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Reduced basis approximation of parametrized optimal flow control problems for the Stokes equations☆

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Abstract

This paper extends the reduced basis method for the solution of parametrized optimal control problems presented in [1] to the case of noncoercive (elliptic) equations, such as the Stokes equations. We discuss both the theoretical properties – with particular emphasis on the stability of the resulting double nested saddle-point problems and on aggregated error estimates – and the computational aspects of the method. Then, we apply it to solve a benchmark vorticity minimization problem for a parametrized bluff body immersed in a two or a three-dimensional flow through boundary control, demonstrating the effectivity of the methodology.

Keywords: reduced basis method, optimal flow control, saddle-point problems, PDE-constrained optimization, a posteriori error estimates
2010 MSC: 49J20, 65K10, 65M15, 65N12, 76B75

1. Introduction

In this paper we extend and deepen the Reduced Basis (RB) method for the solution of parametrized quadratic optimization problems presented in [1] in order to deal with Stokes equations, as a particular case of noncoercive elliptic state system. In this respect, several model order reduction techniques have already been employed to speed up different kinds of PDE-constrained optimization problems: for instance, proper orthogonal decomposition (see e.g. [2, 3] and references therein) and reduced basis method [4, 5, 6, 7, 8, 1, 9]. However, none of the previous works treated the case of parametrized optimization problems constrained by the Stokes equations and featuring an infinite-dimensional control variable. Indeed, after the first pioneering works by Ito and Ravindran [4, 10], in [2] a POD-based reduced model for the optimal control of unsteady Navier-Stokes flows has been proposed, dealing with a space-time distributed control variable, but neither parameter dependence, nor efficient error bounds, were considered.

Let us first introduce in a rather general form the class of problems we are interested in. Let us denote by $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ a vector of $P \geq 1$ input parameters representing either physical or geometrical quantities, $\mathbf{y}$ the state variable, $\mathbf{u}$ the control variable, $\mathcal{J}$ the cost functional, $\mathbf{E}(\cdot, \cdot; \boldsymbol{\mu})$ the residual of the state equation. The full-order parametrized optimal control problem reads: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{\mathbf{y},\mathbf{u}} \mathcal{J}(\mathbf{y}, \mathbf{u}; \boldsymbol{\mu}) \qquad \text{subject to} \qquad \mathbf{E}(\mathbf{y}, \mathbf{u}; \boldsymbol{\mu}) = \mathbf{0}. \tag{1}$$

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In this work we deal with problems featuring a quadratic cost functional and the steady Stokes equations as constraint, so that $\mathbf{E}$ results from, e.g., the Finite Elements discretization of the Stokes problem. The state variables are thus given by velocity and pressure, while control variables may act either as distributed source terms, or as Dirichlet and/or Neumann data prescribed on the boundary.

We remark that the parameters $\boldsymbol{\mu}$ represent an input of the optimization problem – for instance defining the configuration of the state system – rather than design (control) variables. For this reason, in order to possibly compare different scenarios or to evaluate the sensitivity and robustness of the optimal solution with respect to the parameters, we are required to solve the optimization problem many times. This task requires a computational effort which is not only (often) unaffordable, due to the significant challenges that the solution of the full-order problem (1) presents, but which may also be worthless, since the possible smooth dependence of the solution with respect to the parameters is not taken into account.

Our goal is to develop a suitable Reduced Order Model (ROM) that takes advantage of – rather than ignoring – the parametric dependence of the solution. Since the control variable $\mathbf{u}$ results from the discretization of an infinite-dimensional function, the construction of a suitable ROM requires a simultaneous reduction of the state and control spaces. This is why, in [1] we proposed to construct suitable RB spaces for the state, control and adjoint variables by solving the optimality system associated to (1) for suitably chosen parameter values. Then, the RB approximation is obtained by projecting the optimality system onto the low dimensional RB spaces. Here we extend this framework in several directions building it upon the preliminary works [11, 12]:

(i) We consider as state constraint the Stokes equations, as an example of noncoercive equations. In particular, their saddle-point structure introduce further difficulties in designing a stable RB approximation of the optimization problem.

(ii) Inspired by the optimize-then-discretize and discretize-then-optimize strategies (see e.g. [13, 14, 15]) for PDE constrained optimization, we propose two possible approaches to build a ROM, namely the *optimize-then-reduce* and the *reduce-then-optimize* approach. Then, we show that defining a common aggregated RB space for the state and adjoint variables [5, 1, 16, 9], we obtain a ROM which is consistent and preserves the Lagrangian structure of the problem.

(iii) By casting the optimality system in the framework of saddle-point problems, we prove that the above mentioned RB spaces (suitably enriched by supremizer solutions [17, 18]) lead to a stable approximation.

(iv) We review two different a posteriori estimates for the error between the full and reduced-order solutions. First, we show how the estimate proposed in [2, 16, 19] for the error in the control variable can be obtained by formulating the optimization problem through the so-called *reduced-space* setting (see e.g. [20, 15]), and then exploiting the coercivity of the reduced Hessian operator. Since the resulting estimate involves a *non-sparse* and *non-linearly parametrized* residual operator, we move to a *full-space* setting in order to derive a combined estimate for the state, control and adjoint variables [1], which requires the computation of the dual norm of the residual of the optimality system (a *sparse* and *affinely parametrized* operator) and its stability factor. For the latter, we propose a cheap yet accurate approximation rather than a rigorous lower bound.

(v) The above framework is applied to a benchmark Dirichlet boundary control problem of vorticity minimization behind a bluff body immersed in a two and a three-dimensional flow (see e.g. [21, 22] for similar problems) In particular, this latter case represents the first example of a three-dimensional optimal flow control problem (featuring both geometrical and physical parameters) solved within the RB context.

We also remark that in [23] the above framework has been exploited to develop a multi-level and weighted reduced basis approximation to the stochastic version of problem (1).

The plan of the paper is as follows. In the remainder of this section we introduce a general formulation for parametrized optimization problems governed by the Stokes equations. In Sect. 2 we introduce the optimality system formulated as a saddle-point problem and we briefly discuss its Finite Element (FE) approximation. In Sect. 3 we introduce its RB approximation, discussing two possible (reduce-then-optimize and optimize-then-reduce) approaches, as well as the definition of a stable pair of RB aggregated spaces for the state and adjoint variables. Then, in Sect. 4 we derive suitable a posteriori error estimates. Finally, in Sect. 5 we present some numerical results dealing with the Dirichlet boundary control problem already mentioned. Technical proofs are detailed in the Appendix Appendix A.

1.1. Problem description

Let $\mathcal{D} \subset \mathbb{R}^P$ be a prescribed P-dimensional compact set of parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P)$. We consider the following parametrized optimal control problem ($\boldsymbol{\mu}$-OCP): find a triple $(\mathbf{v}, \pi, \mathbf{u})$ such that the cost functional

$$\mathcal{J}_o(\mathbf{v}, \pi, \mathbf{u}; \boldsymbol{\mu}) = \mathcal{F}_o(\mathbf{v}, \pi; \boldsymbol{\mu}) + \mathcal{G}_o(\mathbf{u}; \boldsymbol{\mu}) \tag{2}$$

be minimized, subject to the steady Stokes equations:

$$\begin{aligned} -\nu \Delta \mathbf{v} + \nabla \pi &= \rho_1 \mathbf{u}_1 && \text{in } \Omega_o(\boldsymbol{\mu}) \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega_o(\boldsymbol{\mu}) \\ \mathbf{v} &= \rho_2 \mathbf{u}_2 && \text{on } \Gamma^o_D(\boldsymbol{\mu}) \\ -\pi \mathbf{n} + \nu (\nabla \mathbf{v}) \mathbf{n} &= \rho_3 \mathbf{u}_3 && \text{on } \Gamma^o_N(\boldsymbol{\mu}). \end{aligned} \tag{3}$$

Here $\Omega_o(\boldsymbol{\mu}) \subset \mathbb{R}^d$ $(d = 2, 3)$ is a parametrized spatial domain with Lipschitz boundary $\partial\Omega_o = \Gamma^o_D \cup \Gamma^o_N$, where Γ^o_D and Γ^o_N are the Dirichlet and the Neumann portions of the boundary; the state variables $(\mathbf{v}, \pi)$ denote the velocity and pressure fields, respectively, while $\nu > 0$ is the kinematic viscosity. The Boolean variables ρ_i satisfy $\rho_1 + \rho_2 + \rho_3 = 1$, so that the control variable $\mathbf{u}$ may represent either a source term $(\mathbf{u}_1)$, a boundary velocity on Γ^o_D $(\mathbf{u}_2)$ or a Neumann flux on Γ^o_N $(\mathbf{u}_3)$.
In the parametrized cost functional $\mathcal{J}_o$, $\mathcal{F}_o(\mathbf{v}, \pi; \boldsymbol{\mu})$ represents the objective to be minimized, while $\mathcal{G}_o(\mathbf{u}; \boldsymbol{\mu})$ is a regularization term ensuring the well-posedness of the problem [13]; here we consider both a vorticity type and an energy type functional:

$$\mathcal{F}_o(\mathbf{v}; \boldsymbol{\mu}) = \frac{1}{2} \int_{\Omega_o(\boldsymbol{\mu})} |\nabla \times \mathbf{v}|^2 \, d\Omega_o, \qquad \mathcal{F}_o(\mathbf{v}; \boldsymbol{\mu}) = \frac{\nu}{2} \int_{\Omega_o(\boldsymbol{\mu})} |\nabla \mathbf{v}|^2 \, d\Omega_o.$$

1.2. The optimal control problem

The $\boldsymbol{\mu}$-OCP (2)-(3) thus reads (see e.g. [24]): given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{x \in X_o} \mathcal{J}_o(x; \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{E}_o(x) = 0 \ \text{ in } Q'_o, \tag{4}$$

where X_o and Q_o are two Hilbert spaces, along with their dual X'_o and Q'_o respectively, $x = (\mathbf{v}, \pi, \mathbf{u}) \in X_o$ is the optimization variable and the operator $\mathcal{E}_o \colon X_o \to Q'_o$ describes the state equation. In particular, let us denote by $V_o = V(\Omega_o)$ the velocity space,

$$V_o = \mathbf{H}^1_D(\Omega_o) := \Big\{ \mathbf{v} \in [H^1(\Omega_o)]^d \colon \mathbf{v} = 0 \text{ on } \Gamma^o_D \Big\},$$

by $M_o = L^2(\Omega_o)$ the pressure space and by $\mathbf{H}^{1/2}(\Gamma^o) = [H^{1/2}(\Gamma^o)]^d$. Moreover, we assume the control space U_o to be a Hilbert space[1], whereas we choose as state space $Y_o = V_o \times M_o$. Then, we set $X_o = Y_o \times U_o$ (endowed with the usual ℓ^2-norm) and we set $Q_o = Y_o$, since the Stokes operator (3) can be considered with values in $Q'_o = Y'_o$.

To take into account a possible Dirichlet control $\mathbf{u}_2$, we split the velocity field as $\mathbf{v} = \mathbf{v}_0 + \rho_2 \mathbf{R}(\mathbf{u}_2)$, where $\mathbf{v}_0 \in \mathbf{H}^1_D(\Omega_o)$, $\mathbf{R} \colon \mathbf{H}^{1/2}(\Gamma^o_D) \to \mathbf{H}^1(\Omega_o)$ is a bounded extension operator such that $\mathbf{R}(\mathbf{u}_2) \in \mathbf{H}^1(\Omega_o)$ and $\mathbf{R}(\mathbf{u}_2)|_{\Gamma^o_D} = \mathbf{u}_2$. For the sake of simplicity, we still denote in the following $\mathbf{v}_0$ with $\mathbf{v}$, as no ambiguity occurs, while we use the notation below for test and trial functions, respectively:

trial:	$x = (y, u) \in X_o$,	$y = (\mathbf{v}, \pi) \in Y_o$,	$p = (\boldsymbol{\lambda}, \eta) \in Q_o$,
test:	$\delta x = (\delta y, \delta u) \in X_o$,	$\delta y = (\delta \mathbf{v}, \delta \pi) \in Y_o$,	$\delta p = (\delta \boldsymbol{\lambda}, \delta \eta) \in Q_o$.

[1]For example, if we consider a distributed forcing term $\mathbf{u}_1$ as control variable, the natural choice is $U_o = \mathbf{L}^2(\Omega_o)$, whereas if we consider as control variable $\mathbf{u}_3$, $U_o = \mathbf{L}^2(\Gamma^o_N)$. In Sect. 5, a deeper discussion will be devoted to the case of a Dirichlet control $\mathbf{u}_2$, where the natural choice $U_o = \mathbf{H}^{1/2}(\Gamma^o_D)$ reveals to be troublesome from a numerical standpoint.

Thus, the optimization variable $x = (y, u)$ denotes the aggregated state and control variables, the former being given by velocity $\mathbf{v}$ and pressure π. We define the bilinear form $S_o(\cdot,\cdot)\colon Y_o \times Q_o \to \mathbb{R}$ associated to the Stokes operator [25, 26]:

$$S_o(y, \delta p) = a_o(\mathbf{v}, \delta\boldsymbol{\lambda}) + b_o(\delta\boldsymbol{\lambda}, \pi) + b_o(\mathbf{v}, \delta\eta), \tag{5}$$

where $a_o(\cdot,\cdot)\colon V_o \times V_o \to \mathbb{R}$ and $b_o(\cdot,\cdot)\colon V_o \times M_o \to \mathbb{R}$ are defined by

$$a_o(\mathbf{v}, \delta\boldsymbol{\lambda}) = \int_{\Omega_o(\boldsymbol{\mu})} \nu \nabla \mathbf{v} \cdot \nabla \delta\boldsymbol{\lambda} \, d\Omega_o, \qquad b_o(\mathbf{v}, \delta\eta) = -\int_{\Omega_o(\boldsymbol{\mu})} \delta\eta \nabla \cdot \mathbf{v} \, d\Omega_o.$$

The bilinear form $C_o(\cdot,\cdot)\colon U_o \times Q_o \to \mathbb{R}$ associated to the control variable is given by (after integration by parts and introducing the lifting operator, see e.g. [13])

$$C_o(u, \delta p) \;=\; \rho_1 \int_{\Omega_o(\boldsymbol{\mu})} \mathbf{u}_1 \cdot \delta\boldsymbol{\lambda} \, d\Omega_o \;+\; \rho_3 \int_{\Gamma_N^o(\boldsymbol{\mu})} \mathbf{u}_3 \cdot \delta\boldsymbol{\lambda} \, d\Gamma^o \;-\; \rho_2 a_o(\mathbf{R}(\mathbf{u}_2), \delta\boldsymbol{\lambda}) \;-\; \rho_2 b_o(\mathbf{R}(\mathbf{u}_2), \delta\eta). \tag{6}$$

Then, the state operator $\mathcal{E}_o\colon X_o \to Q_o'$ and the associated bilinear form $\mathcal{B}_o(\cdot,\cdot)\colon X_o \times Q_o \to \mathbb{R}$ are given by

$${}_{Q_o'}\langle \mathcal{E}_o(x), \delta p\rangle_{Q_o} = \mathcal{B}_o(x, \delta p) := S_o(y, \delta p) - C_o(u, \delta p), \qquad \forall\, \delta p \in Q_o. \tag{7}$$

Finally, let us express the quadratic functionals $\mathcal{F}$ and $\mathcal{G}$ appearing in (2) as

$$\mathcal{F}_o(\mathbf{v}; \boldsymbol{\mu}) = \frac{1}{2} m_o(\mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu}), \mathbf{v} - \mathbf{v}_d(\boldsymbol{\mu})) \qquad \mathcal{G}_o(\mathbf{u}; \boldsymbol{\mu}) = \frac{\sigma}{2} n_o(\mathbf{u}, \mathbf{u}),$$

where $\mathcal{Z}_o \supset V_o$ is a Hilbert (observation) space, $m_o(\cdot,\cdot)\colon\ \mathcal{Z}_o \times \mathcal{Z}_o \to \mathbb{R}$ is a symmetric, continuous, non-negative bilinear form, and $\mathbf{v}_d(\boldsymbol{\mu}) \in \mathcal{Z}_o$ is a given observation function. Furthermore, $\sigma > 0$ is a given constant (which can be viewed as the cost needed to implement the control), while $n_o(\cdot,\cdot)\colon U_o \times U_o \to \mathbb{R}$ is a symmetric, bounded and coercive bilinear form[2].

We can equivalently formulate the $\boldsymbol{\mu}$-OCP problem (4) as follows: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{x \in X_o} \ \mathcal{J}_o(x; \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{B}_o(x, \delta p) = 0 \quad \forall\, \delta p \in Q_o, \tag{8}$$

where the cost functional can be expressed in the (more usual) quadratic form

$$\mathcal{J}_o(x; \boldsymbol{\mu}) = \frac{1}{2} \mathcal{A}_o(x, x; \boldsymbol{\mu}) - \langle F_o(\boldsymbol{\mu}), x\rangle, \tag{9}$$

where $F_o(\boldsymbol{\mu}) \in X_o'$ is such that $\langle F_o(\boldsymbol{\mu}), x\rangle = m_o(\mathbf{v}_d(\boldsymbol{\mu}), \mathbf{v})$ and the bilinear form $\mathcal{A}_o(\cdot,\cdot) : X_o \times X_o \to \mathbb{R}$ defined as follows:

$$\mathcal{A}_o(x, \delta x; \boldsymbol{\mu}) = m_o(\mathbf{v}, \delta\mathbf{v}) + \sigma n_o(\mathbf{u}, \delta\mathbf{u}).$$

2. Parametrized formulation of the optimal control problem

Whenever $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ characterizes the geometrical configuration, we assume that the original domain $\Omega_o = \Omega_o(\boldsymbol{\mu})$ can be obtained as the image of a reference $\boldsymbol{\mu}$-independent domain $\Omega = \Omega_o(\bar{\boldsymbol{\mu}})$ (for some reference $\bar{\boldsymbol{\mu}}$) through a parametrized diffeomorphism $T(\cdot; \boldsymbol{\mu})\colon \mathbb{R}^d \times \mathcal{D} \to \mathbb{R}^d$ such that $\Omega_o(\boldsymbol{\mu}) = T(\Omega; \boldsymbol{\mu})$. The

[2] Note that in case of Dirichlet control the functional $\mathcal{F}_o(\mathbf{v}; \boldsymbol{\mu})$ is slightly modified. Indeed, by recalling that the velocity field is $\mathbf{v} = \mathbf{v}_0 + \mathbf{R}(\mathbf{u}_2)$, we can rewrite $\mathcal{F}_o$ as

$$\mathcal{F}_o(\mathbf{v}, \mathbf{u}; \boldsymbol{\mu}) = \frac{1}{2} m_o(\mathbf{v}_0 - \mathbf{v}_d(\boldsymbol{\mu}), \mathbf{v}_0 - \mathbf{v}_d(\boldsymbol{\mu})) + \frac{1}{2} m_o(\mathbf{R}(\mathbf{u}_2), \mathbf{R}(\mathbf{u}_2)) + m_o(\mathbf{v}_0 - \mathbf{v}_d(\boldsymbol{\mu}), \mathbf{R}(\mathbf{u}_2)).$$

parametrized formulation of the optimal control problem can be easily derived by tracing problem (8) back on the reference domain Ω (as shown e.g. in [12, 18]): given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\min_{x(\boldsymbol{\mu})\in X} \mathcal{J}(x(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{B}(x(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}) = 0 \quad \forall \delta p \in Q, \tag{10}$$

where the functional spaces $X = X(\Omega)$ and $Q = Q(\Omega)$ are defined on the reference domain, the bilinear form representing the state operator is given by

$$\mathcal{B}(x(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}) = S(y(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}) - C(u(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}),$$

where the bilinear forms $S(\cdot, \cdot; \boldsymbol{\mu})$ and $C(\cdot, \cdot; \boldsymbol{\mu})$ are linked to the "original" ones through the Jacobian of the mapping $T(\cdot; \boldsymbol{\mu})$. We can operate similarly for $m(\cdot, \cdot; \boldsymbol{\mu})$, $n(\cdot, \cdot; \boldsymbol{\mu})$, $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ and $\mathcal{F}(\cdot; \boldsymbol{\mu})$, so that the cost functional is now given by

$$\mathcal{J}(x(\boldsymbol{\mu}); \boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(x(\boldsymbol{\mu}), x(\boldsymbol{\mu}); \boldsymbol{\mu}) - \langle F(\boldsymbol{\mu}), x(\boldsymbol{\mu}) \rangle.$$

In the following section we recall some results dealing with the well-posedness of problem (10), exploiting the Lagrange multipliers framework [24, 27].

2.1. Existence, uniqueness and optimality conditions

By introducing the Lagrangian functional $\mathcal{L}(\cdot, \cdot; \boldsymbol{\mu}) \colon X \times Q \to \mathbb{R}$ as

$$\mathcal{L}(x, \delta p; \boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(x, x; \boldsymbol{\mu}) - \langle F(\boldsymbol{\mu}), x \rangle + \mathcal{B}(x, \delta p; \boldsymbol{\mu}). \tag{11}$$

the constrained problem (10) turns into the unconstrained problem of finding a saddle point $(x(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$ of (11). Existence and uniqueness of a solution follow from Brezzi theorem [28, 27], under the following conditions:

(H1) the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ is continuous over $X \times X$, symmetric, non-negative over X and coercive on $X^0 = \{\delta x \in X \colon \mathcal{B}(\delta x, \delta p; \boldsymbol{\mu}) = 0 \ \forall \delta p \in Q\}$, i.e.

$$\exists\, \alpha_0 > 0 : \quad \alpha(\boldsymbol{\mu}) = \inf_{x \in X^0} \frac{\mathcal{A}(x, x; \boldsymbol{\mu})}{\|x\|_X^2} \geq \alpha_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}; \tag{12}$$

(H2) the bilinear form $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ is continuous over $X \times Q$ and satisfies the following inf-sup condition:

$$\exists\, \beta > 0 : \quad \beta(\boldsymbol{\mu}) = \inf_{\delta p \in Q} \sup_{\delta x \in X} \frac{\mathcal{B}(\delta x, \delta p; \boldsymbol{\mu})}{\|\delta x\|_X \|\delta p\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{13}$$

If (H1) and (H2) hold, problem (10) has a unique solution for any $\boldsymbol{\mu} \in \mathcal{D}$, which can be determined by solving the following saddle-point problem (i.e. the optimality system): given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in X \times Q$ such that

$$\begin{cases} \mathcal{A}(x(\boldsymbol{\mu}), \delta x; \boldsymbol{\mu}) + \mathcal{B}(\delta x, p(\boldsymbol{\mu}); \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \delta x \rangle & \forall \delta x \in X, \\ \mathcal{B}(x(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}) = 0 & \forall \delta p \in Q, \end{cases} \tag{14}$$

where $p(\boldsymbol{\mu})$ is the Lagrange multiplier (i.e. the adjoint variable) associated to the constraint expressed by the state equation. In fact, for the problem at hand it is possible to show (see Appendix Appendix A.1) the following

Proposition 1. *The bilinear forms $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ and $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ satisfy the assumptions (H1)-(H2).*

Remark 1. The optimality system (14) can be equivalently formulated as a variational problem in the product space $\mathcal{X} = X \times Q$: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in \mathcal{X}$ such that

$$B((x,p),(\delta x,\delta p);\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \delta x\rangle, \qquad \forall(\delta x,\delta p) \in \mathcal{X}. \tag{15}$$

with

$$B((x,p),(\delta x,\delta p);\boldsymbol{\mu}) = \mathcal{A}(x,x;\boldsymbol{\mu}) + \mathcal{B}(\delta x,p;\boldsymbol{\mu}) + \mathcal{B}(x,\delta p;\boldsymbol{\mu}).$$

Since assumptions (H1)-(H2) for problem (14) are equivalent [29, 30] to the hypotheses required by Neças-Babuška's theorem applied to problem (15), Proposition 1 implies that there exists $\hat{\beta}_0 > 0$ such that

$$\hat{\beta}(\boldsymbol{\mu}) := \inf_{(x,p)\in\mathcal{X}} \sup_{(\delta x,\delta p)\in\mathcal{X}} \frac{B((x,p),(\delta x,\delta p);\boldsymbol{\mu})}{\|(x,p)\|_{\mathcal{X}}\|(\delta x,\delta p)\|_{\mathcal{X}}} \geq \hat{\beta}_0 > 0 \quad \forall\boldsymbol{\mu}\in\mathcal{D}. \tag{16}$$

In Sect. 4 this property will play a crucial role in order to develop an a posteriori error bound for the RB approximation of the optimal control problem.

Remark 2. By using the definition of the bilinear forms $\mathcal{A}$ and $\mathcal{B}$, we can also rewrite the optimality system (14) as: find $(y,u,p) \in Y \times U \times Q$ such that

$$\begin{array}{cc|ccc l} m(\mathbf{v},\delta\mathbf{v};\boldsymbol{\mu}) & 0 & S(\delta y,p;\boldsymbol{\mu}) & = & m(\mathbf{v}_d(\boldsymbol{\mu}),\delta\mathbf{v};\boldsymbol{\mu}) & \forall\delta y\in Y \\ 0 & \sigma n(u,\delta u;\boldsymbol{\mu}) & -C(\delta u,p;\boldsymbol{\mu}) & = & 0 & \forall\delta u\in U \\ \hline S(y,\delta p;\boldsymbol{\mu}) & -C(u,\delta p;\boldsymbol{\mu}) & 0 & = & 0 & \forall\delta p\in Q. \end{array} \tag{17}$$

The formulation (17) highlights the structure of the optimality system, featuring two nested saddle-point problems: an *outer* saddle-point problem (given by the optimization problem) and an *inner* one (given by the Stokes constraint). The stability properties of the whole system reflect this particular structure. Indeed, there are two distinct sets of inf-sup conditions associated to the optimality system, that is

(i) a *Brezzi inner condition* on $b(\cdot,\cdot;\boldsymbol{\mu})$ and the *Babuška inner condition* (13) involving only the state variables, and related to the unique solvability of the state equation;

(ii) a *Brezzi outer condition* (13) involving instead both state and control variables and the *Babuška outer condition* (16) which involves all the variables simultaneously (state, control and Lagrange multiplier).

This *cascade* of nested stability conditions must be taken into account when designing suitable RB spaces, as shown in Sect. 3.

2.2. Finite element approximation

Our RB method is premised upon a high-fidelity approximation technique, such as the Galerkin-Finite Elements (FE) method. To obtain the FE approximation of the saddle-point problem (14), we first introduce a stable pair of FE spaces $V_h \subset \mathbf{H}^1(\Omega)$ and $M_h \subset L^2(\Omega)$ [25, 26] such that

$$\exists\,\beta_0^b > 0: \quad \beta_h^b(\boldsymbol{\mu}) = \inf_{\pi_h\in M_h} \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h,\pi_h;\boldsymbol{\mu})}{\|\mathbf{v}_h\|_V\|\pi_h\|_M} \geq \beta_0^b, \qquad \forall\boldsymbol{\mu}\in\mathcal{D}. \tag{18}$$

Furthermore, we assume U_h to be a suitable FE subspace of U and we set

$$Y_h = V_h \times M_h, \qquad X_h = Y_h \times U_h, \qquad Q_h = Y_h.$$

We denote by $\mathcal{N} = \mathcal{N}(h)$ the global dimension of the product space $X_h \times Q_h$, i.e. $\mathcal{N} = \mathcal{N}_X + \mathcal{N}_Q$, where $\mathcal{N}_X = \mathcal{N}_Y + \mathcal{N}_U$ and $\mathcal{N}_Y = \mathcal{N}_Q$; here $\mathcal{N}_Y, \mathcal{N}_U, \mathcal{N}_Q$ denote the dimensions of Y_h, U_h and Q_h, respectively.

The *full-order* Galerkin-FE approximation of the optimality system (14) thus reads as follows: given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu})) \in X_h \times Q_h$ such that

$$\begin{cases} \mathcal{A}(x_h, \delta x; \boldsymbol{\mu}) + \mathcal{B}(\delta x, p_h; \boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), \delta x \rangle & \forall\, \delta x \in X_h, \\ \mathcal{B}(x_h, \delta p; \boldsymbol{\mu}) = 0 & \forall\, \delta p \in Q_h. \end{cases} \tag{19}$$

Note that, by discretizing directly the optimality system (14), we are tacitly following an *optimize-then-discretize* approach rather than a *discretize-then-optimize* one, see e.g. [14, 13, 20]. Actually, in this case the two approaches coincide since a pure Galerkin discretization is used; indeed, (19) is also the optimality system for the following finite dimensional problem

$$\min_{x_h(\boldsymbol{\mu}) \in X_h} \mathcal{J}(x_h(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{B}(x_h(\boldsymbol{\mu}), \delta p; \boldsymbol{\mu}) = 0 \quad \forall \delta p \in Q_h. \tag{20}$$

Provided that $Y_h = Q_h$, by using the same arguments[3] as in Proposition 1 we can show that the bilinear form $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ is still continuous over $X_h \times X_h$ and coercive over $X_h^0 = \{\delta x \in X_h \colon \mathcal{B}(\delta x, \delta p; \boldsymbol{\mu}) = 0\ \forall\, \delta p \in Q_h\}$, with continuity constant $\gamma_h^a(\boldsymbol{\mu})$ and coercivity constant $\alpha_h^a(\boldsymbol{\mu})$, respectively. In the same way, $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ remains continuous and inf-sup stable over $X_h \times Q_h$, i.e.

$$\exists\, \beta_0 > 0: \quad \beta_h(\boldsymbol{\mu}) = \inf_{\delta p \in Q_h} \sup_{\delta x \in X_h} \frac{\mathcal{B}(\delta x, \delta p; \boldsymbol{\mu})}{\|\delta x\|_X \|\delta p\|_Q} \geq \beta_0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{22}$$

In particular, we obtain the estimate

$$\beta_h(\boldsymbol{\mu}) \geq \beta_h^S(\boldsymbol{\mu}),$$

where $\beta_h^S(\boldsymbol{\mu})$ is the Babuška inf-sup constant of the bilinear form $S(\cdot, \cdot; \boldsymbol{\mu})$. Therefore, thanks to Brezzi theory, also the FE approximation (19) is well-posed.

At the algebraic level, (19) leads to the following linear system:

$$\underbrace{\begin{pmatrix} A_h(\boldsymbol{\mu}) & B_h^T(\boldsymbol{\mu}) \\ B_h(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_h(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_h(\boldsymbol{\mu}) \\ \mathbf{p}_h(\boldsymbol{\mu}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{F}_h(\boldsymbol{\mu}) \\ 0 \end{pmatrix}}_{\mathbf{g}_h(\boldsymbol{\mu})}, \tag{23}$$

where $\mathbf{x}_h(\boldsymbol{\mu})$ and $\mathbf{p}_h(\boldsymbol{\mu})$ denote the vectors of the coefficients in the expansion of $x_h(\boldsymbol{\mu})$ and $p_h(\boldsymbol{\mu})$ with respect to the FE basis.
In order to develop an efficient offline-online stratagem, we assume $K_h(\boldsymbol{\mu})$ and $\mathbf{g}_h(\boldsymbol{\mu})$ to be affine in the parameter $\boldsymbol{\mu}$, i.e. they can be expressed as

$$K_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_k} \Theta_k^q(\boldsymbol{\mu})\, K_h^q, \qquad \mathbf{g}_h(\boldsymbol{\mu}) = \sum_{q=1}^{Q_g} \Theta_g^q(\boldsymbol{\mu})\, \mathbf{g}_h^q, \tag{24}$$

for some given finite Q_k, Q_g, smooth $\boldsymbol{\mu}$-dependent function $\Theta_*^q(\boldsymbol{\mu})$ and $\boldsymbol{\mu}$-independent matrices K_h^q and vectors $\mathbf{g}_h^q$.

[3] The stability assumption (18) on the velocity and pressure spaces implies the fulfillment of the discrete counterpart of the Babuška inf-sup condition (A.2) on $S(\cdot, \cdot; \boldsymbol{\mu})$, i.e.

$$\exists\, \beta_0^S > 0: \quad \beta_h^S(\boldsymbol{\mu}) := \inf_{y_1 \in Y_h} \sup_{y_2 \in Y_h} \frac{S(y_1, y_2; \boldsymbol{\mu})}{\|y_1\|_Y \|y_2\|_Y} = \inf_{y_2 \in Y_h} \sup_{y_1 \in Y_h} \frac{S(y_1, y_2; \boldsymbol{\mu})}{\|y_1\|_Y \|y_2\|_Y} \geq \beta_0^S. \tag{21}$$

3. The reduced basis approximation

The idea behind the RB method is to efficiently approximate the solution $x_h(\boldsymbol{\mu})$ of the full-order optimization problem (20) by using spaces made up of well-chosen solutions of the problem itself. The main assumption is that the parameters to solution map $\boldsymbol{\mu} \mapsto x_h(\boldsymbol{\mu})$ is smooth[4] and low dimensional, and therefore approximable by a linear combination of few full-order solutions (obtained for different values of the parameters). For the problem at hand, since the optimal solutions $x_h(\boldsymbol{\mu})$ of (20) are characterized as the solutions of the optimality system (19), two possible approaches to build a ROM can be pursued – analogous to the discretize-then-optimize and optimize-then-discretize approaches introduced in Sect. 2.2. A first approach, that we call *reduce-then-optimize*, consists in building a ROM for the state equation and then solving the resulting low dimensional problem:

$$\min_{x_N(\boldsymbol{\mu})\in X_N} \mathcal{J}_N(x_N(\boldsymbol{\mu});\boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{B}(x_N(\boldsymbol{\mu}),\delta p;\boldsymbol{\mu}) = 0 \quad \forall \delta p \in Q_N, \tag{25}$$

where X_N and Q_N are suitably defined RB spaces of dimension $N \ll \mathcal{N}$. The optimality system associated to (25) reads as follows: find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}),\delta x;\boldsymbol{\mu}) + \mathcal{B}(\delta x, p_N(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}),\delta x\rangle & \forall \delta x \in X_N,\\ \mathcal{B}(x_N(\boldsymbol{\mu}),\delta p;\boldsymbol{\mu}) = 0 & \forall \delta p \in Q_N. \end{cases} \tag{26}$$

In a second approach, that we call *optimize-then-reduce*, the two stages are inverted, and (possibly different) ROMs are built for the state, adjoint and optimality equations. In this case, the reduced optimality system reads as follows: find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \times Q_N$ such that

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}),\delta x;\boldsymbol{\mu}) + \mathcal{B}(\delta x, p_N(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}),\delta x\rangle & \forall \delta x \in X_N^t,\\ \mathcal{B}(x_N(\boldsymbol{\mu}),\delta p;\boldsymbol{\mu}) = 0 & \forall \delta p \in Q_N^t, \end{cases} \tag{27}$$

where we denote by X_N^t and Q_N^t two suitably defined test spaces.

3.1. *Should we* reduce-then-optimize *or* optimize-then-reduce*?*

The goal of this section is to compare these two approaches, highlight their pros and cons, and eventually motivate the use of aggregated approximation spaces. To this end, let us consider the following full-order optimization problem:[5]

$$\min \frac{1}{2}\mathbf{y}_h^T \mathrm{F}_h \mathbf{y}_h + \frac{\sigma}{2}\mathbf{u}_h^T \mathrm{G}_h \mathbf{u}_h \quad \text{s.t.} \quad \mathrm{S}_h\mathbf{y}_h = \mathrm{C}_h\mathbf{u}_h + \mathbf{f}_h, \tag{28}$$

whose optimality system is given by

$$\begin{pmatrix} \mathrm{F}_h & 0 & \mathrm{S}_h^T \\ 0 & \sigma\mathrm{G}_h & -\mathrm{C}_h^T \\ \mathrm{S}_h & -\mathrm{C}_h & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_h \\ \mathbf{u}_h \\ \mathbf{p}_h \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_h \end{pmatrix}. \tag{29}$$

Let us first consider the *reduce-then-optimize* paradigm. In this case, we seek an approximate solution of the form

$$\mathbf{y}_h \approx \mathrm{V}_y \mathbf{y}_N, \qquad \mathbf{u}_h \approx \mathrm{U}\mathbf{u}_N,$$

[4]For instance, assuming that the $\Theta_*^q(\boldsymbol{\mu})$ functions are Lipschitz continuous w.r.t. the parameters, it can be easily proved that the solution map is Lipschitz continuous as well.

[5]We state the problem directly in its algebraic formulation, where the matrices F_h, G_h, S_h and C_h are those obtained from the FE discretization of the bilinear forms introduced in Sect. 1.1. We omit the explicit dependence from the parameters $\boldsymbol{\mu}$ for the sake of clarity.

where $V_y, U \in \mathbb{R}^{\mathcal{N}\times N}$ are the trial bases, which contain as columns N basis vectors[6], and $\mathbf{y}_N, \mathbf{u}_N \in \mathbb{R}^N$ denote two vectors made by the solution components w.r.t. the reduced bases V_y, U, respectively. We also introduce a test basis $W_y \in \mathbb{R}^{\mathcal{N}\times N}$, thus yielding the reduced model of (28) as

$$\min \frac{1}{2}\mathbf{y}_N^T F_N \mathbf{y}_N + \frac{\sigma}{2}\mathbf{u}_N^T G_N \mathbf{u}_N \quad \text{s.t.} \quad S_N \mathbf{y}_N = C_N \mathbf{u}_N + \mathbf{f}_N, \tag{30}$$

where $\mathbf{f}_N = W_y^T \mathbf{f}_h$. Here the reduced matrices are given by

$$S_N = W_y^T S_h V_y, \qquad C_N = W_y^T C_h U, \qquad F_N = V_y^T F_h V_y, \qquad G_N = U^T G_h U,$$

while the reduced optimality system associated to (30) reads[7],

$$\begin{pmatrix} F_N & 0 & S_N^T \\ 0 & \sigma G_N & -C_N^T \\ S_N & -C_N & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_N \\ \mathbf{u}_N \\ \mathbf{p}_N \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_N \end{pmatrix}. \tag{31}$$

In this scenario, the approximation of the adjoint variable is determined by the choice of the test space used in the projection of the state equation, i.e. we seek for an approximate adjoint variable $\mathbf{p}_h \approx W_y \mathbf{p}_N$. Furthermore, we remark that we are testing the adjoint equation onto the reduced state space, indeed the adjoint equation reads:

$$V_y^T S_h^T W_y \mathbf{p}_N = -V_y^T F_h V_y \mathbf{y}_N.$$

If the state reduction is performed via a Galerkin projection, then the test space is the same of the trial space, i.e. $W_y = V_y$. Therefore, we are approximating the adjoint variable as a linear combination of state basis vectors. This kind of reduction turns out to be somehow *inconsistent*: in fact, the reduced scheme is not guaranteed to recover any full-order adjoint solution, since the adjoint snapshots are not contained in V_y.

Let us now consider the *optimize-then-reduce* approach, where we seek directly an approximate solution of the full-order optimality system (29). Therefore, we approximate the state, control and adjoint variables as

$$\mathbf{y}_h \approx V_y \mathbf{y}_N, \qquad \mathbf{u}_h \approx U \mathbf{u}_N, \qquad \mathbf{p}_h \approx V_p \mathbf{p}_N,$$

where $V_p \in \mathbb{R}^{\mathcal{N}\times N}$ is a trial basis for the adjoint variable. Then, we define a test basis W_p for the adjoint equation and we enforce the orthogonality of the residual of (29) to the product basis $W_p \times U \times W_y$ to obtain the reduced optimality system:

$$\begin{pmatrix} W_p^T F_h V_y & 0 & W_p^T S_h^T V_p \\ 0 & \sigma U^T G_h U & -U^T C_h V_p \\ W_y^T S_h V_y & -W_y^T C_h U & 0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_N \\ \mathbf{u}_N \\ \mathbf{p}_N \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_N \end{pmatrix}. \tag{32}$$

While this approximation is automatically consistent, depending on the choice of W_y and W_p, the reduction of the optimality system may lead to a nonsymmetric linear system; however, in this case, there is no optimization problem for which (32) is the corresponding optimality system.

A possible way to simultaneously exploit the advantages of both approaches and overcome their drawbacks (namely inconsistent approximations and nonsymmetric optimality systems), is to define suitable

[6] In the following we will define the columns of V_y (respectively U) as suitably orthogonalized state (resp. control) solutions of (20) computed in some selected points $\boldsymbol{\mu} \in \mathcal{D}$.

[7] Obtained by differentiating the following reduced Lagrangian:

$$\mathcal{L}_N(\mathbf{y}_N, \mathbf{u}_N, \mathbf{p}_N) = \frac{1}{2}\mathbf{y}_N^T F_N \mathbf{y}_N + \frac{\sigma}{2}\mathbf{u}_N^T G_N \mathbf{u}_N + \mathbf{p}_N^T (S_N \mathbf{y}_N - C_N \mathbf{u}_N - \mathbf{f}_N),$$

aggregated trial spaces [5, 1, 16, 9] for the state and adjoint variables, and to employ a Galerkin projection to find the reduced approximation. Indeed, by defining a common basis

$$\mathrm{V} \equiv \mathrm{W} = [\mathrm{V}_y \, \mathrm{V}_p] \tag{33}$$

made of both state and adjoint full-order snapshots, we have that:

1. following the *reduce-then-optimize* approach, we obtain a *consistent* approximation of the optimality system, meaning that full-order state, control and adjoint solutions are recovered by the reduced scheme if the corresponding snapshots are contained in the reduced spaces;
2. following the *optimize-then-reduce* approach, we obtain a symmetric optimality system which preserves the Lagrangian structure (a crucial property in order to obtain an a posteriori error estimate on the cost functional, see Prop. 4).

Moreover, thanks to the choice (33) the two approaches lead to the same ROM.

3.2. Construction of aggregated RB spaces

Following the discussion of the previous Section, we now define a stable pair of aggregated RB spaces X_N, Q_N for state, control and adjoint variables. To take into account the double (nested) saddle-point structure, we have first to ensure the stability (by fulfilling a suitable inf-sup condition) of the RB approximation of the Stokes state operator (see the proof of Proposition 1). Then, we have to verify the stability of the whole optimality system, i.e. we have to guarantee the coercivity of the bilinear form $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ over $X_N^0 = \{\delta x \in X_N \colon \mathcal{B}(\delta x, \delta p; \boldsymbol{\mu}) = 0, \, \forall \delta p \in Q_N\}$, and the fulfillment of an equivalent RB inf-sup condition on $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$, i.e.

$$\exists\, \beta_0 > 0 : \quad \beta_N(\boldsymbol{\mu}) = \inf_{\delta p \in Q_N} \sup_{\delta x \in X_N} \frac{\mathcal{B}(\delta x, \delta p; \boldsymbol{\mu})}{\|\delta x\|_X \|\delta p\|_Q} \geq \beta_0, \qquad \forall \boldsymbol{\mu} \in \mathcal{D}. \tag{34}$$

This also implies the fulfillment of an equivalent RB Babuška inf-sup condition on the whole optimality system. We employ the following strategy: we achieve stability of the Stokes operator by enriching the velocity space with suitably defined *supremizer solutions* [17, 18]. Then to guarantee the stability of the optimality system, we define suitable aggregated spaces for state and adjoint variables [1, 5, 16]. The two recipes are combined together as described below.
Let us denote, for given $N \in [1, N_{\max}]$, a finite set of parameter values $S_N = \{\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N\}$ and consider the corresponding full-order solutions $(x_h(\boldsymbol{\mu}^n), \, p_h(\boldsymbol{\mu}^n))$, $1 \leq n \leq N$. Let us introduce the *supremizer functions* $\mathbf{T}_\pi^{\boldsymbol{\mu}} \in V_h$ defined as [31, 17]

$$(\mathbf{T}_\pi^{\boldsymbol{\mu}}, \delta \mathbf{v})_V = b(\delta \mathbf{v}, \pi; \boldsymbol{\mu}), \qquad \forall \delta \mathbf{v} \in V_h, \tag{35}$$

i.e. as the Riesz representers of the linear functional $b(\cdot, \pi; \boldsymbol{\mu})$. In this way, we can define an aggregated (state and adjoint) pressure space M_N:

$$M_N = \mathrm{span}\{\pi_h(\boldsymbol{\mu}^n), \, \eta_h(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}, \tag{36}$$

and an aggregated (state and adjoint) velocity space V_N, including also the corresponding supremizer snapshots:

$$V_N^{\boldsymbol{\mu}} = \mathrm{span}\big\{\mathbf{v}_h(\boldsymbol{\mu}^n), \mathbf{T}^{\boldsymbol{\mu}}_{\pi_h(\boldsymbol{\mu}^n)}, \boldsymbol{\lambda}_h(\boldsymbol{\mu}^n), \mathbf{T}^{\boldsymbol{\mu}}_{\eta_h(\boldsymbol{\mu}^n)}, \quad n = 1, \dots, N\big\}. \tag{37}$$

Then, we define the RB space for the control variable

$$U_N = \mathrm{span}\{u_h(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\}. \tag{38}$$

Finally, let us define

$$Y_N = V_N^{\boldsymbol{\mu}} \times M_N, \qquad X_N = Y_N \times U_N, \qquad Q_N = Y_N, \tag{39}$$

as the RB spaces for the state, state and control, and adjoint variables, respectively. We end up with the following RB approximation of problem (20):

$$\min_{x_N(\boldsymbol{\mu})\in X_N} \mathcal{J}_N(x_N(\boldsymbol{\mu});\boldsymbol{\mu}) \quad \text{s.t.} \quad \mathcal{B}(x_N(\boldsymbol{\mu}),\delta p;\boldsymbol{\mu})=0 \quad \forall \delta p\in Q_N, \tag{40}$$

and the following optimality system: find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))\in X_N\times Q_N$ such that

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}),\delta x;\boldsymbol{\mu})+\mathcal{B}(\delta x,p_N(\boldsymbol{\mu});\boldsymbol{\mu})=\langle F(\boldsymbol{\mu}),\delta x\rangle & \forall \delta x\in X_N,\\ \mathcal{B}(x_N(\boldsymbol{\mu}),\delta p;\boldsymbol{\mu})=0 & \forall \delta p\in Q_N. \end{cases} \tag{41}$$

For the selection of the sample points $S_N=\{\boldsymbol{\mu}^1,\dots,\boldsymbol{\mu}^N\}$ we rely on the standard *greedy algorithm* [32, 33]. At each iteration N, $\boldsymbol{\mu}^{N+1}$ is chosen as the maximizer of $\Delta_N(\boldsymbol{\mu})$ over a training sample $\Xi_{\text{train}}\subset\mathcal{D}$, where $\Delta_N(\boldsymbol{\mu})$ is an efficiently computable a posteriori estimate for the error on the state, control and adjoint variables, whose definition is postponed to Sect. 4.

3.3. Stability properties

In order to analyze the stability of the proposed scheme (41), let us introduce the following RB inf-sup constant on the Stokes operator

$$\beta_N^S(\boldsymbol{\mu})=\inf_{\delta y\in Y_N}\sup_{\delta p\in Q_N}\frac{S(\delta y,\delta p;\boldsymbol{\mu})}{\|\delta y\|_Y\|\delta p\|_Q}=\inf_{\delta p\in Q_N}\sup_{\delta y\in Y_N}\frac{S(\delta y,\delta p;\boldsymbol{\mu})}{\|\delta y\|_Y\|\delta p\|_Q}, \tag{42}$$

and the following RB inf-sup constant on the optimality system

$$\hat\beta_N(\boldsymbol{\mu}):=\inf_{(x,p)\in\mathcal{X}_N}\sup_{(\delta x,\delta p)\in\mathcal{X}_N}\frac{B((x,p),(\delta x,\delta p);\boldsymbol{\mu})}{\|(x,p)\|_{\mathcal{X}}\|(\delta x,\delta p)\|_{\mathcal{X}}}. \tag{43}$$

The well-posedness of the RB approximation problem (41) is ensured by the following proposition (see Appendix Appendix A.2).

Proposition 2. *If X_N and Q_N are chosen accordingly to (36)-(39), then $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ satisfy the assumptions of Brezzi theorem over X_N, Q_N. Moreover, the following inequalities for the RB stability factors hold, for any $\boldsymbol{\mu}\in\mathcal{D}$:*

$$\beta_N(\boldsymbol{\mu})\ge\beta_N^S(\boldsymbol{\mu})\ge\frac{\alpha_h^a(\boldsymbol{\mu})}{1+\left(\gamma_h^a(\boldsymbol{\mu})/\beta_h^b(\boldsymbol{\mu})\right)^2}, \tag{44}$$

$$\alpha_N(\boldsymbol{\mu})\ge\frac{\sigma}{2}\alpha_h^n(\boldsymbol{\mu})\min\Big\{1,\Big(\frac{\beta_N^S(\boldsymbol{\mu})}{\gamma_h^C(\boldsymbol{\mu})}\Big)^2\Big\}, \tag{45}$$

$$\hat\beta_N(\boldsymbol{\mu})\ge\frac{\alpha_N(\boldsymbol{\mu})}{1+\left(\gamma_h^{\mathcal{A}}(\boldsymbol{\mu})/\beta_N(\boldsymbol{\mu})\right)^2}, \tag{46}$$

where $\gamma_h^{\mathcal{A}}$ is the continuity constant of $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ over $X_h\times X_h$, and $\alpha_N(\boldsymbol{\mu})$ is the coercivity constant of $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ over $X_N^0\times X_N^0$.

Remark 3. Because of the definition of the supremizer solutions $\mathbf{T}_\pi^{\boldsymbol{\mu}}$, the RB velocity space $V_N^{\boldsymbol{\mu}}$ (and therefore also the spaces Y_N and Q_N) still depends on $\boldsymbol{\mu}$, thus requiring to include $2\tilde{Q}_bN$ additional basis functions besides the $2N$ pure velocity snapshots. For this reason, following [18, 34], we rather define

$$V_N=\text{span}\big\{\mathbf{v}_h(\boldsymbol{\mu}^n),\mathbf{T}^{\boldsymbol{\mu}^n}_{\pi_h(\boldsymbol{\mu}^n)},\boldsymbol{\lambda}_h(\boldsymbol{\mu}^n),\mathbf{T}^{\boldsymbol{\mu}^n}_{\eta_h(\boldsymbol{\mu}^n)},\quad n=1,\dots,N\big\}, \tag{47}$$

so that we consider only $2N$ parameter independent supremizer snapshots. This enables a full decoupling of the Offline/Online stages (and thus substantial computational savings). We cannot rigorously demonstrate that the approximation stability of the Stokes operator is preserved, despite being numerically verified [18, 34]. Thus, we obtain a RB velocity space V_N of dimension $4N$ and a RB pressure space Q_N of dimension $2N$. Therefore, since $Q_N=Y_N$, the RB state and adjoint spaces Y_N and Q_N have dimension $6N$, while the control space U_N has dimension N.

Remark 4. If the state operator is a general noncoercive operator (different from the Stokes operator) we can still employ the strategy above to define stable RB spaces. To this goal, it is sufficient to combine a stable approximation for the state equation with the definition of aggregated state/adjoint spaces.

3.4. Algebraic formulation and offline-online computational procedure

The algebraic counterpart of (41) is given by the following reduced linear system:

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ 0 \end{pmatrix}}_{\mathbf{g}_N(\boldsymbol{\mu})}, \tag{48}$$

where $\mathbf{x}_N(\boldsymbol{\mu}) \in \mathbb{R}^{7N}$ and $\mathbf{p}_N(\boldsymbol{\mu}) \in \mathbb{R}^{6N}$ denote the vectors of the coefficients in the expansions of $x_N(\boldsymbol{\mu})$ and $p_N(\boldsymbol{\mu})$ with respect to the reduced basis. The matrix K_N is still symmetric, features a saddle-point structure, however it is dense (rather than sparse, as in the FE case) and has dimension $13N \times 13N$. To keep the condition number of K_N under control, we adopt the Gram-Schmidt orthonormalization [33] separately on velocity, pressure and control basis functions.

Thanks to the assumption of affine parameter dependence, we can decouple the assembling of the matrix $K_N(\boldsymbol{\mu})$ in two stages, by exploiting a suitable offline/online stratagem, that enables the efficient resolution of the system (48) for each new parameter value $\boldsymbol{\mu}$.

4. *A posteriori* error estimates

A posteriori error bounds are an essential ingredient to enable the efficient exploration of the parameter space in the greedy algorithm, and thus the construction of low dimensional RB spaces. For the problem at hand, different error bounds can be derived depending on its formulation: either the full-space or the reduced-space formulation[8] (see e.g. [20, 15]).

Let us briefly recall these two formulations by considering again the optimization problem (28) we discussed in Sect. 3.1. Solving (28) employing a full-space approach – as we have done so far – requires to directly solve its optimality system (29), treated as a monolithic, coupled linear system. Instead, employing a reduced-space approach, we end up with a linear system (of the same dimension as the control variable) involving the so-called reduced Hessian:

$$\mathrm{H}_h \mathbf{u}_h = \mathbf{b}_h, \tag{49}$$

where $\mathrm{H}_h = \sigma \mathrm{G}_h + \mathrm{C}_h^T \mathrm{S}_h^{-T} \mathrm{F}_h \mathrm{S}_h^{-1} \mathrm{C}_h$ is the Schur complement of the matrix in (29) w.r.t. the control variable, while $\mathbf{b}_h = -\mathrm{C}_h^T \mathrm{S}_h^{-T} \mathrm{F}_h \mathrm{S}_h^{-1} \mathbf{f}_h$. Let us remark that (49) represents the optimality system for the reduced optimization problem

$$\min_{\mathbf{u}_h} \hat{\mathcal{J}}(\mathbf{u}_h) := \mathcal{J}(\mathbf{y}_h(\mathbf{u}_h), \mathbf{u}_h), \tag{50}$$

where we denoted by $\mathbf{u}_h \mapsto \mathbf{y}_h(\mathbf{u}_h)$ the solution operator of the state equation. In fact, since $\mathbf{y}_h = \mathrm{S}_h^{-1} \mathrm{C}_h \mathbf{u}_h$, we can readily see that

$$\hat{\mathcal{J}}(\mathbf{u}_h) = \frac{1}{2} \mathbf{u}_h^T \mathrm{H}_h \mathbf{u}_h - \mathbf{u}_h^T \mathbf{b}_h.$$

[8] Here *reduced* must not be understood in the sense of *reduced order model*. In particular the *reduced* Hessian matrix we are going to introduce is not related to the RB approximation.

4.1. Reduced-space *a posteriori error estimates*

Since the reduced- and full-space are equivalent formulations of the same problem, we could build a ROM in the reduced-space setting, rather than in the full-space. To this end, we identify at least two possible alternatives, described below.

1. We first note that (50) is now an unconstrained optimization problem, involving just the control variable $\mathbf{u}_h$. This motivates us to seek an approximate control

$$\mathbf{u}_h \approx \mathrm{U}\mathbf{u}_N,$$

   and to minimize the reduced-reduced cost functional:

$$\min_{\mathbf{u}_N} \hat{\mathcal{J}}(\mathbf{u}_N), \tag{51}$$

   whose optimality system is given by

$$\mathrm{H}_N \mathbf{u}_N = \mathbf{b}_N, \tag{52}$$

   where $\mathrm{H}_N = \mathrm{U}^T \mathrm{H}_h \mathrm{U}$. Here we are employing a reduce-then-optimize approach on the optimization problem formulated in the reduced-space. The same ROM can be obtained by employing an optimize-then-reduce approach in the reduced-space, i.e. by left and right-multiplying (49) by U.

2. An alternative would be to build (formally) a ROM in the full-space (i.e. to construct a basis for state, control and adjoint variables) and then rewrite the problem in the reduced-space. In this case, the reduced-reduced Hessian would be given by:

$$\mathrm{H}_N = \sigma \mathrm{G}_N + \mathrm{C}_N^T \mathrm{S}_N^{-T} \mathrm{F}_N \mathrm{S}_N^{-1} \mathrm{C}_N,$$

   where the matrices $\mathrm{C}_N, \mathrm{F}_N, \mathrm{G}_N, \mathrm{S}_N$ are defined as in Sect. 3.1.

In both cases, in order to obtain an a posteriori estimate for the error $\|\mathbf{u}_h - \mathrm{U}\mathbf{u}_N\|_U$, the key point is the following property fulfilled by the Hessian matrix H_h:

Proposition 3. *The Hessian matrix H_h is symmetric positive definite, since the associated continuous operator $\mathcal{H} : U \times U \to \mathbb{R}$ is symmetric and coercive over $U_h \times U_h$. In particular, if the U-norm is chosen such that $\|\mathbf{u}_h\|_U^2 = \mathbf{u}_h^T \mathrm{G}_h \mathbf{u}_h$, then the following estimate for the coercivity constant holds:*

$$\alpha_H := \inf_{\delta u \in U_h} \frac{\mathcal{H}(\delta u, \delta u)}{\|\delta u\|_U^2} = \inf_{\mathbf{v} \in \mathbb{R}^{\mathcal{N}_U}} \frac{\mathbf{v}^T \mathrm{H}_h \mathbf{v}}{\|\mathbf{v}\|_U^2} \geq \sigma. \tag{53}$$

Proof. The continuity of the operator $\mathcal{H}(\cdot,\cdot)$ follows from the assumptions we made in Sect. 1.1. We prove its coercivity over $U_h \times U_h$: for any $\mathbf{v} \in \mathbb{R}^{\mathcal{N}_U}$,

$$\mathbf{v}^T \mathrm{H}_h \mathbf{v} = \sigma \mathbf{v}^T \mathrm{G}_h \mathbf{v} + \mathbf{v}^T \mathrm{C}_h^T \mathrm{S}_h^{-T} \mathrm{F}_h \mathrm{S}_h^{-1} \mathrm{C}_h \mathbf{v} = \sigma \|\mathbf{v}\|_U^2 + \underbrace{\|\mathrm{F}_h^{1/2} \mathrm{S}_h^{-1} \mathrm{C}_h \mathbf{v}\|_2^2}_{\geq 0} \geq \sigma \|\mathbf{v}\|_U^2,$$

where the existence of $\mathrm{F}_h^{1/2}$ is ensured since F_h is positive semi-definite (see Sect. 2). If the chosen norm does not satisfy our assumption $\|\mathbf{u}_h\|_U^2 = \mathbf{u}_h^T \mathrm{G}_h \mathbf{u}_h$, but rather $\mathbf{v}^T \mathrm{G}_h \mathbf{v} \geq \alpha_G \|\mathbf{v}\|_U^2$ for any $\mathbf{v} \in \mathbb{R}^{\mathcal{N}_U}$ (where α_G is the coercivity constant of the bilinear form associated to G_h), then (53) becomes $\alpha_H \geq \alpha_G \sigma$. Let us finally remark that the coercivity of $\mathcal{H}$ is equivalent to the assumption (H1) of Sect. 2. □

Since problems (49)-(52) are coercive, we can employ the Lax-Milgram lemma to obtain the usual residual-based RB error bound:

$$\|\mathbf{u}_h - \mathrm{U}\mathbf{u}_N\|_U \leq \frac{1}{\alpha_H} \|\mathbf{R}_h(\mathbf{u}_N)\|_{U'}, \tag{54}$$

where $\mathbf{R}_h(\mathbf{u}_N) = \mathbf{b}_h - \mathrm{H}_h\mathbf{u}_N$ is the residual of (49). Then, holding the assumption of Proposition 3, we end up with the inequality

$$\|\mathbf{u}_h - \mathrm{U}\mathbf{u}_N\|_U \leq \frac{1}{\sigma}\|\mathbf{R}_h(\mathbf{u}_N)\|_{U'}, \tag{55}$$

that was formerly obtained by Hinze and Volkwein in [2, 16] (see also the related work by Tröltzsch and Volkwein in [19]). However, the computation of the norm of the residual in (55) requires a forward (S_h^{-1}) and an adjoint (S_h^{-T}) full-order solve. Indeed, the main disadvantage of working in the control space is that the involved operators become:

1. *non-sparse*: H_h cannot be formed explicitly, rather we can define its action on a vector in the control space by performing a state and an adjoint solve;
2. *non-linearly parametrized* (as opposed to affinely parametrized): considering the parametric dependence, even if all matrices in (28) admit an affine decomposition, the matrix H_h cannot be expressed as, e.g., in (24).

In particular, the residual $\mathbf{R}_h$ becomes a *non-sparse* and *non-linearly parametrized* operator, so that the usual offline-online computational procedure cannot be performed. A possible way to obtain an efficiently computable error estimate is to bound the norm of the residual in terms of quantities depending explicitly on the full and reduced order state and adjoint solutions, as done in [7, 9]. An alternative strategy consists of using a full-space approach, where (i) a combined estimate for state, control and adjoint variables can be easily obtained and (ii) the involved operators remains *sparse* and *affinely parametrized*, without requiring any further treatment.

4.2. Full-space *a posteriori error estimate*

We generalize the a posteriori error analysis – based on the so-called Babuška framework for noncoercive problems – proposed in [1] to treat the case of control problems featuring elliptic coercive constraints. This way, we can construct suitable error bounds for the error on the optimization and adjoint variables, as well as for the error on the cost functional. These error bounds are based on two main ingredients, described below.

(i) The calculation of the Babuška inf-sup constant $\hat{\beta}_h(\boldsymbol{\mu})$ of the FE approximation of the optimality system,

$$\hat{\beta}_h(\boldsymbol{\mu}) := \inf_{(x,p)\in\mathcal{X}_h} \sup_{(\delta x,\delta p)\in\mathcal{X}_h} \frac{B((x,p),(\delta x,\delta p);\boldsymbol{\mu})}{\|(x,p)\|_{\mathcal{X}}\|(\delta x,\delta p)\|_{\mathcal{X}}}. \tag{56}$$

(ii) The computation of the dual norm of the residual $R(\cdot;\boldsymbol{\mu}) \in \mathcal{X}_h'$ of the optimality system (19),

$$\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}_h'} = \sup_{(\delta x,\delta p)\in\mathcal{X}_h} \frac{R((\delta x,\delta p);\boldsymbol{\mu})}{\|(\delta x,\delta p)\|_{\mathcal{X}}}, \tag{57}$$

where the residual is defined as

$$R((\delta x,\delta p);\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}),\delta x\rangle - B((x_N,p_N),(\delta x,\delta p);\boldsymbol{\mu}) \qquad \forall(\delta x,\delta p)\in\mathcal{X}_h.$$

Note that, in contrast to the *reduced-space* residual defined in Sect. 4.1, the *full-space* residual $R(\cdot,\boldsymbol{\mu})$ is a *sparse* and *affinely parametrized* operator. Therefore, the evaluation of its dual norm can be performed through an offline-online decomposition (see e.g. [33, 1]). As a result, given $\boldsymbol{\mu} \in \mathcal{D}$, the online evaluation of $\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}'}$ requires $O((13NQ_k + Q_g)^2)$ operations, independent of $\mathcal{N}$.

By using the same arguments as in [1], we can prove the following

Proposition 4. *For any given $\boldsymbol{\mu} \in \mathcal{D}$, $N \in [1, N_{max}]$, the following estimate holds*

$$\left(\|x_h(\boldsymbol{\mu}) - x_N(\boldsymbol{\mu})\|_X^2 + \|p_h(\boldsymbol{\mu}) - p_N(\boldsymbol{\mu})\|_Q^2\right)^{1/2} \leq \frac{\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}_h'}}{\hat{\beta}_h(\boldsymbol{\mu})}. \tag{58}$$

Furthermore, the error on the cost functional is bounded by

$$|\mathcal{J}(x_h(\boldsymbol{\mu});\boldsymbol{\mu}) - \mathcal{J}(x_N(\boldsymbol{\mu});\boldsymbol{\mu})| \leq \frac{1}{2}\frac{\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}_h'}^2}{\hat{\beta}_h(\boldsymbol{\mu})}. \tag{59}$$

In order to turn (58) and (59) into efficiently computable error estimates, we have to get rid of the dependency from $\hat{\beta}_h(\boldsymbol{\mu})$, since its online evaluation has complexity which depends on $\mathcal{N}$. A first option consists in relying on a $\boldsymbol{\mu}$-dependent lower bound $\hat{\beta}_{\mathrm{LB}}(\boldsymbol{\mu}) : \mathcal{D} \to \mathbb{R}$ whose construction can can be carried out by using the Successive Constraint Method, see e.g. [35]. However, as we already pointed out in [1], due to the intrinsic complexity of the optimal control problem (and in particular to the spectral properties of the matrix $K_h(\boldsymbol{\mu})$), the SCM algorithm can exhibit a low convergence rate, thus requiring unaffordable Offline computational costs. Rather, as recently proposed in [8, 1], we rely on an interpolant approximation $\hat{\beta}_I(\boldsymbol{\mu})$ of $\hat{\beta}_h(\boldsymbol{\mu})$, whose construction will be detailed in Sect. 5. As a result, we estimate the errors on the solution and on the cost functional employing the following error indicators, respectively:

$$\Delta_N(\boldsymbol{\mu}) = \frac{\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}_h'}}{\hat{\beta}_I(\boldsymbol{\mu})}, \qquad \Delta_N^{\mathcal{J}}(\boldsymbol{\mu}) = \frac{1}{2}\frac{\|R(\cdot;\boldsymbol{\mu})\|_{\mathcal{X}_h'}^2}{\hat{\beta}_I(\boldsymbol{\mu})}. \tag{60}$$

5. Numerical results

In this section we apply our methodology to a parametrized flow control problem, in order to show its computational performances. We deal with a problem of vorticity minimization through suction/injection of fluid on the downstream portion of a bluff body, see e.g. [21, 22]. In particular, we consider a body embedded in a viscous flow at low Reynolds number governed by the steady incompressible Stokes equations. The goal consists in minimizing the vorticity (and thus the drag, as a result) in the wake of the body, by regulating the flow across a portion of its boundary Γ_C. In particular, we minimize the following cost functional[9],

$$\mathcal{J}(\mathbf{v}, u; \boldsymbol{\mu}) = \frac{1}{2}\int_{\Omega_{\mathrm{obs}}(\mu_1)} |\nabla \times \mathbf{v}|^2 \, d\Omega + \frac{1}{2\mu_3}\,\mathcal{G}(u;\boldsymbol{\mu}), \tag{61}$$

subject to the steady Stokes equations,

$$\begin{aligned}
-\nu\Delta\mathbf{v} + \nabla\pi &= \mathbf{0} && \text{in } \Omega(\mu_1)\\
\operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega(\mu_1)\\
\mathbf{v}\cdot\mathbf{t} = 0,\ \mathbf{v}\cdot\mathbf{n} &= u && \text{on } \Gamma_C(\mu_1)\\
\mathbf{v} &= \mu_2\mathbf{t} && \text{on } \Gamma_{in}\\
\mathbf{v} &= \mathbf{0} && \text{on } \Gamma_w\\
\mathbf{v}\cdot\mathbf{n} = \mathbf{0},\ (\nabla\mathbf{v})\mathbf{n}\cdot\mathbf{t} &= \mathbf{0} && \text{on } \Gamma_s(\mu_1)\\
-\pi\mathbf{n} + \nu(\nabla\mathbf{v})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N,
\end{aligned} \tag{62}$$

where $\mathbf{n}$ and $\mathbf{t}$ are the outward normal and tangential unit vectors to the boundary. We impose an horizontal constant velocity profile on the inflow boundary Γ_{in}, no-slip conditions on Γ_w, symmetry conditions on Γ_s,

[9] We omit here the subscript "o" to denote the parametrized domain and boundaries.

no-stress conditions on Γ_N and Dirichlet conditions on the control boundary Γ_C.
In particular, we consider suction/injection of fluid through the control boundary only in the normal direction, while we impose a no-slip condition in the tangential one. We first consider a two-dimensional flow and then a three-dimensional flow, see Figures 1 and 5, respectively, for the details about the domain and boundaries. The parameters are given by: the length μ_1 of the control boundary, the magnitude μ_2 of the inflow velocity profile and the inverse of the penalization factor μ_3.

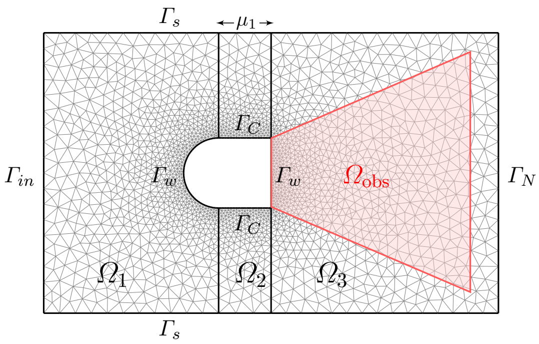

Figure 1: Domain, boundaries, observation region (in red) and computational mesh (less refined than the one actually used in the computations) for the 2D problem.

The penalization term $\mathcal{G}(u;\boldsymbol{\mu})$ deserves a special consideration in this case. In fact, the "natural" choice for the control space would be $U = H^{1/2}(\Gamma_C)$, so that we should penalize the $H^{1/2}(\Gamma_C)$-norm of the control variable, i.e.

$$\mathcal{G}(u;\boldsymbol{\mu}) = \|u\|^2_{H^{1/2}(\Gamma_C)}.$$

However, the realization of the $H^{1/2}(\Gamma_C)$-inner product would introduce undesirable computational complexities, see e.g. [36] for further details. Several alternatives have been proposed to overcome this issue, for instance:

(i) considering weaker solutions by choosing $U = L^2(\Gamma_C)$, which requires to cast the state equation in the ultra-weak variational formulation, see e.g. [37, 38];

(ii) adopting a penalty approach [39], in which the Dirichlet condition is approximated by a Robin condition which allows to set $U = L^2(\Gamma_C)$;

(iii) requiring further regularity by choosing $U = H^1_0(\Gamma_C)$, see e.g. [40].

Here we follow the third approach, i.e. we assume the control variable to belong to $U = H^1_0(\Gamma_C)$, and then we treat the Dirichlet control employing a lifting approach; an alternative would be to employ a Lagrange multiplier approach, see e.g. [40, 22]. As a result, the penalization term in the cost functional is now given by

$$\mathcal{G}(u;\boldsymbol{\mu}) = \int_{\Gamma_C(\mu_1)} \left|\nabla u \cdot \mathbf{t}\right|^2 d\Gamma.$$

5.1. Two-dimensional flow

We first deal with the two-dimensional version of the problem, i.e. we take $\Omega(\mu_1) \subset \mathbb{R}^2$ and consider a two-dimensional profile for the body, see Figure 1. The length of the control boundary can vary in the range $\mu_1 \in [0.1, 0.35]$, the magnitude of the inflow velocity profile $\mu_2 \in [0.5, 5]$, while the penalization constant $\mu_3 \in [1, 10^3]$. To handle the geometric parametrization and provide an affine decomposition of the problem, we divide the domain $\Omega(\mu_1)$ into three subdomains Ω_1, $\Omega_2(\mu_1)$ and $\Omega_3(\mu_1)$, see Figure 1. Provided this decomposition of the domain, we can easily build an affine geometrical mapping such that, by tracing the problem back to the reference domain $\Omega(\bar{\mu}_1)$ with $\bar{\mu}_1 = 0.15$, we obtain the parametrized formulation (10), where the affine decomposition is made by $Q_k = 14$ and $Q_g = 9$ terms.

For the FE discretization, we use a $\mathbb{P}^2$-$\mathbb{P}^1$ approximation for the velocity and pressure variables, respectively, and a $\mathbb{P}^2$ approximation (obtained as the restriction on Γ_C of the velocity FE space) for the control variable. The total number of degrees of freedom, i.e. the size of the full-order optimality system (23), is $\mathcal{N} = 99\,288$, obtained using a mesh of 11 055 triangular elements.

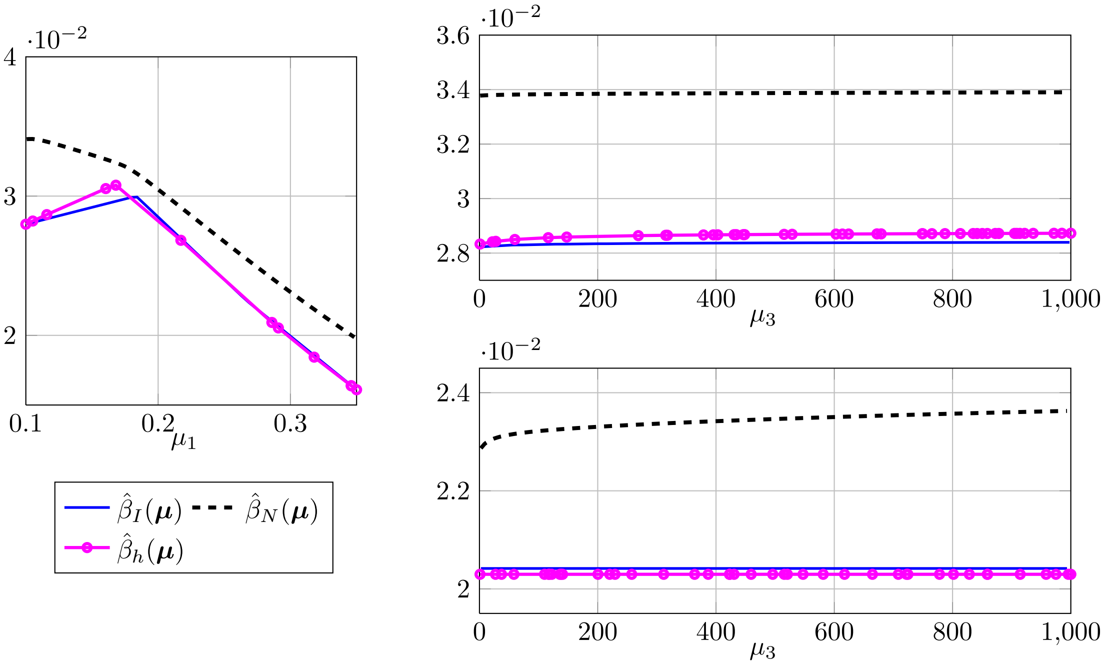

Figure 2: 2D flow. Comparison between the FE stability factor $\hat{\beta}_h(\boldsymbol{\mu})$ (the *dots* represent computed values), the interpolant surrogate $\hat{\beta}_I(\boldsymbol{\mu})$ and the RB stability factor $\hat{\beta}_N(\boldsymbol{\mu})$. Left: stability factors as functions of μ_1, $\mu_3 = 100$ fixed. Right: stability factors as functions of μ_3 with $\mu_1 = 0.1$ (top) and $\mu_1 = 0.3$ (bottom).

As previously mentioned in Sect. 4 we employ an alternative strategy to SCM in order to compute a lower bound of the inf-sup constant $\hat{\beta}_h(\boldsymbol{\mu})$. As recently proposed in [8, 1], we consider an approximation of $\hat{\beta}_h(\boldsymbol{\mu})$ given by an interpolation procedure. We select a (possibly *small*) set of interpolation points $\Xi_\beta \subset \mathcal{D}$ and compute the inf-sup constant $\hat{\beta}_h(\boldsymbol{\mu})$ by solving the related eigenproblem for each $\boldsymbol{\mu} \in \Xi_\beta$. Then, we compute a suitable *interpolant surrogate* $\hat{\beta}_I(\boldsymbol{\mu})$ such that $\hat{\beta}_I(\boldsymbol{\mu}) = \hat{\beta}_h(\boldsymbol{\mu})$, $\forall \boldsymbol{\mu} \in \Xi_\beta$. Here, since the parameter μ_2 does not affect the value of $\hat{\beta}_h(\boldsymbol{\mu})$, we perform a two dimensional linear interpolation with respect to the parameters μ_1 and μ_3, using an equally spaced grid of 4×12 interpolation points. In Figure 2 we report the interpolant surrogate $\hat{\beta}_I(\boldsymbol{\mu})$, which demonstrates to be a sharp approximation of the stability factor $\hat{\beta}_h(\boldsymbol{\mu})$ (despite not being a rigorous lower bound, as can be seen in the figure).

The greedy procedure for the construction of the RB spaces selects $N_{max} = 31$ sample points with a fixed tolerance $\varepsilon_{tol} = 5 \cdot 10^{-3}$ so that $\Delta_{N_{max}}(\boldsymbol{\mu}) \leq \varepsilon_{tol}$ $\forall \boldsymbol{\mu} \in \Xi_{\text{train}}$, where Ξ_{train} is a training set of $4 \cdot 10^4$ random points. In Figure 2, the RB stability factor $\hat{\beta}_N(\boldsymbol{\mu})$ defined in (43) is also reported: we can observe that $\hat{\beta}_N(\boldsymbol{\mu}) \geq \hat{\beta}_h(\boldsymbol{\mu})$, thus confirming numerically the good stability properties of the RB approximation.

In Figure 3 we compare the error estimate $\Delta_N(\boldsymbol{\mu})$ with the true error between the FE and RB approximations, as well as the error estimate $\Delta_N^{\mathcal{J}}(\boldsymbol{\mu})$ with the error on the cost functional $|\mathcal{J}_h(\boldsymbol{\mu}) - \mathcal{J}_N(\boldsymbol{\mu})|$. In Figure 4 some representative RB solutions are shown; as expected (see e.g. [22] for a comparison), the optimal controls correspond to an aspiration of the flow through the control boundary.

As regards the computational performances, the time spent in the offline computations is about two hours[10], while the solution of the reduced optimality system (of dimension 403×403) requires only 0.03 s (see Table 1). Let us remark that only a small fraction (less than 10%) of the time spent performing the

[10]The code we use in this work has been developed in the MATLAB® environment. All the full-order optimality systems

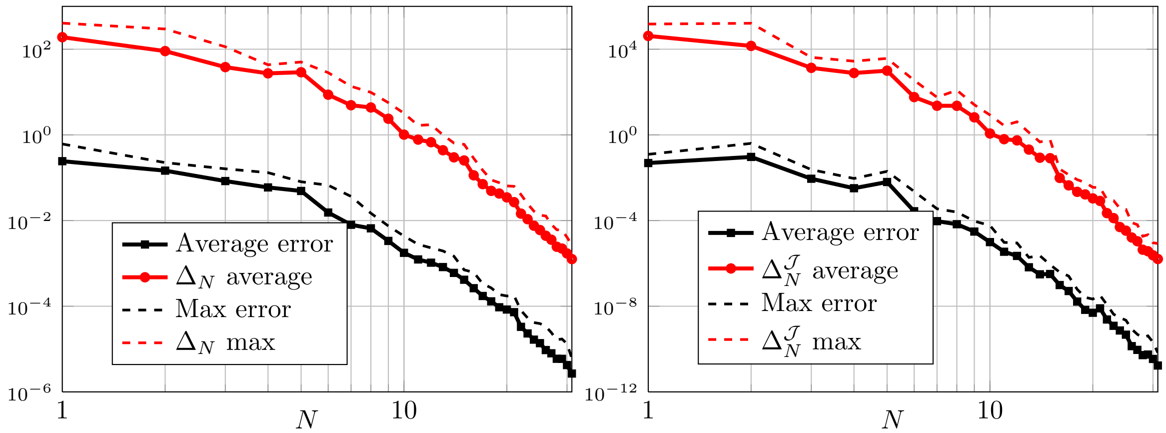

Figure 3: 2D flow. Left: average and max computed relative errors and bound $\Delta_N(\boldsymbol{\mu})$ between the full-order FE solution and the RB approximation. Right: average and max relative error and bound $\Delta_N^{\mathcal{J}}(\boldsymbol{\mu})$ between $\mathcal{J}_h(\boldsymbol{\mu})$ and $\mathcal{J}_N(\boldsymbol{\mu})$. Computations have been performed over a test sample set of 300 random points.

greedy procedure is devoted to actually solving the full-order optimization problem, rather most of the time is spent computing the quantities required to evaluate the dual norm of the residual and then evaluating the error estimate over the training set.

Approximation data		Computational performances	
Number of FE dofs $\mathcal{N}$	99 288	Number of RB dofs	403
Number of parameters P	3	Dofs reduction	248:1
Error tolerance greedy ε_{tol}	$5 \cdot 10^{-3}$	Stability factor interp. time	415 s
Affine operator components Q_k	14	Offline greedy time	6 095 s
Affine rhs components Q_g	9	RB online solution	0.03 s
FE solution time	≈ 10 s	RB online estimation	0.32 s

Table 1: Numerical details for the 2D-flow. The RB spaces have been built by means of the greedy procedure and $N = 31$ samples points have been selected.

5.2. Three-dimensional flow

We now consider the three-dimensional version of the problem, whose geometry is obtained by extruding the 2D-geometry in the orthogonal direction to the plane (x_1, x_2); we introduce suitable symmetry boundary conditions so that only a quarter of the geometry is meshed, see Figure 5. In this case, we consider a slightly different cost functional, minimizing the viscous energy dissipation rather than the vorticity: the two functionals serve the same purpose, but the latter would lead to an even larger affine decomposition compared to the two-dimensional case.

The parameters and their range of variations are the same as in the previous problem, except that for the geometrical parameter μ_1, which can now vary in a smaller range, $\mu_1 \in [0.1, 0.3]$. We employ a similar decomposition of the geometry into three subdomains to obtain an affine decomposition with $Q_k = 14$ and $Q_g = 9$. In this case, even using a rather coarse mesh, the dimension of the full-order optimality system rapidly grows when using stable $\mathbb{P}^2$-$\mathbb{P}^1$ or $\mathbb{P}^1_b$-$\mathbb{P}^1$ FE spaces for the velocity and pressure variables. Therefore,

are solved in one-shot using the sparse direct solver provided by MATLAB. Parallelism is exploited to speed up the assembly of the FE matrices, the evaluation of the stability factor, the calculation of the terms required to compute the dual norm of the residual and the evaluation of the a posteriori error estimate. For the 2D problem, computations have been performed using 8 cores on a node of the SuperB cluster at EPF Lausanne.

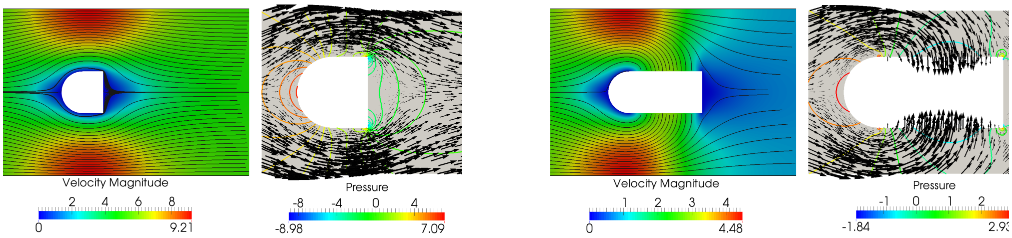

Figure 4: 2D flow. RB optimal state and control solutions for $\boldsymbol{\mu} = (0.1, 5, 10)$ (left) and $\boldsymbol{\mu} = (0.35, 2.5, 100)$ (right). For each case, we report the velocity magnitude with the streamlines on the left, the pressure contours and the velocity vectors around the body on the right. For the second configuration, the high value of μ_3 allows a significant flow suction on the control boundary, thus resulting in a low velocity profile at the outflow.

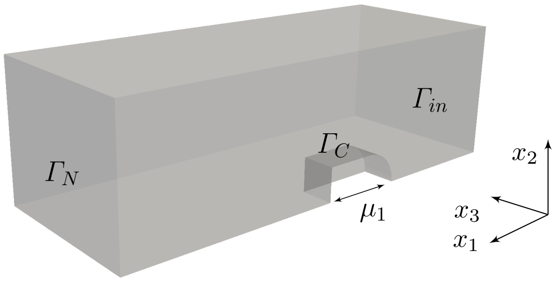

Figure 5: Domain and boundaries for the 3D problem. On the body boundary we impose a no-slip condition except that on the control region; on the top, bottom and lateral boundaries of the domain we impose symmetry conditions.

in order to alleviate the offline computational effort, we use low order $\mathbb{P}^1$-$\mathbb{P}^1$ spaces with Dohrmann-Bochev stabilization [41, 42]; the control variable is now discretized with $\mathbb{P}^1$ finite elements. In this way, using a mesh of 91 394 tetrahedral elements, the total number of degrees of freedom is $\mathcal{N} = 125\,266$ (for a comparison, using a $\mathbb{P}^1_b$ or $\mathbb{P}^2$ approximation for the velocity we would have $\mathcal{N} \approx 600\,000$).

As in the previous case, in order to compute the *interpolant surrogate* $\hat{\beta}_I(\boldsymbol{\mu})$ of the stability factor, we perform a two dimensional linear interpolation with respect to μ_1 and μ_3, using an equally spaced grid of 4×6 interpolation points. The greedy algorithm for the construction of the RB spaces selects $N_{max} = 20$ sample points with a fixed tolerance $\varepsilon_{tol} = 5 \cdot 10^{-3}$ (here Ξ_{train} is a set of $2 \cdot 10^4$ random points). The online evaluation of the errors and the corresponding estimates are reported in Figure 6. In Figure 7 we show some optimal control and state solutions obtained for different values of the parameters; note that in this case the control variable $u = u(x_1, x_3)$ is distributed over the surface $\Gamma_C(\mu_1)$.

Approximation data		Computational performances	
Number of FE dofs $\mathcal{N}$	125 266	Number of RB dofs	260
Number of parameters P	3	Dofs reduction	481:1
Error tolerance greedy ε_{tol}	$5 \cdot 10^{-3}$	Stability factor interp. time	592 s
Affine operator components Q_k	14	Offline greedy time	4296 s
Affine rhs components Q_g	9	RB online solution	0.026 s
FE solution time	≈ 35 s	RB online estimation	0.3 s

Table 2: Numerical details for the 3D flow. The RB spaces have been built by means of the greedy procedure and $N = 20$ samples points have been selected.

As regards the computational performances, the time spent offline to build the RB spaces is about 1.5 hours[11], while the solution of the reduced optimality system (of dimension 260×260) requires only $0.026\,s$,

[11] In this case we have used 12 cores on a node of the SuperB cluster at EPF Lausanne.

yet providing a good accuracy (see Table 2). As a result, in the online stage, we can perform the optimization in different scenarios, and thus evaluate the dependence of the optimal solution w.r.t the parameters in a very rapid way. For instance, in Figure 8 we report the value of the cost functional $\mathcal{J}_N(\boldsymbol{\mu})$ as a function of μ_1 and μ_3, $\mu_2 = 3$ being fixed; using the ROM, it takes only 20 seconds to solve the optimization problem and to evaluate the cost functional on a grid of 40×15 points in the parameter space.

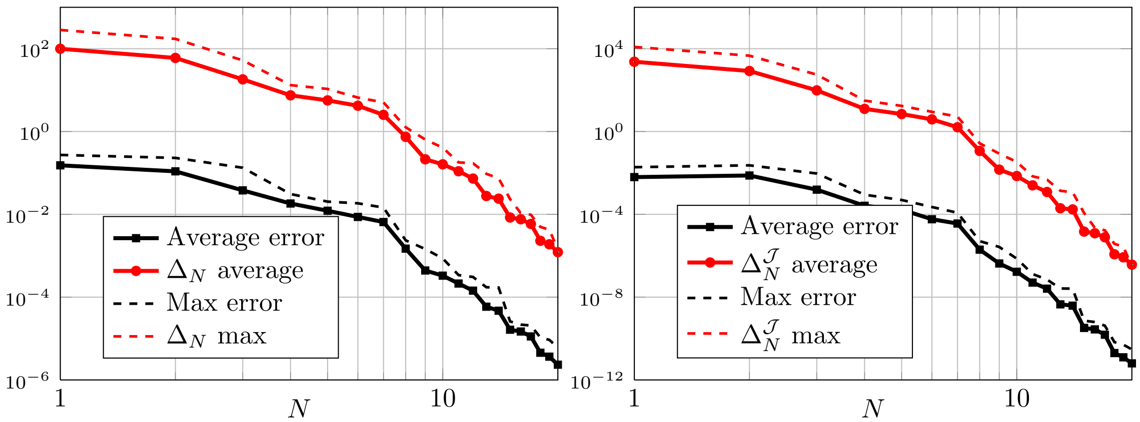

Figure 6: 3D flow. Left: average and max computed relative errors and bound $\Delta_N(\boldsymbol{\mu})$ between the full-order FE solution and the RB approximation. Right: average and max relative error and bound $\Delta_N^{\mathcal{J}}(\boldsymbol{\mu})$ between $\mathcal{J}_h(\boldsymbol{\mu})$ and $\mathcal{J}_N(\boldsymbol{\mu})$. Computations have been performed over a test sample set of 200 random points.

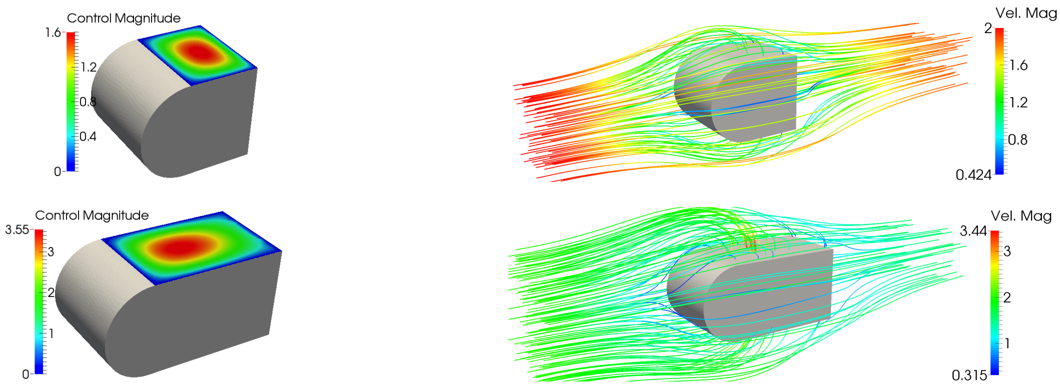

Figure 7: 3D flow. RB control and state solutions for $\boldsymbol{\mu} = (0.15, 2, 600)$ (top) and $\boldsymbol{\mu} = (0.3, 2, 1000)$ (bottom); for each case, we report the optimal control $u = u(x_1, x_3)$ on the upper control boundary (left) and the resulting state velocity streamlines around the body (right).

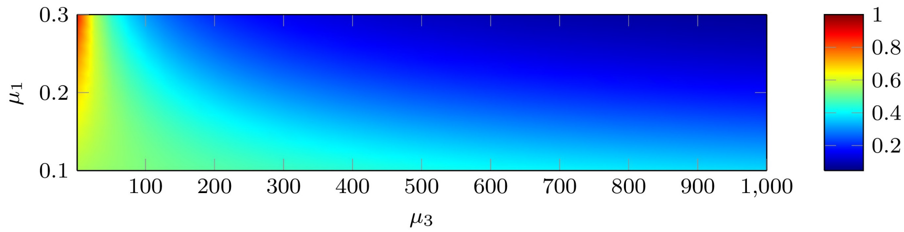

Figure 8: 3D flow. We report the value of the cost functional $\mathcal{J}_N(\boldsymbol{\mu})$ w.r.t (μ_1, μ_3), where $\mu_2 = 3$ is fixed; $\mathcal{J}_N(\boldsymbol{\mu})$ here is normalized to its maximum. As expected, $\mathcal{J}_N(\boldsymbol{\mu})$ decreases as the length of the control boundary μ_1 increases and the penalization factor $1/\mu_3$ decreases.

6. Conclusions

We have extended the reduced basis method for the solution of PDE-constrained parametrized optimization problems proposed in [1, 11] by considering the Stokes equations as a relevant case of noncoercive PDE. Two approaches for the reduction of approximate control variables and corresponding state and adjoint variables were proposed: the *reduce-then-optimize* and the *optimize-then-reduce* approach.

The discussion of these two paradigms led us to motivate the use of aggregated reduced spaces for the state and adjoint variables, which were also proved to be an inf-sup stable pair of approximation spaces for the optimality system. Formulating this latter as a coupled variational problem, and applying standard arguments based on Babuška stability theory, suitable a posteriori error estimates are provided. The method has been applied to a Dirichlet boundary control problem of vorticity minimization behind a body immersed in a two and a three-dimensional flow. This numerical test represents an important benchmark case in (parametrized infinite-dimensional) optimal flow control, and it provides important indications about computational performances, error estimates and stability factors approximation. In both 2D and 3D cases, the proposed methodology yields a reduction of at least two orders of magnitude in the number of variables, thus enabling an almost real-time solution of the optimal control problem.

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Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

It is sufficient to exploit the properties/assumptions stated above. For the complete proof we refer to [27, Ch. 11]; here we briefly recall its basic ingredients. The continuity of $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ and $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu})$ follows from the continuity of the bilinear forms $S(\cdot,\cdot;\boldsymbol{\mu})$, $C(\cdot,\cdot;\boldsymbol{\mu})$, $m(\cdot,\cdot;\boldsymbol{\mu})$ and $n(\cdot,\cdot;\boldsymbol{\mu})$. To prove the coercivity of $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ on X_0 it is sufficient to exploit the weak coercivity of the Stokes operator, the coercivity of $n(\cdot,\cdot;\boldsymbol{\mu})$ and the non-negativeness of $m(\cdot,\cdot;\boldsymbol{\mu})$. In particular, for any prescribed $\boldsymbol{\mu} \in \mathcal{D}$, for all $x = (y,u) \in X^0$ the following estimate holds:

$$\|y\|_Y \leq \frac{\gamma^C(\boldsymbol{\mu})}{\beta^S(\boldsymbol{\mu})}\|u\|_U, \tag{A.1}$$

where $\gamma^C(\boldsymbol{\mu})$ is the continuity constant of the bilinear form $C(\cdot,\cdot;\boldsymbol{\mu})$ and $\beta^S(\boldsymbol{\mu})$ defined as the Babuška inf-sup constant of the Stokes operator, i.e.

$$\beta^S(\boldsymbol{\mu}) := \inf_{y_1\in Y}\sup_{y_2\in Y}\frac{S(y_1,y_2;\boldsymbol{\mu})}{\|y_1\|_Y\|y_2\|_Y} = \inf_{y_2\in Y}\sup_{y_1\in Y}\frac{S(y_1,y_2;\boldsymbol{\mu})}{\|y_1\|_Y\|y_2\|_Y}. \tag{A.2}$$

Denoting with $\alpha^n(\boldsymbol{\mu})$ the coercivity constant of $n(\cdot,\cdot;\boldsymbol{\mu})$ and exploiting (A.1), we obtain that, for any $x \in X^0$

$$\mathcal{A}(x,x;\boldsymbol{\mu}) \geq \sigma n(u,u;\boldsymbol{\mu}) \geq \frac{\sigma}{2}\alpha^n(\boldsymbol{\mu})\|u\|_U^2 + \frac{\sigma\alpha^n(\boldsymbol{\mu})}{2}\left(\frac{\beta^S(\boldsymbol{\mu})}{\gamma^C(\boldsymbol{\mu})}\right)^2\|y\|_Y^2 \geq C(\boldsymbol{\mu})\|x\|_X^2,$$

where

$$C(\boldsymbol{\mu}) = \frac{\sigma}{2}\alpha^n(\boldsymbol{\mu})\min\left\{1,\left(\frac{\beta^S(\boldsymbol{\mu})}{\gamma^C(\boldsymbol{\mu})}\right)^2\right\}.$$

Finally, by exploiting again the weak coercivity of the Stokes operator and the fact that $Y \equiv Q$, also the inf-sup condition in (H2) is in fact fulfilled, since:

$$\begin{aligned}\sup_{0\neq\delta x\in X}\frac{\mathcal{B}(\delta x,\delta p;\boldsymbol{\mu})}{\|\delta x\|_X} &= \sup_{0\neq(\delta y,\delta u)\in Y\times U}\frac{S(\delta y,\delta p;\boldsymbol{\mu}) - C(\delta u,\delta p;\boldsymbol{\mu})}{(\|\delta y\|_Y^2+\|\delta u\|_U^2)^{1/2}}\\ &\geq \sup_{0\neq\delta y\in Y}\frac{S(\delta y,\delta p;\boldsymbol{\mu})}{\|\delta y\|_Y} \geq \beta^S(\boldsymbol{\mu})\|\delta p\|_Y = \beta^S(\boldsymbol{\mu})\|\delta p\|_Q.\end{aligned}$$

Appendix A.2. Proof of Proposition 2

The continuity of $\mathcal{A}(\cdot,\cdot:\boldsymbol{\mu})$ and $\mathcal{B}(\cdot,\cdot:\boldsymbol{\mu})$ are automatically fulfilled, since $X_N \subset X_h$ and $Q_N \subset Q_h$. Then, thanks to the enrichment of the RB velocity space $V_N^{\boldsymbol{\mu}}$ by supremizer solutions and to the fact that $Y_N = Q_N$, the RB approximation of the Stokes operator satisfies the assumptions of Brezzi theorem [17], which implies [29, 30] the existence of a constant $\beta^S_{N0} > 0$ such that

$$\beta_N^S(\boldsymbol{\mu}) = \inf_{\delta y\in Y_N}\sup_{\delta p\in Q_N}\frac{S(\delta y,\delta p;\boldsymbol{\mu})}{\|\delta y\|_Y\|\delta p\|_Q} = \inf_{\delta p\in Q_N}\sup_{\delta y\in Y_N}\frac{S(\delta y,\delta p;\boldsymbol{\mu})}{\|\delta y\|_Y\|\delta p\|_Q} \geq \beta^S_{N0};$$

in other words $S(\cdot,\cdot;\boldsymbol{\mu})$ is inf-sup stable on $Y_N \times Q_N$. By exploiting this property, we can use the same arguments of Proposition 1 in order to prove the coercivity of $\mathcal{A}(\cdot,\cdot;\boldsymbol{\mu})$ over X_N^0, and the fulfillment of the inf-sup condition (34).

To obtain the inequality (44), we first apply to $\beta_N^S(\boldsymbol{\mu})$ the estimate

$$\beta_N^S(\boldsymbol{\mu}) \geq \frac{\alpha_N^a(\boldsymbol{\mu})}{1+\left(\gamma_N^a(\boldsymbol{\mu})/\beta_N^b(\boldsymbol{\mu})\right)^2} \qquad \forall\boldsymbol{\mu}\in\mathcal{D},$$

recently proved in [43, Th. 1], where $\gamma_N^a(\boldsymbol{\mu})$ and $\alpha_N^a(\boldsymbol{\mu})$ are the RB approximation of the continuity and coercivity constants of $a(\cdot,\cdot;\boldsymbol{\mu})$. Then, since $\gamma_N^a(\boldsymbol{\mu}) \leq \gamma_h^a(\boldsymbol{\mu})$, $\alpha_N^a(\boldsymbol{\mu}) \geq \alpha_h^a(\boldsymbol{\mu})$ and $\beta_N^b(\boldsymbol{\mu}) \geq \beta_h^b(\boldsymbol{\mu})$ (see e.g. [18, 17] for the proof), (44) directly follows. To prove (45), we proceed as we did in the proof of Proposition 1, to obtain

$$\alpha_N(\boldsymbol{\mu}) \geq \frac{\sigma}{2}\alpha_N^n(\boldsymbol{\mu})\min\left\{1,\left(\frac{\beta_N^S(\boldsymbol{\mu})}{\gamma_N^C(\boldsymbol{\mu})}\right)^2\right\}.$$

Since $\gamma_N^C(\boldsymbol{\mu}) \leq \gamma_h^C(\boldsymbol{\mu})$ and $\alpha_N^n(\boldsymbol{\mu}) \geq \alpha_h^n(\boldsymbol{\mu})$, (45) directly follows. Finally, we apply once again the inequality provided in [43, Th. 1] to get

$$\hat{\beta}_N(\boldsymbol{\mu}) \geq \frac{\alpha_N(\boldsymbol{\mu})}{1+\left(\gamma_N^{\mathcal{A}}(\boldsymbol{\mu})/\beta_N(\boldsymbol{\mu})\right)^2}, \qquad \forall\boldsymbol{\mu}\in\mathcal{D},$$

from which (46) direcly follows, since $\gamma_N^{\mathcal{A}}(\boldsymbol{\mu}) \leq \gamma_h^{\mathcal{A}}(\boldsymbol{\mu})$.

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