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Ph. Blanc

http://mathicse.epfl.ch

Phone: +41 21 69 37648

Address:
EPFL - SB - MATHICSE (Bâtiment MA)
Station 8 - CH-1015 - Lausanne - Switzerland

Fax: +41 21 69 32545

Lower bound for the maximum of some derivative of Hardy's function

Philippe Blanc*

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Haute École d'Ingénierie et de Gestion
CH-1400 Yverdon-les-Bains

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Abstract

Under the Riemann hypothesis, we use the distribution of zeros of the zeta function to get a lower bound for the maximum of some derivative of Hardy's function.

1 Introduction and main results

To situate the problem we address here, we recall some classical results on the zeros of the Riemann zeta function.

We denote as usual by Z the Hardy function whose real zeros coincide with the zeros of ζ located on the line of real part $\frac{1}{2}$. If the Riemann hypothesis is true, what we assume from now on, then the number of zeros of Z in the interval $]0,t]$ is given by [9]

$$N(t) = \frac{t}{2\pi}\log\left(\frac{t}{2\pi e}\right) + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right) \tag{1.1}$$

where $S(t) = \dfrac{1}{\pi}\arg\zeta(\dfrac{1}{2}+it)$ if t is not a zero of Z and $\arg\zeta(\frac{1}{2}+it)$ is defined by continuous variation along the straight lines joining 2, $2+it$ and $\frac{1}{2}+it$ starting with the initial value $\arg\zeta(2)=0$. If t is a zero of Z we set $S(t) = \lim_{\epsilon\to 0_+} S(t+\epsilon)$. It is well known that

$$S(t) = O\Big(\frac{\log t}{\log\log t}\Big).$$

Given T such that $Z(T) \neq 0$ we denote by γ_k the real zeros of Hardy's function numbered so that $\ldots \leqslant \gamma_{-2} \leqslant \gamma_{-1} < T < \gamma_1 \leqslant \gamma_2 \leqslant \ldots$ Using the bound [3]

$$|S(t+h) - S(t)| \leqslant \left(\frac{1}{2} + o(1)\right)\frac{\log t}{\log\log t}$$

for $0 < h \leqslant \sqrt{t}$ where t is sufficiently large, we infer from (1.1) that

$$\mid \gamma_{\pm k} - T \mid \leqslant (k-1)\frac{\pi}{\log\sqrt{\frac{T}{2\pi}}} + \Big(\pi + o(1)\Big)\frac{1}{\log\log T} \quad \text{for } k = 1, 2, \cdots, l \tag{1.2}$$

where $l = \lfloor\sqrt{T}\rfloor$ and T is large. As a consequence of this relation and without using any other properties of the zeta function we prove :

*E-mail: philippe.blanc@heig-vd.ch

Theorem 1.1. If the Riemann hypothesis holds, then for any fixed $C > (2\log 2)^{-1}$ and any sufficiently large T, there exists $k \in \{1, 3, 5, \ldots, 2m-1, 2m\}$, where $m = \lfloor C \log T \log\log T \rfloor$, such that

$$\max_{t\in[T-2\pi,T+2\pi]} |Z^{(k)}(t)| \geqslant \Big(1 - \frac{\log\log\log T}{\log\log T}\Big)^k \Big(\log\sqrt{\frac{T}{2\pi}}\,\Big)^k |Z(T)|\,.$$

Theorem 1.1 is a consequence of the following result.

Theorem 1.2. Let $f \in C^\infty(\mathbb{R})$ satisfying $f(0) = 1$ and vanishing at x_k where the x_k are numbered taking into account their multiplicity and $x_{-n} \leqslant \ldots \leqslant x_{-1} < 0 < x_1 \leqslant \ldots \leqslant x_n$ and let s such that

$$|x_{\pm k}| \leqslant (k-1)\pi + s \quad \text{for} \quad k = 1, 2, \ldots, n.$$

We assume that there exists a constant $0 < c < 1$ such that $-a < x_{-n} \leqslant \ldots \leqslant x_n < a$ where $a = (n - \frac{1}{2})\frac{\pi}{c}$ and

$$|f^{(2j-1)}(\pm a)| \leqslant c^{2j-1} \quad \text{for} \quad j = 1, \ldots, m \quad \text{and} \quad \max_{x\in[-a,a]} |f^{(2m)}(x)| \leqslant c^{2m}$$

for some integer $m \geqslant n\log n$. Then for any sufficiently large n and any $0 < \epsilon < \log 2$ there exists $0 < c_\epsilon < 1$ depending only on ϵ such that if $c_\epsilon < c < 1 - \frac{1}{2n}$ then

$$s \geqslant (\log 2 - \epsilon)\,\frac{1-c}{|\log(1-c)|}\,n\,\pi\,. \tag{1.3}$$

This work stems from an observation of A.Ivič [6] about the values of the derivatives of Z in a neighborood of points where $|Z|$ reaches a large value. In [1] we made a first step toward the proof of Theorem 1.2 by solving a simpler problem of the same nature.
The organization of this paper is as follows : In Section 2 we prove the key identity, a property of the derivatives of Bernoulli polynomials and preparatory lemmas. The proofs of Theorem 1.1 and 1.2 are given in Section 3.
The notations used in this paper are standard : we denote by $\lfloor x \rfloor$ the usual floor function and we set $\{x\} = x - \lfloor x \rfloor$. As usual $B_n(x)$ and $T_n(x)$ stand for Bernoulli and Chebyshev polynomial of degree n.

2 Preliminary results

We first prove an identity which will be used later to establish a relation between the value of a function $f \in C^{2m}[-a,a]$ at 0, the zeros of f and the values of its derivatives of odd order on the boundaries of the interval.

Lemma 2.1. Let $-a < x_{-n} < \ldots < x_{-1} < x_0 < x_1 < \ldots < x_n < a$ and for $l = 1, 2, \ldots$ let Ψ_{2l-1} be the function defined on $[-a,a]$ by

$$\Psi_{2l-1}(x) = \frac{(4a)^{2l-1}}{(2l)!}\sum_{k=-n}^{n} \mu_k \Big(B_{2l}\big(\frac{1}{2} + \frac{x+x_k}{4a}\big) + B_{2l}\big(\Big\{\frac{x-x_k}{4a}\Big\}\big)\Big)$$

where $\sum_{k=-n}^{n} \mu_k = 0$. Then for $f \in C^{2m}[-a,a]$ where $m \geqslant 1$ we have the identity

$$\sum_{k=-n}^{n} \mu_k f(x_k) = \sum_{k=1}^{m} f^{(2k-1)}(a)\Psi_{2k-1}(a) - \sum_{k=1}^{m} f^{(2k-1)}(-a)\Psi_{2k-1}(-a) - \int_{-a}^{a} f^{(2m)}(x)\Psi_{2m-1}(x)\,dx. \tag{2.1}$$

Proof. By definition the function Ψ_{2m-1} is C^{2m-2}, piecewise polynomial and the relation $B_l'(x) = lB_{l-1}(x)$ for $l = 1, 2, \dots$ leads to

$$\Psi_{2m-1}^{(j)}(x) = \frac{(4a)^{2m-j-1}}{(2m-j)!} \sum_{k=-n}^{n} \mu_k \left(B_{2m-j}\big(\frac{1}{2} + \frac{x+x_k}{4a}\big) + B_{2m-j}\big(\big\{\frac{x-x_k}{4a}\big\}\big)\right)$$

for $j = 1, \dots, 2m-1$ and $x \neq x_k$ if $j = 2m-1$. This implies that

$$\Psi_{2m-1}^{(2m-2j)} = \Psi_{2j-1} \text{ for } j = 1, 2, \dots, m \tag{2.2}$$

and that

$$\begin{aligned}
\Psi_{2m-1}^{(2m-2j+1)}(\pm a) &= \frac{(4a)^{2j-2}}{(2j-1)!} \sum_{k=-n}^{n} \mu_k \left(B_{2j-1}\big(\frac{1}{2} + \frac{\pm a + x_k}{4a}\big) + B_{2j-1}\big(\big\{\frac{\pm a - x_k}{4a}\big\}\big)\right) \\
&= \frac{(4a)^{2j-2}}{(2j-1)!} \sum_{k=-n}^{n} \mu_k \left(B_{2j-1}\big(\frac{1}{2} + \frac{\pm a + x_k}{4a}\big) + B_{2j-1}\big(\frac{1}{2} - \frac{\pm a + x_k}{4a}\big)\right) \\
&= 0 \text{ for } j = 1, 2, \dots, m
\end{aligned} \tag{2.3}$$

since $B_{2l-1}(\frac{1}{2} + x) = -B_{2l-1}(\frac{1}{2} - x)$ for $l = 1, 2, \dots$ Further for $x \neq x_k$ we have

$$\begin{aligned}
\Psi_{2m-1}^{(2m-1)}(x) &= \sum_{k=-n}^{n} \mu_k \left(B_1\big(\frac{1}{2} + \frac{x+x_k}{4a}\big) + B_1\big(\big\{\frac{x-x_k}{4a}\big\}\big)\right) \\
&= \sum_{k=-n}^{n} \mu_k \left(\frac{x+x_k}{4a} + \big\{\frac{x-x_k}{4a}\big\} - \frac{1}{2}\right)
\end{aligned} \tag{2.4}$$

and as $\sum_{k=-n}^{n} \mu_k = 0$ the function $\Psi_{2m-1}^{(2m-1)}$ is piecewise constant. Explicitly, for $x \in]x_j, x_{j+1}[$ we get

$$\Psi_{2m-1}^{(2m-1)}(x) = \sum_{k=-n}^{j} \mu_k \big(\frac{x}{2a} - \frac{1}{2}\big) + \sum_{k=j+1}^{n} \mu_k \big(\frac{x}{2a} + 1 - \frac{1}{2}\big) = \sum_{k=j+1}^{n} \mu_k = -\sum_{k=-n}^{j} \mu_k$$

which leads to

$$\int_{x_j}^{x_{j+1}} f'(x)\Psi_{2m-1}^{(2m-1)}(x)\,dx = -\Big(\sum_{k=-n}^{j} \mu_k\Big)(f(x_{j+1}) - f(x_j)) \text{ for } j = -n, \dots, n-1.$$

Summing these equalities and using that $\Psi_{2m-1}^{(2m-1)} = 0$ on the intervals $[-a, x_{-n}[$ and $]x_n, a]$, which follows from (2.4), we have

$$\sum_{k=-n}^{n} \mu_k f(x_k) = \int_{-a}^{a} f'(x)\Psi_{2m-1}^{(2m-1)}(x)\,dx$$

and we complete the proof by integrating $2m-1$ times the right-hand side by parts taking into account relations (2.2) and (2.3). □

For further use we recall some elementary facts concerning the divided differences.

Lemma 2.2. Let $f \in C^{2n}]-T, T[$ and let g be the function defined for pairwise distinct numbers $t_{-n}, \dots, t_n \in]-T, T[$ by

$$g(t_{-n}, \dots, t_n) = \sum_{k=-n}^{n} \frac{f(t_k)}{\prod\limits_{\substack{-n \leqslant j \leqslant n \\ j \neq k}} (t_k - t_j)}.$$

Then

a) The function g has a continuous extension g^* defined for $t_{-n}, \dots, t_n \in]-T, T[$.

b) There exist $\eta = \eta(t_{-n}, \dots, t_n)$ and $\xi = \xi(t_{-n}, \dots, t_n) \in]-T, T[$ such that

$$g^*(t_{-n}, \dots, t_n) = \frac{f^{(2n)}(\eta)}{(2n)!} \quad \text{and} \quad \frac{\partial}{\partial t_i}\Big((t_i - t_j) g^*(t_{-n}, \dots, t_n)\Big) = \frac{f^{(2n)}(\xi)}{(2n)!} \quad \text{if} \quad i \neq j.$$

Moreover if $f \in C^{2n+1}]-T, T[$ there exists $\tau = \tau(t_{-n}, \dots, t_n) \in]-T, T[$ such that

$$\frac{\partial}{\partial t_i} g^*(t_{-n}, \dots, t_n) = \frac{f^{(2n+1)}(\tau)}{(2n+1)!}.$$

c) Let $y_0, y_1, \dots, y_l$ be the distinct values of $t_{-n}, \dots, t_n$ considered as fixed and let r_k be the number of index j such that $t_j = y_k$. Then there exist $\alpha_{k,i}$ depending on $y_0, y_1, \dots, y_l$ such that

$$g^*(t_{-n}, \dots, t_n) = \sum_{k=0}^{l} \sum_{i=0}^{r_k - 1} \alpha_{k,i} f^{(i)}(y_k).$$

Proof.

a) This is a consequence of the representation formula

$$g(t_{-n}, \dots, t_n) = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} f^{(2n)}\Big(t_{-n} + \sum_{k=1}^{2n} \tau_k (t_{-n+k} - t_{-n+k-1})\Big)\, d\tau_{2n}\,. \tag{2.5}$$

b) The first and last assertions follow from relation (2.5) together with the mean value theorem. Since divided differences are invariant by permutation it is sufficient to prove the second assertion for $i = n$ and $j = n - 1$. Multiplying (2.5) by $t_n - t_{n-1}$ and integrating with respect to τ_{2n} we have

$$(t_n - t_{n-1}) g^*(t_{-n}, \dots, t_n) =$$
$$\int_0^1 d\tau_1 \cdots \int_0^{\tau_{2n-2}} f^{(2n-1)}\Big(t_{-n} + \sum_{k=1}^{2n-1} \tau_k (t_{-n+k} - t_{-n+k-1}) + \tau_{2n-1}(t_n - t_{n-1})\Big) d\tau_{2n-1} -$$
$$\int_0^1 d\tau_1 \cdots \int_0^{\tau_{2n-2}} f^{(2n-1)}\Big(t_{-n} + \sum_{k=1}^{2n-1} \tau_k (t_{-n+k} - t_{-n+k-1})\Big) d\tau_{2n-1}$$

and therefore

$$\frac{\partial}{\partial t_n}\Big((t_n - t_{n-1}) g^*(t_{-n}, \dots, t_n)\Big) =$$
$$\int_0^1 d\tau_1 \cdots \int_0^{\tau_{2n-2}} f^{(2n)}\Big(t_{-n} + \sum_{k=1}^{2n-1} \tau_k (t_{-n+k} - t_{-n+k-1}) + \tau_{2n-1}(t_n - t_{n-1})\Big) \tau_{2n-1}\, d\tau_{2n-1}\,.$$

The use of the mean value theorem completes the proof.

c) The proof is given in [7].

□

In the next lemma we indicate the choice of coefficients μ_k for which the identity of Lemma 2.1 is of practical use for large values of a. The main reason of this choice will appear in the proof of c) of Lemma 2.7.

Lemma 2.3. Let ω and Ω be the sets defined by

$$\omega = \{(x_{-n},\dots,x_{-1},x_1,\dots,x_n) \in \mathbb{R}^{2n} \mid -a < x_{-n} < \dots < x_{-1} < 0 < x_1 < \dots < x_n < a\}$$

and

$$\Omega = \{(x_{-n},\dots,x_{-1},x_1,\dots,x_n) \in \mathbb{R}^{2n} \mid -a < x_{-n} \leqslant \dots \leqslant x_{-1} < 0 < x_1 \leqslant \dots \leqslant x_n < a\}$$

and let further Ψ_{2l-1} be the function defined on $\omega \times [-a,a]$ by

$$\Psi_{2l-1}(x_{-n},\dots,x_{-1},x_1,\dots,x_n,x) = \frac{(4a)^{2l-1}}{(2l)!}\sum_{k=-n}^{n}\mu_k\Big(B_{2l}\big(\frac{1}{2}+\frac{x+x_k}{4a}\big)+B_{2l}\big(\Big\{\frac{x-x_k}{4a}\Big\}\big)\Big)$$

where $x_0 = 0$, $\mu_k = \dfrac{\alpha_k}{\alpha_0}$ and

$$\alpha_k = \frac{1}{\prod\limits_{\substack{-n\leqslant j\leqslant n\\ j\neq k}}\Big(\sin(\pi\frac{x_k}{2a})-\sin(\pi\frac{x_j}{2a})\Big)} \qquad \text{for } k=-n,\dots,n.$$

Then

a) For $l \geqslant 1$ the functions $\Psi_{2l-1}(\cdot,\dots,\cdot,\pm a)$ have continuous extensions $\Psi^*_{2l-1}(\cdot,\dots,\cdot,\pm a)$ to Ω.

b) For $l \geqslant n+1$ the function Ψ_{2l-1} has a continuous extension Ψ^*_{2l-1} to $\Omega\times[-a,a]$.

c) Let $m \geqslant n+1$ and $f \in C^{2m}[-a,a]$ a function defined on $[-a,a]$ which vanishes at x_k where $a < x_{-n} \leqslant \dots \leqslant x_{-1} < 0 < x_1 \leqslant \dots \leqslant x_n < a$ and the x_k are numbered taking into account their multiplicity. Then we have the identity

$$f(0) = \sum_{k=1}^{m} f^{(2k-1)}(a)\Psi^*_{2k-1}(a) - \sum_{k=1}^{m} f^{(2k-1)}(-a)\Psi^*_{2k-1}(-a) - \int_{-a}^{a} f^{(2m)}(x)\Psi^*_{2m-1}(x)\,dx \qquad (2.6)$$

where for short $\Psi^*_{2k-1}(\pm a)$ and $\Psi^*_{2m-1}(x)$ stand for $\Psi^*_{2k-1}(\cdot,\dots,\cdot,\pm a)$ and $\Psi^*_{2m-1}(\cdot,\dots,\cdot,x)$.

Proof.

a) Introducing the function h defined by

$$h(t,x) = \frac{(4a)^{2l-1}}{(2l)!}\Big(B_{2l}\big(\frac{1}{2}+\frac{x}{4a}+\frac{1}{2\pi}\operatorname{Arcsin} t\big)+B_{2l}\big(\Big\{\frac{x}{4a}-\frac{1}{2\pi}\operatorname{Arcsin} t\Big\}\big)\Big)$$

we have

$$\Psi_{2l-1}(x_{-n},\dots,x_n,\pm a) = \frac{1}{\alpha_0}\sum_{k=-n}^{n}\alpha_k\, h(\sin(\pi\frac{x_k}{2a}),\pm a)$$

for $(x_{-n},\dots,x_n)\in\omega$ and the conclusion holds since the functions $h(\cdot,\pm a)$ belong to $C^\infty]-1,1[$.

b) By definition the function h belongs to $C^{2l-2}\big(]-1,1[\times[-a,a]\big)$ and the assertion is a consequence of the representation formula (2.5) since $2l-2 \geqslant 2n$.

c) For $(x_{-n},\dots,x_n)\in\omega$ the left-hand side of identity (2.1) writes

$$\frac{1}{\alpha_0}\sum_{k=-n}^{n}\alpha_k f\big(\frac{2a}{\pi}\operatorname{Arcsin}(\sin(\pi\frac{x_k}{2a}))\big)$$

and thanks to Lemma 2.2 this expression and hence the identity (2.1) extend to $(x_{-n},\dots,x_n)\in\Omega$. One completes the proof by observing, thanks to Lemma 2.2, that the left-hand side reduces to $f(0)$ when the x_k are zeros of multiplicity r_k of f.

□

The results stated in Lemma 2.4 play a central role in the proof of main properties of functions $\Psi^*_{2l-1}(\cdot,\dots,\cdot,\pm a)$.

Lemma 2.4. For all $m, k \in \mathbb{N}^*$ we have the inequality

$$(-1)^{m+1}\frac{d^k}{dx^k}B_{2m}(\,\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Arcsin}\sqrt{x}\,)>0 \quad \text{ for } x\in[0,1[\,.$$

The proof of Lemma 2.4 requires two technical results given in Sublemmas 2.5 and 2.6.

Sublemma 2.5. For all $k \in \mathbb{N}$ we have the Taylor expansion

$$(\mathrm{Arcsin}\, x)^{2k} = \sum_{l=0}^{\infty}\frac{(2k)!}{(2l)!}2^{2l-2k}\,b_{k\,,\,l}\,x^{2l} \quad \text{ for } x\in[-1,1]$$

where $b_{k\,,\,l}$ are integers defined recursively by

$$\begin{cases} b_{0\,,\,0}=1 \text{ and } b_{k\,,\,0} \;=\; b_{0\,,\,l}=0 & \text{for } k,l\geqslant 1\\ b_{k+1\,,\,l+1} \;=\; b_{k\,,\,l} \;+\; l^2 b_{k+1\,,\,l} & \text{for } k,l\geqslant 0\,. \end{cases}$$

Proof. We note first that the functions $f_{2k}(x) \stackrel{def}{=} (\mathrm{Arcsin}\,x)^{2k}$ satisfy

$$(1-x^2)f''_{2k+2}(x) - x\,f'_{2k+2}(x) - (2k+2)(2k+1)f_{2k}(x) = 0 \quad \text{ for } x\in\,]-1,1[.$$

From the definition of f_{2k} and the above equality it follows that numbers $c_{k\,,\,l}$ defined by

$$f_{2k}(x) = \sum_{l=0}^{\infty} c_{k\,,\,l}\,x^{2l} \quad \text{ for } \quad x\in[-1,1]$$

are uniquely determined by the recurrence relations

$$\begin{cases} c_{0\,,\,0} \;=\; 1 \text{ and } c_{k\,,\,0} \;=\; c_{0\,,\,l} \;=\; 0 \quad \text{for } k,\, l\geqslant 1\\ (2l+2)(2l+1)c_{k+1\,,\,l+1} - 4l^2 c_{k+1\,,\,l} - (2k+2)(2k+1)c_{k\,,\,l} = 0 \;\text{ for } k,\, l\geqslant 0\,. \end{cases}$$

A simple check shows that $c_{k\,,\,l} = \dfrac{(2k)!}{(2l)!}2^{2l-2k}b_{k\,,\,l}$. ☐

Sublemma 2.6. Let $b_{k\,,\,l}$ be the numbers defined in Sublemma 2.5. Then

$$\lim_{l\to\infty}\frac{b_{k\,,\,l}}{((l-1)!)^2} = \frac{\pi^{2k-2}}{(2k-1)!} \quad \text{for all } k\geqslant 1. \tag{2.7}$$

Proof. From the definition of numbers $b_{k\,,\,l}$ we infer that $b_{1\,,\,l}=((l-1)!)^2$ for $l\geqslant 1$. Thus relation (2.7) is trivially true for $k=1$. We then assume $k\geqslant 2$. As $b_{j\,,\,1}=0$ for $j\geqslant 2$ the numbers $d_{j\,,\,l}$ defined for $j,\,l\geqslant 1$ by $d_{j\,,\,l}=\dfrac{b_{j\,,\,l}}{((l-1)!)^2}$ satisfy the recurrence relations

$$\begin{cases} d_{j\,,\,1} \;=\; 0 \text{ and } d_{1\,,\,l} \;=1 & \text{for } j\geqslant 2 \text{ and } l\geqslant 1,\\ d_{j+1\,,\,l+1} = \dfrac{1}{l^2}d_{j\,,\,l} + d_{j+1\,,\,l} & \text{for } j,\, l\geqslant 1\,. \end{cases}$$

Using the fact that $d_{j-1\,,\,l} \;=\; 0$ for $l=1,\cdots,j-2$ we get first for $j\geqslant 2$ the equality

$$d_{j\,,\,n_j} = \sum_{n_{j-1}=j-1}^{n_j-1}\frac{1}{n_{j-1}^2}d_{j-1\,,\,n_{j-1}}$$

which we iterate to obtain

$$d_{k\,,l} = \sum_{n_{k-1}=k-1}^{l-1} \frac{1}{n_{k-1}^2} \sum_{n_{k-2}=k-2}^{n_{k-1}-1} \frac{1}{n_{k-2}^2} \cdots \sum_{n_2=2}^{n_3-1} \frac{1}{n_2^2} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^2}\,.$$

This leads to

$$\lim_{l\to\infty} d_{k\,,l} = \sum_{n_{k-1}>n_{k-2}>\cdots>n_2>n_1>0} \prod_{j=1}^{k-1} \frac{1}{n_j^2}$$

and we recognize in the right-hand side the number $\zeta(\{2\}_{(k-1)})$ whose value, given in [2], is equal to the right-hand side of (2.7) . ☐

Proof of Lemma 2.4. It suffices to prove that the numbers $e_{m\,,l}$ defined by

$$(-1)^{m+1} B_{2m}(\frac{1}{2} + \frac{1}{\pi}\text{Arcsin}\, x) = \sum_{l=0}^{\infty} e_{m\,,l}\, x^{2l} \tag{2.8}$$

satisfy $e_{m,\,l} > 0$ for all $m, l \in \mathbb{N}^*$. Using Taylor's formula and the evenness of function $B_{2m}(\frac{1}{2} + \frac{t}{\pi})$ we have

$$B_{2m}(\frac{1}{2} + \frac{t}{\pi}) = \sum_{k=0}^{m} \frac{1}{(2k)!} B_{2m}^{(2k)}(\frac{1}{2})\,(\frac{t}{\pi})^{2k} = \sum_{k=0}^{m} \binom{2m}{2k} B_{2m-2k}(\frac{1}{2})\,(\frac{t}{\pi})^{2k}$$

and the Taylor expansion of $(\text{Arcsin}\, x)^{2k}$ given in Sublemma 2.5 leads to

$$\begin{aligned} B_{2m}(\frac{1}{2} + \frac{1}{\pi}\text{Arcsin}\, x) &= \sum_{k=0}^{m} \left(\binom{2m}{2k} B_{2m-2k}(\frac{1}{2})\,\pi^{-2k} \sum_{l=0}^{\infty} \frac{(2k)!}{(2l)!} 2^{2l-2k} b_{k\,,l} x^{2l} \right) \\ &= \frac{(2m)!}{(2\pi)^{2m}} \sum_{k=0}^{m} \left(\frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2}) \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} b_{k\,,l} x^{2l} \right)\,. \end{aligned}$$

We then change the order of summation to get

$$(-1)^{m+1} B_{2m}(\frac{1}{2} + \frac{1}{\pi}\text{Arcsin}\, x) = \frac{(2m)!}{(2\pi)^{2m}} \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} f_{m\,,l} x^{2l} \tag{2.9}$$

where

$$f_{m\,,l} = (-1)^{m+1} \sum_{k=0}^{m} \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2})\, b_{k\,,l}\,.$$

We prove by recurrence over m that $f_{m\,,l} > 0$ for $m, l \geqslant 1$. To this end we set $g_{m,l} = \dfrac{f_{m,l}}{((l-1)!)^2}$ for $m, l \geqslant 1$ and since $b_{0\,,l} = 0$ for $l \geqslant 1$ we have

$$\begin{aligned} g_{m+1\,,l+1} &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2})\, b_{k\,,l+1} \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2}) \left(b_{k-1\,,l} + l^2\, b_{k\,,l} \right) \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}(\frac{1}{2})\, b_{k-1\,,l} + g_{m+1\,,l} \\ &= -\frac{(-1)^{m+1}}{(l!)^2} \sum_{k=0}^{m} \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k}(\frac{1}{2})\, b_{k\,,l} + g_{m+1\,,l} \\ &= -\frac{1}{l^2} g_{m\,,l} + g_{m+1\,,l} \end{aligned}$$

and this implies that

$$g_{m+1\,,\,l+1}+\frac{1}{l^2}g_{m\,,\,l}=g_{m+1\,,\,l}\quad\text{for } l\geqslant 1.$$

We have $g_{1\,,\,l}=f_{1\,,\,l}=1$ for all $l\geqslant 1$. Let us suppose that $g_{m\,,\,l}>0$ for all $l\geqslant 1$. Then $g_{m+1\,,\,l+1}<g_{m+1\,,\,l}$ and it follows that $g_{m+1\,,\,l}>\lim\limits_{l\to\infty} g_{m+1\,,\,l}$. Thanks to Sublemma 2.6 we have

$$\begin{aligned}\lim_{l\to\infty} g_{m+1\,,\,l} &= (-1)^{m+2}\sum_{k=1}^{m+1}\frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!}B_{2m+2-2k}(\frac{1}{2})\,\frac{\pi^{2k-2}}{(2k-1)!}\\ &= (-1)^{m+2}\pi^{2m}\sum_{k=1}^{m+1}\frac{2^{2m+2-2k}}{(2m+2-2k)!(2k-1)!}B_{2m+2-2k}(\frac{1}{2})\end{aligned}\tag{2.10}$$

and using $B_j(\frac{1}{2})=0$ for all odd j and the formula

$$B_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}B_j(x)y^{n-j}$$

we check that the sum which appears in (2.10) is equal to

$$\begin{aligned}\sum_{j=0}^{2m+1}\frac{2^j}{j!(2m+1-j)!}B_j(\frac{1}{2}) &= \frac{2^{2m+1}}{(2m+1)!}\sum_{j=0}^{2m+1}\binom{2m+1}{j}B_j(\frac{1}{2})\left(\frac{1}{2}\right)^{2m+1-j}\\ &= \frac{2^{2m+1}}{(2m+1)!}B_{2m+1}(1)=0\,.\end{aligned}$$

Hence $g_{m\,,\,l}>0$ for $m,l\geqslant 1$ and this implies, thanks to (2.9), that the numbers $e_{m\,,\,l}$ defined by (2.8) are positive for $m,l\geqslant 1$. □

We are now in position to prove main properties of functions $\Psi^*_{2l-1}(\cdot,\ldots,\cdot,\pm a)$.

Lemma 2.7. Let $\Psi^*_{2l-1}(\cdot,\ldots,\cdot,\pm a)$ be the functions defined in Lemma 2.3. Then

a) $(-1)^{n+l+1}\Psi^*_{2l-1}(x_{-n},\ldots,x_n,\pm a)>0$ for $(x_{-n},\ldots,x_n)\in\Omega$.

b) $(-1)^{n+l+1}\dfrac{\partial}{\partial x_i}\Psi^*_{2l-1}(x_{-n},\ldots,x_n,\pm a)$ is negative if $i=-n,\ldots,-1$ and positive if $i=1,\ldots,n$ for $(x_{-n},\ldots,x_n)\in\Omega$. [1]

c) For $(x_{-n},\ldots,x_n)\in\omega$ and c such that $0<c\,a<n\pi$ and $c\,a\neq j\,\pi$ where $j=1,\ldots,n-1$ we have the upper bound [2]

$$\begin{aligned}&\sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\ldots,x_n,a)c^{2k-1}+\sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\ldots,x_n,-a)c^{2k-1}\\ &\leqslant\Big((-1)^n\prod_{\substack{-n\leqslant j\leqslant n\\ j\neq 0}}\sin(\pi\frac{x_j}{2a})\Big)\Big(\frac{-1}{\sin(c\,a)}\sum_{k=-n}^{n}\frac{\cos(c\,x_k)}{\prod\limits_{\substack{-n\leqslant j\leqslant n\\ j\neq k}}\Big(\sin(\pi\frac{x_k}{2a})-\sin(\pi\frac{x_j}{2a})\Big)}\Big)\,.\end{aligned}\tag{2.11}$$

[1] By $\dfrac{\partial}{\partial x_i}\Psi^*_{2l-1}(\cdot,\ldots,\cdot,\pm a)$ we mean the continuous extension of $\dfrac{\partial}{\partial x_i}\Psi_{2l-1}(\cdot,\ldots,\cdot,\pm a)$ to Ω.

[2] It follows from the proof that the singularities of the right-hand side when $c\,a=j\,\pi$ for $j=1,\ldots,n-1$ are removable.

Proof.

a) For $(x_{-n},\ldots,x_n)\in\omega$ we have

$$\Psi^*_{2l-1}(x_{-n},\ldots,x_n,\pm a)=2\frac{(4a)^{2l-1}}{(2l)!}\sum_{k=-n}^{n}\mu_k\,B_{2l}\big(\frac{1}{2}+\frac{\pm a+x_k}{4a}\big)$$

since the function $B_{2m}\big(\frac{1}{2}+t\big)$ is even and then

$$(-1)^{n+l+1}\,\Psi^*_{2l-1}(x_{-n},\ldots,x_n,\pm a)$$

$$=2\frac{(4a)^{2l-1}}{(2l)!}\left(\frac{(-1)^n}{\alpha_0}\right)\sum_{k=-n}^{n}\alpha_k(-1)^{l+1}B_{2l}\big(\frac{1}{2}+\frac{\pm a+x_k}{4a}\big)\,.$$

The first two terms of the right-hand side are positive and the third term writes $\sum_{k=-n}^{n}\alpha_k h(\sin(\pi\frac{x_k}{2a}))$ where

$$h(t)=(-1)^{l+1}B_{2l}(\frac{3}{4}\pm\frac{1}{2\pi}\mathrm{Arcsin}\,t)\ \text{ for }\ t\in[-1,1].$$

The identity

$$\frac{3}{4}\pm\frac{1}{2\pi}\mathrm{Arcsin}\,t=\frac{1}{2}+\frac{1}{\pi}\mathrm{Arcsin}\sqrt{\frac{1\pm t}{2}}\quad\text{for }\ t\in[-1,1]$$

together with Lemma 2.4 show that $h^{(2n)}$ is positive on $]-1,1[$ and the conclusion holds by Lemma 2.2.

b) For $(x_{-n},\ldots,x_n)\in\omega$ and with the notations of a) we have

$$(-1)^{n+l+1}\frac{\partial}{\partial x_i}\Psi^*_{2l-1}(x_{-n},\ldots,x_n,\pm a)$$

$$=2\frac{(4a)^{2l-1}}{(2l)!}\left((-1)^n\prod_{\substack{-n\leqslant j\leqslant n\\ j\neq 0,i}}\sin(\pi\frac{x_j}{2a})\right)\frac{\partial}{\partial x_i}\big(\sin(\pi\frac{x_i}{2a})\sum_{k=-n}^{n}\alpha_k\,h(\sin(\pi\frac{x_k}{2a}))\big).$$

The first and the third term of the right-hand side are positive whereas the second is negative if $i\leqslant -1$ and positive if $i\geqslant 1$. The conclusion holds by Lemma 2.2.

c) The use of the Fourier series expansion

$$B_{2l}(x)=(-1)^{l+1}2((2l)!)\sum_{j=1}^{\infty}\frac{1}{(2j\pi)^{2l}}\cos(2j\pi x)\qquad\text{for }x\in[0,1]$$

and the identity $\cos\alpha+\cos\beta=2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ lead to the expression

$$\Psi^*_{2l-1}(x_{-n},\ldots,x_n,x)$$

$$=(-1)^{l+1}\,2\,\frac{(2a)^{2l-1}}{\alpha_0\pi^{2l}}\sum_{j=1}^{\infty}\frac{1}{j^{2l}}\Big(\sum_{k=-n}^{n}\alpha_k\,\cos(j\pi(\frac{1}{2}+\frac{x_k}{2a}))\Big)\cos(j\pi(\frac{1}{2}+\frac{x}{2a}))\,.$$

Using the identity $\cos(j\pi(\frac{1}{2}+y))=(-1)^jT_j(\sin(\pi y))$ and setting
$a_{j,k}=(-1)^j\,T_j(\sin(\pi\frac{x_k}{2a}))$ we have $\sum_{k=-n}^{n}\alpha_k\,a_{j,k}=0$ for $j=1,\ldots,2n-1$ and therefore

$$\Psi^*_{2l-1}(x_{-n},\ldots,x_n,x)=(-1)^{l+1}\,2\,\frac{(2a)^{2l-1}}{\alpha_0\pi^{2l}}\sum_{j=2n}^{\infty}\frac{1}{j^{2l}}\Big(\sum_{k=-n}^{n}\alpha_k\,a_{j,k}\Big)\cos(j\pi(\frac{1}{2}+\frac{x}{2a}))\,.\qquad(2.12)$$

It follows that

$$\sum_{k=1}^{\infty}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\ldots,x_n,\pm a)c^{2k-1}$$
$$=\frac{(-1)^n}{\alpha_0}\,2\sum_{p=2n}^{\infty}\sum_{j=-n}^{n}\alpha_j\;a_{p\,,\,j}\,(\mp 1)^p\,\frac{1}{2ac}\sum_{k=1}^{\infty}\left(\frac{2ac}{p\pi}\right)^{2k}$$

and by a) we have

$$\sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\ldots,x_n,a)c^{2k-1}+\sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\ldots,x_n,-a)c^{2k-1}$$

$$\begin{aligned}
&\leqslant \frac{(-1)^n}{\alpha_0}\,2\sum_{p=2n}^{\infty}\sum_{j=-n}^{n}\alpha_j\,a_{p\,,\,j}\,((-1)^p+1)\,\frac{2ac}{p^2\pi^2-4a^2c^2}\\
&= \frac{(-1)^n}{\alpha_0}\,2\sum_{q=n}^{\infty}\sum_{j=-n}^{n}\alpha_j\,a_{2q\,,\,j}\,\frac{ac}{q^2\pi^2-a^2c^2}\\
&= \frac{(-1)^n}{\alpha_0}\,2\sum_{q=1}^{\infty}\sum_{j=-n}^{n}\alpha_j\,a_{2q\,,\,j}\,\frac{ac}{q^2\pi^2-a^2c^2}\\
&= \frac{(-1)^n}{\alpha_0}\,2\sum_{q=1}^{\infty}\sum_{j=-n}^{n}\alpha_j\;\cos(2q\pi(\frac{1}{2}+\frac{x_j}{2a}))\,\frac{ac}{q^2\pi^2-a^2c^2}\\
&= \frac{(-1)^{n+1}}{\alpha_0}\,2\sum_{q=1}^{\infty}\sum_{j=-n}^{n}\alpha_j\,(-1)^{(q-1)}\;\cos(q\frac{\pi x_j}{a})\,\frac{ac}{q^2\pi^2-a^2c^2}
\end{aligned}$$

where we use successively the absolute convergence to change the order of summation and the relations $\displaystyle\sum_{j=-n}^{n}\alpha_j a_{2q\,,\,j}=0$ for $q=1,\cdots,n-1$.
We finally make use of the identity ([2], formula (17.3.10))

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2r^2-s^2}\cos(kx)=\frac{\pi}{2rs}\frac{\cos(\frac{s}{r}x)}{\sin(\pi\frac{s}{r})}-\frac{1}{2s^2}\quad\text{where}\quad\frac{s}{r}\notin\mathbb{Z}\quad\text{and}\quad-\pi\leqslant x\leqslant\pi$$

to check that the last term of the above equalities is equal to the right-hand side of (2.11).

□

The next step is to bound the right-hand side of (2.11) for particular values of x_k.

Lemma 2.8. For $\epsilon\in]0,\log 2[$ and $c\in]\frac{3}{4},1[$ we introduce $\delta=1-c$, $\eta=(\log 2-\epsilon)\frac{\delta}{|\log\delta|}$, $a=(n-\frac{1}{2})\frac{\pi}{c}$ and $s^*=\eta\,a$. Further let $x^*_0=0$ and $(x^*_{-n},\ldots,x^*_n)\in\omega$ such that

$$|x^*_{\pm k}|=(k-1)\pi+s^*\quad\text{for}\quad k=1,\ldots,n.$$

Then there exists $0<c_\epsilon<1$ such that for $c_\epsilon<c<1-\dfrac{1}{2n}$ we have

$$0<(-1)^n\prod_{\substack{-n\leqslant j\leqslant n\\ j\neq 0}}\sin(\pi\,\frac{x^*_j}{2a})\;<2^{-2n}\tag{2.13}$$

and

$$0<\frac{-1}{\sin(c\,a)}\sum_{k=-n}^{n}\frac{\cos(c\,x^*_k)}{\displaystyle\prod_{\substack{-n\leqslant j\leqslant n\\ j\neq k}}\Big(\sin(\pi\frac{x^*_k}{2a})-\sin(\pi\frac{x^*_j}{2a})\Big)}<2^{2n-1}\,.\tag{2.14}$$

The proof of (2.14) requires preliminary results which we state in the next sublemmas.

Sublemma 2.9.

a) Let $n \geqslant 2$ be even and let $0 \leqslant t_0 < t_1 < \ldots < t_n \leqslant 1$ and $0 \leqslant t_0^* < t_1^* < \ldots < t_n^* \leqslant 1$ such that

$$t_{\frac{n}{2}+j} + t_{\frac{n}{2}-j} = 1 \quad \text{and} \quad t^*_{\frac{n}{2}+j} + t^*_{\frac{n}{2}-j} = 1 \quad \text{for} \quad j = 0, 1, \ldots, \frac{n}{2}$$

and

$$t_{\frac{n}{2}+j} \leqslant t^*_{\frac{n}{2}+j} \quad \text{for} \quad j = 1, 2, \ldots, \frac{n}{2}.$$

Let further $f \in C[0,1] \bigcap C^{n+2}[0,1[$ such that $f^{(n+2)} \geqslant 0$ on $[0,1[$. Then

$$\sum_{k=0}^{n} \frac{f(t_k)}{\prod\limits_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (t_k - t_j)} \leqslant \sum_{k=0}^{n} \frac{f(t_k^*)}{\prod\limits_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (t_k^* - t_j^*)}.$$

b) Under the same assumptions on f, a similar result holds for odd integers n.

Proof. The proofs of a) and b) are similar so we only prove a). Let g be the function defined for pairwise distinct $x_0, \ldots, x_n \in [0,1]$ by

$$g(x_0, \ldots, x_n) = \sum_{k=0}^{n} \frac{f(x_k)}{\prod\limits_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (x_k - x_j)}.$$

Since divided differences are invariant by permutation and since f is continuous on $[0,1]$ it is sufficient to prove that

$$\frac{\partial}{\partial t_n} g(\frac{1}{2}, 1 - t_{\frac{n}{2}+1},\, t_{\frac{n}{2}+1}, 1 - t_{\frac{n}{2}+2},\, t_{\frac{n}{2}+2}, \ldots, 1 - t_{n-1},\, t_{n-1}, 1 - t_n,\, t_n) \geqslant 0$$

holds for pairwise distinct $t_{\frac{n}{2}+1}, \ldots, t_n \in]\frac{1}{2}, 1[$. We complete the proof using the representation formula and integrating by parts to get

$$\frac{\partial}{\partial t_n} g(\frac{1}{2}, 1 - t_{\frac{n}{2}+1},\, t_{\frac{n}{2}+1}, 1 - t_{\frac{n}{2}+2},\, t_{\frac{n}{2}+2}, \ldots, 1 - t_{n-1},\, t_{n-1}, 1 - t_n,\, t_n) =$$

$$\int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} f^{(n+1)}\,(\frac{1}{2} + \tau_1(\frac{1}{2} - t_{\frac{n}{2}+1}) + \cdots + \tau_{n-1}(1 - t_n - t_{n-1}) + \tau_n(2t_n - 1))(2\tau_n - \tau_{n-1})\, d\tau_n \quad =$$

$$\int_0^1 d\tau_1 \cdots \int_0^{\tau_{n-1}} f^{(n+2)}\,(\frac{1}{2} + \tau_1(\frac{1}{2} - t_{\frac{n}{2}+1}) + \cdots + \tau_n(2t_n - 1))(2t_n - 1)\tau_n(\tau_{n-1} - \tau_n)\, d\tau_n \quad \geqslant 0\,.$$

□

Sublemma 2.10. Let n be a positive integer and $t_k^* = \sin^2(k\frac{\pi}{2n})$ for $k = 0, 1, \ldots, n$. Then

$$\Big| \sum_{k=0}^{n} \frac{\cos((2n-1)\mathrm{Arcsin}\sqrt{t_k^*}\,)}{\prod\limits_{\substack{0 \leqslant j \leqslant n \\ j \neq k}} (t_k^* - t_j^*)} \Big| \leqslant 2^{2n-1}. \tag{2.15}$$

Proof. Setting $\gamma_k = \Big(\prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}} (t_k^* - t_j^*)\Big)^{-1}$ we have $\gamma_0 = (-1)^n \dfrac{2^{2n-2}}{n}$ and $\gamma_n = (-1)^n \gamma_0$.

Further for $k = 1, 2, \ldots, n-1$, using the identity $\sin^2 a - \sin^2 b = \sin(a-b)\sin(a+b)$, we have

$$\begin{aligned}
\sin(k\frac{\pi}{n})\frac{1}{\gamma_k} &= \prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}} \sin((k-j)\frac{\pi}{2n}) \prod_{0\leqslant j\leqslant n} \sin((k+j)\frac{\pi}{2n})\\
&= (-1)^{n-k} \prod_{1\leqslant l\leqslant k} \sin(l\frac{\pi}{2n}) \prod_{1\leqslant l\leqslant n-k} \sin(l\frac{\pi}{2n}) \prod_{k\leqslant l\leqslant n} \sin(l\frac{\pi}{2n}) \prod_{n-k<j\leqslant n} \sin(\pi-(k+j)\frac{\pi}{2n})\\
&= \Big((-1)^{n-k} \prod_{1\leqslant l\leqslant n} \sin^2(l\frac{\pi}{2n})\Big)\Big(\sin(k\frac{\pi}{2n})\,\sin((n-k)\frac{\pi}{2n})\Big) = \sin(k\frac{\pi}{n})\frac{(-1)^k}{2\gamma_0}
\end{aligned}$$

and therefore $\gamma_k = (-1)^k 2\gamma_0$. We bound trivially the right-hand side of (2.15) to complete the proof. □

Sublemma 2.11. Let n be a positive integer and $f(t) = \cos((2n-1)\mathrm{Arcsin}\sqrt{t}\,)$. Then

$$(-1)^n f^{(k)} > 0 \ \text{ on } \ [0,1[\ \text{ for } \ k = n, n+1, \ldots$$

Proof. We have $f(t) = F(n-\frac{1}{2}, -n+\frac{1}{2}, \frac{1}{2}, t)$ where F is Gauss' hypergeometric function [4] and this implies that $f(t) = \sum_{k=0}^{\infty} \dfrac{\alpha_k}{(2k)!}\, t^k$ where

$$\alpha_k = \prod_{j=0}^{k-1} (4j^2 - (2n-1)^2)\,.$$

Hence $(-1)^n \alpha_k > 0$ for $k \geqslant n$ and this leads to $(-1)^n f^{(k)} > 0$ on $[0,1[$ for $k \geqslant n$. □

Proof of Lemma 2.8. To prove (2.13) we introduce the functions $g(t) = \log\sin(\frac{\pi}{2}t)$ and $G(t) = \int_0^t g(\tau)\,d\tau$. Since $g' \geqslant 0$ and $g'' < 0$ on $]0,1]$ we have

$$\begin{aligned}
\sum_{k=1}^{n} g(\frac{(k-1)\pi + s^*}{a}) &\leqslant \int_0^{n-\frac{1}{2}} g(\frac{x\pi + s^*}{a})\,dx + \frac{1}{2}\,g(\frac{s^*}{a}) = \frac{a}{\pi}\int_\eta^{1-\delta+\eta} g(t)\,dt + \frac{1}{2}\,g(\eta)\\
&= \frac{a}{\pi}(G(1-\delta+\eta) - G(\eta)) + \frac{1}{2}\,g(\eta)
\end{aligned}$$

and therefore

$$\begin{aligned}
\log\Big((-1)^n \prod_{\substack{-n\leqslant k\leqslant n\\ k\neq 0}} \sin(\frac{\pi}{2}\frac{x_k^*}{a})\Big) &\leqslant 2\sum_{k=1}^{n} g(\frac{(k-1)\pi + s^*}{a}) \leqslant \frac{2a}{\pi}\,(G(1-\delta+\eta) - G(\eta)) + g(\eta)\\
&= \frac{2n-1}{c}\,(G(1-\delta+\eta) - G(\eta)) + g(\eta)\\
&= \frac{2n-1}{c}\,(c\log 2 + G(1-\delta+\eta) - G(\eta)) - (2n-1)\log 2 + g(\eta)\\
&= \frac{2n-1}{c}\ h(\delta) - (2n-1)\log 2 + g(\eta)
\end{aligned}$$

where

$$h(\delta) = (1-\delta)\log 2 + G(1-\delta + (\log 2 - \epsilon)\frac{\delta}{|\log\delta|}) - G((\log 2 - \epsilon)\frac{\delta}{|\log\delta|}).$$

As $G(1) = -\log 2$ we have $\lim_{\delta\to 0_+} h(\delta) = 0$ and $\lim_{\delta\to 0_+} h'(\delta) = -\epsilon$ and there exists $0 < \delta_\epsilon < \frac{1}{4}$ such that $h < 0$ on $]0, \delta_\epsilon[$. Moreover $g(\eta) < g(\frac{1}{8}) < \log\frac{1}{2}$ and the bound (2.13) holds with $c_\epsilon = 1 - \delta_\epsilon$.
To check (2.14) we set $t_k(s) = \sin^2(\pi\frac{(k-1)\pi+s}{2a})$ for $s > 0$ and $k = 1, 2, \ldots, n$ and we introduce the function

$$\phi(y_0, \ldots, y_n) = (-1)^n \sum_{k=0}^{n} \frac{\cos((2n-1)\mathrm{Arcsin}\sqrt{y_k})}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}} \Big(y_k - y_j\Big)}$$

defined for $0 \leqslant y_0 < y_1 < \ldots < y_n \leqslant 1$.
Using Lemma 2.2 and Sublemma 2.11 we easily check that $\phi > 0$ and ϕ is increasing in each argument. Further $x^*_{-k} = -x^*_k$ and $cx^*_k = \dfrac{2ca}{\pi}\mathrm{Arcsin}\sqrt{t_k(s^*)} = (2n-1)\mathrm{Arcsin}\sqrt{t_k(s^*)}$ for $k = 1, \ldots, n$ and we have

$$\frac{-1}{\sin(c\,a)} \sum_{k=-n}^{n} \frac{\cos(c\,x^*_k)}{\prod\limits_{\substack{-n\leqslant j\leqslant n\\ j\neq k}} \Big(\sin(\pi\frac{x^*_k}{2a}) - \sin(\pi\frac{x^*_j}{2a})\Big)} = \frac{-1}{\sin(c\,a)} \sum_{k=0}^{n} \frac{\cos(c\,x^*_k)}{\prod\limits_{\substack{0\leqslant j\leqslant n\\ j\neq k}} \Big(\sin^2(\pi\frac{x^*_k}{2a}) - \sin^2(\pi\frac{x^*_j}{2a})\Big)}$$

$$= \phi(0, t_1(s^*), \ldots, t_n(s^*)) \leqslant \phi(0, t_1(s^*), \ldots, t_{n-1}(s^*), 1)$$

since $\sin(ca) = (-1)^{n+1}$.
Assume that n is even. Then

$$t_{\frac{n}{2}-j}(s^*) + t_{\frac{n}{2}+j}(s^*) = 1 - \cos(\pi\frac{(\frac{n}{2}-1)\pi + s^*}{a})\cos(\pi\frac{j\pi}{a}) < 1 \quad \text{for} \quad j = 0, 1, \ldots, \frac{n}{2} - 1$$

since

$$\frac{(\frac{n}{2}-1)\pi + s^*}{a} < \frac{c}{2} + \frac{s^*}{a} < \frac{1}{2} - \frac{\delta}{2} + \log 2\,\frac{\delta}{|\log\delta|} = \frac{1}{2} + \frac{\delta}{2}(\frac{2\log 2}{|\log\delta|} - 1) < \frac{1}{2}$$

as $0 < \delta < \frac{1}{4}$. Now we choose s^{**} such that

$$\frac{(\frac{n}{2}-1)\pi + s^{**}}{a} = \frac{1}{2}$$

and since $s^* < s^{**}$ and $t_j(s^*) < t_j(s^{**})$ we get

$$\phi(0, t_1(s^*), \ldots, t_{n-1}(s^*), 1) < \phi(0, t_1(s^{**}), \ldots, t_{n-1}(s^{**}), 1). \tag{2.16}$$

As $c < 1 - \frac{1}{2n}$ we have $a > n\pi$ and $t_{\frac{n}{2}+j}(s^{**}) = \sin^2(\pi(\frac{1}{4} + j\frac{\pi}{2a})) < t^*_{\frac{n}{2}+j}$ for $j = 1, 2, \ldots, \frac{n}{2} - 1$ where $t^*_k = \sin^2(k\frac{\pi}{2n})$. We complete the proof using Sublemmas 2.9 and 2.10 to bound the right-hand side of (2.16). If n is an odd integer a similar proof holds. □

The last point is to bound the integral which appears in the right-hand side of the identity (2.6). This is the content of Lemma 2.13.

Sublemma 2.12. Let $b_{n,l}$ the numbers defined for integers $n \geq 1$ and $l \geq 0$ by

$$b_{n,l} = \Big(\frac{2n}{2n+l}\Big)^{2n\log n - 1}\binom{4n+l-1}{l}.$$

Then there exists a constant C such that $\sum\limits_{l=0}^{\infty} b_{n,l} \leqslant C$ for all $n \geq 10$.

Proof. Since

$$b_{n,l} = \Big(\frac{2n}{2n+l}\Big)^{2n\log n - 1} \frac{\Gamma(4n+l)}{\Gamma(l+1)\Gamma(4n)}$$

we have, using Stirling's formula

$$b_{n,l} = O_n(l^{-2n\log n + 4n}) = O_n(l^{-2})$$

for $n \geqslant 10$ and there exists a constant C such that $\sum_{l=0}^{\infty} b_{10,l} \leqslant C$. We now show that $b_{n,l} \leqslant b_{10,l}$ for $n \geqslant 10$. We have $\log b_{n,l} = g(n,l)$ where the function g is defined for $(x,y) \in [1,\infty[\times[0,\infty[$ by

$$g(x,y) = (2x\log x - 1)\log(\frac{2x}{2x+y}) + \log\Gamma(4x+y) - \log\Gamma(y+1) - \log\Gamma(4x).$$

Straightforward computations lead to

$$\frac{\partial g}{\partial x}(x,y) = (2\log x + 2)\log(\frac{2x}{2x+y}) + (2x\log x - 1)\frac{y}{x(2x+y)} + 4\Psi(4x+y) - 4\Psi(4x)$$

and

$$\frac{\partial^2 g}{\partial y \partial x}(x,y) = -\frac{2(1+2x+y+y\log x)}{(2x+y)^2} + 4\Psi'(4x+y)$$

where Ψ is the derivative of $\log\Gamma$. From now on we assume that $(x,y) \in [10,\infty[\times[0,\infty[$. We have $\frac{\partial g}{\partial x}(x,0) = 0$ and since $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$ we get $\Psi'(z) \leqslant \frac{1}{z} + \frac{1}{z^2}$ for $z > 0$ and therefore

$$\begin{aligned}\frac{\partial^2 g}{\partial y \partial x}(x,y) &\leqslant -\frac{2(1+2x+y+y\log x)}{(2x+y)^2} + \frac{4}{4x+y} + \frac{4}{(4x+y)^2} \\ &= \frac{2(-8x^2+y^2+8x^2y+6xy^2+y^3)}{(2x+y)^2(4x+y)^2} - \frac{2y\log x}{(2x+y)^2} \leqslant \frac{2y(1-\log x)}{(2x+y)^2} \leqslant 0\,.\end{aligned}$$

Hence $\frac{\partial g}{\partial x}(x,y) \leqslant \frac{\partial g}{\partial x}(x,0) = 0$ and this implies that $g(x,y) \leqslant g(10,y)$ and hence $b_{n,l} \leqslant b_{10,l}$ for $n \geqslant 10$. □

Lemma 2.13. Let n and m be integers such that $n \geqslant 10$ and $m \geqslant n\log n$ and let further Ψ^*_{2l-1} be the function defined in Lemma 2.3. Then there exists a constant C such that

$$\Big|\Psi^*_{2m-1}(x_{-n},\ldots,x_n,x)\Big| \leqslant \frac{2^{2n-1}}{\alpha_0\, a}\Big(\frac{a}{n\pi}\Big)^{2m} C$$

for $(x_{-n},\ldots,x_n,x) \in \Omega \times [-a,a]$.

Proof. For $(x_{-n},\ldots,x_n,x) \in \omega \times [-a,a]$ we use relation (2.12) and Lemma 2.2 to get

$$\Big|\Psi^*_{2m-1}(x_{-n},\ldots,x_n,x)\Big| \leqslant 2\,\frac{(2a)^{2m-1}}{\alpha_0\pi^{2m}} \sum_{j=2n}^{\infty} \frac{1}{j^{2m}} \frac{\Big|T_j^{(2n)}(\tau_j)\Big|}{(2n)!}$$

for some $\tau_j \in]-1,1[$. It is well known that [8]

$$\max_{-1\leqslant x\leqslant 1}\Big|T_j^{(2n)}(x)\Big| = T_j^{(2n)}(1) = 2^{2n-1}\,(2n-1)!\,j\binom{2n+j-1}{j-2n}$$

for $j = 2n, 2n+1, \ldots$ and therefore

$$\Big|\Psi^*_{2m-1}(x_{-n},\ldots,x_n,x)\Big| \leqslant \frac{2^{2n-1}}{\alpha_0\, a}\Big(\frac{a}{n\pi}\Big)^{2m} \sum_{j=2n}^{\infty}\Big(\frac{2n}{j}\Big)^{2m-1}\binom{2n+j-1}{j-2n}.$$

We set $j = 2n + l$ and since $m \geqslant n \log n$ we have

$$\begin{aligned}\left|\Psi^*_{2m-1}(x_{-n}, \ldots, x_n, x)\right| &\leqslant \frac{2^{2n-1}}{\alpha_0\, a}\Big(\frac{a}{n\pi}\Big)^{2m} \sum_{l=0}^{\infty} \Big(\frac{2n}{2n+l}\Big)^{2m-1} \binom{4n+l-1}{l} \\ &\leqslant \frac{2^{2n-1}}{\alpha_0\, a}\Big(\frac{a}{n\pi}\Big)^{2m} \sum_{l=0}^{\infty} b_{n,l} \leqslant \frac{2^{2n-1}}{\alpha_0\, a}\Big(\frac{a}{n\pi}\Big)^{2m} C\end{aligned}$$

thanks to Sublemma 2.12. □

3 Proofs of Theorems and Conclusion

Proof of Theorem 1.1. Assume that Theorem 1.1 is not true. Then there exist $(2\log 2)^{-1} < C < 1$ and arbitrary large T such that

$$\max_{t\in[T-2\pi,T+2\pi]} |Z^{(k)}(t)| < \Big(1 - \frac{\log\log\log T}{\log\log T}\Big)^k \Big(\log\sqrt{\frac{T}{2\pi}}\,\Big)^k |Z(T)|$$

for $k = 1, 3, 5, \ldots, 2m-1,\ 2m$ where $m = \lfloor C \log T \log\log T\rfloor$. For such a T we have $Z(T) \neq 0$ and we introduce the function f defined by

$$f(t) = \frac{Z(T + \dfrac{t}{\log\sqrt{\frac{T}{2\pi}}})}{Z(T)}\,.$$

It satisfies $f(0) = 1$ and its zeros $x_{\pm k}$ numbered such that $\ldots \leqslant x_{-1} < 0 < x_1 \leqslant \ldots$ are such that

$$|\, x_{\pm k} \,| \leqslant (k-1)\,\pi + \Big(\frac{\pi}{2} + o(1)\Big)\frac{\log T}{\log\log T} \quad \text{for } k = 1, 2, \cdots, \lfloor\sqrt{T}\rfloor.$$

One easily checks that function f satisfies the assumptions of Theorem 1.2 for large T with

$$n = \lfloor C \log T\rfloor\,, \qquad c = 1 - \frac{\log\log\log T}{\log\log T} \qquad s = \Big(\frac{\pi}{2} + o(1)\Big)\frac{\log T}{\log\log T}$$

and relation (1.3) leads to

$$\Big(\frac{\pi}{2} + o(1)\Big)\frac{\log T}{\log\log T} \geqslant (\log 2 - \epsilon)\frac{\lfloor C\log T\rfloor}{\log\log T}\pi$$

which is a contradiction for ϵ sufficiently small and T sufficiently large. □

Proof of Theorem 1.2. Let f be a function satisfying the assumptions of Theorem 1.2 for the parameters n, c and s. We fix $0 < \epsilon < \log 2$ and assume that $c_\epsilon < c < 1 - \dfrac{1}{2n}$ where c_ϵ is defined in Lemma 2.8. We set $a = (n - \frac{1}{2})\frac{\pi}{c}$ and we suppose that $s < s^*$ where

$$s^* = (\log 2 - \epsilon)\frac{1-c}{|\log(1-c)|}\, a$$

to get a contradiction. Introducing $x^*_k = \operatorname{sgn}(k)((|k|-1)\pi + s^*)$ with $\operatorname{sgn}(0) = 0$ and using identity (2.6) and Lemmas 2.8 and 2.13 we have

$$\begin{aligned} f(0) &\leqslant \sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\dots,x_n,a)c^{2k-1} + \sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x_{-n},\dots,x_n,-a)c^{2k-1} \\ &+ \Big|\int_{-a}^{a} f^{(2m)}(x)\Psi^*_{2m-1}(x_{-n},\dots,x_n,x)\,dx\Big| \\ &\leqslant \sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x^*_{-n},\dots,x^*_n,a)c^{2k-1} + \sum_{k=1}^{m}(-1)^{n+k+1}\Psi^*_{2k-1}(x^*_{-n},\dots,x^*_n,-a)c^{2k-1} \\ &+ 2^{2n}\Big|\prod_{\substack{-n\leqslant j\leqslant n\\ j\neq 0}}\sin(\pi\frac{x^*_j}{2a})\Big|\Big(1-\frac{1}{2n}\Big)^{2n\log n}C \\ &< \frac{1}{2}+\Big(1-\frac{1}{2n}\Big)^{2n\log n}C<1 \end{aligned}$$

for n sufficiently large. This is a contradiction since $f(0)=1$ and therefore

$$s \geqslant s^* \geqslant (\log 2-\epsilon)\frac{1-c}{|\log(1-c)|}\,n\,\pi$$

as $a > n\pi$ since $c < 1-\frac{1}{2n}$. □

The bound given in Lemma 2.13 does not take into account the repartition of x_k and Theorem 1.2 should hold under the weaker assumption $m \geqslant n$. It should also be possible to use deep properties of the argument of the zeta function to get a stronger version of Theorem 1.1.

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