Well-posedness, regularity, and convergence analysis of the finite element approximation of a generalized Robin boundary value problem

Takahito Kashiwabara, Claudia M. Colciago, Luca Dedè, Alfio Quarteroni

WELL-POSEDNESS, REGULARITY, AND CONVERGENCE ANALYSIS OF THE FINITE ELEMENT APPROXIMATION OF A GENERALIZED ROBIN BOUNDARY VALUE PROBLEM

TAKAHITO KASHIWABARA[†], CLAUDIA MARIA COLCIAGO[‡], LUCA DEDÈ[‡], AND ALFIO QUARTERONI[‡§]

Abstract. In this paper, we propose the mathematical and finite element analysis of a second order Partial Differential Equation endowed with a generalized Robin boundary condition which involves the Laplace–Beltrami operator, by introducing a function space $H^1(\Omega; \Gamma)$ of $H^1(\Omega)$-functions with $H^1(\Gamma)$-traces, where $\Gamma \subseteq \partial\Omega$. Based on a variational method, we prove that the solution of the generalized Robin boundary value problem possesses a better regularity property on the boundary than in the case of the standard Robin problem. We numerically solve generalized Robin problems by means of the finite element method with the aim of validating the theoretical rates of convergence of the error in the norms associated to the space $H^1(\Omega; \Gamma)$.

Key words. generalized Robin boundary conditions; Laplace-Beltrami operator; Poisson equation; well-posedness; regularity of solution; finite element method; isoparametric analysis; a-priori error estimation

AMS subject classifications. 35J15, 65N30

1. Introduction. When modelling physical phenomena through Partial Differential Equations (PDEs), the introduction of non-standard boundary conditions may be required. This is the case for example of the following Poisson equation, endowed with a boundary condition involving the Laplace-Beltrami operator:

$$
\begin{cases}
-\Delta u = f & \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \boldsymbol{n}} + \alpha u - \beta \Delta_\Gamma u = h & \text{on} \quad \Gamma = \partial\Omega,
\end{cases}
\tag{1.1}
$$

where α, β are positive constants and f, h are given functions. We refer to problem (1.1) as a generalized Poisson (GR-P) problem, which is the focus of the paper. We remark that $(1.1)_2$ is the usual Robin boundary condition augmented with an extra boundary stiffness term of second order (higher-order versions could also be considered); we refer to such a boundary condition as a *generalized* Robin one.

The problem described by (1.1) has some backgrounds in physical modeling and mathematical analysis. For example, in [26] it is proposed, among other possibilities, an unsteady version problem of (1.1) to model a heat conduction process with a heat source on the boundary. The resulting boundary condition is known as the Goldstein-Wentzell condition, which is studied in [45, 47], in the simplified case $f \equiv 0$ (note that this assumption also influences the boundary condition). When f is arbitrary, the unsteady version of (1.1) is analyzed only in the framework of maximal regularity for parabolic equations with dynamic boundary conditions (see [13, 40]). Generalized Robin boundary conditions are also used in [24] in the context of Schröedinger operators. A mathematical analysis of (1.1) is proposed by J. L. Lions in [34] for

---

[†]Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan (tkashiwa@ms.u-tokyo.ac.jp).

[‡]CMCS–MATHICSE–SB, École Polytechnique Fédérale de Lausanne, Av. Picard, Station 8, Lausanne, CH–1015, Switzerland
(claudia.colciago@epfl.ch, luca.dede@epfl.ch, alfio.quarteroni@epfl.ch).

[§]MOX, Department of Mathematics, Politecnico di Milano, Piazza L. da Vinci 32, Milano, 20133, Italy (on leave).

a computational domain representing the half space $\mathbb{R}^N \cap \{x_N > 0\}$ (see Example III.2.6 of [38]), featuring a flat boundary. However, to the best of our knowledge, a full theoretical analysis of the finite element approximation of problem (1.1) has never been considered in literature.

Another motivation for the study of the (GR-P) (1.1) consists in the fact that it represents a simplified version of reduced fluid-structure interaction problems arising for example in hemodynamics applications. When modeling the blood flow in arteries the structure is represented by the vessel wall which delimitates the computational domain of the fluid. In this framework, under the assumptions of a thin wall structure, small displacements and linear stress-strain constitutive relation, reduced FSI models, e.g. the so-called coupled momentum method (see [19]), have been developed by using membrane models for the structure defined on the boundary of the computational domain of the fluid. More specifically, by the coupling conditions between fluid velocity and structural displacement it is possible to write the structural stress tensor in terms of fluid dynamics quantities. The resulting reduced model is a set of fluid dynamics equations for which the structural membrane model is incorporated by means of high-order generalized Laplace-Beltrami boundary conditions, defined by the constitutive stress-strain relation. The reduced model involves therefore generalized Robin boundary conditions of which $(1.1)_2$ represents a simplified version. So far, both full and reduced FSI models have been successfully used to solve problems in hemodynamics (see [2, 9, 11, 14, 18, 33, 37, 46, 41]). The theoretical analysis of FSI models with a membrane structure has been addressed in [7, 16, 17, 28, 29] where the equations of the fluid and structural models are treated in two separate partial differential equations. At the best of our knowledge, the theoretical analysis of the fluid dynamics equations with the embedding of the structural model as a generalized Robin boundary conditions have not been carried out yet. Thus, in this framework, we believe that the theoretical and numerical study of (GR-P) provides an insight into the properties of such reduced FSI problems.

Generalized Robin boundary conditions have a relevance also in the context of domain decomposition methods (see [23, 36, 44]) and, in particular, in the so-called Schwarz waveform relaxation algorithm (see [21, 30]). Different types of transmission conditions between subdomains have been studied in order to speed up the convergence of the subdomain iterative algorithm (see e.g. [22]). As a matter of fact, one of the most effective transmission conditions involve Robin-type boundary terms including tangential gradient and Laplace-Beltrami operators evaluated at the interfaces between subdomains. The mathematical analysis of the subproblem arising from the Schwartz waveform relaxation algorithm has been proposed in [4] but only for the half space domain with flat boundary. Moreover, in [4] the convergence of the subdomain iterative algorithm is studied, however no analysis of the finite element discretization on the subdomains was carried out.

In this paper, we propose a theoretical analysis of the (GR-P) (1.1) with generalized Robin boundary conditions in terms of existence, uniqueness and regularity of a weak solution, and convergence of the finite element approximation. The theoretical results are verified by means of numerical tests. In this analysis, we are facing the difficulty of dealing with the generalized Robin condition which involves higher-order derivatives on Γ resulting from the Laplace-Beltrami operator. Consequently, the $H^{1/2}(\Gamma)$-regularity assured by the standard trace theorem for the Sobolev space $H^1(\Omega)$ is insufficient to define a suitable weak formulation associated with (1.1). For this reason, we introduce non-standard Sobolev spaces where functions admit equal-

order regularity both in Ω and on Γ, that is

$$H^m(\Omega; \Gamma) = \{v \in H^m(\Omega) : \ v|_\Gamma \in H^m(\Gamma)\}, \quad m = 1, 2, \ldots. \tag{1.2}$$

We show that this class of function spaces is well-suited to prove the well-posedness and regularity of problem (1.1). Moreover, our choice of the regularity allows us to properly define a finite element approximation of problem (1.1), for which the convergence result follows by a standard argument. The function spaces of type (1.2) are already used in [34] when Ω is the half space $\mathbb{R}^N \cap \{x_N > 0\}$, such that, the boundary Γ is the hyperplane $\{x_N = 0\}$. In this work, we generalize the results of [34] by taking into account domains with a curved boundary Γ. In [45, 47] these spaces are also considered, but there the assumption $f \equiv 0$ is essential to prove the results. For the treatment of the general case we need to introduce results for tangential derivatives on the boundary together with the function spaces $H^m(\Omega; \Gamma)$.

Our analysis could extend to the case where the Laplacian in (1.1) is replaced with a more general second-order elliptic operator, that is,

$$\begin{cases} Lu := -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} + cu = f & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}_L} + \alpha u - \beta \Delta_\Gamma u = h & \text{on} \quad \Gamma, \end{cases}$$

($\frac{\partial u}{\partial \boldsymbol{n}_L}$ means $\sum_{i,j=1}^N n_j a_{ij}\frac{\partial u}{\partial x_i}$) provided that the associated bilinear form satisfies suitable coercivity conditions. In this paper, however, we restrict ourselves to problem (1.1) in order to simplify notation, statement of results, and proofs of theorems.

The paper is organized as follows. In Sect. 2 we introduce basic notation and auxiliary lemmas, especially explaining tangential derivatives. Sect. 3 and Sect. 4 are devoted to the theoretical analysis of problem (1.1) in terms of existence, uniqueness and regularity of weak solutions, as well as a-priori error estimates for the finite element approximation. Finally, in Sect. 5 we verify the validity of the theoretical results by means of numerical examples. Conclusions follow.

2. Preliminaries.

2.1. Basic notation. We use the notational convention for which lightface italics indicates scalar quantities and boldface italics vectors or second-order tensors. Let us present a detailed setting to study problem (1.1). We let Ω be a bounded domain in $\mathbb{R}^N$ with boundary Γ, whose smoothness properties will be specified later. The outer unit normal on Γ is denoted by $\boldsymbol{n}$; Δ_Γ indicates the Laplace-Beltrami operator, defined in terms of tangential derivatives (see Sect. 2.2). The parameters $\alpha > 0$ and $\beta \geq 0$ are given real constants. We refer to problem (1.1) with $\beta > 0$ and $\beta = 0$ as the generalized Robin boundary value problem (GR-P) and the standard Robin boundary value problem (SR-P), respectively. We also use the notation GRBC and SRBC, respectively, in order to indicate only the boundary conditions.

Let $C > 0$ denote a generic constant depending only on Ω, Γ and the dimension N of the space $\mathbb{R}^N$, unless otherwise stated. If other dependecies quantities need to be specified, we will indicate them case by case, e.g. we will write $C(f, h)$, $C(\alpha, \beta)$ etc. The i-th component of a vector $\boldsymbol{u}$, with respect to the standard basis in $\mathbb{R}^N$, is written as $u_i = (\boldsymbol{u})_i$. We assume the boundary is at least Lipschitz continuous, i.e., $\Gamma \in C^{0,1}$. The additional hypotheses on Γ will be specified later depending on our statements.

We employ the standard Lebesgue and Sobolev spaces: $L^2(\Omega) = \{v : \Omega \to \mathbb{R} : \ \|v\|_{L^2(\Omega)} = (\int_\Omega |v|^2 \, d\boldsymbol{x})^{1/2} < \infty\}$ and $H^m(\Omega) = \{v \in L^2(\Omega) : \ \|v\|_{H^m(\Omega)} =$

$(\sum_{|\alpha|\le m}\|\partial^\alpha v\|^2_{L^2(\Omega)})^{1/2}<\infty\}$ for $m=0,1,\dots$. We also use $L^\infty(\Omega)$ and $W^{m,\infty}(\Omega)$ with their norms. Those spaces are defined on Γ as well, and moreover we need fractional Sobolev spaces $H^s(\Gamma)$ for $s>0$. It is well known that the trace operator is surjective from $H^m(\Omega)$ onto $H^{m-1/2}(\Gamma)$ if $\Gamma\in C^{m-1,1}$ ($m\ge 1$), admitting its right continuous inverse (see [38, Theorems 2.5.5 and 2.5.8]). The trace of u is denoted by $u|_\Gamma$ or simply by u. $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, which is known to coincide with the set of the $H^1(\Omega)$-functions with zero-traces. The dual space of a Banach space X is denoted by X', and the duality pairing between X' and X is indicated by $\langle\cdot,\cdot\rangle_X$.

A localization argument using a partition of unity will be important in the study of regularity. Let K be a cylinder:

$$K=\{\boldsymbol{\xi}=(\boldsymbol{\xi}',\xi^N)=(\xi^1,\dots,\xi^N)\in\mathbb{R}^N\ :\ |\boldsymbol{\xi}'|\le 1,\ |\xi^N|\le 1\},$$

and let $K_+=K\cap\{\xi^N>0\}$, $K_0=K\cap\{\xi^N=0\}$. If Ω is a C^m (resp. $C^{m,1}$) domain ($m\ge 1$), then at each $\boldsymbol{x}\in\Gamma$ there exist a neighborhood U and a map $\boldsymbol{\phi}:U\to K$ such that $\boldsymbol{\phi}$ is a C^m (resp. $C^{m,1}$)-diffeomorphism and $\boldsymbol{\phi}(\Omega\cap U)=K_+$, $\boldsymbol{\phi}(\Gamma\cap U)=K_0$. The inverse of $\boldsymbol{\phi}$ is denoted by $\boldsymbol{\psi}$. We refer to $\boldsymbol{\xi}=\boldsymbol{\phi}(\boldsymbol{x})$ and $\boldsymbol{x}=\boldsymbol{\psi}(\boldsymbol{\xi})$ as a local coordinate and a parametrization respectively.

Because Ω is bounded, Γ can be covered by finite number of such neighborhoods U_r ($r=1,\dots,r_0$), with $\boldsymbol{\phi}_r,\boldsymbol{\psi}_r$ being the local coordinate and parametrization associated with U_r. They, supplemented by some open set U_0 such that $\overline{U}_0\subset\Omega$, form a covering of Ω. By standard arguments on partition of unity (see [6, Lemma 9.3]), we see that there exist functions $\theta_r\in C^\infty(\mathbb{R}_x)$ ($r=0,\dots,r_0$) such that $\mathrm{supp}\,\theta_r\subset U_r$ is compact and $\sum_{r=0}^{r_0}\theta_r\equiv 1$.

Let $u:\Omega\to\mathbb{R}$ be a (reasonably smooth) function. Its local representation on each U_r is defined by $\tilde{u}(\boldsymbol{\xi})=u(\boldsymbol{\psi}_r(\boldsymbol{\xi}))$. The Jacobi matrix of $\boldsymbol{\phi}_r$ is denoted by $\boldsymbol{\Phi}_r$, that is, $(\boldsymbol{\Phi}_r)_{ij}=\partial(\boldsymbol{\phi}_r)_i/\partial x_j$, and we put $J_r=|\det(\boldsymbol{\Phi}_r^{-1})|>0$. Then we see that

$$\frac{\partial u}{\partial x_i}=\sum_{j=1}^N(\boldsymbol{\Phi}_r)_{ji}\frac{\partial\tilde{u}}{\partial\xi^j}\quad(i=1,\dots,N),\qquad\int_{\Omega\cap U_r}u\,d\boldsymbol{x}=\int_{K_+}\tilde{u}\,|J_r|\,d\boldsymbol{\xi}.\qquad(2.1)$$

2.2. Tangential derivatives. In this subsection, assuming $\Gamma\in C^1$, we fix one local coordinate $\boldsymbol{\phi}_r:U_r\to K$ and omit the subscript r. We let Greek ($\alpha,\beta,\dots$) and Latin ($i,j,\dots$) indices take their values in $\{1,\dots,N-1\}$ and $\{1,\dots,N\}$ respectively.[1] We employ the summation convention.

Let us define the bases of the tangent space $T_{\boldsymbol{x}}\Gamma$ by (cf. [20, p. 109]) $\boldsymbol{g}_\alpha=\frac{\partial\boldsymbol{\psi}}{\partial\xi^\alpha}\big|_{\xi^N=0}$. If Γ is regarded as an $N-1$ dimensional manifold embedded in $\mathbb{R}^N$, then $\boldsymbol{g}_\alpha$ can be identified with the abstract tangent vector $\frac{\partial}{\partial\xi^\alpha}$.

Covariant and contravariant components of a metric tensor $g_{\alpha\beta}$, $g^{\alpha\beta}$, and contravariant vectors $\boldsymbol{g}^\alpha$ are obtained through the following relationships:

$$g_{\alpha\beta}=\boldsymbol{g}_\alpha\cdot\boldsymbol{g}_\beta,\quad(g^{\alpha\beta})=(g_{\alpha\beta})^{-1},\quad\boldsymbol{g}^\alpha=g^{\alpha\beta}\boldsymbol{g}_\beta,$$

where the dot means the inner product in $\mathbb{R}^N$. Then it follows that $\boldsymbol{g}^\alpha\cdot\boldsymbol{g}^\beta=g^{\alpha\beta}$, $\boldsymbol{g}_\alpha\cdot\boldsymbol{g}^\beta=\delta_\alpha^\beta$ (the Kronecker delta), and $g_{\alpha\beta}\boldsymbol{g}^\beta=\boldsymbol{g}_\alpha$.

Next we introduce a projection matrix $\boldsymbol{P}=\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}=(\delta_{ij}-n_in_j)_{i,j=1}^N$, where $\boldsymbol{a}\otimes\boldsymbol{b}=(a_ib_j)_{ij}$ is the dyadic product. For a scalar function $u:\Omega\to\mathbb{R}$, we define

---

[1] Note that the greek letters α and β represent here indices and not the coefficients defining the generalized Robin boundary condition $(1.1)_2$.

its tangential gradient by the matrix-vector product $\nabla_\Gamma u = \boldsymbol{P} \nabla u =: (\partial_{i,\Gamma} u)_{i=1}^N$. In particular, one has $\partial_{i,\Gamma} = (\delta_{ij} - n_i n_j)\partial_j$. Notice that $\partial_{i,\Gamma} \partial_{j,\Gamma} \neq \partial_{j,\Gamma} \partial_{i,\Gamma}$ in general. The tangential gradient of a vector-valued function $\boldsymbol{u} : \Omega \to \mathbb{R}^N$ is defined by the matrix-matrix product $\nabla_\Gamma \boldsymbol{u} = \nabla \boldsymbol{u}\, \boldsymbol{P} = (\partial_{j,\Gamma} u_i)_{i,j=1,\ldots,N}$.

The above representations of ∇_Γ are expressed in the $\boldsymbol{x}$-coordinate and hence they are global. The way they can be represented terms of the local coordinate $\boldsymbol{\xi}$ is addressed in the next lemma. The result itself is not new (e.g. page 55 of [12]); nevertheless it is of critical importance in our theory, especially when proving regularity, so that we give a complete proof.

LEMMA 2.1. *The following local representations of $\boldsymbol{P}$ and $\nabla_\Gamma u$ hold:*

(i) $\boldsymbol{P} = \boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha$,

(ii) $\nabla_\Gamma u = \boldsymbol{g}^\alpha \frac{\partial \tilde{u}}{\partial \xi^\alpha}$.

Proof. (i) It suffices to prove $\boldsymbol{I} = \boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha + \boldsymbol{n} \otimes \boldsymbol{n}$. In fact, an application of the right-hand side matrix to the vectors $\boldsymbol{g}^\beta$ and $\boldsymbol{n}$, which form a basis of $\mathbb{R}^N$, produces the same vectors, and thus the desired equality follows.

(ii) By virtue of (i) and a chain rule, we have

$$\nabla_\Gamma u = (\boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha)\nabla u = \boldsymbol{g}^\alpha(\boldsymbol{g}_\alpha \cdot \nabla u) = \boldsymbol{g}^\alpha \left(\frac{\partial(\boldsymbol{\psi})_i}{\partial \xi^\alpha} \frac{\partial u}{\partial x_i} \right) = \boldsymbol{g}^\alpha \frac{\partial \tilde{u}}{\partial \xi^\alpha}. \;\square$$

REMARK 2.2. *(i) It also follows that $\boldsymbol{P} = \boldsymbol{g}_\alpha \otimes \boldsymbol{g}^\alpha$, hence $\boldsymbol{P}$ is symmetric.*

(ii) One can write $\partial_{i,\Gamma} = (\boldsymbol{g}^\alpha)_i \frac{\partial}{\partial \xi^\alpha}$. This shows that the tangential derivatives are determined only by the information on Γ. In fact, to apply $\partial_{i,\Gamma}$ to $u : \Gamma \to \mathbb{R}$ one has to, at first, extend u in a smooth way to a neighbourhood of Γ, but the result is independent of such extensions.

For a vector-valued function $\boldsymbol{A} : \Gamma \to \mathbb{R}^N$, the tangential divergence operator is defined as $\mathrm{div}_\Gamma \boldsymbol{A} := \mathrm{Tr}(\nabla_\Gamma \boldsymbol{A}) = \partial_{i,\Gamma} A_i$, where $A_i = (\boldsymbol{A})_i$. The Laplace-Beltrami operator is then defined by $\Delta_\Gamma u = \mathrm{div}_\Gamma(\nabla_\Gamma u)$.

By a direct computation, we can verify the following formulas concerning tangential derivatives for product:

LEMMA 2.3. *For smooth functions $\theta, u, \boldsymbol{u}$ defined on Γ, we have*

(i) $\nabla_\Gamma(\theta u) = (\nabla_\Gamma \theta)\, u + \theta\, \nabla_\Gamma u$.

(ii) $\nabla_\Gamma(\theta \boldsymbol{u}) = \nabla_\Gamma \theta \otimes \boldsymbol{u} + \theta\, \nabla_\Gamma \boldsymbol{u}$.

(iii) $\mathrm{div}_\Gamma(\theta \boldsymbol{u}) = \nabla_\Gamma \theta \cdot \boldsymbol{u} + \theta\, \mathrm{div}_\Gamma \boldsymbol{u}$.

We conclude this subsection with change of variables formulas for calculus on Γ (cf. (2.1)). In view of Lemma 2.1(ii) and the surface element given by $ds = \sqrt{|g|}d\boldsymbol{\xi}'$ where $|g| = \det(g_{\alpha\beta}) > 0$, it follows that

$$\partial_{i,\Gamma} u = g_i^\alpha \frac{\partial \tilde{u}}{\partial \xi^\alpha} \quad (i = 1, \ldots, N), \qquad \int_{\Gamma \cap U_r} u\, ds = \int_{K_0} \tilde{u}\sqrt{|g|}\, d\boldsymbol{\xi}'. \qquad (2.2)$$

2.3. Space $H^m(\Omega; \Gamma)$. First we introduce another expression for the norm of $H^m(\Gamma)$ under $\Gamma \in C^{m-1,1}$ if $m \geq 2$ or $\Gamma \in C^1$ if $m = 1$. In the existing literature, as in [35, 38], it is usually defined in the local coordinates after truncation by a partition of unity, that is,

$$\|u\|_{H^m(\Gamma)} = \left(\sum_{r=1}^{r_0} \|\widetilde{\theta_r u}\|_{H^m(K_0)}^2 \right)^{1/2}.$$

This is well-defined as far as mathematics is concerned. However, from a numerical point of view, it is not suitable for real computation. Instead, we consider

$$\|u\|_{H^m(\Gamma)} = \left(\|u\|^2_{H^{m-1}(\Gamma)} + \|\nabla_\Gamma u\|^2_{\boldsymbol{H}^{m-1}(\Gamma)} \right)^{1/2},$$

which can be defined inductively in m. The global representation of ∇_Γ then enables us to compute it as easily as we calculate the $H^m(\Omega)$-norm. The two norms are equivalent as shown in the next lemma.

LEMMA 2.4. *The above two norms are equivalent for $u \in H^m(\Gamma)$.*

Proof. Consider any piece $\theta_r u$ truncated by the partition of unity θ_r ($r = 1, \ldots, r_0$), and write u instead of $\theta_r u$ for simplicity. By induction with respect to m, it suffices to prove that

$$\frac{1}{C} \sum_{\alpha=1}^{N-1} \left\| \frac{\partial \tilde{u}}{\partial \xi^\alpha} \right\|^2_{L^2(K_0)} \le \sum_{i=1}^{N} \|\partial_{i,\Gamma} u\|^2_{L^2(\Gamma)} \le C \sum_{\alpha=1}^{N-1} \left\| \frac{\partial \tilde{u}}{\partial \xi^\alpha} \right\|^2_{L^2(K_0)}. \qquad (2.3)$$

For that purpose, we note that (recall the first relation of (2.2))

$$\partial_{i,\Gamma} u(\boldsymbol{x}) = g_i^\alpha(\boldsymbol{\phi}(\boldsymbol{x})) \frac{\partial \tilde{u}}{\partial \xi^\alpha}(\boldsymbol{\phi}(\boldsymbol{x})), \qquad\qquad \boldsymbol{x} \in \Gamma \cap U,$$

$$\frac{\partial \tilde{u}}{\partial \xi^\alpha}(\boldsymbol{\xi}', 0) = g_{\alpha i}(\boldsymbol{\xi}) \partial_{i,\Gamma} u(\boldsymbol{\psi}(\boldsymbol{\xi}', 0)), \qquad\qquad (\boldsymbol{\xi}', 0) \in K_0.$$

Then, because $\sqrt{|g|}$ is bounded from above and below by some $C > 0$, we obtain (2.3) using the second relation of (2.2). $\square$

We are ready to define our fundamental space $H^m(\Omega; \Gamma) = \{v \in H^m(\Omega) \ ; \ v \in H^m(\Gamma)\}$ (the same as in (1.2)), endowed with the norm

$$\|u\|_{H^m(\Omega;\Gamma)} = \left(\|u\|^2_{H^m(\Omega)} + \|u|_\Gamma\|^2_{H^m(\Gamma)} \right)^{1/2} \qquad (2.4)$$

LEMMA 2.5. *$H^m(\Omega; \Gamma)$ is a Hilbert space.*

Proof. Obviously $H^m(\Omega; \Gamma)$ admits an inner product which induces (2.4). It remains to show the completeness. Let $\{v_n\}$ be a Cauchy sequence in $H^m(\Omega; \Gamma)$. Since $H^m(\Omega; \Gamma) \subset H^m(\Omega)$, there exists some $v \in H^m(\Omega)$ such that $v_n \to v$ in $H^m(\Omega)$. Similarly, there exists some $w \in H^m(\Gamma)$ such that $v_n|_\Gamma \to w$ in $H^m(\Gamma)$. However, by the standard trace theorem one knows $v_n|_\Gamma \to v|_\Gamma$ in $H^{m-1/2}(\Gamma)$, so that $v|_\Gamma = w$. Hence $v \in H^m(\Omega; \Gamma)$ and $v_n \to v$ in $H^m(\Omega; \Gamma)$.$\square$

REMARK 2.6. *(i) As is pointed out in [34], $H^m(\Omega; \Gamma)$ is not closed in $H^m(\Omega)$, its closure being identical to $H^m(\Omega)$. As a result, it is not trivial whether $C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega; \Gamma)$ with respect to the norm (2.4).*

(ii) $H^m(\Omega) \cap H^1_0(\Omega)$ is a closed subspace of $H^m(\Omega; \Gamma)$. In particular, when $m = 1$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega; \Gamma)$ coincides with $H^1_0(\Omega)$.

We will also need the spaces $H^m(\omega; \gamma) = \{v \in H^m(\omega) : v|_\gamma \in H^m(\gamma)\}$, where $\omega \subset \mathbb{R}^N$ is a domain with only piecewise smooth boundary $\partial\omega$ and $\gamma \subset \partial\omega$ is a smooth portion of $\partial\omega$. They will be used in the proof of Theorems 3.3–3.4.

3. Analysis of (GR-P).

3.1. Weak solution. We derive a suitable weak formulation and establish its well-posedness for (1.1). First recall the following integration by parts formula on Γ:

$$-\int_\Gamma \operatorname{div}_\Gamma \boldsymbol{A}\, v\, ds = \int_\Gamma \boldsymbol{A} \cdot \nabla_\Gamma v\, ds - \int_\Gamma \kappa(\boldsymbol{A} \cdot \boldsymbol{n})\, v\, ds, \qquad (3.1)$$

where $\boldsymbol{A}$ is a vector-valued function and $\kappa = \operatorname{div}_\Gamma \boldsymbol{n}$ (see Lemma 16.1 of [25]).

REMARK 3.1. *For (3.1) to be valid, we need at least the regularity $\Gamma \in C^{1,1}$, because otherwise (3.1) does not hold globally. For instance, let Ω be a plane polygon. Then on each side Γ_j we have (note that $\kappa = 0$ on Γ_j)*

$$-\int_{\Gamma_j} \operatorname{div}_\Gamma \boldsymbol{A}\, v\, ds = \int_{\Gamma_j} \boldsymbol{A} \cdot \nabla_\Gamma v\, ds + \big[(\boldsymbol{A} \cdot \boldsymbol{\nu}_{\partial\Gamma_j})\, v\big]_{\partial\Gamma_j},$$

where $\boldsymbol{\nu}_{\partial\Gamma_j}$ denotes the outer unit normal to $\partial\Gamma_j$. Because $\boldsymbol{\nu}_{\partial\Gamma_j}$'s for adjacent two sides are not equal at the corner point shared by them, the extra "boundary of boundary" terms do not cancel out even if we add up the equalities above.

Let u be a sufficiently smooth solution of (1.1). Multiplying the first equation of (1.1) by a test function v and integrating in Ω, one finds that

$$\int_\Omega \nabla u \cdot \nabla v\, d\boldsymbol{x} - \int_\Gamma \frac{\partial u}{\partial \boldsymbol{n}}\, v\, ds = \int_\Omega fv\, d\boldsymbol{x}.$$

Substituting the GRBC of (1.1) into the above equation and applying (3.1) with $\boldsymbol{A} = \nabla_\Gamma u$ (note that $\nabla_\Gamma u \cdot \boldsymbol{n} = 0$), we arrive at

$$\int_\Omega \nabla u \cdot \nabla v\, d\boldsymbol{x} + \alpha \int_\Gamma uv\, ds + \beta \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v\, ds = \int_\Omega fv\, d\boldsymbol{x} + \int_\Gamma hv\, ds.$$

The above equation motivates us to propose the following variational problem: given $f \in H^1(\Omega)'$ and $h \in H^1(\Gamma)'$, find $u \in H^1(\Omega; \Gamma)$ such that

$$a(u, v) := a_\Omega(u, v) + a_\Gamma(u, v) = \langle F, v\rangle_{H^1(\Omega;\Gamma)}, \qquad \forall v \in H^1(\Omega; \Gamma). \qquad (3.2)$$

Here, we have defined bilinear forms a_Ω and a_Γ by

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v\, d\boldsymbol{x}, \qquad\qquad\qquad u, v \in H^1(\Omega),$$

$$a_\Gamma(u, v) = \alpha \int_\Gamma uv\, ds + \beta \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v\, ds, \qquad u, v \in H^1(\Gamma),$$

and a linear form F by

$$\langle F, v\rangle_{H^1(\Omega;\Gamma)} = \langle f, v\rangle_{H^1(\Omega)} + \langle h, v\rangle_{H^1(\Gamma)}, \qquad v \in H^1(\Gamma).$$

THEOREM 3.2. *Let $\alpha, \beta > 0$, then there exists a unique solution u of (3.2) which fulfills*

$$\|u\|_{H^1(\Omega;\Gamma)} \leq C(\alpha, \beta)(\|f\|_{H^1(\Omega)'} + \|h\|_{H^1(\Gamma)'}). \qquad (3.3)$$

We call this u a weak solution of (1.1).

Proof. By the Schwartz inequality a is bounded (and symmetric) on $H^1(\Omega; \Gamma)$. Combining the two coercivities

$$\int_\Omega |\nabla v|^2 \, d\boldsymbol{x} + \int_\Gamma |v|^2 \, ds \geq C\|v\|^2_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega), \qquad (3.4)$$

$$\int_\Gamma |\nabla_\Gamma v|^2 \, d\boldsymbol{x} + \int_\Gamma |v|^2 \, ds = \|v\|^2_{H^1(\Gamma)}, \qquad \forall v \in H^1(\Gamma), \qquad (3.5)$$

one also finds that a is coercive on $H^1(\Omega; \Gamma)$, i.e.,

$$a(v, v) \geq C(\alpha, \beta)\|v\|^2_{H^1(\Omega; \Gamma)}, \quad \forall v \in H^1(\Omega; \Gamma).$$

Finally, F is bounded on $H^1(\Omega; \Gamma)$. In fact, it is immediate to see

$$|\langle F, v\rangle_{H^1(\Omega; \Gamma)}| \leq (\|f\|_{(H^1(\Omega))'} + \|h\|_{(H^1(\Gamma))'})\|v\|_{H^1(\Omega; \Gamma)}, \quad \forall v \in H^1(\Omega; \Gamma).$$

The theorem is now only a consequence of the celebrated Lax-Milgram theorem. □

Let us see the way u recovers (1.1) in a weak sense. Restricting test functions to $H^1_0(\Omega)$ in (3.2), we obtain $-\Delta u = f$ in $H^1_0(\Omega)'$. Then $\frac{\partial u}{\partial \boldsymbol{n}} \in H^{1/2}(\Gamma)'$ is characterized by

$$\left\langle \frac{\partial u}{\partial \boldsymbol{n}}, v \right\rangle_{H^{1/2}(\Gamma)} = a_\Omega(u, v) - \langle f, v\rangle_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega).$$

Combining this with (3.2) we have

$$\left\langle \frac{\partial u}{\partial \boldsymbol{n}}, v \right\rangle_{H^{1/2}(\Gamma)} + a_\Gamma(u, v) = \langle h, v\rangle_{H^1(\Gamma)}, \qquad (3.6)$$

at least for all $v \in C^\infty(\Gamma)$. Since $C^\infty(\Gamma)$ is dense in $H^1(\Gamma)$, (3.6) is valid for all $v \in H^1(\Gamma)$, so that $\frac{\partial u}{\partial \boldsymbol{n}} + \alpha u - \beta \Delta_\Gamma u = h$ in $H^1(\Gamma)'$.

3.2. Regularity. Let us prove that the weak solution in Theorem 3.2 can in fact be more regular, depending on the smoothness of the data. The main ingredient is the method of difference quotient combined with truncation and localization arguments. In doing so, for $\alpha = 1, ..., N-1$ and $\delta \in \mathbb{R} \setminus \{0\}$, we introduce a shift operator s_δ^α and a tangential difference quotient operator D_δ^α by

$$s_\delta^\alpha \tilde{v}(\boldsymbol{\xi}) = \tilde{v}(\xi^1, ..., \xi^\alpha + \delta, ..., \xi^N), \qquad D_\delta^\alpha \tilde{v} = \frac{s_\delta^\alpha \tilde{v} - \tilde{v}}{\delta}.$$

It is easy to see that

$$D_\delta^\alpha(\tilde{u}\tilde{v}) = (D_\delta^\alpha \tilde{u})\tilde{v} + (s_\delta^\alpha \tilde{u})\tilde{v}, \qquad (3.7)$$

and that (if $\operatorname{supp} \tilde{u} \subset K$, $\operatorname{supp} \tilde{v} \subset K$ and $|\delta|$ is sufficiently small)

$$\int_{K_+} \tilde{u}(D_{-\delta}^\alpha \tilde{v}) \, d\boldsymbol{\xi} = \int_{K_+} (D_\delta^\alpha \tilde{u})\tilde{v} \, d\boldsymbol{\xi}, \quad \int_{K_0} \tilde{u}(D_{-\delta}^\alpha \tilde{v}) \, d\boldsymbol{\xi}' = \int_{K_0} (D_\delta^\alpha \tilde{u})\tilde{v} \, d\boldsymbol{\xi}'. \qquad (3.8)$$

We also have (see [6, Proposition 9.3])

$$\|D_\delta^\alpha \tilde{v}\|_{L^2(K_+)} \leq \left\|\frac{\partial \tilde{v}}{\partial \xi^\alpha}\right\|_{L^2(K_+)}, \quad \|D_\delta^\alpha \tilde{v}\|_{L^2(K_0)} \leq \left\|\frac{\partial \tilde{v}}{\partial \xi^\alpha}\right\|_{L^2(K_0)}, \qquad (3.9)$$

together with their analogies in L^∞-spaces.

Now we begin with the case $m = 2$.

THEOREM 3.3. *Let $\alpha, \beta > 0$. Assume $\Gamma \in C^{1,1}$, $f \in L^2(\Omega)$, $h \in L^2(\Gamma)$, and let $u \in H^1(\Omega; \Gamma)$ be a weak solution of (1.1). Then we have $u \in H^2(\Omega; \Gamma)$ and*

$$\|u\|_{H^2(\Omega;\Gamma)} \le C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\Gamma)}),$$

where $C = C(\alpha, \beta)$.

Proof. We divide the proof into five steps.

Step 1: Take any piece $\theta_r u$ supported in U_r, which is obtained from the partition of unity. We concentrate on the so-called "estimates near the boundary", that is, $1 \le r \le r_0$, since the interior case $r = 0$ is easier to study (extend $\theta_0 u$ to the whole space and discuss by the method of difference quotient). In the sense of a weak solution, we obtain (recall Lemma 2.3)

$$-\Delta(\theta_r u) = \theta_r f - 2\nabla\theta_r \cdot \nabla u - \Delta\theta_r u =: F_r \in L^2(\Omega),$$

$$\frac{\partial(\theta_r u)}{\partial \boldsymbol{n}} + \alpha\theta_r u - \beta\Delta_\Gamma(\theta_r u) = \theta_r h + \frac{\partial\theta_r}{\partial\boldsymbol{n}} - 2\beta\nabla_\Gamma\theta_r \cdot \nabla_\Gamma u - \Delta_\Gamma\theta_r u =: H_r \in L^2(\Gamma).$$

For notational simplicity, we will omit the the subscript r and write u instead of $\theta_r u$ until Step 5.

It follows that $u \in H^1(\Omega \cap U; \Gamma \cap U)$, with $\operatorname{supp} u \subset U$, and that

$$\int_{\Omega \cap U} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\boldsymbol{x} + \alpha\int_{\Gamma \cap U} uv\, ds + \beta\int_{\Gamma \cap U} \partial_{i,\Gamma}u\, \partial_{i,\Gamma}v\, ds$$

$$= \int_{\Omega \cap U} Fv\, d\boldsymbol{x} + \int_{\Gamma \cap U} Hv\, ds, \qquad \forall v \in H^1(\Omega \cap U; \Gamma \cap U),\ \operatorname{supp} v \subset U,$$

where the summation convention is employed.

Step 2: Moving to the local coordinate, we see that $\tilde u \in H^1(K_+; K_0)$, with $\operatorname{supp}\tilde u \subset K$, and that (recall (2.1) and (2.2))

$$\int_{K_+} \Phi_{ji}\frac{\partial\tilde u}{\partial\xi^j}\Phi_{ki}\frac{\partial\tilde v}{\partial\xi^k}|J|\, d\boldsymbol{\xi} + \alpha\int_{K_0} \tilde u\tilde v\sqrt{|g|}d\boldsymbol{\xi}' + \beta\int_{K_0} g_i^\alpha\frac{\partial\tilde u}{\partial\xi^\alpha}g_i^\beta\frac{\partial\tilde v}{\partial\xi^\beta}\sqrt{|g|}d\boldsymbol{\xi}'$$

$$= \int_{K_+} \tilde F\tilde v|J|\, d\boldsymbol{\xi} + \int_{K_0} \tilde H\tilde v\sqrt{|g|}d\boldsymbol{\xi}', \qquad \forall\tilde v \in H^1(K_+; K_0),\ \operatorname{supp}\tilde v \subset K.$$

Here, let us redefine the quantities multiplied by $\Phi_{ji}, g_i^\alpha, |J|, \sqrt{|g|}, \alpha, \beta$, which appear above, as single symbols $\boldsymbol{A}, \boldsymbol{A}', L, \hat F, \hat H$. Then, the above equation can be rewritten as

$$\int_{K_+} A_{jk}\frac{\partial\tilde u}{\partial\xi^j}\frac{\partial\tilde v}{\partial\xi^k}\, d\boldsymbol{\xi} + \int_{K_0} L\tilde u\tilde v\, d\boldsymbol{\xi}' + \int_{K_0} A'_{\alpha\beta}\frac{\partial\tilde u}{\partial\xi^\alpha}\frac{\partial\tilde v}{\partial\xi^\beta}\, d\boldsymbol{\xi}'$$

$$= \int_{K_+} \hat F\tilde v\, d\boldsymbol{\xi} + \int_{K_0} \hat H\tilde v\, d\boldsymbol{\xi}', \qquad \forall\tilde v \in H^1(K_+; K_0),\ \operatorname{supp}\tilde v \subset K. \qquad (3.10)$$

Here note that $\boldsymbol{A} \in \boldsymbol{W}^{1,\infty}(K_+), \boldsymbol{A}' \in \boldsymbol{W}^{1,\infty}(K_0), L \in W^{1,\infty}(K_0)$ and that

$$C\|\tilde v\|^2_{H^1(K_+;K_0)} \le \int_{K_+} A_{jk}\frac{\partial\tilde v}{\partial\xi^j}\frac{\partial\tilde v}{\partial\xi^k}\, d\boldsymbol{\xi} + \int_{K_0} L|\tilde v|^2\, d\boldsymbol{\xi}' + \int_{K_0} A'_{\alpha\beta}\frac{\partial\tilde v}{\partial\xi^\alpha}\frac{\partial\tilde v}{\partial\xi^\beta}\, d\boldsymbol{\xi}' \quad (3.11)$$

for all $\tilde v \in H^1(K_+; K_0)$ such that $\operatorname{supp}\tilde v \subset K$, as a result of (3.4)–(3.5).

Step 3: Fix $1 \leq \gamma \leq N-1$ and take $\tilde{v} = D_{-\delta}^\gamma D_\delta^\gamma \tilde{u}$ in (3.10), with $|\delta|$ small enough. We omit the superscript γ. Then it follows from (3.7)–(3.8) that

$$
\int_{K_+} A_{jk} \frac{\partial D_\delta \tilde{u}}{\partial \xi^j} \frac{\partial D_\delta \tilde{u}}{\partial \xi^k} \, d\boldsymbol{\xi} + \int_{K_0} L|D_\delta \tilde{u}|^2 \, d\boldsymbol{\xi}' + \int_{K_0} A'_{\alpha\beta} \frac{\partial D_\delta \tilde{u}}{\partial \xi^\alpha} \frac{\partial D_\delta \tilde{u}}{\partial \xi^\beta} \, d\boldsymbol{\xi}'
$$
$$
= \int_{K_+} \hat{F} D_{-\delta} D_\delta \tilde{u} \, d\boldsymbol{\xi} + \int_{K_0} \hat{H} D_{-\delta} D_\delta \tilde{u} \, d\boldsymbol{\xi}' - \int_{K_+} D_\delta A_{jk} \frac{\partial s_\delta \tilde{u}}{\partial \xi^j} \frac{\partial D_\delta \tilde{u}}{\partial \xi^k} \, d\boldsymbol{\xi}
$$
$$
- \int_{K_0} D_\delta L \, s_\delta \tilde{u} \, D_\delta \tilde{u} \, d\boldsymbol{\xi}' - \int_{K_0} D_\delta A'_{\alpha\beta} \frac{\partial s_\delta \tilde{u}}{\partial \xi^j} \frac{\partial D_\delta \tilde{u}}{\partial \xi^k} \, d\boldsymbol{\xi}'.
$$

This combined with (3.11) and (3.9) leads to

$$
C(\|D_\delta \tilde{u}\|_{H^1(K_+)}^2 + \|D_\delta \tilde{u}\|_{H^1(K_0)}^2)
$$
$$
\leq \|\hat{F}\|_{L^2(K_+)} \|D_\delta \tilde{u}\|_{H^1(K_+)} + \|\hat{H}\|_{L^2(K_0)} \|D_\delta \tilde{u}\|_{H^1(K_0)}
$$
$$
+ \|A\|_{\boldsymbol{W}^{1,\infty}(K_+)} \|\tilde{u}\|_{H^1(K_+)} \|D_\delta \tilde{u}\|_{H^1(K_+)} + \|L\|_{\boldsymbol{W}^{1,\infty}(K_0)} \|\tilde{u}\|_{L^2(K_0)} \|\tilde{u}\|_{H^1(K_0)}
$$
$$
+ \|A'\|_{\boldsymbol{W}^{1,\infty}(K_0)} \|\tilde{u}\|_{H^1(K_0)} \|D_\delta \tilde{u}\|_{H^1(K_0)}.
$$

Since we already know (3.3), the above estimate implies

$$
\|D_\delta \tilde{u}\|_{H^1(K_+)}^2 + \|D_\delta \tilde{u}\|_{H^1(K_0)}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Gamma)}^2).
$$

Letting $\delta \to 0$, we deduce $\partial \tilde{u}/\partial \xi^\gamma \in H^1(K_+)$ and $\partial \tilde{u}/\partial \xi^\gamma \in H^1(K_0)$ for $\gamma \neq N$; especially we have shown $\tilde{u}|_{K_0} \in H^2(K_0)$.

Step 4: Restricting test functions to $H_0^1(K_+)$ in (3.6) (and thus neglecting the integrals on K_0), we get $-\frac{\partial}{\partial \xi^k}(A_{jk} \frac{\partial \tilde{u}}{\partial \xi^j}) = \hat{F}$ in $H_0^1(K_+)'$. Therefore,

$$
-A_{NN} \frac{\partial^2 \tilde{u}}{\partial (\xi^N)^2} = \hat{F} + \sum_{(j,k) \neq (N,N)} \frac{\partial}{\partial \xi^k} \left(A_{jk} \frac{\partial \tilde{u}}{\partial \xi^j} \right) + \frac{\partial A_{NN}}{\partial \xi^N} \frac{\partial \tilde{u}}{\partial \xi^N} \in L^2(K_+).
$$

Since $A_{NN} = \sum_{i=1}^N |\Phi_{Ni}|^2 > 0$, this implies that $\partial \tilde{u}/\partial \xi^N$ is also in $H^1(K_+)$, so that $\tilde{u} \in H^2(K_+)$. Moreover, we obtain the estimate

$$
\|\tilde{u}\|_{H^2(K_+)} + \|\tilde{u}\|_{H^2(K_0)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\Gamma)}).
$$

Step 5: We return to the global coordinate and retrieve the notation $\theta_r u$ which was denoted by u up to here. From the conclusions of Steps 3–4 it follows that $\theta_r u \in H^2(\Omega; \Gamma)$. We conclude that $u = \sum_{r=0}^{r_0} \theta_r u \in H^2(\Omega; \Gamma)$ and that $\|u\|_{H^2(\Omega;\Gamma)} \leq C(\|f\|_{L^2(\Omega)} + \|h\|_{L^2(\Gamma)})$. $\square$

Theorem 3.3 does not follow from a classical theory for general elliptic boundary value problems. For example, in [35, 38] it is assumed that the differential order of a boundary operator must be *strictly* less than that of the interior one, which is not the case in (GR-P); the a priori estimate of Agmon-Douglis-Nirenberg (see [1]) only applies to $u \in H^3(\Omega)$ or higher-order spaces. Neither Theorem 3.3 is covered by the preceding results in [40, 45, 47] for parabolic problems. In fact, the author of [40] works with the setting such that $h \in H^{1/2}(\Gamma)$ and $u|_\Gamma \in H^{5/2}(\Gamma)$, and thus the maximal regularity space in [40], for the steady case, is different from the one in Theorem 3.3. Our choice of regularity, where only integer orders appear, is more preferable from numerical point of view. The analysis in [45, 47] essentially relies on the assumption $f \equiv 0$, which is not available in our setting.

Next let us establish the general case including $m \geq 3$.

THEOREM 3.4. *Let* $\alpha, \beta > 0$ *and* $m \geq 2$. *Assume* $\Gamma \in C^{m-1,1}$, $f \in H^{m-2}(\Omega)$, $h \in H^{m-2}(\Gamma)$, *and let* u *be a weak solution of (1.1). Then,* $u \in H^m(\Omega; \Gamma)$ *and*

$$\|u\|_{H^m(\Omega;\Gamma)} \leq C(\|f\|_{H^{m-2}(\Omega)} + \|h\|_{H^{m-2}(\Gamma)}), \qquad (3.12)$$

where $C = C(\alpha, \beta, m)$.

Proof. As in the proof of Theorem 3.3, we discuss only the estimates near the boundary. Suppose we have localized our analysis as before and arrived at (3.10). We observe that $\tilde{u}$ is a weak solution of the following boundary value problem:

$$\begin{cases} -\dfrac{\partial}{\partial \xi^k}\left(A_{jk}\dfrac{\partial \tilde{u}}{\partial \xi^j}\right) = \hat{F} & \text{in } K_+, \quad \operatorname{supp} \tilde{u} \subset K, \\ -A_{jN}\dfrac{\partial \tilde{u}}{\partial \xi^j} + L\tilde{u} - \dfrac{\partial}{\partial \xi^\beta}\left(A'_{\alpha\beta}\dfrac{\partial \tilde{u}}{\partial \xi^\alpha}\right) = \hat{H} & \text{on } K_0. \end{cases}$$

To prove the theorem, it suffices to show, by the induction in m, that $\tilde{u} \in H^m(K_+; K_0)$ under the conditions $\Gamma \in C^{m-1,1}$, $\hat{F} \in H^{m-2}(K_+)$, $\hat{H} \in H^{m-2}(K_0)$. Because the statement for $m = 2$ is already proved in Theorem 3.3, we only need to establish the case for $m \geq 3$, assuming the one for $m - 1$.

For $1 \leq \gamma \leq N - 1$, by differentiating the above equations with respect to ξ^γ, one finds that $\partial \tilde{u}/\partial \xi^\gamma$ satisfies the same equations, the right-hand sides being replaced by some $F^* \in H^{m-3}(\Omega)$ and $H^* \in H^{m-3}(\Gamma)$. For a rigorous justification of this point, it suffices to take $\partial \tilde{v}/\partial \xi^\gamma$ as a test function in (3.10) with smooth $\tilde{v}$, perform integration by parts, and discuss in a similar way as we did after Theorem 3.2.

According to the inductive assumption, we deduce $\partial \tilde{u}/\partial \xi^\gamma \in H^{m-1}(K_+; K_0)$ together with the estimate

$$\left\|\frac{\partial \tilde{u}}{\partial \xi^\gamma}\right\|_{H^{m-1}(K_+; K_0)} \leq C(\|F^*\|_{H^{m-3}(K_+)} + \|H^*\|_{H^{m-3}(K_0)})$$
$$\leq C(\|f\|_{H^{m-2}(\Omega)} + \|h\|_{H^{m-2}(\Gamma)}).$$

As a consequence, we have $\partial \tilde{u}/\partial \xi^\gamma \in H^{m-1}(K_+)$ for $\gamma \neq N$ and $\tilde{u} \in H^m(K_0)$. In a manner similar to Step 4 of the proof of Theorem 3.3, we obtain $\partial \tilde{u}/\partial \xi^N \in H^{m-1}(K_+)$. Therefore, $\tilde{u} \in H^m(K_+; K_0)$ and it also follows that $\|\tilde{u}\|_{H^m(K_+; K_0)} \leq C(\|f\|_{H^{m-2}(\Omega)} + \|h\|_{H^{m-2}(\Gamma)})$. This proves the induction statement for m, and thus completes the proof. $\square$

REMARK 3.5. *In (SR-P), that is,* $\beta = 0$, *a well known regularity result (see e.g. Sect. 6.3 in [15]) combined with a trace theorem yields*

$$\|u\|_{H^m(\Omega)} + \|u\|_{H^{m-1/2}(\Gamma)} \leq C(\alpha)(\|f\|_{H^{m-2}(\Omega)} + \|h\|_{H^{m-3/2}(\Gamma)}). \qquad (3.13)$$

Comparing (3.12) and (3.13), we find that GRBC possesses a better smoothing effect on the boundary than SRBC. In fact, we have proved that the solution u *for (GR-P) is in* $H^m(\Gamma)$ *if* $h \in H^{m-2}(\Gamma)$, *whereas the one for (SR-P) is expected to be at best in* $H^{m-1/2}(\Gamma)$ *even if* $h \in H^{m-3/2}(\Gamma)$ *(see e.g. Sect. 6.1.3 in [43]).*

4. Finite element approximation.

4.1. Hypotheses. Whenever we consider the finite element method (FEM), we assume $N = 2, 3$ and that Γ is either class of $C^{1,1}$ or a polyhedral domain. Furthermore, Γ is assumed to consist of two open subsets Γ_0 and $\Gamma_1 \neq \emptyset$, which are mutually

disjoint. We designate the $N - 2$ dimensional portion $\partial \Gamma_1 = \partial \Gamma_0 = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$ as the "boundary of boundary". Then, we consider the following Dirichlet-generalized Robin mixed boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma_0, \\ \frac{\partial u}{\partial \boldsymbol{n}} + \alpha u - \beta \Delta_\Gamma u = h & \text{on} \quad \Gamma_1. \end{cases} \tag{4.1}$$

Because the H^1 space will be employed not only in Ω but also on Γ, it is natural to supplement (4.1) with the compatibility condition

$$u = 0 \quad \text{on} \ \partial \Gamma_1, \tag{4.2}$$

which we call the "boundary of boundary" condition.

On the one hand, if $\Gamma \in C^{1,1}$, then $\Gamma_0 = \emptyset$ is allowed and one can impose the GRBC on the whole Γ. On the other hand, if Γ is a polygon or polyhedron, then it is not appropriate to impose it entirely on Γ at least in a naive way. This is because the integration-by-parts formula (3.1) fails to hold (see Remark 3.1), leaving non-trivial Neumann conditions at corners or edges. Therefore, in this case, we assume that Γ_1 is only one side ($N = 2$) or face ($N = 3$) of $\overline{\Omega}$ and that the remaining part Γ_0 is subjected to the homogeneous Dirichlet condition.

REMARK 4.1. *Our analysis easily extends to the case when Ω is polyhedral and Γ_1 consists of a finite number of sides or faces, as far as we impose suitable Dirichlet "boundary of boundary" conditions at corners or on edges contained in $\overline{\Gamma}_1$.*

REMARK 4.2. *If $\Gamma_0 \neq \emptyset$, the mixed-boundary condition would cause regularity loss (see [27]), which will not be covered by our regularity theorem. Therefore, we cannot avoid assuming some regularity of an exact solution in deriving error estimates.*

Under these settings, we introduce a Hilbert space:

$$H^m_{\Gamma_0}(\Omega; \Gamma_1) = \{v \in H^m(\Omega) : \ v|_{\Gamma_0} = 0, \ v|_{\Gamma_1} \in H^m(\Gamma_1) \cap H^1_0(\Gamma_1)\},$$

where $m \geq 1$ and $H^1_0(\Gamma_1)$ denotes the closure of $C^\infty_0(\Gamma_1)$ in $H^1(\Gamma_1)$. We equip $H^m_{\Gamma_0}(\Omega; \Gamma_1)$ with the norm

$$\|u\|_{H^m_{\Gamma_0}(\Omega; \Gamma_1)} = \left(\|u\|^2_{H^m(\Omega)} + \|u|_{\Gamma_1}\|^2_{H^m(\Gamma_1)} \right)^{1/2},$$

and set $V := H^1_{\Gamma_0}(\Omega; \Gamma_1)$.

Let $\mathcal{T}_h$ be a mesh of $\overline{\Omega}$ into element domains. Namely, 1) $\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$; 2) each $T \in \mathcal{T}_h$ is a closed subset with non-empty interior; 3) if $T, T' \in \mathcal{T}_h$ are distinct, $\text{int} \, T \cap \text{int} \, T' = \emptyset$; 4) each $T \in \mathcal{T}_h$ has a piecewise smooth boundary. Here, the family index h represents the mesh size defined by[2]

$$h = h_\Omega := \max\{h_T \ ; \ T \in \mathcal{T}_h\}, \quad h_T = \text{diam} \, T = \{|\boldsymbol{x} - \boldsymbol{y}| \ ; \ \boldsymbol{x}, \boldsymbol{y} \in T\}.$$

In addition, we set

$$\mathcal{S}_h = \{S \subset \Gamma \ ; \ \text{there exists some } T \in \mathcal{T}_h \text{ such that } S = T \cap \Gamma\},$$

---

[2] Note that the symbol h represents here the mesh size and not the right-hand side function of the generalized Robin boundary condition $(4.1)_3$.

and assume that $\mathcal{S}_h$ also satisfies the requirements to be a mesh on Γ. We call $\mathcal{S}_h$ the boundary mesh inherited from $\mathcal{T}_h$. The boundary mesh size h_Γ is defined in a manner similar to h_Ω; we see that $h_\Gamma \leq h_\Omega$.

Let us introduce an abstract finite element space equipped with an ability to approximate V.

DEFINITION 4.3. *Let $V_h \subset V$ be a finite dimensional subspace. We say that V_h approximates V with degree k $(k = 1, 2, \dots)$ if for all $m = 2, 3, \dots$ and $v \in H^m_{\Gamma_0}(\Omega; \Gamma_1)$ there exists some $v_h \in V_h$ such that*

$$\|v - v_h\|_V \leq C(k) h^{\min\{m-1,k\}} \|v\|_{H^m_{\Gamma_0}(\Omega;\Gamma_1)}. \tag{4.3}$$

Examples of V_h approximating V with degree k are given in the next subsection.

4.2. Examples of a finite element space. In the first example, we consider the case when Ω is polyhedral. Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ by flat triangles or tetrahedra. We assume $\{\mathcal{T}_h\}_{h\downarrow 0}$ is regular, i.e., $h_T \leq C\rho_T$ for all $T \in \mathcal{T}_h$, where ρ_T denotes the diameter of the inscribed ball of T. The boundary meshes $\{\mathcal{S}_h\}_{h\downarrow 0}$ are also assumed to be regular (its definition is similar as above).

For $k = 1, 2, \dots$, we define the standard P_k finite element space by

$$X_h^k = \{v_h \in C^0(\overline{\Omega}) : \ v_h|_T \in P_k(T) \ (\forall T \in \mathcal{T}_h)\} \subset H^1(\Omega),$$

where $P_k(T)$ denotes the space of the polynomials of degree $\leq k$ on T. We notice that $\{v_h|_{\overline{\Gamma}_1} ; v_h \in X_h^k\} \subset H^1(\Gamma_1)$ is identical to the standard P_k finite element space given on $\overline{\Gamma}_1$. Furthermore, we let $V_h^k = X_h^k \cap V$ be a conforming approximation to V, which is well-defined because any $v_h \in X_h^k$ such that $v_h = 0$ at the nodes in $\overline{\Gamma}_1$ exactly satisfies the Dirichlet conditions incorporated in V.

If we denote by $\mathcal{I}_h^k : C^0(\overline{\Omega}) \to X_h^k$ the standard P_k Lagrange interpolation operator, we see that $\mathcal{I}_h^k(V \cap C^0(\overline{\Omega})) = V_h^k$. By the Sobolev embedding $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ (recall $N = 2, 3$), $\mathcal{I}_h$ is well-defined on $H^m_{\Gamma_0}(\Omega; \Gamma_1)$ if $m \geq 2$. Then, well-known interpolation error estimates (see e.g.[5, 43, 42]), applied to either Ω or Γ_1, give $\|u - \mathcal{I}_h^k u\|_{H^1(\Omega)} \leq C h_\Omega^{\min\{m-1,k\}} \|u\|_{H^m(\Omega)}$ and $\|u - \mathcal{I}_h^k u\|_{H^1(\Gamma_1)} \leq C h_\Gamma^{\min\{m-1,k\}} \|u\|_{H^m(\Gamma_1)}$, where $C = C(m, k)$. Since $h_\Gamma \leq h_\Omega = h$, this implies that

$$\|u - \mathcal{I}_h^k u\|_V \leq C(m,k) h^{\min\{m-1,k\}} \|u\|_{H^m_{\Gamma_0}(\Omega;\Gamma_1)}. \tag{4.4}$$

Consequently, V_h^k approximates V with degree k.

In the second example, we let $\Gamma \in C^{1,1}$. In this case we should ideally use computational meshes capable of exactly representing the boundary Γ. This can be achieved by using isoparametric finite elements (see e.g. [8]) in the case of polynomial boundaries Γ; alternatively, if Γ corresponds to a conics section represented by NURBS, as typical in several applications, Isogeometric Analysis (see e.g. [3, 10, 32]) can be used with this aim.

In the latter case, we assume that Ω admits a NURBS parametrization $\boldsymbol{F}$: $(0,1)^N \to \Omega$ such that $\boldsymbol{F}$ is invertible, with smooth inverse on each element $Q \in \mathcal{Q}_h$, being $\mathcal{Q}_h$ the mesh associated with $(0,1)^N$. Here, we suppose that $\boldsymbol{F}$ is obtained from NURBS basis functions built from piecewise polynomials of degree k (globally $C^{1,1}$), and that $\{\mathcal{Q}_h\}_{h\downarrow 0}$ is a shape-regular family of a rectangle mesh of $(0,1)^N$. The mesh in the physical space is $T = \{\boldsymbol{F}(Q) : Q \in \mathcal{Q}_h\}$ and the NURBS function space defined in Ω is denoted by V_h. Then, by Theorem 3.2 of [3], for all $v \in H^m(\Omega)$ there

exists some NURBS interpolant $\mathcal{I}_h v \in V_h$ such that

$$\sum_{T \in \mathcal{T}_h} \|v - \mathcal{I}_h v\|_{H^1(T)}^2 \leq C(\boldsymbol{F}) \sum_{T \in \mathcal{T}_h} h_T^{2(l-1)} \|v\|_{H^l(T)}^2,$$

where $l = \min\{m, k+1\}$. Since $\boldsymbol{F}$ depends only on the shape of Ω and not its size, we have $\|v - \mathcal{I}_h v\|_{H^1(\Omega)} \leq C h^{l-1} \|v\|_{H^l(\Omega)}$. The analysis of [3] can be possibly extended to boundary norms, leading to a similar estimate on Γ. As a conclusion, we remark that V_h approximates V with degree k if $\Gamma_0 = \emptyset$; the same result is obtained when $\Gamma_0 \neq \emptyset$, assuming that Γ_0 is the union of element edges or faces on $\partial\Omega$ in virtue of Theorem 3.3 of [3].

4.3. Convergence of FEM for (GR-P). Recall that we are going to solve the problem (4.1)–(4.2). Using the formula

$$-\int_{\Gamma_1} \Delta_\Gamma u\, v\, ds = \int_{\Gamma_1} \nabla_\Gamma u \cdot \nabla_\Gamma v\, ds + \int_{\partial\Gamma_1} (\nabla_\Gamma u \cdot \boldsymbol{\nu}_{\partial\Gamma_1})\, v\, d\ell, \qquad (4.5)$$

($\boldsymbol{\nu}_{\partial\Gamma_1}$ is the outer unit normal of $\partial\Gamma_1$) we see that a smooth solution of (4.1) fulfills

$$\int_\Omega \nabla u \cdot \nabla v\, d\boldsymbol{x} + \alpha \int_{\Gamma_1} uv\, ds + \beta \int_{\Gamma_1} \nabla_\Gamma u \cdot \nabla_\Gamma v\, ds = \int_\Omega fv\, d\boldsymbol{x} + \int_{\Gamma_1} hv\, ds,$$

for all $v \in V = H^1_{\Gamma_0}(\Omega; \Gamma_1)$. Note that, when $\Gamma_0 \neq \emptyset$, the condition $v = 0$ on $\partial\Gamma_1$ circumvents the extra term on $\partial\Gamma_1$ in (4.5).

We omit the proof of the next result since it follows immediately from the Lax-Milgram theorem as in Theorem 3.2.

PROPOSITION 4.4. *Given $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_1)$, there exists a unique $u \in V$ such that*

$$a(u, v) := a_\Omega(u, v) + a_{\Gamma_1}(u, v) = (f, v)_{L^2(\Omega)} + (h, v)_{L^2(\Gamma_1)}, \qquad \forall v \in V, \qquad (4.6)$$

where

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v\, d\boldsymbol{x}, \quad a_{\Gamma_1}(u, v) = \alpha \int_{\Gamma_1} uv\, ds + \beta \int_{\Gamma_1} \nabla_\Gamma u \cdot \nabla_\Gamma v\, ds.$$

Letting $V_h \subset V$ be a finite dimensional subspace, we propose an approximate problem for (4.6) as follows: find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h)_{L^2(\Omega)} + (h, v_h)_{L^2(\Gamma_1)}, \qquad \forall v \in V_h \qquad (4.7)$$

(The reader will pardon the use of h with two different meanings: the subindex refers to the finite element gridsize, whereas h on the last inner product on the right hand side denotes the given Robin data in equation (1.1)). The existence of a unique u_h is again an immediate consequence of the Lax-Milgram theorem. We are ready to give a convergence result for the finite element approximation of (4.1).

THEOREM 4.5. *Let u and u_h be solutions of (4.6) and (4.7) respectively. If V_h approximates V with degree k, and $u \in H^m_{\Gamma_0}(\Omega; \Gamma_1)$ ($m \geq 2$), then we have*

$$\|u - u_h\|_V \leq C h^{\min\{k, m-1\}} \|u\|_{H^m_{\Gamma_0}(\Omega; \Gamma_1)}, \qquad (4.8)$$

where $C = C(\alpha, \beta, m, k)$.

Proof. According to the coercivity and boundedness of a and to the conformity $V_h \subset V$, we see from Céa's lemma (see Sect. 4.2 in [42]) that $\|u - u_h\|_V \leq C \inf_{v_h \in V_h^k} \|u - v_h\|_V$. Then (4.3) concludes the desired result. $\square$

REMARK 4.6. *If an auxiliary adjoint problem admits a suitable $H_{\Gamma_0}^2(\Omega; \Gamma_1)$-regularity result, then the Aubin-Nitsche trick can be applied to derive the L^2-estimate, which tells us that*

$$\|u - u_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Gamma_1)} \leq O(h^{\min\{k+1, m\}}).$$

Let us compare the rate of convergence, especially on the boundary, for (GR-P) $(\beta > 0)$ with that for (SR-P) $(\beta = 0)$. Suppose $C = C(\alpha, \beta, m, k)$ below. In the former case, as shown above, we get

$$\|u - u_h\|_{H^1(\Gamma_1)} \leq Ch^{\min\{k, m-1\}}(\|u\|_{H^m(\Omega)} + \|u\|_{H^m(\Gamma_1)}). \tag{4.9}$$

In the latter case, as far as we work with L^2-based Sobolev spaces, we expect to have only

$$\|u - u_h\|_{H^1(\Gamma_1)} \leq Ch^{\min\{k, m-1\}-1/2}\|u\|_{H^m(\Omega)}, \tag{4.10}$$

in view of the trace theorem combined with an inverse inequality between $H^1(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$ and accordingly to the interpolation error estimates in [48, 49]. On the other hand, if an $L^\infty(\Omega)$-error estimate as in the Dirichlet problem (see Ch. 8 of [5]) is available, then we have

$$\|u - u_h\|_{H^1(\Gamma_1)} \leq C\|u - u_h\|_{W^{1,\infty}(\Gamma_1)} \leq C\|u - u_h\|_{W^{1,\infty}(\Omega)}$$
$$\leq Ch^{\min\{k, m-1\}}\|u\|_{W^{m,\infty}(\Omega)}.$$

REMARK 4.7. *If an exact solution u is smooth enough to admit $W^{m,\infty}(\Omega)$-regularity, both the cases (GR-P) and (SR-P) would give an optimal rate of convergence. However, when $u \in H^m(\Omega; \Gamma_1)$ but $u \notin W^{m,\infty}(\Omega)$, we still have an optimal convergence rate in (GR-P), whereas we expect only sub-optimal one in (SR-P). Such behavior is indeed observed in our numerical experiment shown in Sect. 5.1.*

5. Numerical examples. Let us define the notation that we are going to use in this section. In our numerical tests, we build a sequence of N_T meshes $\{\mathcal{T}_h^i\}_{i=0}^{N_T}$, each one characterised with a mesh step h_i. The index i represents indeed the *refinement step* of the sequence of meshes. The numerical error with respect to the exact solution is addressed with the notation e_h^i. For each triangulation $\mathcal{T}_h^i$, we compute the following error norms: $\|e_h^i\|_{L^2(\Omega)}$, $\|e_h^i\|_{L^2(\Gamma)}$, $\|e_h^i\|_{H^1(\Omega)}$, and $\|e_h^i\|_{H^1(\Gamma)}$. We also define the order of convergence of a generic norm $\|\cdot\|_*$ as follows:

$$\rho_*^i = \log\left(\frac{\|e_h^i\|_*}{\|e_h^{i-1}\|_*}\right) \Big/ \log\left(\frac{h^i}{h^{i-1}}\right) \qquad \forall i = 1, .., N_T. \tag{5.1}$$

To indicate exact or numerical solutions of (GR-P) and (SR-P), we add the superscripts GR and SR, respectively.

5.1. Regularity. In Remark 4.7, we have highlighted the difference between the convergence rate for $\|u^{GR} - u_h^{GR}\|_{H^1(\Gamma_1)}$ and that for $\|u^{SR} - u_h^{SR}\|_{H^1(\Gamma_1)}$. In this subsection we give a numerical example where such a difference is actually attained.

With this aim, we perform a two-dimensional test using the software `Freefem++` (see [31]). We choose as domain Ω the square $(0,1)^2$ and, as boundary Γ_1, the upper side of Ω (see Figure 5.2(a)). We set the data of the problem as follows:

$$h(x,y) = \begin{cases} 20 & \text{if } x > 1/2 \\ 0 & \text{if } x \leq 1/2 \end{cases} \quad \text{and} \quad f(x,y) = 1,$$

α is set to 1, and, in the case of generalized Robin conditions, β is set equal to 10. As exact solutions, we use the numerical solutions computed with P2 finite element on a fine mesh with the mesh step $1/300$. In the following, we suppose that the mixed boundary conditions do not cause a singularity.

The regularity of h defined above is $h \in W^{s,p}(\Gamma_1)$ such that $sp < 1, s > 0, p > 1$; especially, if $p = 2$, then $h \in H^{1/2-\epsilon}(\Gamma_1)$ for arbitrary small $\epsilon > 0$. In view of the regularity theorem for (GR-P) and (SR-P) (see (3.12) and (3.13)), we expect to have $u^{GR} \in H^2(\Omega; \Gamma_1)$ and $u^{SR} \in H^{2-\epsilon}(\Omega)$. Note that u^{SR} can never be in $W^{2,\infty}(\Omega)$. As a result, (4.8)–(4.10) tell us that

$$\|u^{GR} - u_h^{GR}\|_{H^s(\Omega)} = O(h^{2-s}), \qquad \|u^{GR} - u_h^{GR}\|_{H^s(\Gamma_1)} = O(h^{2-s}),$$
$$\|u^{SR} - u_h^{SR}\|_{H^s(\Omega)} = O(h^{2-s-\epsilon}), \qquad \|u^{SR} - u_h^{SR}\|_{H^s(\Gamma_1)} = O(h^{3/2-s-\epsilon}).$$

with $s = 0, 1$. When $s = 0$, we assume that the regularity results for the Aubin-Nitsche technique in L^2-norms in Ω and on Γ_1 is applicable and we have used $\|v\|_{L^2(\Gamma_1)} \leq C\|v\|_{L^2(\Omega)}^{1/2}\|v\|_{H^1(\Omega)}^{1/2}$ to estimate $\|u^{SR} - u_h^{SR}\|_{L^2(\Gamma_1)}$.

The predictions made above are consistent with the numerical results reported in Figure 5.1. In fact, Figure 5.1 shows that the expected orders of convergence are attained, which confirms $u^{GR} \in H^2(\Omega; \Gamma_1)$ and $u^{SR} \in H^{2-\epsilon}(\Omega)$. In other words, our theoretical result that the GRBC provides more regularity on the boundary than the SRBC does, if given data are the same, is numerically confirmed.
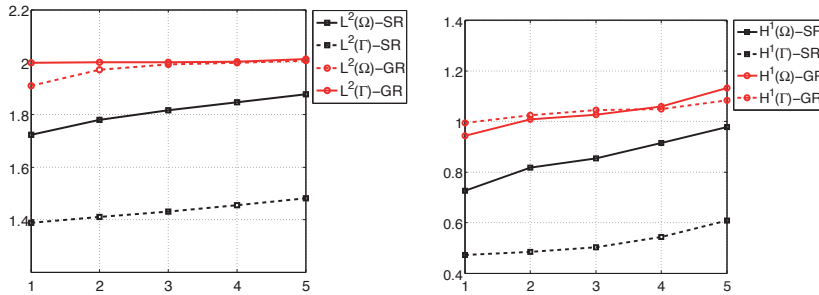
(a) $\rho_{L^2(\Omega)}$ and $\rho_{L^2(\Gamma_1)}$, P1 basis functions. (b) $\rho_{H^1(\Omega)}$ and $\rho_{H^1(\Gamma_1)}$, P1 basis functions.

FIGURE 5.1. *Rate of convergence on the refinement step.*

5.2. Convergence. In this section we aim at providing numerical evidence for the rates of convergence of the errors associated to the finite element approximation of the (GR-P) (4.1), as reported in Theorem 4.5. Firstly, we consider a problem for which the computational domain and its boundary are exactly represented by the computational mesh of the finite element (the mesh). Then, we repeat our analysis for a curved domain (a conic section) for which the triangulation introduces a geometrical

error associated to the polygonal representation of the domain and the boundary; we remark that this situation often occurs in fluid-structure-interaction problems of practical interest, as in hemodynamics. Specifically, we aim at highlighting the influence of the geometrical approximation on the rates of convergence of the errors for for the finite element approximation of (GR-P).

We also compare the rates of convergence of the errors predicted by Theorem 4.5 for the finite element method with those obtained when considering Isogeometric Analysis (see [10, 32]), a numerical approximation method for PDEs which preserves the exactness of the geometrical representation of a computational domain through the whole h-refinement procedure when the latter is described by B-splines or NURBS (see [39]), as it is the case for conic shapes. We observe that in such cases Isogeometric Analysis also allows the exact evaluation of the Laplace-Beltrami term appearing in the GRBC. Finally, we remark that the interpolation error estimates for functions in a NURBS space possess the same convergence orders of their polynomial counterpart of degree k (see [3]); therefore, the result of Theorem 4.5 holds also when NURBS-based Isogeometric Analysis is considered as approximation method for the PDEs. We observe that the rates of convergence of the errors predicted by Theorem 4.5 eventually hold also when considering an isoparametric finite element method (see [8]), being the error associated to the geometrical approximation convergent at least at the same order of the interpolation error.

The expected rates of convergence of the errors for the finite element method and Isogeometric Analysis are summarized in Table 5.1.

Meth.	Degree	Γ_1	$\rho_{H^s(\Omega;\Gamma_1)}$	Meth.	Degree	Γ_1	$\rho_{H^s(\Omega;\Gamma_1)}$
FE	k	Flat	k+1-s	IGA	k	Flat	k+1-s
FE	k	Curved	2-s	IGA	k	Curved	k+1-s

TABLE 5.1

Expected rates of convergence of the errors in norm $L^2(\Omega;\Gamma)$ ($s = 0$) and $H^1(\Omega;\Gamma)$ ($s = 1$) for problem (4.1) for the finite element method (FE) and NURBS-based Isogeometric Analysis (IGA) of polynomial degree k; comparison for domains with boundary Γ_1 "flat" or "curved".

We solve problem (4.1) and we set the data so that the exact solution is $u_{ex} = y\cos(\pi y)\sin(\pi x)$ and we set the data $f(x,y)$ and $h(x,y)$ so that u_{ex} is the exact solution of problem (4.1). In (GR-P) we use $\beta = 1$. We perform two tests: for the first case, a domain with a flat boundary Γ_1 is selected and, in particular, Ω and Γ_1 are the same as in Figure 5.2(a); for the second case, we perform the test, with the same exact solution, using a geometry with a curved boundary and we select Γ_1 as it is shown is Figure 5.2(b).
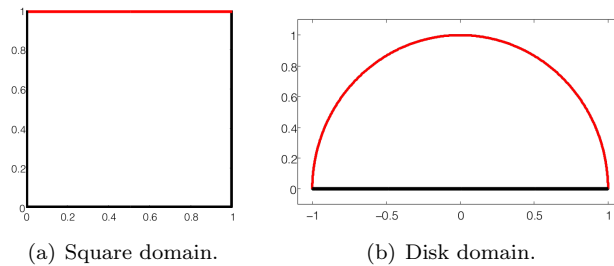

(a) Square domain.   (b) Disk domain.

FIGURE 5.2. *Computational domains Ω. The subset $\Gamma_1 \subset \Gamma$ is indicated in red.*

The results obtained with the finite element method are reported in Figures 5.3 and 5.4, while those obtained with Isogeometric Analysis are reported in Figures 5.5 and 5.6. We use the notation -F and -C to address the Flat boundary case (square) and, respectively, the Curved one (disk). The results obtained with a SRBC are addressed with the notation (SR), otherwise, when it is not specified, a GRBC is involved.
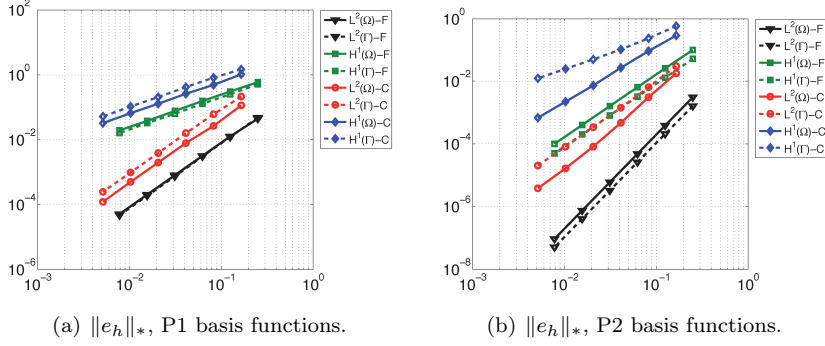

(a) $\|e_h\|_*$, P1 basis functions.   (b) $\|e_h\|_*$, P2 basis functions.

FIGURE 5.3. *Error norms vs the mesh size, computed for the 2D test cases with the finite element method. The notations $-F$ and $-C$ stand for, respectively, Flat and Curved Γ_1.*
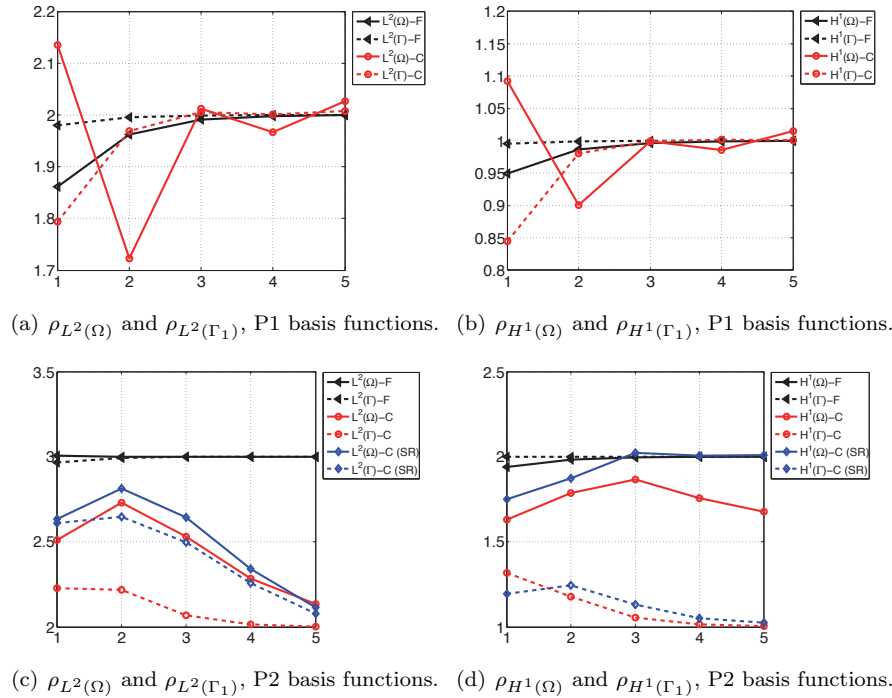

(a) $\rho_{L^2(\Omega)}$ and $\rho_{L^2(\Gamma_1)}$, P1 basis functions. (b) $\rho_{H^1(\Omega)}$ and $\rho_{H^1(\Gamma_1)}$, P1 basis functions.

(c) $\rho_{L^2(\Omega)}$ and $\rho_{L^2(\Gamma_1)}$, P2 basis functions. (d) $\rho_{H^1(\Omega)}$ and $\rho_{H^1(\Gamma_1)}$, P2 basis functions.

FIGURE 5.4. *Rate of convergence vs the refinement step, computed for the 2D test cases with the finite element method. The notations $-F$ and $-C$ stand for Flat and Curved surface Γ_1, respectively.*

Regarding the finite element approximation, in Figure 5.3 we display the computed errors for the linear case (Fig. 5.3(a)) and for the quadratic one (Fig. 5.3(b)).
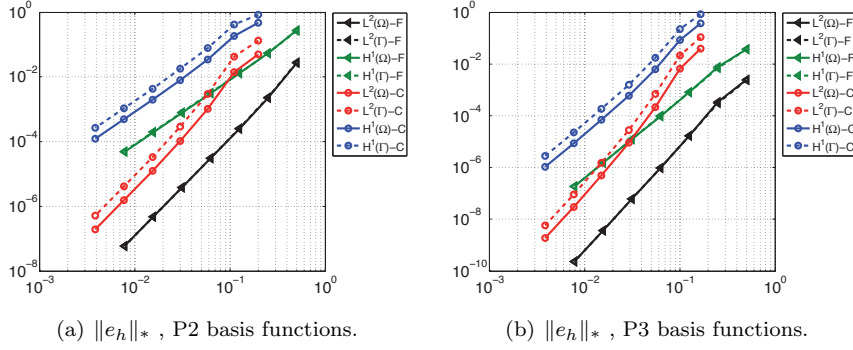
(a) $\|e_h\|_*$, P2 basis functions.   (b) $\|e_h\|_*$, P3 basis functions.

FIGURE 5.5. *Error norms vs the mesh step computed for the 2D test cases with the Isogemetric Analysis discretization. The notations $-F$ and $-C$ stand for Flat and Curved Γ_1, respectively.*
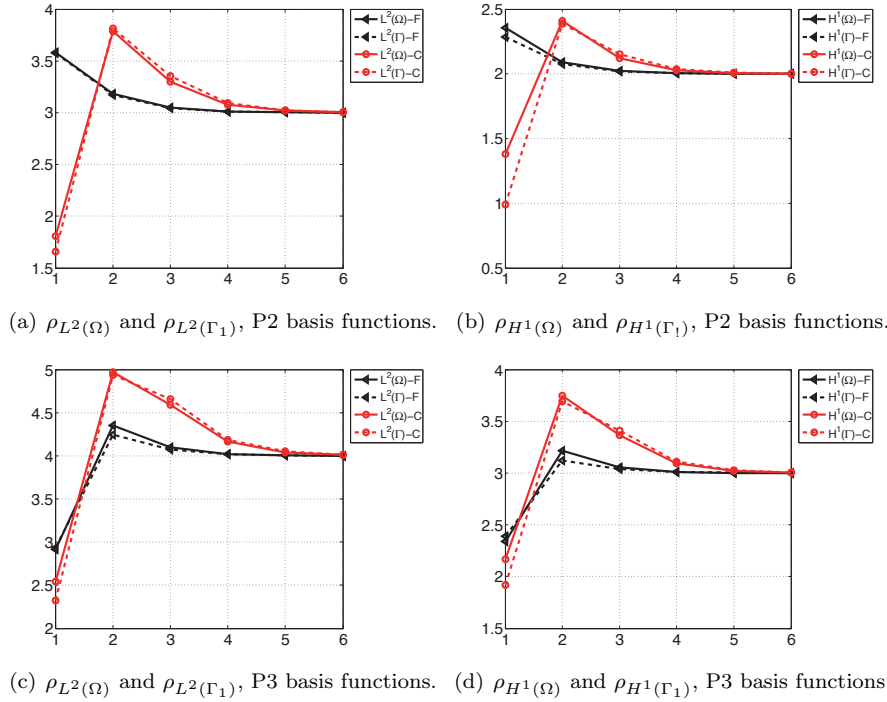

(a) $\rho_{L^2(\Omega)}$ and $\rho_{L^2(\Gamma_1)}$, P2 basis functions.   (b) $\rho_{H^1(\Omega)}$ and $\rho_{H^1(\Gamma_1)}$, P2 basis functions.

(c) $\rho_{L^2(\Omega)}$ and $\rho_{L^2(\Gamma_1)}$, P3 basis functions.   (d) $\rho_{H^1(\Omega)}$ and $\rho_{H^1(\Gamma_1)}$, P3 basis functions.

FIGURE 5.6. *Rate of convergence on the refinement step computed for the 2D test cases with the Isogemetric Analysis discretization. The notations $-F$ and $-C$ stand for Flat and Curved surface Γ_1, respectively.*

In Figure 5.4 the rate of convergence ρ_* are shown for the norms of $L^2(\Omega)$, $L^2(\Gamma_1)$, $H^1(\Omega)$ and $H^1(\Gamma_1)$. From Figures 5.4(a) and 5.4(b) we observe that the rate of convergence is always optimal when using P1 basis functions, no matter if the boundary Γ_1 is flat or curved. On the contrary, when we use P2-FE basis functions the order of convergence are optimal only in the flat case as it is clear in Figures 5.4(c) and 5.4(d). In the curved boundary case, the polygonal approximation of Γ_1 affects the convergence rate for both the errors inside the domain and on the surface. Moreover,

we observe that when using a SRBC, instead of a GRBC, the order of convergence is faster. In our opinion, this is due to the fact that, in the case of SRBC, the approximation of the boundary and, thus, of its normal, affects only the part of the domain and surface where the integrals are computed. On the other hand, in (GR-P), the approximation of the normal affects directly the projection operator and thus the left hand side term of the finite element problem. When Ω is a polygonal (or polyhedral) domain, the sequence $\{\mathcal{T}_h^i\}_{i=0}^{N_T}$ is composed of nested meshes. On the contrary, in the case where a part of $\partial\Omega$ is curved and a polygonal approximation of the geometry is used, the computational domain changes at each refinement step and the triangulation are not nested. In our opinion, the fact that the mesh are not nested can explain the oscillating behaviour of the rate of convergence in case of a curved boundary (see Figs. 5.4(a) and 5.4(b)).

The results obtained by considering NURBS-based Isogeometric Analysis are reported in Figures 5.5 and 5.6. In Figure 5.5(a) and 5.5(b) we display the behavior of the errors vs. the mesh size h obtained with NURBS bases of quadratic ($k = 2$) and cubic ($k = 3$) polynomial degrees, respectively; in Figure 5.6 we report the convergence rates of the errors in $L^2(\Omega)$, $L^2(\Gamma_1)$, $H^1(\Omega)$ and $H^1(\Gamma_1)$ norms. We observe that the rates of convergence always coincide with the polynomial degree k in the cases of H^1 norms, while with $k + 1$ in the cases of L^2 norms.

We conclude that if we are interested in a high order finite element approximation of the exact solution of (4.1), it is preferable to use a consistent approximation of the geometrical domain that yields a convergence rate of the geometrical error at least equivalent to the one expected by the choice of the finite dimensional function space.

6. Conclusions. In this work we considered the analysis of a steady Poisson problem endowed with a generalized Robin boundary condition that involves a Laplace-Beltrami operator on the boundary. We proposed a mathematical analysis of the problem and, in particular, we proved the well-posedness of the weak formulation, regularity of the solution and convergence of the finite element error. In particular, we proved that, in case of generalized Robin conditions, the regularity of the solution on the boundary is higher with respect to the standard Robin case. We showed that the convergence analysis of the finite element discretization hold for domain with both planar and curved boundaries, under the condition that the numerical domain coincides with the exact one.

We provided numerical evidence to the proposed theoretical results through two-dimensional finite element simulations. In particular, we compared numerical results obtained for discretizations with and without the introduction of geometrical approximation errors. In the case of domain with curved boundaries we used Isogeometric Analysis, a generalization of the finite element method, in order to exactly represent the domain. We noticed that, when a geometrical error is introduced, the rates of convergence are sub-optimal, on the contrary, when the domain geometry is exactly reproduced, the observed numerical rates coincide with those predicted by the theory.

Possible extensions of the current work include the analysis of vectorial unsteady and saddle-point problems endowed with generalized Robin boundary conditions.

REFERENCES

[1] S. AGMON, A. DOUGLIS, AND L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math., 12 (1959), pp. 623–727.

[2] S. BADIA, A. QUAINI, AND A. QUARTERONI, *Splitting methods based on algebraic factorization for fluid-structure interaction*, SIAM J. Sci. Comput., 30 (2008), pp. 1778–1805.

[3] Y. BAZILEVS, L. BEIRÃO DA VEIGA, J. A. COTTRELL, T. J. R. HUGHES, AND G. SANGALLI, *Isogeometric Analysis: approximation, stability and error estimates for h-refined meshes*, Math. Model. Meth. Appl. Sci., 16 (2006), pp. 1031–1090.

[4] D. BENNEQUIN, M. J. GANDER, AND L. HALPERN, *A homographic best approximation problem with application to optimized schwarz waveform relaxation.*, Math. Comput., 78 (2009), pp. 185–223.

[5] S. C. BRENNER AND L. R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer, 3rd ed., 2007.

[6] H. BREZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

[7] M. BUKAC, S. CANIC, R. GLOWINSKI, J. TAMBACA, AND A. QUAINI, *Fluid-structure interaction in blood flow capturing non-zero longitudinal structure displacement*, J. Comput. Phys., 228 (2012), pp. 515–541.

[8] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, 1978.

[9] C. M. COLCIAGO, S. DEPARIS, AND A. QUARTERONI, *Comparisons between reduced order models and full 3D models for fluid-structure interaction problems in haemodynamics*, J. Comput. Appl. Math., (2013 in press).

[10] J. A. COTTRELL, T. J. R. HUGHES, AND Y. BAZILEVS, *Isogeometric Analysis: toward integration of CAD and FEA*, Wiley, 2009.

[11] P. CROSETTO, P. REYMOND, S. DEPARIS, D. KONTAXAKIS, N. STERGIOPULOS, AND A. QUARTERONI, *Fluid-structure interaction simulation of aortic blood flow*, Comput. & Fluids, 43 (2011), pp. 46–57.

[12] M. C. DELFOUR AND J. P. ZOLÉSIO, *Differential equations for linear shells : comparison between intrinsic and classical models*, CRM Proc. Lecture Notes, 11 (1997), pp. 41–124.

[13] R. DENK, J. PRÜSS, AND R. ZACHER, *Maximal l_p-regularity of parabolic problems with boundary dynamics of relaxation type*, J. Funct. Anal., 255 (2008), pp. 3149–3187.

[14] S. DEPARIS, M. DISCACCIATI, G. FOURESTEY, AND A. QUARTERONI, *Fluid-structure algorithms based on Steklov-Poincaré operators*, Comput. Meth. Appl. Mech. Eng., 195 (2006), pp. 5797–5812.

[15] L.C. EVANS, *Partial Differential Equation*, American Mathematical Soc., 2010.

[16] M. A. FERNANDEZ, *Incremental displacement-correction schemes for incompressible fluid-structure interaction. stability and convergence analysis.*, Numer. Math., 123 (2013), pp. 21–65.

[17] M. A. FERNANDEZ, J.F. GERBEAU, AND C. GRANDMONT, *A projection semi-implicit scheme for the coupling of an elastic structure with an incompressible fluid*, Int. J. Numer. Meth. Eng., 64 (2007), pp. 794–821.

[18] C. A. FIGUEROA, S. BAEK, C. A. TAYLOR, AND J. D. HUMPHREY, *A computational framework for fluid-solid-growth modeling in cardiovascular simulations*, Comput. Meth. Appl. Mech. Eng., 198 (2009), pp. 3583–3601.

[19] C. A. FIGUEROA, I. E. VIGNON-CLEMENTEL, K. E. JANSEN, T. J.R. HUGHES, AND C. A. TAYLOR, *A coupled momentum method for modeling blood flow in three-dimensional deformable arteries*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 5685–5706.

[20] L. FORMAGGIA, A. QUARTERONI, AND A. VENEZIANI EDS, *Cardiovascular Mathematics: Modeling and Simulation of the Circulatory System*, Springer, 2009.

[21] M. J. GANDER AND L. HALPERN, *Optimized schwarz waveform relaxation methods for advection reaction diffusion problems.*, SIAM J. Numer. Anal., 45 (2007), pp. 666–697.

[22] M. J. GANDER, L. HALPERN, AND F. NATAF, *Optimal Convergence for Overlapping and Non-Overlapping Schwarz Waveform Relaxation*, 11th International Conference on Domain Decomposition Methods, 1999, pp. 27–36. ID: unige:8286.

[23] L. Gerardo-Giorda, F. Nobile, and C. Vergara, *Analysis and optimization of Robin-Robin partitioned procedures in fluid-structure interaction problems*, SIAM J. Numer. Anal., 48 (2012), pp. 2091–2116.

[24] F. Gesztesy and M. Mitrea, *Generalized robin boundary conditions, Robin-to-Dirichlet Maps, and Krein-Type resolvent formulas for Schrödinger Operators on bounded lipschitz domains*, ArXiv e-prints, (2008).

[25] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[26] G. R. Goldstein, *Derivation and physical interpretation of general boundary conditions*, Adv. Differential Equations, 11 (2006), pp. 457–480.

[27] P. Grisvard, *Elliptic Problems in Non Smooth Domains*, Pitman, 1985.

[28] G. Guidoboni, R. Glowinski, N. Cavallini, and S. Canic, *Stable loosely-coupled-type algorithm for fluid-structure interaction in blood flow*, J. Comput. Phys., 228 (2009), pp. 6916–6937.

[29] G. Guidoboni, R. Glowinski, N. Cavallini, S. Canic, and S. Lapin, *A kinematically coupled time-splitting scheme for fluid-structure interaction in blood flow*, J. Comput. Phys., 22 (2009), pp. 684–688.

[30] L. Halpern, *Optimized schwarz waveform relaxation: Roots, blossoms and fruits*, in Domain Decomposition Methods in Science and Engineering XVIII, vol. 70 of Lecture Notes in Computational Science and Engineering, Springer Berlin Heidelberg, 2009, pp. 225–232.

[31] F. Hecht, O. Pironneau, F. Le Hyaric, and K. Ohtsuka, *Freefem++*, available online at www.freefem.org.

[32] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Comput. Meth. Appl. Mech. Eng., 194 (2005), pp. 4135–4195.

[33] H. J. Kim, I. E. Vignon-Clementel, J. S. Coogan, C. A. Figueroa, K. E. Jansen, and C. A. Taylor, *Patient-specific modeling of blood flow and pressure in human coronary arteries*, Ann. Biomed. Eng., 38 (2010), pp. 3195–3209.

[34] J. L. Lions, *Lectures on Elliptic Partial Differential Equations*, Tata Institute of Fundamental Research, Bombay, 1957.

[35] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer, 1972.

[36] P.L. Lions, *On the Schwarz alternating method iii: A variant for nonoverlapping subdomains*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, 1990.

[37] P. Moireau, N. Xiao, M. Astorino, C. A. Figueroa, D.Chapelle, C. A. Taylor, and J.F. Gerbeau, *External tissue support and fluid-structure simulation in blood flows*, Biomech. Model. Mechanobiol., 11 (2012), pp. 1–18.

[38] J. Nečas, *Direct Methods in the Theory of Elliptic Equations*, Springer, 2012.

[39] L. Piegel and W. Tiller, *The NURBS Book*, Springer–Verlag, New York, 1997.

[40] J. Prüss, *Maximal regularity for abstract parabolic problems with inhomogeneous boundary data in L_p spaces*, Math. Bohem., 127 (2002), pp. 311–327.

[41] A. Quaini and A. Quarteroni, *A semi-implicit approach for fluid-structure interaction based on an algebraic fractional step method*, Math. Models Methods Appl. Sci., 17 (2007), pp. 957–983.

[42] A. Quarteroni, *Numerical Models for Differential Problems*, vol. 8 of MS&A, Springer, 2nd ed., 2013.

[43] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Series in Computational Mathematics, Springer, 2008.

[44] ———, *Domain Decomposition Methods for Partial Differential Equation*, vol. 10, Clarendon Press, 99.

[45] S. Romanelli, *Goldstein-Wentzell boundary conditions: recent results with Jerry and Gisèle Goldstein*, Discrete Contin. Dyn. Syst., 34 (2014), pp. 749–760.

[46] A. Tagliabue, *Isogeometric analysis for reduced fluid-structure interaction models in haemodynamic applications*, master's thesis, University of Insubria, 2012.

[47] J. L. Vázquez and E. Vitillaro, *Heat equation with dynamical boundary conditions of reactive-diffusive type*, J. Differential Equations, 250 (2011), pp. 2143–2161.

[48] R. Verfürth, *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Advances in numerical mathematics, Wiley-Teubner, 1996.

[49] R. Verfürth, *Error estimates for some quasi-interpolation operators*, ESAIM, Math. Model. Numer. Anal., 33 (1999), pp. 695–713.

37.2013 S. DEPARIS, D. FORTI, A. QUARTERONI:
A rescaled localized radial basis functions interpolation on non-cartesian and non-conforming grids

38.2013 A. GIZZI, R. RUIZ-BAIER, S. ROSSI, A. LAADHARI, C. CHERUBINI, S. FILIPPI:
A three-dimensional continuum model of active contraction in single cardiomyocytes

39.2013 P. TRICERRI, L. DEDÈ, A. QUARTERONI, A. SEQUEIRA:
Numerical validation of isotropic and transversely isotropic constitutive models for healthy and unhealthy cerebral arterial tissues

40.2013 D. KRESSNER, M. STEINLECHNER, A. USCHMAJEW:
Low-rank tensor methods with subspace correction for symmetric eigenvalue problems

41.2013 A. ABDULLE, O. BUDÁC:
An adaptive finite element heterogeneous multiscale method for Stokes flow in porous media

42.2013 C. JÄGGLI, L.IAPICHINO, G. ROZZA:
An improvement on geometrical parametrizations by transfinite maps

43.2013 P. PACCIARINI, G. ROZZA:
Stabilized reduced basis method for parametrized advection-diffusion PDEs

44.2013 A. KOSHAKJI, A. QUARTERONI, G. ROZZA:
Free form deformation techniques applied to 3D shape optimization problems

45.2013 J. E. CATRILLON-CANDAS, F. NOBILE, R. F. TEMPONE:
Analytic regularity and collocation approximation for PDEs with random domain deformations

_____

01.2014 GIOVANNI MIGLIORATI:
Multivariate Markov-type and Nicolskii-type inequalities for polynomials associated with downward closed multi-index sets

02.2014 FEDERICO NEGRI, ANDREA MANZONI, GIANLUIGI ROZZA:
Certified reduced basis method for parametrized optimal control problems governed by the Stokes equations

03.2014 CEDRIC EFFENBERGER, DANIEL KRESSNER:
On the residual inverse iteration for nonlinear eigenvalue problems admitting a Rayleigh functional

04.2014 TAKAHITO KASHIWABARA, CLAUDIA M. COLCIAGO, LUCA DEDÈ, ALFIO QUARTERONI:
Numerical Well-posedness, regularity, and convergence analysis of the finite element approximation of a generalized Robin boundary value problem