly to the error estimate of Proposition 6.3, the terms of higher order are neglected.

Proposition 6.5. *Under the assumptions of Proposition 6.3, there exist constants C_1, C_3 and C_4 independent of h and ε and $s \in (0,1]$ such that*

$$|||\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h}||| \leq \sqrt{2C}\left(C_1\eta_h^2 + \hat{\eta}_\varepsilon^2\right)^{\frac{1}{2}} + \sqrt{C}(C_3\varepsilon^2 + C_4 h^s \varepsilon) \quad with \quad \hat{\eta}_\varepsilon^2 = \sum_{j=1}^{L} \hat{\eta}_j^2, \qquad (46)$$

where η_h is as in (42) and

$$\hat{\eta}_j^2 := \frac{1}{\nu}\hat{\eta}_{j,1}^2 + \nu\eta_{j,2}^2$$

with $\eta_{j,2}$ given in (44) and $\hat{\eta}_{j,1}^2 := \varepsilon^2\|\nabla\mathbf{w}_{j,h}\|_{L^2(D)}^2$ for $j = 1,...,L$, and $\mathbf{w}_{j,h} \in V_h$ is the solution of

$$\int_D \nabla\mathbf{w}_{j,h} : \nabla\mathbf{v}_h d\boldsymbol{\xi} = \int_D (tr(\nabla\boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j) \cdot \mathbf{v}_h d\boldsymbol{\xi} - \nu\int_D (\hat{B}_j\nabla)\mathbf{u}_{0,h} : \nabla\mathbf{v}_h d\boldsymbol{\xi} + \int_D p_{0,h}(B_j\nabla)\cdot\mathbf{v}_h d\boldsymbol{\xi}$$
$$- \int_D [(\mathbf{u}_{0,h}\cdot B_j\nabla)\mathbf{u}_{0,h}]\cdot\mathbf{v}_h d\boldsymbol{\xi} \qquad (47)$$

for all $\mathbf{v}_h \in V_h$. Moreover, the constants C_1 and C_4 depend only on the mesh aspect ratio.

Notice that contrary to the error estimate of Proposition 6.3, there is no *internal* constant multiplying $\hat{\eta}_\varepsilon$ in (46), the constant $C_2 = C_2(C_P)$ appearing in (42) being indeed no longer present.

Proof. The proof is similar to the one of Proposition 6.3. The only difference is the estimation of the term $r(\mathbf{v}; \mathbf{y}) := R_1(\mathbf{v}; \mathbf{y}) - R_1(\mathbf{v}; \mathbf{y}_0)$ in the V' norm. We have $\|r(\mathbf{v}; \mathbf{y})\|_{V'} = \|\nabla\mathbf{w}(\mathbf{y})\|_{L^2(D)}$, where $\mathbf{w}$ denote the Riesz representant of r, i.e. $\mathbf{w}(\mathbf{y}) \in V$ is such that $\int_D \nabla\mathbf{w}(\mathbf{y}) : \nabla\mathbf{v} = r(\mathbf{v}; \mathbf{y})$ for all $\mathbf{v} \in V$ and ρ-a.e. in Γ. If we keep only the terms of order ε and use the properties of the random variables Y_j, $j = 1, ..., L$, taking the expected value of $\|r(\cdot; \mathbf{Y})\|_{V'}^2$ we get

$$\mathbb{E}\left[\|r\|_{V'}^2\right] \leq \varepsilon^2 \sum_{j=1}^{L} \|\nabla\mathbf{w}_j\|_{L^2(D)}^2 + \mathcal{O}(\varepsilon^3)$$

where $\mathbf{w}_j$ is the solution of

$$\int_D \nabla\mathbf{w}_j : \nabla\mathbf{v} d\boldsymbol{\xi} = \int_D (tr(\nabla\boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j) \cdot \mathbf{v} d\boldsymbol{\xi} - \nu\int_D (\hat{B}_j\nabla)\mathbf{u}_{0,h} : \nabla\mathbf{v} d\boldsymbol{\xi} + \int_D p_{0,h}(B_j\nabla)\cdot\mathbf{v} d\boldsymbol{\xi}$$
$$- \int_D [(\mathbf{u}_{0,h}\cdot B_j\nabla)\mathbf{u}_{0,h}]\cdot\mathbf{v} d\boldsymbol{\xi}$$

for all $\mathbf{v} \in V$. Obviously, the solution $\mathbf{w}_j$ cannot be computed exactly. However, replacing $\mathbf{w}_j$ by its finite element approximation $\mathbf{w}_{j,h} \in V_h$ introduces an error of higher order, namely an error of order εh^s, where $s \in (0,1]$ depends on the domain D [27]. Indeed, thanks to triangle's inequality we have

$$\varepsilon\|\nabla\mathbf{w}_j\|_{L^2(D)} \leq \varepsilon\|\nabla\mathbf{w}_{j,h}\|_{L^2(D)} + \varepsilon\|\nabla(\mathbf{w}_j - \mathbf{w}_{j,h})\|_{L^2(D)} \leq \varepsilon\|\nabla\mathbf{w}_{j,h}\|_{L^2(D)} + C_4 h^s \varepsilon|\nabla\mathbf{w}_j|_{H^{1+s}(D)}.$$

Notice that if D is convex or ∂D is C^2, then $s = 1$ and the term $\varepsilon\|\nabla(\mathbf{w}_j - \mathbf{w}_{j,h})\|_{L^2(D)}$ is of order εh. $\qquad\square$

Based on Propositions 6.3 and 6.5, we can define two computable error estimators $\eta = \left(\eta_h^2 + \eta_\varepsilon^2\right)^{\frac{1}{2}}$ and $\hat{\eta} = \left(\eta_h^2 + \hat{\eta}_\varepsilon^2\right)^{\frac{1}{2}}$, where η_h and η_ε are given in (42) and $\hat{\eta}_\varepsilon$ is given in (46). From a computational point of view, the computation of $\hat{\eta}$ requires the solution of L additional (linear) problems compared to the cost of getting the error estimator η. However, the gain of the second error estimator is twofold: it does not use the triangle inequality to bound each term of $r(\mathbf{v}; \mathbf{y})$ separately and it does not require the use of the Poincaré inequality. The numerical tests of the next section provide an illustration of the theoretical results obtained so far.

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7 Numerical example: flow past a cylinder

We present now two numerical examples to test the error estimates derived in the previous section. We consider the problem of a flow past a cylinder and consider two different types of perturbation of the domain, namely a perturbation along the vertical axis of the position of the cylinder and a perturbation of its shape. The true error $|||\mathbf{u} - \mathbf{u}_{0,h}, p - p_{0,h}|||$ is approximated with the standard Monte Carlo method using

$$|||\mathbf{v}, q||| \approx \left(\frac{1}{K} \sum_{k=1}^{K} \left\{ \nu \|\nabla \mathbf{v}(\mathbf{y}_k)\|_{L^2(D)}^2 + \frac{1}{\nu} \|q(\mathbf{y}_k)\|_{L^2(D)}^2 \right\} \right)^{\frac{1}{2}}$$

where $\{\mathbf{y}_k\} \in \Gamma$ are i.i.d. realizations of the random vector $\mathbf{Y}$. We choose a sample size of $K = 1000$ in which case the variance of the estimation of the error is at least a factor $2 \cdot 10^{-4}$ smaller than the estimated error in all considered test cases. In what follows, whenever we refer to *error* it should be understood that the true error has been computed by the Monte Carlo procedure. Finally, the approximate solution $(\mathbf{u}_{0,h}, p_{0,h})$ is computed using $\mathbb{P}_1 bubble - \mathbb{P}_1$ finite elements and, since the exact solution $(\mathbf{u}, p)$ of the problem is not known, we compute a reference solution using $\mathbb{P}_2 - \mathbb{P}_1$ finite elements on the finest mesh considered.

7.1 First example

For this first problem, based on a well-known benchmark problem described in [41], we consider the geometry presented on Figure 2 and assume that it corresponds to the reference domain D. More precisely, D consists of the rectangle $[a_1, b_1] \times [a_2, b_2]$ with a hole of radius R located at $\mathbf{c} = (c_1, c_2)$. We assume that the rectangle is fixed and that the center $\mathbf{c}$ of the cylinder is randomly moved along the vertical axis, namely that it is given by $(c_1, c_2 + \varepsilon Y)$ in D_ω with Y a uniform random variable in $[-1, 1]$. We take $\tilde{\mathbf{f}} = \mathbf{0}$ and we prescribe the following inflow and outflow (parabolic) velocity profile on
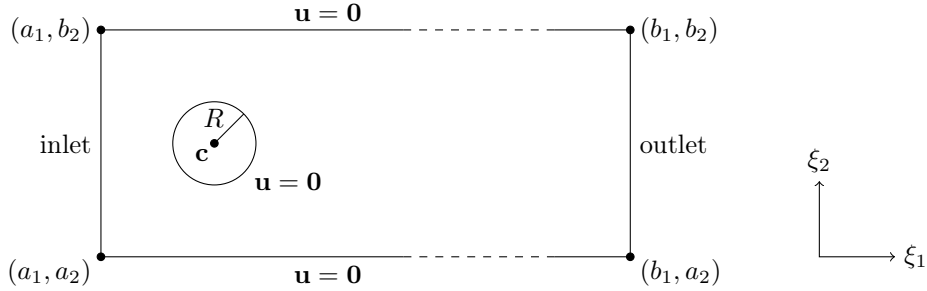

Figure 2: Geometry with prescribed boundary conditions for the first example.

the inlet and outlet part of ∂D_ω

$$\tilde{\mathbf{u}}(a_1, x_2) = \tilde{\mathbf{u}}(b_1, x_2) = (4\tilde{U}(x_2 - a_2)(b_2 - x_2)/(b_2 - a_2)^2, 0)^T \quad \text{for } a_2 \leq x_2 \leq b_2,$$

with a maximum velocity $\tilde{U} = 0.3$ achieved at $x_2 = \frac{a_2 + b_2}{2}$. We impose homogeneous Dirichlet boundary conditions on the remaining parts of the boundary. The Reynolds number is then given by $\frac{2}{3}\tilde{U}(2R)\nu^{-1}$, where $\frac{2}{3}\tilde{U}$ corresponds to the mean velocity.

We choose a mapping $\mathbf{x}_\omega$, consistent with the perturbation mentioned above, such that all the boundary nodes are fixed. In such a case, the boundary conditions for the equivalent problem on D are the same than the ones on D_ω. More precisely, we consider the mapping $\mathbf{x}_\omega : D \to D_\omega$ given component-wise by:

$$\begin{bmatrix} x_1 &=& \xi_1 \\ x_2 &=& \xi_2 + \varepsilon \varphi_1(\xi_1)\varphi_2(\xi_2)Y(\omega), \end{bmatrix}$$

where for $i = 1, 2$

$$\varphi_i(\xi_i) = \begin{cases} \frac{\xi_i - a_i}{c_i - R - a_i} - \tau \frac{(\xi_i - a_i)(\xi_i - c_i + R)}{(c_i - R - a_i)^2} & \text{if } \xi_i \in [a_i, c_i - R[\\ 1 & \text{if } \xi_i \in [c_i - R, c_i + R] \\ \frac{\xi_i - b_i}{c_i + R - b_i} - \tau \frac{(\xi_i - b_i)(\xi_i - c_i - R)}{(c_i + R - b_i)^2} & \text{if } \xi_i \in]c_i + R, b_i], \end{cases} \tag{48}$$

which can be written under the form (17) as $\mathbf{x}(\boldsymbol{\xi}, \omega) = \boldsymbol{\xi} + \varepsilon \boldsymbol{\varphi}(\boldsymbol{\xi}) Y(\omega)/\sqrt{3}$ with Y a uniform random variable in $[-\sqrt{3}, \sqrt{3}]$ and $\boldsymbol{\varphi}(\boldsymbol{\xi}) = (0, \varphi_1(\xi_1)\varphi_2(\xi_2))^T$. The function φ_2 alone fits the required perturbation of the domain but we use the function φ_1 to fix the nodes on the inlet and outlet boundaries. Moreover, the parameter $\tau \in \{0, 1\}$ is used to control the regularity of the mapping. Indeed, choosing $\tau = 1$ implies that all the functions appearing in the jacobian matrix A^{-1} of the mapping $\mathbf{x}_\omega$ are continuous. From now on, according to [41], we fix the value of the various geometry parameters to $a_1 = a_2 = 0$, $b_1 = 2.2$, $b_2 = 0.41$, $c_1 = c_2 = 0.2$ and $R = 0.05$, and we choose $\tau = 1$. The functions φ_1 and φ_2 for these values of the various geometrical parameters are given on Figure 3.
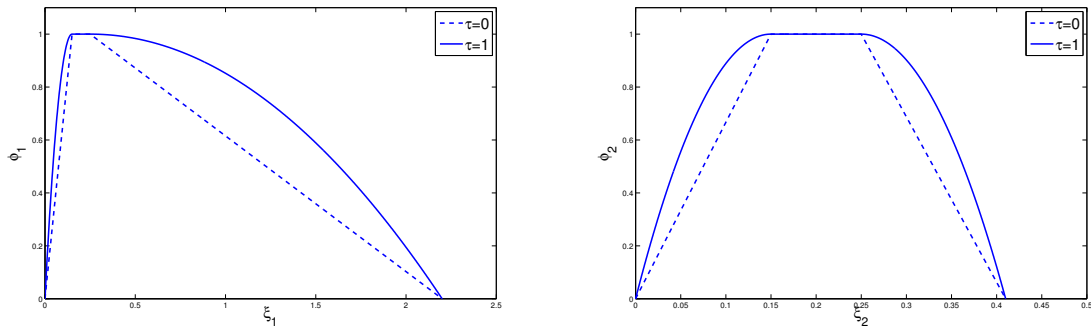

Figure 3: Functions $\varphi_1(\xi_1)$, $\xi_1 \in [0, 2.2]$ (left) and $\varphi_2(\xi_2)$, $\xi_2 \in [0, 0.41]$ (right) defined in (48).

The numerical tests are performed using FreeFem++ 3.19.1-1 [31]. The mesh is constructed with a Delaunay triangulation using n equispaced points on the left and right boundaries, $5n$ on the upper and lower boundaries and $2n$ on the hole. The mesh size is then given by $h \approx (\sqrt{2}n)^{-1}$ while the number of elements and vertices are about $12n^2$ and $7n^2$, respectively. Finally, we recall that the error estimates derived in the sections 6.1 and 6.2 are valid for homogeneous Dirichlet boundary conditions. In the case of inhomogeneous conditions, as considered here, an additional term due to the approximation of the Dirichlet data should be included. However, thanks to the fact that the later is not affected by the mapping, it is a higher order term in h (see for instance [6]) and thus we do not take it into account in the numerical results.

Deterministic case

We first consider the deterministic case, namely when ε is set to zero. The reference values in [41] include the drag (c_D) and lift (c_L) coefficients and the pressure difference $\Delta p = p(0.15, 0.2) - p(0.25, 0.2)$ between the value at the front and the end point of the cylinder. Using $\mathbb{P}_2 - \mathbb{P}_1$ FE on a mesh with $n = 80$, we obtain the values $c_D = 5.57469$, $c_L = 0.0104584$ and $\Delta p = 0.117525$ which are consistant with the bounds given in [41]. In Table 1, we give the results obtained for various values of n and ν, where err, η and e.i. denote respectively the error, the error estimator $(\eta_h^2 + \eta_\varepsilon^2)^{\frac{1}{2}}$ with η_h and η_ε defined in (42) and the effectivity index, namely the ratio between the error estimator and the error. Notice that $\eta_\varepsilon = 0$ here since $\varepsilon = 0$. We can see that in all cases, for h small enough, the effectivity index is about 2.8.

Random case

We treat now the random case by considering values of ε between 0 and 0.05. With $\varepsilon = 0.05$, the random position of the cylinder on the vertical axis lies between 0.15 and 0.25 with nominal value in

	$\nu = 0.001$			$\nu = 0.01$			$\nu = 0.1$			$\nu = 1$		
n	err	η	e.i.	err	η	e.i.	err	η	e.i.	err	η	e.i.
4	0.136	0.566	4.17	0.158	0.310	1.96	0.514	0.963	1.87	1.628	3.052	1.87
8	0.039	0.150	3.87	0.060	0.135	2.27	0.188	0.415	2.20	0.596	1.312	2.20
16	0.015	0.044	2.87	0.028	0.070	2.55	0.086	0.216	2.52	0.271	0.684	2.52
32	0.007	0.019	2.73	0.013	0.034	2.70	0.039	0.105	2.69	0.124	0.333	2.69
64	0.003	0.009	2.75	0.006	0.017	2.78	0.019	0.052	2.78	0.060	0.166	2.78

Table 1: Error, error estimator and effectivity index for the deterministic case ($\varepsilon = 0$) and various viscosities.

0.2, which is quite a large perturbation considering that the height of the rectangle is equal to 0.41. We give in Table 2 the numerical results obtained for $\nu = 0.001$ and $\nu = 1$ and various values of n and ε.

We recall that we use different FE spaces for the reference and the approximate solution and thus, even in the case where the same mesh is used for both solutions, there is still an error due to space discretization. We can see in Table 2 that the effectivity index tends to the one obtained in Table 1 when the spatial error is dominating while when the statistical error dominates, it is about 13 and 3 for $\nu = 0.001$ and $\nu = 1$, respectively. This highlights the dependence of the error estimate given in Section 6.1 with respect to the input data. However, we can see that when both h and ε are divided by 2 then the effectivity index remains constant, this observation being tempered by the fact that the effectivity index for $\varepsilon = 0$ is not constant for the various meshes considered (see Table 1). For instance, in the case $\nu = 0.001$ and $\varepsilon = (5n)^{-1}$, which corresponds to $h \approx 3.5\varepsilon$, the effectivity index is about 8. We study now the efficiency of the second error estimate with respect to the viscosity. In Figure 4, we give the effectivity index with respect to ν for both error estimators η and $\hat{\eta} = (\eta_h^2 + \hat{\eta}_\varepsilon^2)^{\frac{1}{2}}$, where $\hat{\eta}_\varepsilon$ is given in (46), in the case $\varepsilon = 0.025$, $n = 64$ and $n_{ref} = 64$, which corresponds to a statistical error dominant regime.
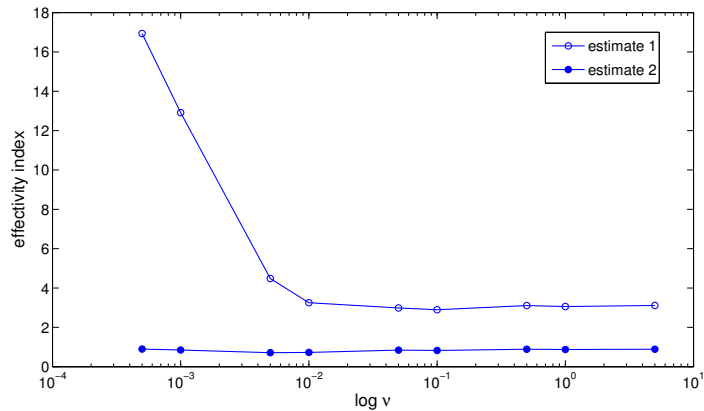

Figure 4: Effectivity index with respect to the viscosity ν for the two error estimators η and $\hat{\eta}$ defined in (42) and (46).

We can see that the effectivity index of the first error estimator η remains constant for viscosities greater than 0.01 while below this value, it starts increasing as ν decreases. The situation is different for the second estimator $\hat{\eta}$ of Section 6.2, whose efficiency is not sensitive to the value of ν.

Remark 7.1. *In order to have the correct balance of the two terms appearing in the error estimator η or $\hat{\eta}$, we could estimate numerically the constants in front of each term η_h and η_ε or $\hat{\eta}_\varepsilon$. The estimation of these constants can also be used to construct a sharp error estimator, namely an error estimator with effectivity index close to 1. According to the results in Table 1, the term η_h should be multiplied*

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		$\nu = 0.001$				$\nu = 1$			
n	ε	err	η_h	η_ε	e.i.	err	η_h	η_ε	e.i.
4	0.05	0.1389	0.5656	1.0649	8.68	1.8881	3.0521	2.4890	2.09
8	0.05	0.0591	0.1503	0.6797	11.78	1.0157	1.3124	2.3458	2.65
16	0.05	0.0452	0.0440	0.5487	12.19	0.8110	0.6839	2.3018	2.96
32	0.05	0.0429	0.0190	0.5288	12.32	0.7713	0.3333	2.2887	3.00
64	0.05	0.0428	0.0091	0.5246	12.25	0.7526	0.1655	2.2856	3.05
4	0.025	0.1361	0.5656	0.5325	5.71	1.6989	3.0521	1.2445	1.94
8	0.025	0.0436	0.1503	0.3399	8.52	0.7159	1.3124	1.1729	2.46
16	0.025	0.0249	0.0440	0.2743	11.15	0.4701	0.6839	1.1509	2.85
32	0.025	0.0205	0.0190	0.2644	12.96	0.3916	0.3333	1.1444	3.04
64	0.025	0.0194	0.0091	0.2623	13.51	0.3831	0.1655	1.1428	3.01
4	0.0125	0.1356	0.5656	0.2662	4.61	1.6458	3.0521	0.6223	1.89
8	0.0125	0.0401	0.1503	0.1699	5.66	0.6291	1.3124	0.5865	2.29
16	0.0125	0.0181	0.0440	0.1372	7.98	0.3310	0.6839	0.5755	2.70
32	0.0125	0.0119	0.0190	0.1322	11.25	0.2264	0.3333	0.5722	2.92
64	0.0125	0.0100	0.0091	0.1311	13.13	0.2056	0.1655	0.5714	2.89
4	0.00625	0.1356	0.5656	0.1331	4.29	1.6324	3.0521	0.3111	1.88
8	0.00625	0.0392	0.1503	0.0850	4.41	0.6043	1.3124	0.2932	2.23
16	0.00625	0.0160	0.0440	0.0686	5.08	0.2872	0.6839	0.2877	2.58
32	0.00625	0.0084	0.0190	0.0661	8.17	0.1559	0.3333	0.2861	2.82
64	0.00625	0.0058	0.0091	0.0656	11.45	0.1117	0.1655	0.2857	2.96
4	0.003125	0.1355	0.5656	0.0666	4.20	1.6324	3.0521	0.1556	1.88
8	0.003125	0.0389	0.1503	0.0425	4.01	0.6043	1.3124	0.1466	2.23
16	0.003125	0.0155	0.0440	0.0343	3.60	0.2872	0.6839	0.1439	2.58
32	0.003125	0.0074	0.0190	0.0330	5.18	0.1328	0.3333	0.1430	2.73
64	0.003125	0.0041	0.0091	0.0328	8.32	0.0760	0.1655	0.1429	2.88

Table 2: The error, the two contributions η_h and η_ε of the error estimator in (42) and the effectivity index for $\nu = 0.001$ and $\nu = 1$.

by a factor $1/2.8$. *For the term due to uncertainty, we obtain that $\hat{\eta}_\varepsilon$ should be multiplied by about 1.5, considering for instance same FE spaces and fine mesh for both the reference and approximate solutions, whereas the constant in front of η_ε depends on the viscosity as seen in Table 2 or Figure 4 (for instance $1/13$ for $\nu = 0.001$ or $1/3$ for $\nu \geq 0.01$).*

To conclude the analysis of this first example, we mention that similar results are obtained if we use homogeneous Neumann boundary conditions on the outlet part of the boundary. Notice that in this case, the jump term should be modified appropriately since it is no longer zero on the boundary edges belonging to the outlet.

7.2 Second example

For this second example, the reference geometry D consists in a square $[-H, H]^2$ with $H = 0.5$ and a circular hole of radius $R = 0.15$ centred at the origin, as depicted on Figure 5 where the prescribed boundary conditions are also indicated. The shape of the hole is given on D by $(\xi_1, \xi_2) = (R\cos(\theta), R\sin(\theta))$
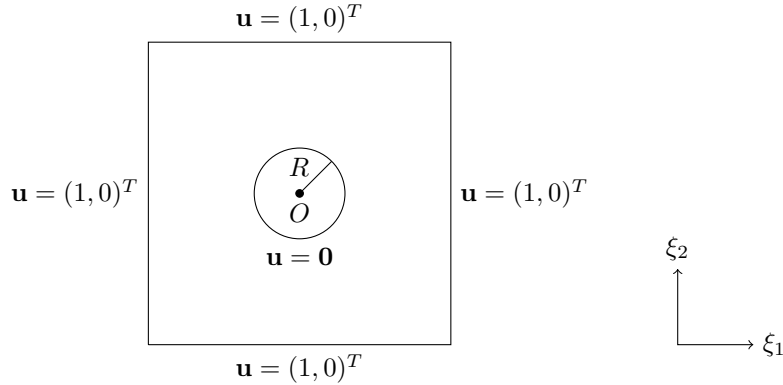

Figure 5: Geometry with prescribed boundary conditions for the second example.

with $\theta \in [0, 2\pi]$. We perturb this hole by modifying its radius with respect to the angle by the formula $R + \varepsilon d_\theta$, where $d_\theta = \sum_{j=1}^{L} \alpha_j \cos(k_j\theta)Y_j$ and Y_j are i.i.d uniform random variables in $[-1, 1]$. The coefficients k_j and α_j control the frequency and the amplitude of each term, respectively. We mention that a similar perturbation is consider in [45], where the mapping is not constructed explicitly but computed through solutions of Laplace equations. We consider here the following mapping $\mathbf{x}_\omega$ from D to D_ω which fits the above perturbation: denoting $r = \sqrt{\xi_1^2 + \xi_2^2}$ and $\theta = \arctan(\frac{\xi_2}{\xi_1})$ the cylindrical coordinates of any point $\boldsymbol{\xi} = (\xi_1, \xi_2)$ of D, we take

$$\mathbf{x} = \boldsymbol{\xi} + \varepsilon \sum_{j=1}^{L} \boldsymbol{\varphi}_j(\boldsymbol{\xi})Y_j(\omega), \quad \boldsymbol{\varphi}_j(\boldsymbol{\xi}) = \alpha_j \cos(k_j\theta)g(\boldsymbol{\xi}) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \tag{49}$$

where the *filter function* g is such that it vanishes at the boundary of the domain and is equal to 1 in the hole, namely we use

$$g(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } r \in [0, R] \\ \frac{(\xi_1^2 - H^2)(\xi_2^2 - H^2)}{(R^2\xi_1^2 r^{-2} - H^2)(R^2\xi_2^2 r^{-2} - H^2)} & \text{otherwise.} \end{cases} \tag{50}$$

The mesh is again built with a Delaunay triangulation using n equispaced points on the boundaries of the square and $2n$ on the hole for various values of n with corresponding mesh size $h \approx 1.5n^{-1}$ and number of elements and vertices of about $3.5n^2$ and $2n^2$, respectively.

Remark 7.2. *Contrary to the previous example, the choice of the boundary conditions on the outlet has an impact on the solution of this problem, due to the fact that the outlet is close to the cylinder.*

This is especially true for small viscosities, in which case some flow is re-entering the domain when homogeneous Neumann conditions are used while the solution presents a boundary layer when Dirichlet conditions are enforced.

For this problem, we give only the results for the non-deterministic case since the conclusions for the case $\varepsilon = 0$ are essentially the same as in the previous example. We consider $L = 1$ random variable, we fix $\alpha_1 = 1$ and $k_1 = 6$ in the definition of d_θ and we let $0 \le \varepsilon \le 0.01$. The vorticity of the velocity **u** and the pressure p in the case $\varepsilon = 0.01$, $\nu = 0.05$ and $Y = 1$ is given on Figure 6, where the solution obtained by solving the problem defined on D_ω as well as the solution for the case $\varepsilon = 0$ are also given for comparison.
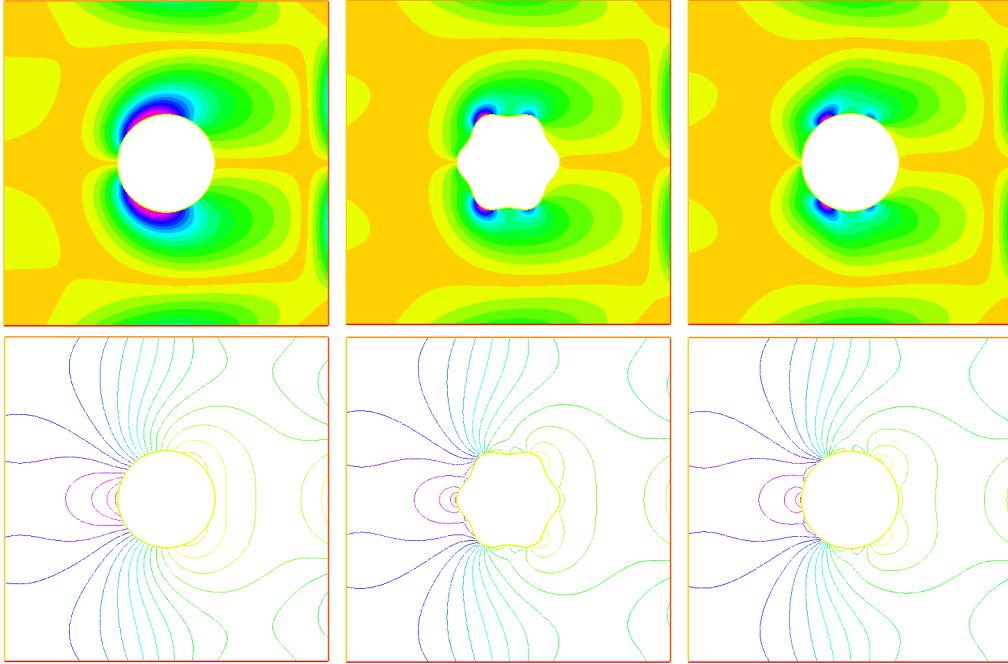

Figure 6: Vorticity of the velocity and pressure for $\nu = 0.05$ in the case $\varepsilon = 0$ (left) and $\varepsilon = 0.01$ with $Y = 1$ computed on D_ω (middle) and on D (right).

We give in Table 3 the numerical results obtained for $\nu = 0.05$ and $\nu = 1$ and various values of n and ε. Similarly to the previous example, we observe that the effectivity index tends to the one obtained for the deterministic case ($\varepsilon = 0$) when the error in h is dominating, while it is about 6 and 1.5 for $\nu = 0.05$ and $\nu = 1$, respectively, when the statistical error dominates. This shows again the sensitivity of the efficiency of the first error estimator with respect to the input data but, as before, the effectivity index remains about constant when both h and ε are divided by 2. Indeed, for instance for $\nu = 0.05$ and $\varepsilon = (10n)^{-1}$, corresponding to $h \approx 15\varepsilon$, it stays between 3.81 and 4.05. Finally, the same behaviour than in the previous example is observed for the efficiency of the second error estimator $\hat{\eta}$ with respect to the viscosity, as can be seen on Figure 7 where the results are given for the case $\varepsilon = 0.005$, $n = 160$ and $n_{ref} = 160$.

Finally, we mention that similar results are obtained when changing the mapping, for instance taking $\alpha_1 = 0.9$ and $k_1 = 14$ in (49) or considering $L = 2$ with $k_1 = 6$, $k_2 = 11$, $\alpha_1 = 1$ and $\alpha_2 = 0.8$.

8   Conclusion

In this paper, we have considered the steady-state incompressible Navier-Stokes equations defined on random domains and we have used the *domain mapping method* to transform them into PDEs on a

		$\nu = 0.05$				$\nu = 1$			
n	ε	err	η_h	η_ε	e.i.	err	η_h	η_ε	e.i.
10	0.01	0.5125	1.1492	1.6181	3.87	2.0403	4.2209	1.4479	2.19
20	0.01	0.3251	0.5785	1.5682	5.14	1.2200	2.0741	1.3862	2.04
40	0.01	0.2625	0.2937	1.5552	6.03	1.0216	1.0524	1.3730	1.69
80	0.01	0.2486	0.1478	1.5519	6.27	1.0040	0.5233	1.3696	1.46
160	0.01	0.2431	0.07279	1.5511	6.39	0.9630	0.2594	1.3687	1.45
10	0.005	0.4859	1.1492	0.8090	2.89	1.9575	4.2209	0.7240	2.19
20	0.005	0.2556	0.5785	0.7841	3.81	0.9477	2.0741	0.6931	2.31
40	0.005	0.1628	0.2937	0.7776	5.11	0.6163	1.0524	0.6865	2.04
80	0.005	0.1340	0.1478	0.7759	5.91	0.5149	0.5233	0.6848	1.67
160	0.005	0.1238	0.0728	0.7755	6.29	0.4891	0.2594	0.6843	1.50
10	0.0025	0.4792	1.1492	0.4045	2.54	1.9363	4.2209	0.3620	2.19
20	0.0025	0.2370	0.5785	0.3921	2.95	0.8602	2.0741	0.3465	2.44
40	0.0025	0.1263	0.2937	0.3888	3.86	0.4538	1.0524	0.3433	2.44
80	0.0025	0.0808	0.1478	0.3880	5.14	0.3085	0.5233	0.3424	2.03
160	0.0025	0.0662	0.0728	0.3878	5.96	0.2584	0.2594	0.3422	1.66
10	0.00125	0.4776	1.1492	0.2023	2.44	1.9317	4.2209	0.1810	2.19
20	0.00125	0.2319	0.5785	0.1960	2.63	0.8399	2.0741	0.1733	2.48
40	0.00125	0.1154	0.2937	0.1944	3.05	0.4098	1.0524	0.1716	2.60
80	0.00125	0.0624	0.1478	0.1940	3.91	0.2237	0.5233	0.1712	2.46
160	0.00125	0.0405	0.0728	0.1939	5.12	0.1517	0.2594	0.1711	2.05
10	0.000625	0.4772	1.1492	0.1011	2.42	1.9304	4.2209	0.0905	2.19
20	0.000625	0.2306	0.5785	0.0980	2.54	0.8347	2.0741	0.0866	2.49
40	0.000625	0.1125	0.2937	0.0972	2.75	0.3977	1.0524	0.0858	2.66
80	0.000625	0.0565	0.1479	0.0970	3.13	0.1987	0.5233	0.0856	2.67
160	0.000625	0.0304	0.0728	0.0970	3.99	0.1101	0.2594	0.0855	2.48

Table 3: The error, the two contributions η_h and η_ε of the estimator in (42) and the effectivity index for $\nu = 0.05$ and $\nu = 1$.
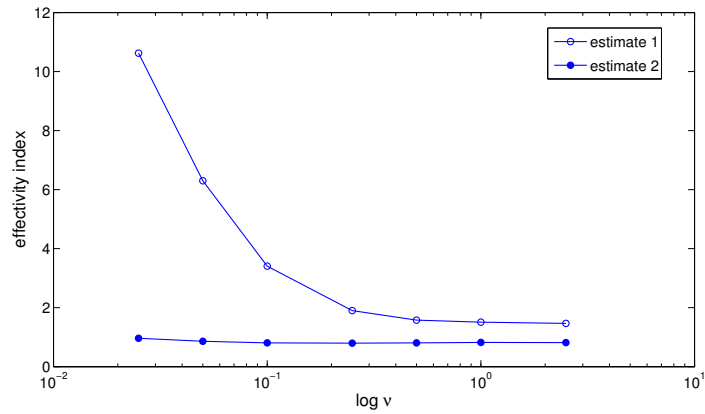

Figure 7: Effectivity index with respect to the viscosity ν for the two error estimators η and $\hat{\eta}$ defined in (42) and (46).

fixed reference domain with random coefficients. We started the analysis by showing the well-posedness of the problem under suitable assumptions on the input data and the mapping, before performing an *a posteriori* error analysis. Using a perturbation method, we obtained two error estimates for the *first order* approximation $(\mathbf{u}, p) \approx (\mathbf{u}_{0,h}, p_{0,h})$. Both estimates are constituted of two parts, namely one part due to space discretization in h and one due to the uncertainty in ε. They already give useful information, especially when the problem contains small uncertainties. They can indeed be used to adaptively find a spatial mesh that balances the two sources of error. Further mesh refinement should then be avoided since it would not decrease the total error, the statistical error being dominant. The latter can only be decreased by adding more terms in the expansion of the solution. Notice that if we want to analyse higher order approximations in ε, then we should impose additional regularity assumptions on f and on the random mapping, namely that the Jacobian matrix $\nabla \boldsymbol{\varphi}_j$ belongs to $\left[W^{1,\infty}(D) \right]^{d \times d}$ for $j = 0, 1, ..., L$ and not only for $j = 0$. Indeed, we have that the residual for the FE approximation $(U_{j,h}, P_{j,h})$ of (U_j, P_j) belongs to $L^2(D)$ for $j = 1, ..., L$, where (U_j, P_j) is the solution of (28) and appears in the second term of the expansion of the solution. The same holds for the residual of the higher order terms.

Each of the two error estimators η and $\hat{\eta}$ that we obtained presents its advantages and drawbacks. The first one can be computed by solving only one nonlinear problem, namely the *standard* Navier-Stokes equations on the reference domain. We have seen however that the sharpness of this estimator might be affected when changing the input data, as predicted by the theory. In the two numerical examples considered here, the effectivity index remains constant for moderate Reynolds number but then starts to increase as the viscosity diminishes. The second error estimator shows promising results, its efficiency being indeed independent of the input data for all the cases we have considered. The extra cost to pay is the resolution of L additional linear problems. Finally, as mentioned in Remark 7.1, the constant in front of the two terms in h and ε can be estimated numerically (once for all for the second estimator) to get a sharp error estimator, that is an estimator with effectivity index close to 1.

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A Proof of Proposition 4.6

We give here the proof of the well-posedness of Problem (12) under the *small data* assumption stated in Proposition 4.6 which uses a fixed point argument.

Proof. In this proof, the explicit dependence of the functions with respect to $\omega \in \Omega$ will not necessarily be indicated, unless ambiguity holds. Moreover, with little abuse of notation we define the space

$$L_P^2(\Omega; V_{\mathrm{div},\omega}) := \{\mathbf{v} \in L_P^2(\Omega; V) : \mathbf{v}(\omega) \in V_{\mathrm{div},\omega} \quad \text{a.s. in } \Omega\}.$$

First of all, we can show that

$$c(\mathbf{u}, \mathbf{v}, \mathbf{v}; \omega) = 0 \quad \forall \mathbf{u} \in V_{\mathrm{div},\omega}, \forall \mathbf{v} \in V, \text{ a.s. in } \Omega. \tag{51}$$

Indeed, if we write $\tilde{\mathbf{u}} = \mathbf{u} \circ \boldsymbol{\xi}_\omega$ and $\tilde{\mathbf{v}} = \mathbf{v} \circ \boldsymbol{\xi}_\omega$ then $\tilde{\mathbf{u}} \in \tilde{V}_{\mathrm{div},\omega}$, $\tilde{\mathbf{v}} \in \tilde{V}_\omega$ and

$$\begin{aligned}
c(\mathbf{u}, \mathbf{v}, \mathbf{v}; \omega) &= \int_D [(\mathbf{u} \cdot A^T \nabla)\mathbf{v}] \cdot \mathbf{v} J_{\mathbf{x}} d\boldsymbol{\xi} = \int_{D_\omega} [(\tilde{\mathbf{u}} \cdot \nabla_{\mathbf{x}})\tilde{\mathbf{v}}] \cdot \tilde{\mathbf{v}} d\mathbf{x} \\
&= -\frac{1}{2} \int_{D_\omega} (\nabla_{\mathbf{x}} \cdot \tilde{\mathbf{u}})|\tilde{\mathbf{v}}|^2 d\mathbf{x} + \frac{1}{2} \int_{\partial D_\omega} (\tilde{\mathbf{u}} \cdot \mathbf{n})|\tilde{\mathbf{v}}|^2 ds = 0
\end{aligned}$$

using the fact that we have imposed homogeneous Dirichlet boundary conditions. Now, for any $\mathbf{u} \in L_P^2(\Omega; V_{\text{div}})$ we define the (pointwise in ω) bilinear form $\mathcal{A}_{\mathbf{u}(\omega)}(\cdot, \cdot; \omega) : V_{\text{div},\omega} \times V_{\text{div},\omega} \to \mathbb{R}$ by

$$\mathcal{A}_{\mathbf{u}(\omega)}(\mathbf{w}, \mathbf{v}; \omega) := a(\mathbf{w}, \mathbf{v}; \omega) + c(\mathbf{u}(\omega), \mathbf{w}, \mathbf{v}; \omega),$$

which is uniformly continuous and coercive (on V and thus on $V_{\text{div},\omega}$) thanks to Proposition 4.3 and relation (51). Since $\|\mathbf{f} J_{\mathbf{x}}\|_{L^2(D)} \leq \sigma_{max}^{d/2} \|\tilde{\mathbf{f}}\|_{L^2(\hat{D})} < +\infty$ a.s. in Ω, in particular $\mathbf{f} J_{\mathbf{x}} \in L_P^2(\Omega; L^2(D))$ and Lax-Milgram's lemma ensures the existence of a unique solution to the problem: for every $\omega \in \Omega$, find $\mathbf{w}(\omega) \in V_{\text{div},\omega}$ such that

$$\mathcal{A}_{\mathbf{u}(\omega)}(\mathbf{w}, \mathbf{v}; \omega) = F(\mathbf{v}; \omega) \quad \forall \mathbf{v} \in V_{\text{div},\omega}, \text{ a.s. in } \Omega. \tag{52}$$

Moreover, taking $\mathbf{v} = \mathbf{w}(\omega)$ in (52) and using the coercivity of $\mathcal{A}_{\mathbf{u}}(\cdot, \cdot; \omega)$ we have a.s. in Ω

$$\nu \sigma_{min}^d \sigma_{max}^{-2} \|\nabla \mathbf{w}\|_{L^2(D)}^2 \quad \leq \quad \mathcal{A}_{\mathbf{u}}(\mathbf{w}, \mathbf{w}; \omega) = F(\mathbf{w}; \omega) \leq C_P \sigma_{max}^{\frac{d}{2}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \|\nabla \mathbf{w}\|_{L^2(D)}$$

and thus

$$\|\nabla \mathbf{w}\|_{L^2(D)} \leq \frac{C_P \sigma_{max}^{\frac{d}{2}+2}}{\nu \sigma_{min}^d} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \leq \frac{C_P \sigma_{max}^{\frac{d}{2}+2}}{\nu \sigma_{min}^d} \|\tilde{\mathbf{f}}\|_{L^2(\hat{D})} < \infty \tag{53}$$

from which we deduce that $\mathbf{w} \in L_P^2(\Omega; V_{\text{div},\omega})$. Notice that a fixed point of the application $\Phi : L_P^2(\Omega; V_{\text{div},\omega}) \to L_P^2(\Omega; V_{\text{div},\omega})$, which maps $\mathbf{u}$ to the unique solution $\mathbf{w}$ of (52), is a solution of Problem (12). Therefore, it only remains to prove that Φ is a strict contraction. Let $\mathbf{w} = \Phi(\mathbf{u})$ with $\mathbf{u} \in L_P^2(\Omega; V_{\text{div},\omega})$. First, using relation (53) we directly get that $\Phi(L_P^2(\Omega; V_{\text{div}})) \subset \mathcal{M}$, where the ball $\mathcal{M} \subset L_P^2(\Omega; V_{\text{div},\omega})$ is defined by

$$\mathcal{M} := \{\mathbf{v} \in L_P^2(\Omega; V_{\text{div},\omega}) : \|\nabla \mathbf{v}\|_{L^2(D)} \leq \frac{C_P \sigma_{max}^{\frac{d}{2}+2}}{\nu \sigma_{min}^d} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \text{ a.s. in } \Omega\}.$$

Finally, we show that Φ is a contraction, i.e. that there exists a constant $0 < k < 1$ such that

$$\|\Phi(\mathbf{u}) - \Phi(\bar{\mathbf{u}})\|_{L_P^2(\Omega; V)} \leq k \|\mathbf{u} - \bar{\mathbf{u}}\|_{L_P^2(\Omega; V)} \quad \forall \mathbf{u}, \bar{\mathbf{u}} \in L_P^2(\Omega; V_{\text{div},\omega}).$$

Let $\mathbf{w} = \Phi(\mathbf{u})$ and $\bar{\mathbf{w}} = \Phi(\bar{\mathbf{u}})$. Since $\mathbf{w}$ and $\bar{\mathbf{w}}$ satisfy Problem (52) with $\mathcal{A}_{\mathbf{u}}(\cdot, \cdot; \omega)$ and $\mathcal{A}_{\bar{\mathbf{u}}}(\cdot, \cdot; \omega)$, respectively, we have

$$a(\mathbf{w} - \bar{\mathbf{w}}, \mathbf{v}; \omega) + c(\mathbf{u}, \mathbf{w}, \mathbf{v}; \omega) - c(\bar{\mathbf{u}}, \bar{\mathbf{w}}, \mathbf{v}; \omega) = 0 \quad \forall \mathbf{v} \in V_{\text{div},\omega}, \text{ a.s. in } \Omega,$$

from which we deduce

$$a(\mathbf{w} - \bar{\mathbf{w}}, \mathbf{v}; \omega) + c(\mathbf{u} - \bar{\mathbf{u}}, \bar{\mathbf{w}}, \mathbf{v}; \omega) + c(\mathbf{u}, \mathbf{w} - \bar{\mathbf{w}}, \mathbf{v}; \omega) = 0,$$

or in other words

$$\mathcal{A}_{\mathbf{u}}(\mathbf{w} - \bar{\mathbf{w}}, \mathbf{v}; \omega) = -c(\mathbf{u} - \bar{\mathbf{u}}, \bar{\mathbf{w}}, \mathbf{v}; \omega).$$

Since $\bar{\mathbf{w}} \in \mathcal{M}$, taking $\mathbf{v} = \mathbf{w} - \bar{\mathbf{w}}$ in the last equation yields a.s. in Ω

$$
\begin{aligned}
\nu \sigma_{min}^d \sigma_{max}^{-2} \|\nabla(\mathbf{w} - \bar{\mathbf{w}})\|_{L^2(D)}^2 \quad &\leq \quad \mathcal{A}_{\mathbf{u}}(\mathbf{w} - \bar{\mathbf{w}}, \mathbf{w} - \bar{\mathbf{w}}; \omega) = -c(\mathbf{u} - \bar{\mathbf{u}}, \bar{\mathbf{w}}, \mathbf{w} - \bar{\mathbf{w}}; \omega) \\
&\leq \quad C_I^2 \sigma_{max}^d \sigma_{min}^{-1} \|\nabla(\mathbf{u} - \bar{\mathbf{u}})\|_{L^2(D)} \|\nabla \bar{\mathbf{w}}\|_{L^2(D)} \|\nabla(\mathbf{w} - \bar{\mathbf{w}})\|_{L^2(D)} \\
&\leq \quad \frac{C_P C_I^2 \sigma_{max}^{\frac{3d}{2}+2}}{\nu \sigma_{min}^{d+1}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \|\nabla(\mathbf{u} - \bar{\mathbf{u}})\|_{L^2(D)} \|\nabla(\mathbf{w} - \bar{\mathbf{w}})\|_{L^2(D)}.
\end{aligned}
$$

Therefore

$$\|\nabla(\mathbf{w} - \bar{\mathbf{w}})\|_{L^2(D)} \leq \frac{C_P C_I^2 \sigma_{max}^{\frac{3d}{2}+4}}{\nu^2 \sigma_{min}^{2d+1}} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \|\nabla(\mathbf{u} - \bar{\mathbf{u}})\|_{L^2(D)} \quad \text{a.s. in } \Omega$$

which proves that Φ is a contraction under the assumption that (13) holds. By the Banach contraction theorem, we know that there exists a unique fixed point $\mathbf{u} = \Phi(\mathbf{u})$, which is solution of Problem (12).

The fact that any solution of (12) is in $\mathcal{M}$ and is a fixed point of Φ achieves the proof of well-posedness of the problem. Finally, recalling that α and $\hat{C}$ are defined in Proposition 4.3, the bound (14) is immediate since

$$\|\nabla \mathbf{u}\|_{L^2(D)} \leq \frac{C_P \sigma_{max}^{\frac{d}{2}+2}}{\nu \sigma_{min}^d} \|\tilde{\mathbf{f}}\|_{L^2(D_\omega)} \leq \theta \frac{\nu \sigma_{min}^{d+1}}{C_I^2 \sigma_{max}^{d+2}} = \theta \frac{\nu \sigma_{max}^{-2} \sigma_{min}^d}{C_I^2 \sigma_{max}^d \sigma_{min}^{-1}} = \theta \frac{\nu \alpha}{\hat{C}}$$

where we have used that $\mathbf{u} \in \mathcal{M}$ for the first inequality and relation (13) for the second one. $\qquad\square$

B Derivation of Problems (27) and (28)

We give here some details about the derivation of the problems (27) and (28) that we need to solve to obtain the first two terms in the expansion of the solution $(\mathbf{u}, p)$, namely $(\mathbf{u}_0, p_0)$ and $(\mathbf{u}_1, p_1)$. These problems are obtained by replacing each term in (7), the problem in strong form for $(\mathbf{u}, p)$, by its expansion with respect to ε and keeping only the appropriate terms. Using relations (24) and (25), we can write

$$
\begin{aligned}
J_\mathbf{x} A A^T &= (1 + \varepsilon tr(A_1) + \mathcal{O}(\varepsilon^2))(I - \varepsilon A_1 + \mathcal{O}(\varepsilon^2))(I - \varepsilon A_1^T + \mathcal{O}(\varepsilon^2)) \\
&= I + \varepsilon(tr(A_1)I - A_1 - A_1^T) + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

and similarly

$$J_\mathbf{x} A^T = I + \varepsilon(tr(A_1)I - A_1^T) + \mathcal{O}(\varepsilon^2).$$

Therefore, considering for instance the convection term, we get

$$
\begin{aligned}
(\mathbf{u} \cdot J_\mathbf{x} A^T \nabla)\mathbf{u} &= ((\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \mathcal{O}(\varepsilon^2)) \cdot (I + \varepsilon(tr(A_1)I - A_1^T) + \mathcal{O}(\varepsilon^2))\nabla)(\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \mathcal{O}(\varepsilon^2)) \\
&= (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \varepsilon \left[(\mathbf{u}_1 \cdot \nabla)\mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_0 \cdot (tr(A_1)I - A_1^T)\nabla)\mathbf{u}_0 \right] + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

Proceeding similarly for all the terms involved in the first equation of (7) and keeping the $\mathcal{O}(1)$ terms with respect to ε we obtain

$$-\nu \Delta \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla p_0 = \mathbf{f}_0$$

which is the first equation of (27). If we collect now the terms of order $\mathcal{O}(\varepsilon)$ we get

$$
\begin{aligned}
-\nu \Delta \mathbf{u}_1 + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_0 + \nabla p_1 &= tr(A_1)\mathbf{f}_0 + \mathbf{f}_1 + \nu \nabla \cdot \left[((tr(A_1)I - A_1 - A_1^T)\nabla)\mathbf{u}_0 \right] \\
&\quad - (\mathbf{u}_0 \cdot (tr(A_1)I - A_1^T)\nabla)\mathbf{u}_0 - ((tr(A_1)I - A_1^T)\nabla)p_0.
\end{aligned}
\tag{54}
$$

Finally, since

$$A_1 = \sum_{j=1}^L \nabla \boldsymbol{\varphi}_j Y_j, \quad \mathbf{f}_1 = \sum_{j=1}^L \mathbf{F}_j Y_j, \quad \mathbf{u}_1 = \sum_{j=1}^L \mathbf{U}_j Y_j \quad \text{and} \quad p_1 = \sum_{j=1}^L P_j Y_j,$$

equation (54) is satisfied if

$$
\begin{aligned}
-\nu \Delta U_j + (\mathbf{u}_0 \cdot \nabla)\mathbf{U}_j + (\mathbf{U}_j \cdot \nabla)\mathbf{u}_0 + \nabla P_j &= tr(\nabla \boldsymbol{\varphi}_j)\mathbf{f}_0 + \mathbf{F}_j + \nu \nabla \cdot \left[((tr(\nabla \boldsymbol{\varphi}_j)I - \nabla \boldsymbol{\varphi}_j - \nabla \boldsymbol{\varphi}_j^T)\nabla)\mathbf{u}_0 \right] \\
&\quad - (\mathbf{u}_0 \cdot (tr(\nabla \boldsymbol{\varphi}_j)I - \nabla \boldsymbol{\varphi}_j^T)\nabla)\mathbf{u}_0 - ((tr(\nabla \boldsymbol{\varphi}_j)I - \nabla \boldsymbol{\varphi}_j^T)\nabla)p_0
\end{aligned}
\tag{55}
$$

for $j = 1, ..., L$, which is the second equation of Problem (28). In fact, relations (54) and (55) are equivalent since the random variables $\{Y_j\}$ are independent, with zero mean and unit variance and thus form an orthonormal set. The second equation of (7), corresponding to the incompressibility constraint, is treated analogously.

C Choice of the norm

We give here three *justifications* about the choice of the norm on the space $V \times Q$ for the couple $(\mathbf{u}, p)$, more precisely about the scaling with respect to the kinematic viscosity ν. We claim that the appropriate scaling is given by

$$|||\mathbf{v}, q|||_k^2 := \nu^k \|\nabla \mathbf{v}\|^2 + \nu^{k-2} \|q\|^2 \quad \text{for any choice } k = 0, 1, 2. \tag{56}$$

First of all, we can perform a dimensional analysis. The dimension unit of the kinematic viscosity is $[\nu] = \frac{m^2}{s}$ while we have, recall that p corresponds to the pressure divided by the density of the fluid,

$$[|\nabla \mathbf{u}|^2] = \left(\frac{1}{m} \cdot \frac{m}{s} \right)^2 = \frac{1}{s^2} \quad \text{and} \quad [p^2] = \left(\frac{N}{m^2} \cdot \frac{m^3}{kg} \right)^2 = \frac{m^4}{s^4},$$

from which we deduce that $[\nu^k |\nabla \mathbf{u}|] = [\nu^{k-2} p^2]$ for all k. This is also the natural choice of scaling that arises when looking at the *a priori* estimates on the solution $(\mathbf{u}, p)$ or when performing *a posteriori* error estimations. For simplicity, let us consider the (deterministic) Stokes problem given under the weak form by: find $(\mathbf{u}, p) \in V \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= F(\mathbf{v}) & \forall \mathbf{v} \in V \\ b(\mathbf{u}, q) &= 0 & \forall q \in Q, \end{aligned}$$

with $V = [H_0^1(D)]^d$, $Q = L_0^2(D)$, $a(\mathbf{u}, \mathbf{v}) = \nu \int_D \nabla \mathbf{u} : \nabla \mathbf{v}$, $b(\mathbf{v}, q) = -\int_D q \nabla \cdot \mathbf{v}$ and $F(\mathbf{v}) = \int_D \mathbf{f} \cdot \mathbf{v}$. The bilinear form a is continuous and coercive on V with constant ν and b is continuous on V with constant 1 and satisfy the inf-sup condition with constant $\beta = \beta(D)$. The problem is thus well-posed (see [9]) and the following *a priori* estimates are satisfied

$$\|\nabla \mathbf{u}\| \leq \frac{1}{\nu} \|\mathbf{f}\|_{V'} \quad \text{and} \quad \|p\| \leq \frac{1}{\beta} \left(\|\mathbf{f}\|_{V'} + \nu \|\nabla \mathbf{u}\| \right) \leq \frac{2}{\beta} \|\mathbf{f}\|_{V'}.$$

Therefore, we have

$$\nu^{k/2} \|\nabla \mathbf{u}\| + \nu^{k/2-1} \|p\| \leq C \nu^{k/2-1} \|\mathbf{f}\|_{V'} \quad \forall k,$$

where $C = (1 + 2/\beta)$ is independent of ν, which is consistant with the scaling (56). Finally, for the *a posteriori* error analysis, denoting $e = \mathbf{u} - \mathbf{u}_h$ and $E = p - p_h$ with $\mathbf{u}_h$ and p_h the finite element approximation of $\mathbf{u}$ and p, respectively, we have for any $(\mathbf{v}, q) \in V \times Q$

$$a(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, E) + b(\mathbf{e}, q) = R_1(\mathbf{v}) + R_2(q), \tag{57}$$

with

$$R_1(\mathbf{v}) := F(\mathbf{v}) - a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) \quad \text{and} \quad R_2(q) := -b(\mathbf{u}_h, q).$$

Using relation (57), Young's inequality and the properties of a and b, we can easily show that

$$\|E\| \leq \frac{1}{\beta} \|R_1\|_{V'} + \frac{\nu}{\beta} \|\nabla \mathbf{e}\| \tag{58}$$

and

$$\nu \|\nabla \mathbf{e}\|^2 \leq \frac{c_1}{\nu} \|R_1\|_{V'}^2 + \frac{c_2 \nu}{\beta^2} \|R_2\|_{Q'}^2 \tag{59}$$

with for instance $c_1 = c_2 = 3$, the value of these constants depending only on how we use Young's inequality. From the last two inequalities, we deduce that the scaling (56) should be used to get

$$\nu^k \|\nabla \mathbf{e}\|^2 + \nu^{k-2} \|E\|^2 \leq C \left(\nu^{k-2} \|R_1\|_{V'}^2 + \nu^k \|R_2\|_{Q'}^2 \right),$$

where C is a constant independent of ν (but which depends on the inf-sup constant β).

We mention that in a diffusion-dominating regime, the choice $k = 0$ yields a total error $\|\mathbf{e}, E\|_0$ which remains constant when ν varies. Indeed, in such a case the velocity error $\|\nabla \mathbf{e}\|$ is constant while the pressure error $\|E\|$ behaves as ν, i.e. $\frac{1}{\nu} \|E\|$ is constant.

D   Proof of some properties

Proposition D.1. *Let $A, B, C \in \mathbb{R}^{n \times n}$ be square matrices with coefficients denoted respectively by a_{ij}, b_{ij} and c_{ij} for $1 \le i, j \le n$, and let $\mathbf{w}$ be any smooth function with value in $\mathbb{R}^n$. We then have*

$$AB : CB = ABB^T : C \tag{60}$$

and

$$(B^T \nabla)\mathbf{w} = \nabla \mathbf{w} B. \tag{61}$$

Proof. We first show (60). For the term on the left-hand side, we have

$$AB : CB = \sum_{i,j=1}^{n} (AB)_{ij}(CB)_{ij} = \sum_{i,j=1}^{n} \left(\sum_{l=1}^{n} a_{il} b_{lj} \right) \left(\sum_{k=1}^{n} c_{ik} b_{kj} \right) = \sum_{i,j,k,l=1}^{n} a_{il} b_{lj} c_{ik} b_{kj},$$

while for the right-hand side, we get

$$ABB^T : C = \sum_{i,k=1}^{n} (ABB^T)_{ik}(C)_{ik} = \sum_{i,k=1}^{n} \sum_{j=1}^{n} (AB)_{ij}(B^T)_{jk}(C)_{ik} = \sum_{i,j,k,l=1}^{n} a_{il} b_{lj} b_{kj} c_{ik}.$$

We now prove (61). From the *definition* of the gradient operator applied to a vector field, we have

$$(B^T \nabla)\mathbf{w} = \begin{pmatrix} ((B^T \nabla)w_1)^T \\ \vdots \\ ((B^T \nabla)w_n)^T \end{pmatrix} = \begin{pmatrix} (B^T \nabla)_1 w_1 & \cdots & (B^T \nabla)_n w_1 \\ \vdots & \ddots & \vdots \\ (B^T \nabla)_1 w_n & \cdots & (B^T \nabla)_n w_n \end{pmatrix}$$

where w_i denotes the ith component of $\mathbf{w}$, and thus

$$\left[(B^T \nabla)\mathbf{w} \right]_{ij} = (B^T \nabla)_j (\mathbf{w})_i.$$

Therefore, the coefficient of the ith-row and j^{th}-column of the $n \times n$ matrix $(B^T \nabla)\mathbf{w}$ is given by

$$\left[(B^T \nabla)\mathbf{w} \right]_{ij} = \sum_{k=1}^{n} (B^T)_{jk}(\nabla)_k w_i = \sum_{k,l=1}^{n} b_{kj} \frac{\partial w_i}{\partial \xi_k} = \sum_{k=1}^{n} (\nabla \mathbf{w})_{ik}(B)_{kj} = (\nabla \mathbf{w} B)_{ij}.$$

$\square$

We now show the relation (6) used in Section 3 to write the strong formulation of the problem on D. It can be proven by an integration by part back on the random domain D_ω or using the Piola identity $\nabla \cdot (J_\mathbf{x} A^T) = \mathbf{0}$ (see [38] for instance). Indeed, we have

$$\int_D q|J_\mathbf{x}|(A^T \nabla) \cdot \mathbf{v} d\boldsymbol{\xi} = \int_{D_\omega} \tilde{q} \nabla_\mathbf{x} \cdot \tilde{\mathbf{v}} dx = -\int_{D_\omega} \nabla_\mathbf{x} \tilde{q} \cdot \tilde{\mathbf{v}} dx = -\int_D |J_\mathbf{x}|(A^T \nabla q) \cdot \mathbf{v} d\boldsymbol{\xi},$$

which yields (6) since $J_\mathbf{x}$ is either positive or negative, depending if the orientation is preserved or not by the mapping. Using the second alternative, since $\nabla \cdot (J_\mathbf{x} A \mathbf{v}) = (\nabla \cdot (J_\mathbf{x} A^T)) \cdot \mathbf{v} + (J_\mathbf{x} A^T \nabla) \cdot \mathbf{v}$ we have

$$\int_D q J_\mathbf{x}(A^T \nabla) \cdot \mathbf{v} d\boldsymbol{\xi} = \int_D q \nabla \cdot (J_\mathbf{x} A \mathbf{v}) d\boldsymbol{\xi} - \int_D \underbrace{(\nabla \cdot (J_\mathbf{x} A^T))}_{=\mathbf{0}} \cdot (q\mathbf{v}) d\boldsymbol{\xi}$$

$$= -\int_D J_\mathbf{x}(A^T \nabla q) \cdot \mathbf{v} d\boldsymbol{\xi}.$$

Be aware that in [38], the divergence operator applied to a tensor field is defined as the divergence applied to its transposed according to the definition used here. Recall that here we defined $\left[\nabla \cdot (J_\mathbf{x} A^T) \right]_i = \sum_{j=1}^{d} \frac{\partial}{\partial \xi_j} (J_\mathbf{x}(A^T)_{ij}) = \sum_{j=1}^{d} \frac{\partial}{\partial \xi_j} (J_\mathbf{x} \frac{\partial(\boldsymbol{\xi}_\omega)_j}{\partial x_i} \circ \mathbf{x}_\omega)$ for $i = 1, ..., d$. Moreover, we mention that the Piola identity, which is easily obtained for smooth functions, say C^2 functions, is still valid (in a weak sense) for less regular functions such as H^1 functions (see for instance [15, 5]).

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