Learning Sparse Representations for Graph Signals

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HIM Workshop on Harmonic Analysis, Graphs and Learning

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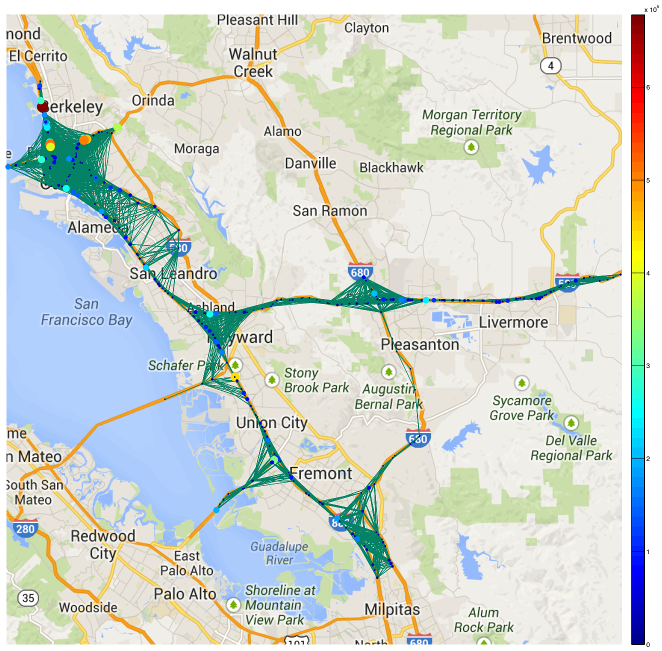
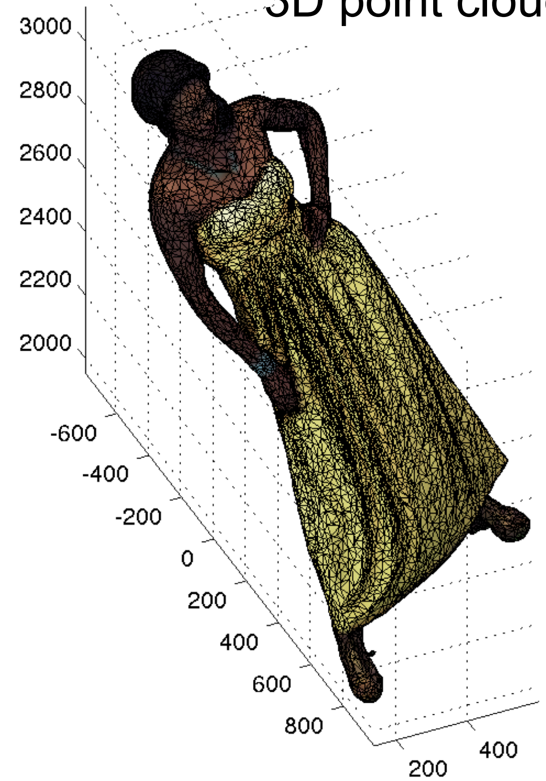
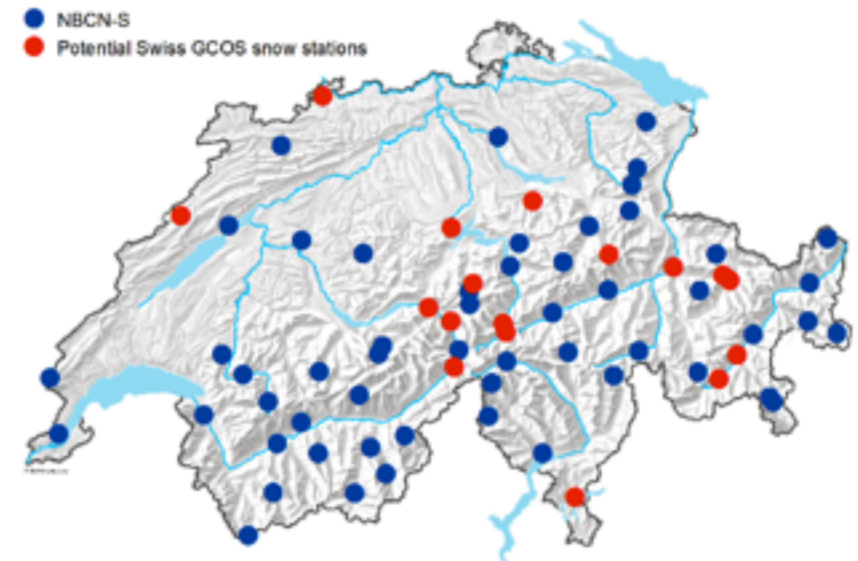
Structured data



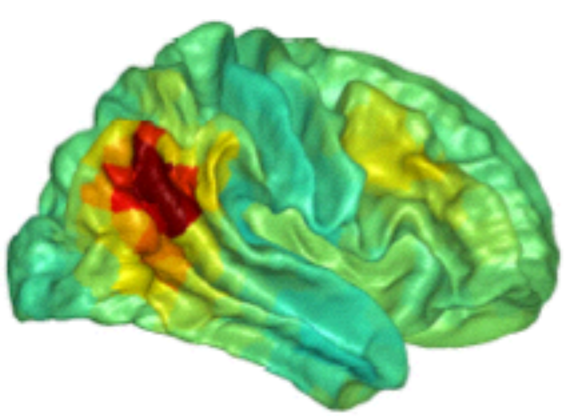
Traffic bottlenecks

3D point clouds



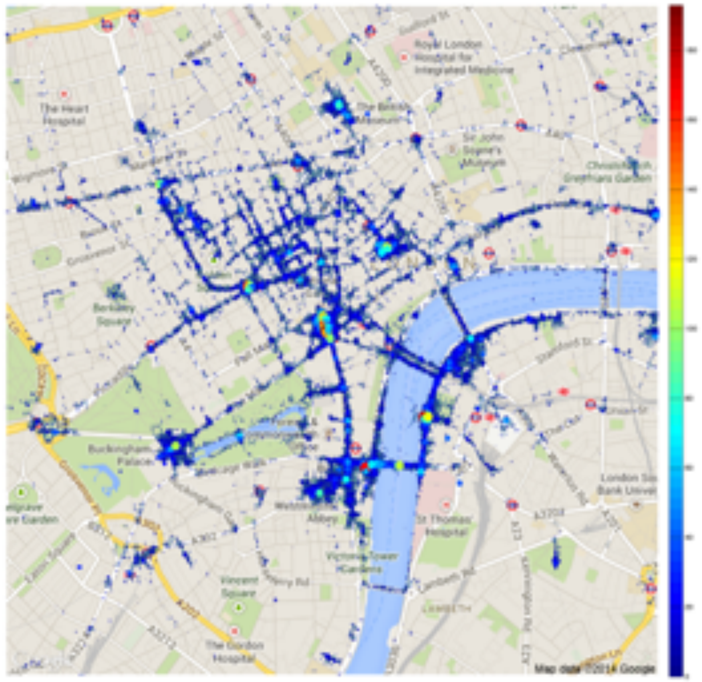


Sensor networks



Brain signals

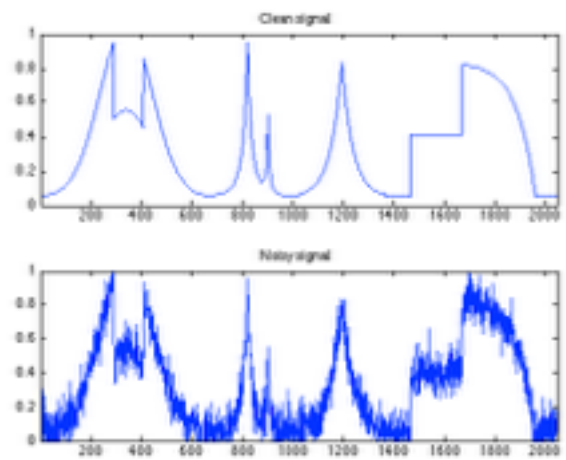
Social networks

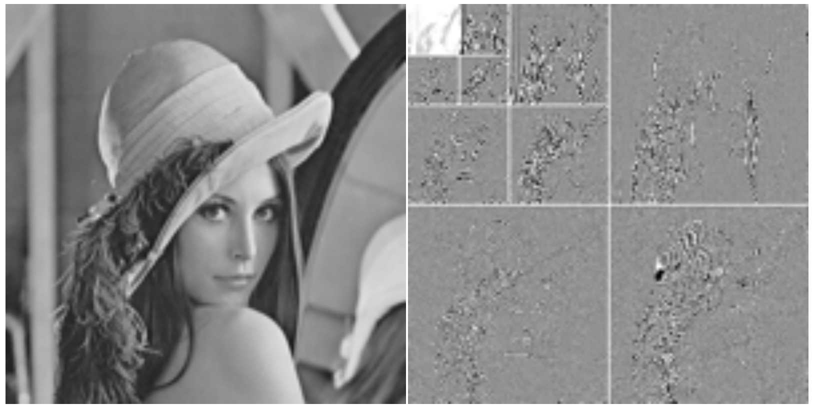


Mobility patterns

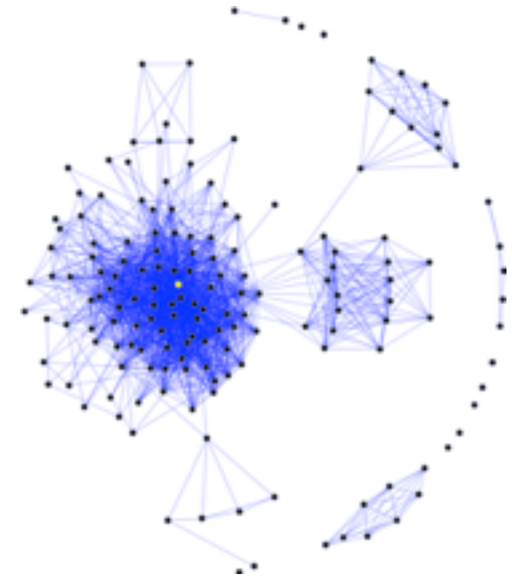
Structured, but irregular data …

- Traditional signal processing in Euclidean space





- Irregular (graph) structures: new challenges for signal processing?



Acknowledgements

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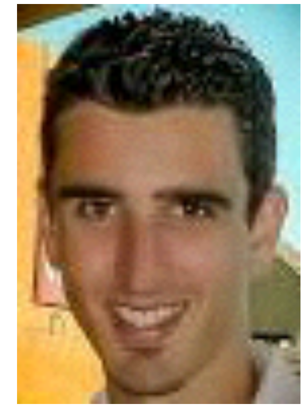

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Pierre Vandergheynst

Phil Chou

Antonio Ortega

Sunil Narang

Agenda

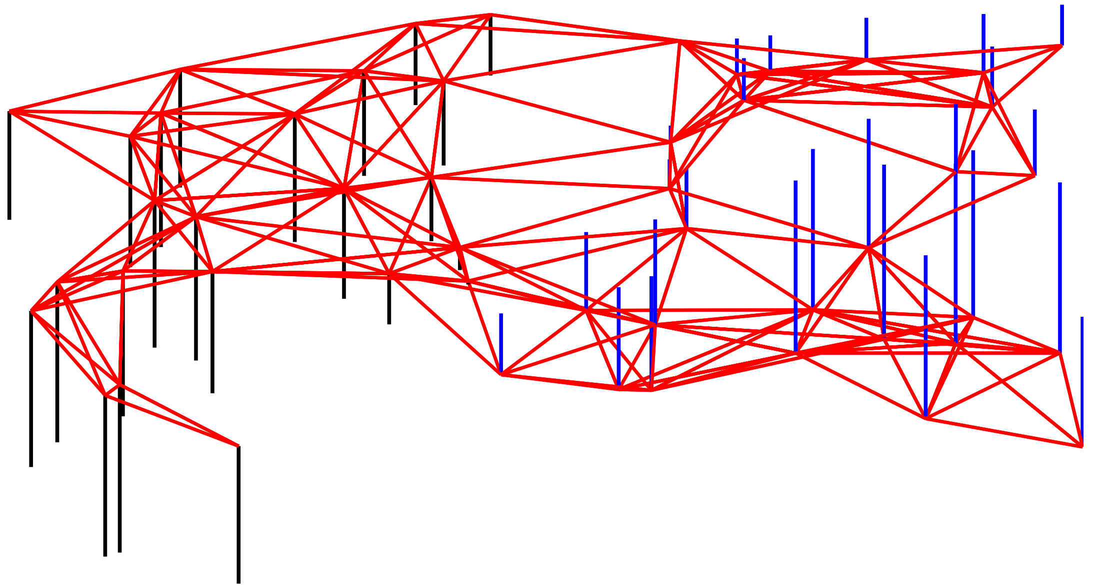
- Graph Signal Processing Basics
  - Main definitions and operators
- Adaptive Graph Signal Representations
  - Graph Spectral Dictionaries
  - Dictionary Learning Algorithm
  - Applications of Graph Spectral Dictionaries
- Inferring Graphs from Observations
  - Factor Analysis Model
  - Graph Learning Algorithm
  - Illustrative Applications

Signals on Graphs

- Connected, undirected, weighted graph $\mathcal{G} = (V, E, W)$
  where $W_{i,j}$ is the weight of the edge $e = (i, j)$

- Graph signal: a function $f : \mathcal{V} \to \mathbb{R}$ that assigns real values to each vertex of the graph

- Graph description:
  - Degree matrix $\mathbf{D}$: diagonal matrix with sum of weights of incident edges
  - Laplacian matrix $\mathcal{L}$: difference operator defined based on $\mathbf{W}$



(Unormalized) Laplacian

- Laplacian is a difference operator $\mathcal{L} := \mathbf{D} - \mathbf{W}$

$$(\mathcal{L}f)(i) = \sum_{j \in \mathcal{N}_i} W_{i,j} [f(i) - f(j)]$$

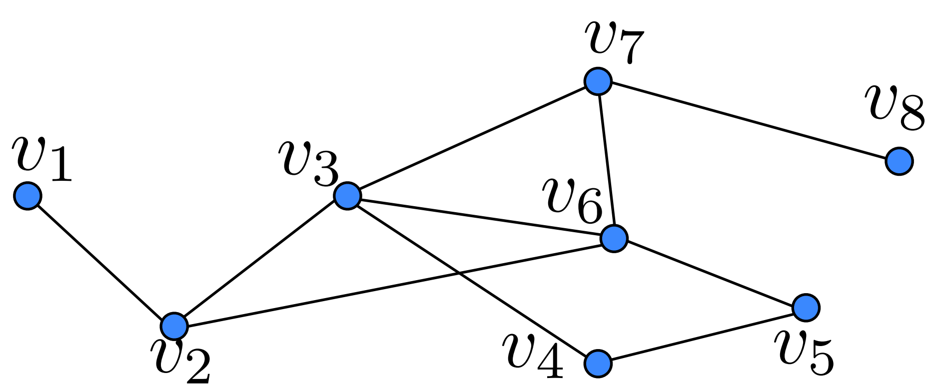
- It is a real symmetric matrix
- It has a complete set of eigenvectors $\{\mathbf{u}_\ell\}_{\ell=0,1,\ldots,N-1}$
- The eigenvectors are associated with real, nonnegative eigenvalues $\{\lambda_\ell\}_{\ell=0,1,\ldots,N-1}$

$$\mathcal{L}\mathbf{u}_\ell = \lambda_\ell \mathbf{u}_\ell, \ \forall \ell = 0, 1, \ldots, N-1$$

- Its spectrum is defined as $\sigma(\mathcal{L}) := \{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}\}$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_{N-1} := \lambda_{\max}$$

Laplacian example



$$G = \{V, E\}$$

$$D = diag(degree(v_1) \quad \dots \quad degree(v_n))$$

$$\mathcal{L} := \mathbf{D} - \mathbf{W}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$\mathbf{W}$ $\qquad$ $\mathcal{L}$

- Symmetric
- Off-diagonal entries non-positive
- Rows sum up to zero
- Has a complete set of orthonormal eigenvectors: $L = \chi \Lambda \chi^T$

  $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_{n-1}$

Normalized Laplacian

- The normalized Laplacian is another popular graph matrix
- Each weight $W_{i,j}$ is normalised by $\frac{1}{\sqrt{d_i d_j}}$

$$\tilde{\mathcal{L}} := \mathbf{D}^{-\frac{1}{2}} \mathcal{L} \mathbf{D}^{-\frac{1}{2}}$$

$$(\tilde{\mathcal{L}} f)(i) = \frac{1}{\sqrt{d_i}} \sum_{j \in \mathcal{N}_i} W_{i,j} \left[\frac{f(i)}{\sqrt{d_i}} - \frac{f(j)}{\sqrt{d_j}} \right]$$

- The set of eigenvalues is $0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_{\max} \leq 2$
- The normalized Laplacian has often stability benefits

Graph Fourier Transform

- The eigenvectors of the graph Laplacian are used for defining the Graph Fourier Transform

GFT

$$\hat{f}(\lambda_\ell) := \langle \mathbf{f}, \mathbf{u}_\ell \rangle = \sum_{i=1}^{N} f(i) u_\ell^*(i)$$

IGFT

$$f(i) = \sum_{\ell=0}^{N-1} \hat{f}(\lambda_\ell) u_\ell(i)$$

- This is analogous to the classical Fourier Transform built on eigenfunctions of the 1-D Laplace operator

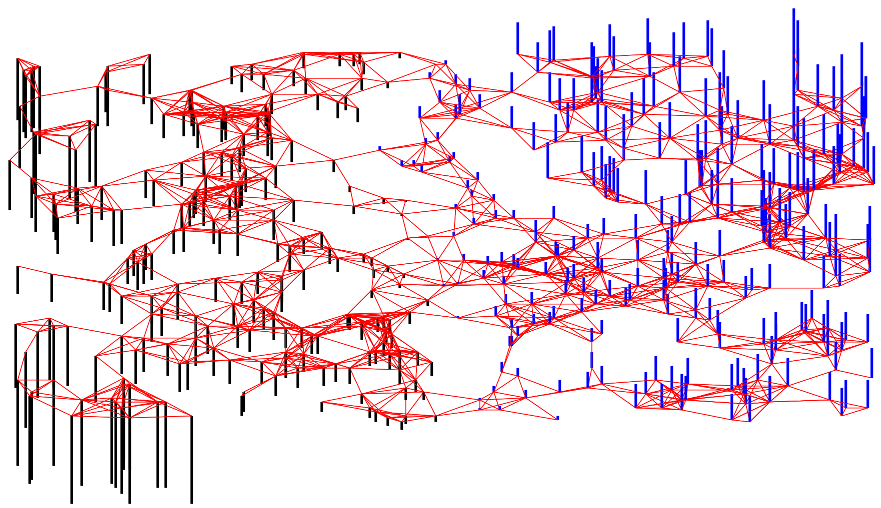
$$\hat{f}(\xi) := \langle f, e^{2\pi i \xi t} \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} dt$$

$$-\Delta(e^{2\pi i \xi t}) = -\frac{\partial^2}{\partial t^2} e^{2\pi i \xi t} = (2\pi\xi)^2 e^{2\pi i \xi t}$$

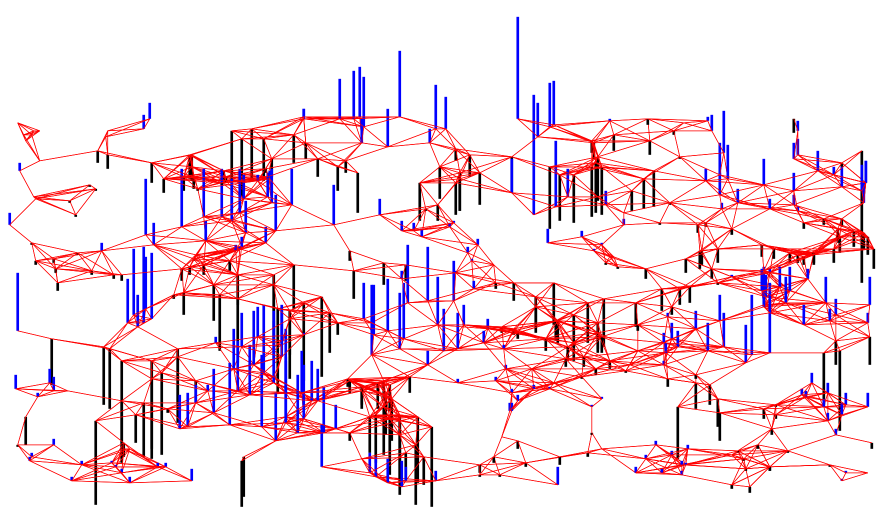
Notion of ‘frequency’

- The graph Laplacian eigenvalues and eigenvectors carry a notion of frequency

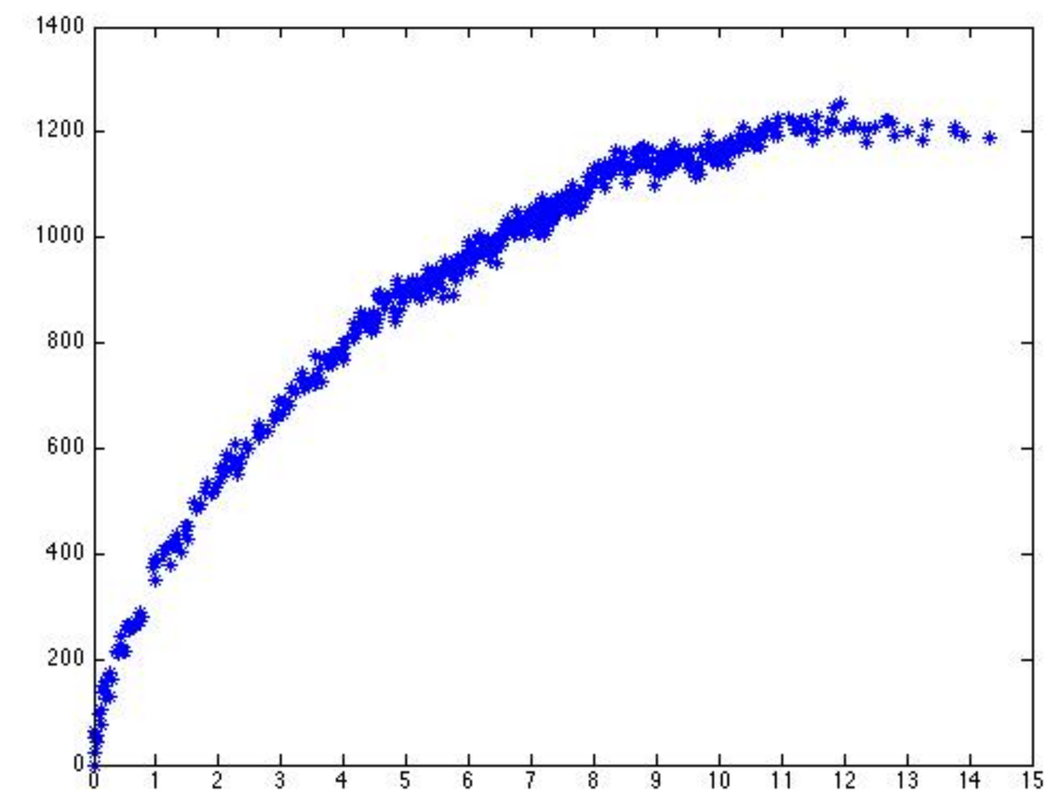
$\mathbf{u}_1$



$\mathbf{u}_{50}$



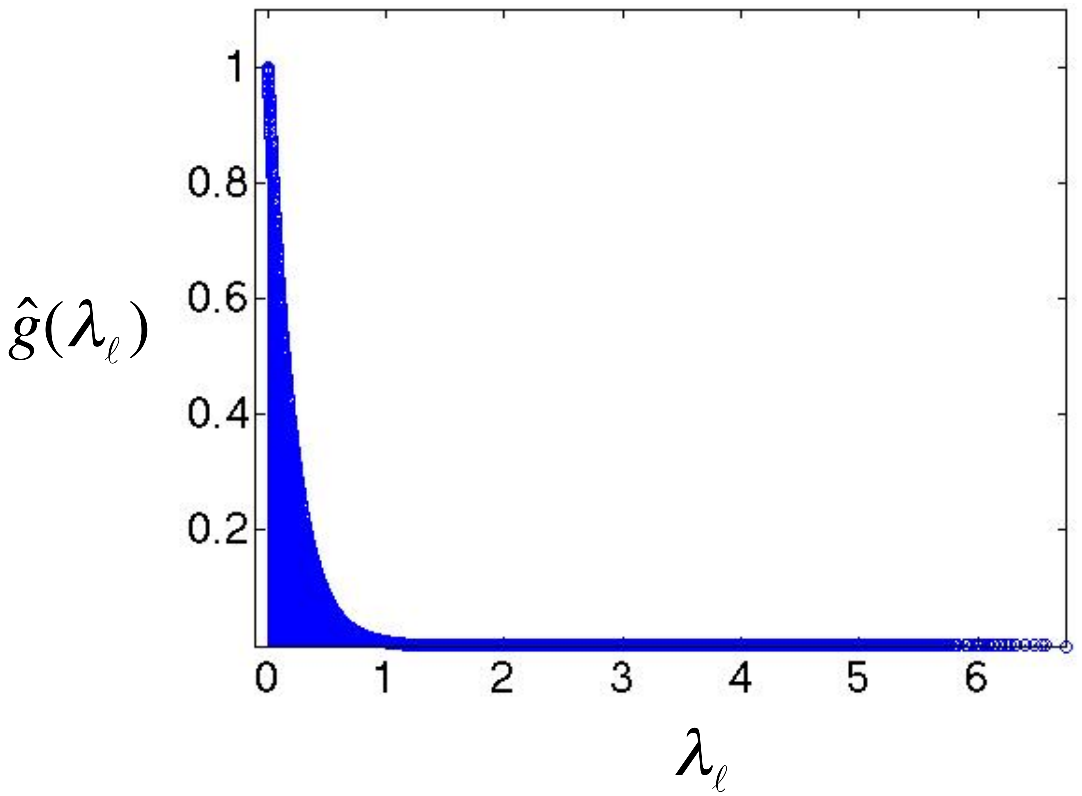
Number of zero crossings



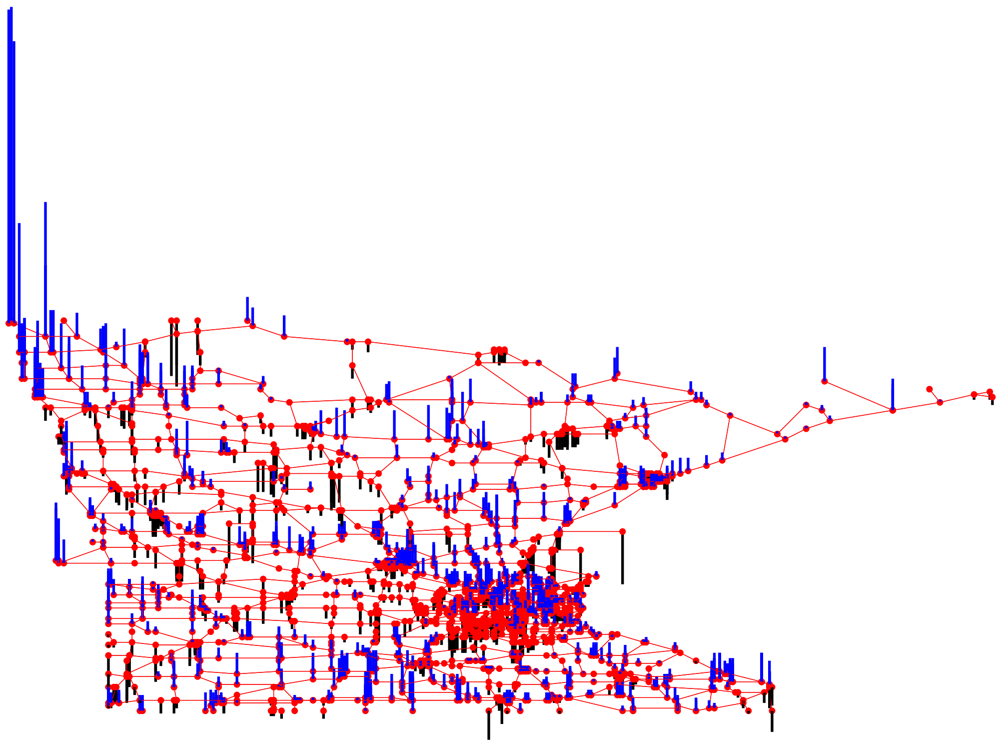
λ_ℓ

Dual representations

- Graph signals represented in either the vertex or the spectral domains (*kernels, or graph Fourier multipliers*)



$$\hat{g}(\lambda_\ell) = e^{-5\lambda_\ell}$$



$$g(n) \xleftarrow{IGFT} \hat{g}(\lambda_\ell)$$

Frequency filtering

- Analogously to classical filtering, one can perform graph spectral filtering with transfer function $\hat{h}(\lambda_\ell)$

$$\hat{f}_{out}(\lambda_\ell) = \hat{f}_{in}(\lambda_\ell)\hat{h}(\lambda_\ell)$$

- Equivalently $$f_{out}(i) = \sum_{\ell=0}^{N-1} \hat{f}_{in}(\lambda_\ell)\hat{h}(\lambda_\ell)u_\ell(i)$$

- In matrix notation: $$\mathbf{f}_{out} = \hat{h}(\mathcal{L})\mathbf{f}_{in}$$

$$\hat{h}(\mathcal{L}) := \mathbf{U} \begin{bmatrix} \hat{h}(\lambda_0) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \hat{h}(\lambda_{N-1}) \end{bmatrix} \mathbf{U}^{\mathrm{T}}$$

Filtering in the vertex domain

- Linear combination of values at neighbour vertices

$$f_{out}(i) = b_{i,i} f_{in}(i) + \sum_{j \in \mathcal{N}(i,K)} b_{i,j} f_{in}(j)$$

  - localized linear transform

- Example: polynomial filter as $\hat{h}(\lambda_\ell) = \sum_{k=0}^{K} a_k \lambda_\ell^k$

$$f_{out}(i) = \sum_{\ell=0}^{N-1} \hat{f}_{in}(\lambda_\ell) \hat{h}(\lambda_\ell) u_\ell(i)$$

$$= \sum_{j=1}^{N} f_{in}(j) \sum_{k=0}^{K} a_k \left(\mathcal{L}^k\right)_{i,j} \rightarrow b_{i,j} := \sum_{k=d_{\mathcal{G}}(i,j)}^{K} a_k \left(\mathcal{L}^k\right)_{i,j}$$

Translation on graphs

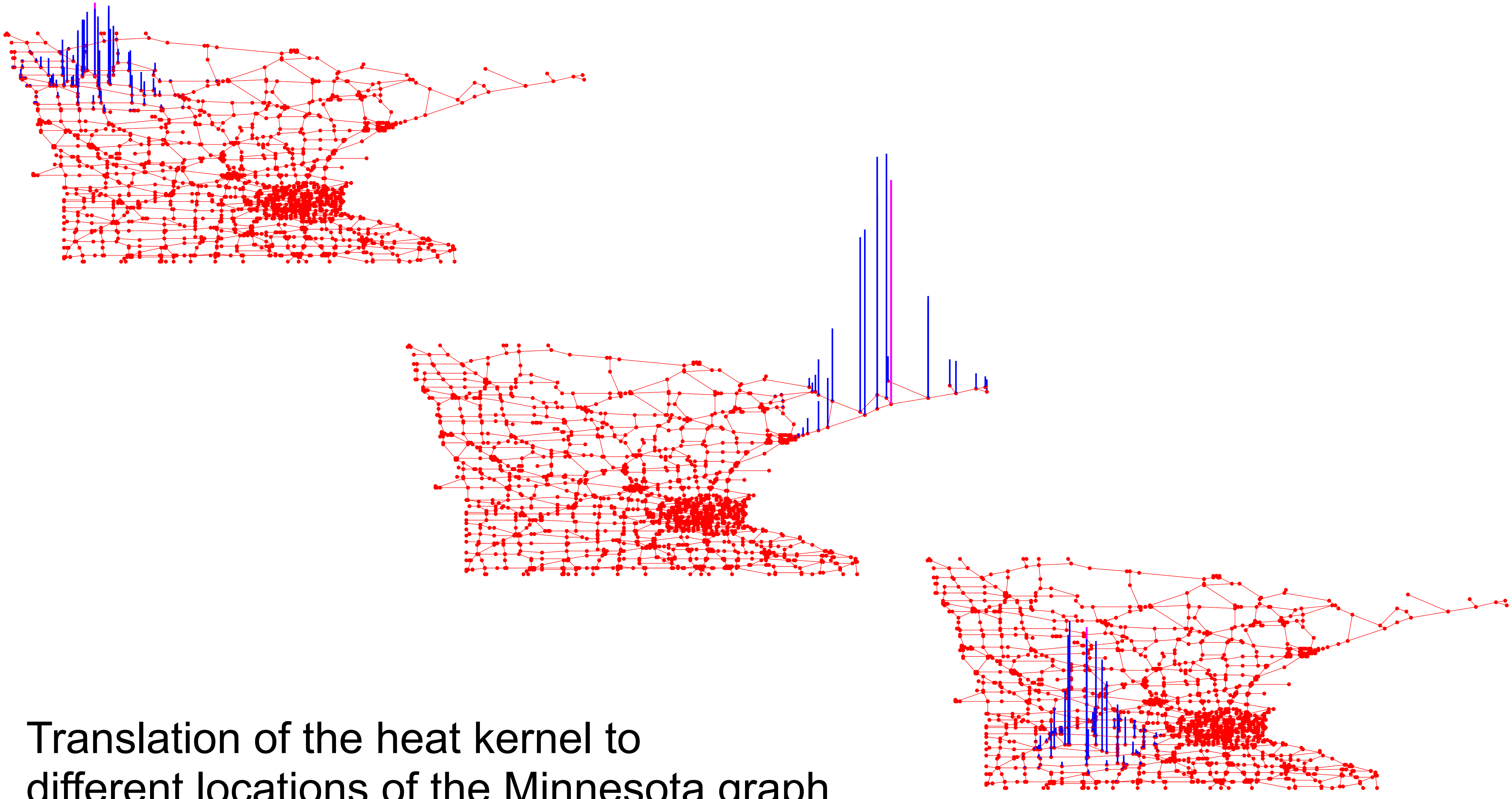
- The classical translation $(T_u f)(t) := f(t-u)$ does not generalise to non-regular graphs
- A generalized translation operator on graphs can still be defined as

$$T_n : \mathbb{R}^N \to \mathbb{R}^N$$

$$(T_n g)(i) := \sqrt{N}(g * \delta_n)(i) = \sqrt{N} \sum_{\ell=0}^{N-1} \hat{g}(\lambda_\ell) u_\ell^*(n) u_\ell(i)$$

$$\delta_n(i) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases} \qquad (f * h)(i) := \sum_{\ell=0}^{N-1} \hat{f}(\lambda_\ell)\hat{h}(\lambda_\ell) u_\ell(i)$$

Translation example



Translation of the heat kernel to
different locations of the Minnesota graph

Transforms on graphs

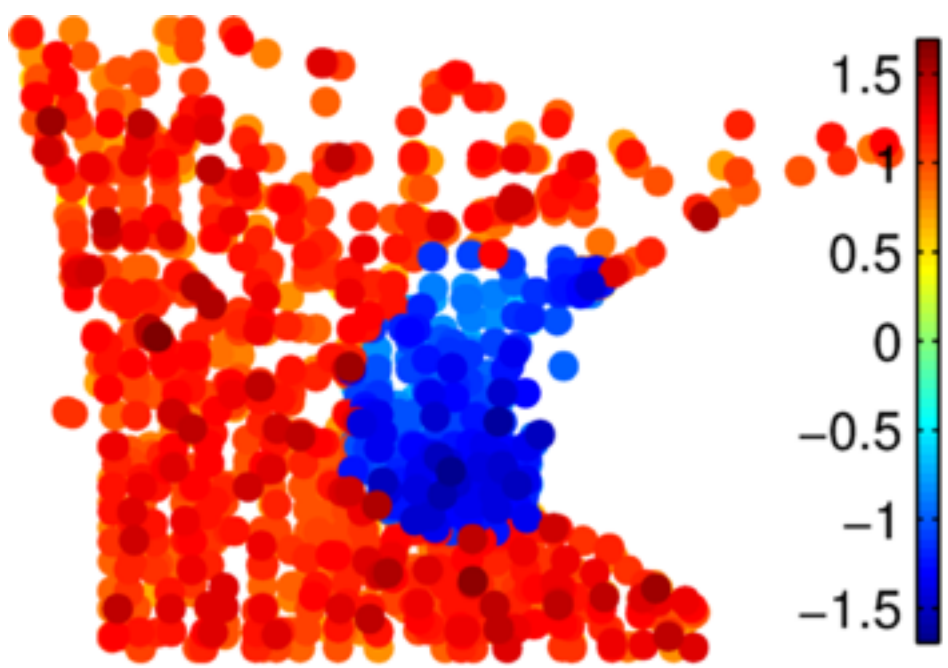
- Localized transforms are ideal to analyse graph signals
  - analysis properties and scalable implementations
  - GFT is unfortunately not a local transform
- Wavelet transforms are particularly interesting
  - localization in both the vertex and spectral domains
  - different designs in the vertex or the spectral domain [Shuman:2013]
  - *example*: Spectral Graph Wavelets [Hammond:2011]

$$\mathbf{\Psi}^{SGWT} : \mathbb{R}^N \rightarrow \mathbb{R}^{N(K+1)} \qquad \mathbf{\Psi}^{SGWT} = [\mathbf{\Psi}^{SGWT}_{scal}; \mathbf{\Psi}^{SGWT}_{t_1}; \ldots ; \mathbf{\Psi}^{SGWT}_{t_K}]$$

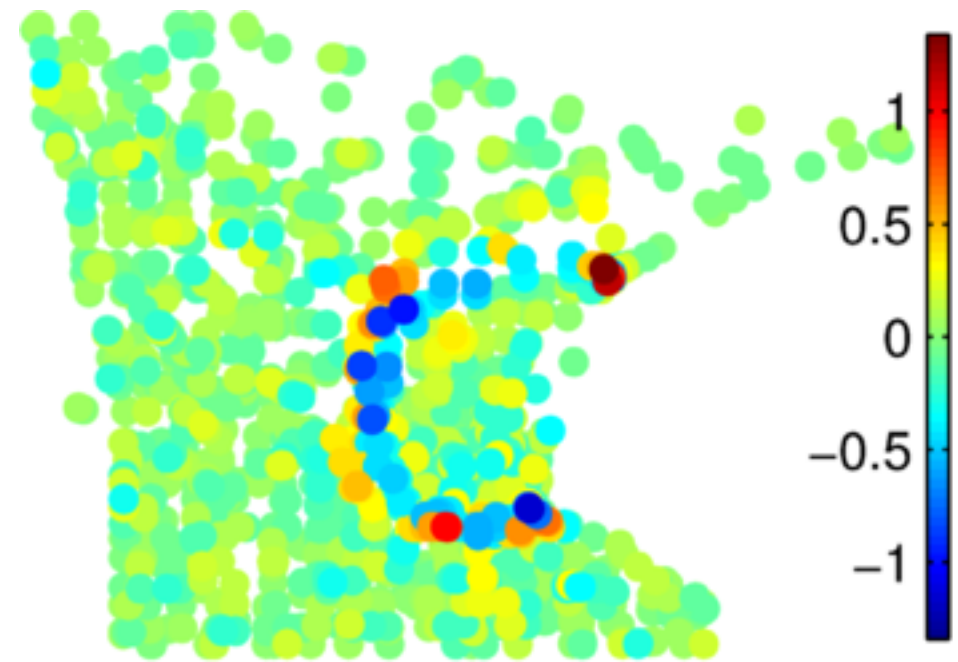
  - Dilations and translations of a band-pass kernel $\psi^{SGWT}_{t_k,i} := T_i \mathcal{D}_{t_k} \mathbf{g} = \widehat{\mathcal{D}_{t_k} g}(\mathcal{L}) \boldsymbol{\delta}_i$
  - Translation of a low-pass kernel $\psi^{SGWT}_{scal,i} := T_i \mathbf{h} = \hat{h}(\mathcal{L}) \boldsymbol{\delta}_i$

- Such transforms do not explicitly adapt to the data :(

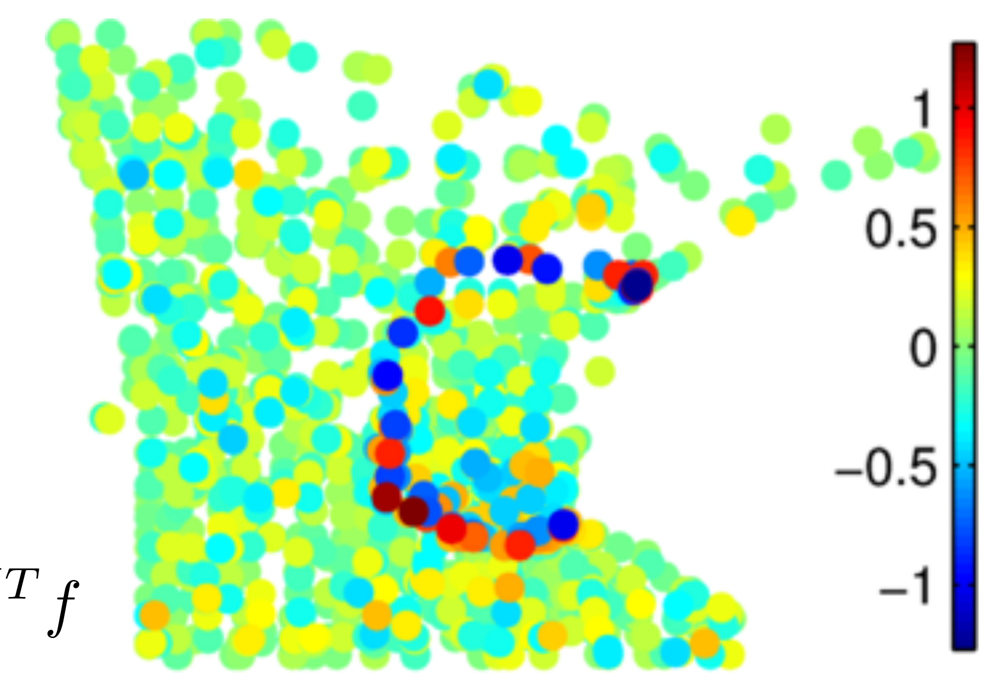
SGWT illustration



f



$\Psi_{t_2}^{SGWT} f$



$\Psi_{t_4}^{SGWT} f$

[Shuman:2013]

Data-adaptive representations

- Sparse graph signal representation

$$\mathbf{y} = \boldsymbol{\Phi}\mathbf{x} \quad \text{s.t.} \quad \|\mathbf{x}\|_0 \leq T_0$$

structure-based methods

$\boldsymbol{\Phi}$ defined
via mathematical structures
(*transforms*):
Fourier, wavelets...

not data-specific!

numerical methods

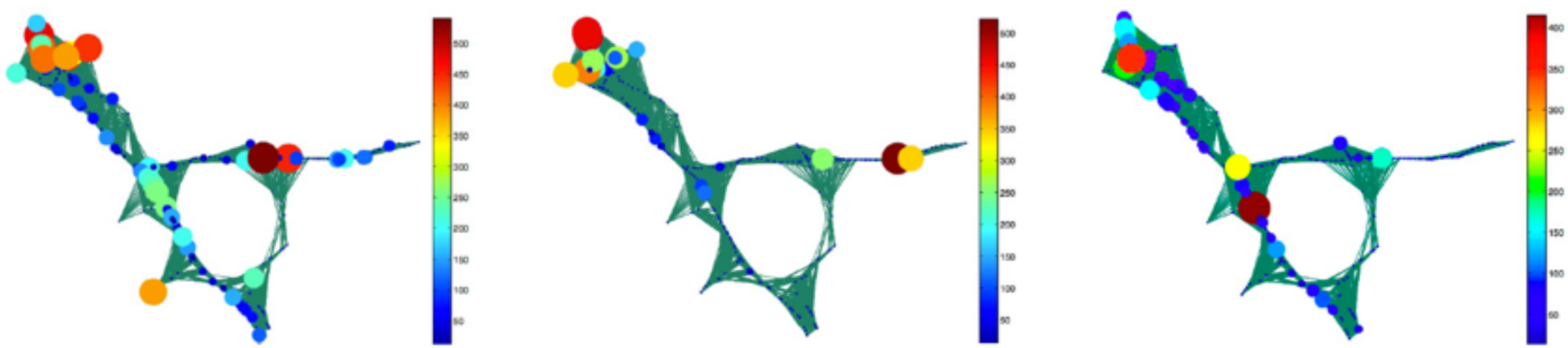
$\boldsymbol{\Phi}$ learned
via training samples
(*dictionary learning*):
sparse coding, MOD, K-SVD...

not graph-specific!

- We want to have an efficient structured representation $\boldsymbol{\Phi}$ that is adapted to data: *graph spectral dictionaries*

Dictionary for Graph Signals

- Our objective: meaningful graph signal representations that
  - ✓ reveal relevant structural properties of the graph signals/extract important features on graphs
  - ✓ sparsely represent different classes of signals on graphs

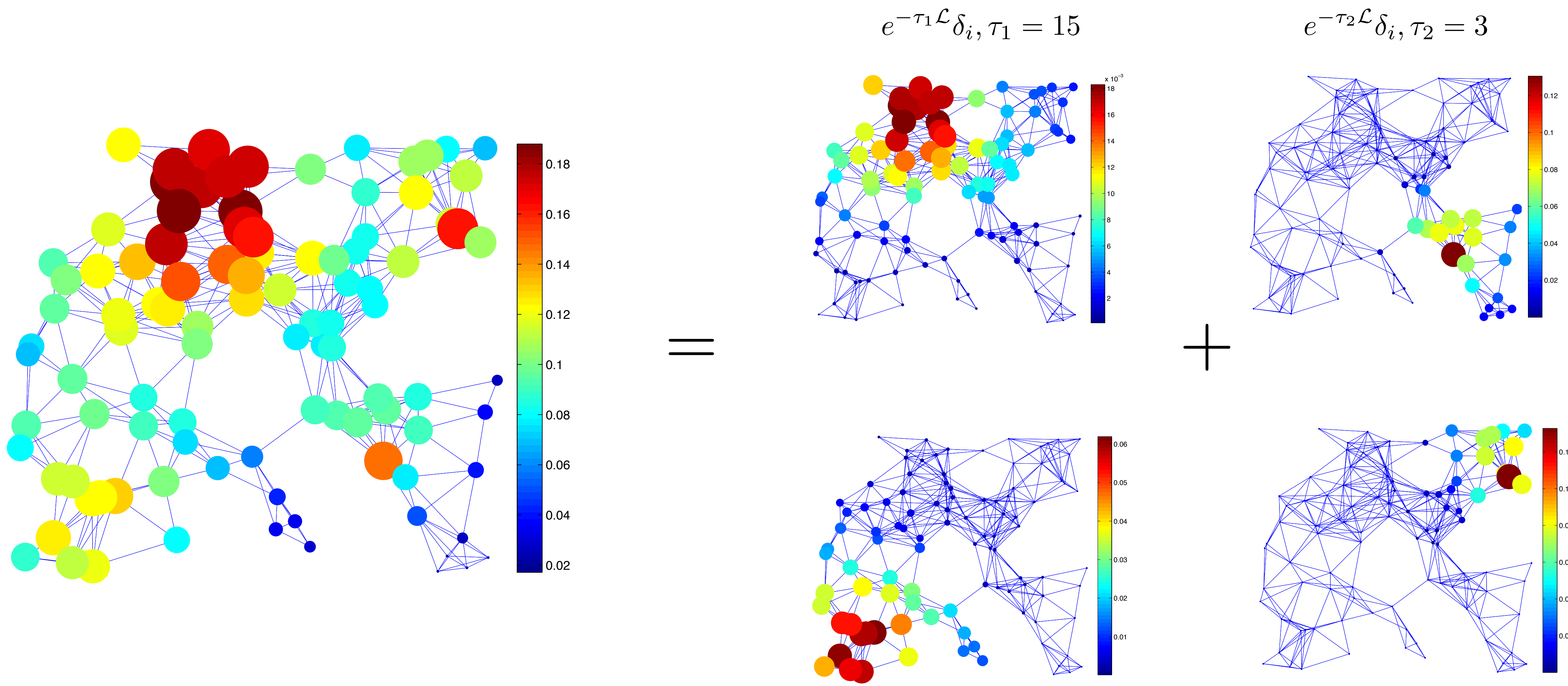


(a) Day 1 (b) Day 2 (c) Day 3

How can we define atoms on graphs?

Sparse signal model

- Graph signals can be approximated by a small number of localized components
  - e.g., multiple processes started at different vertices



Parametric graph atoms

- A set of generating kernels $\{\widehat{g_s}(\cdot)\}_{s=1,2,\ldots,S}$ represent the spectral characteristics of the signals
- The kernels are chosen to be smooth polynomial of degree K in order to form localized graph features

$$\hat{g}(\lambda_\ell) = \sum_{k=0}^{K} \alpha_k \lambda_\ell^k, \quad \ell = 0, \ldots, N-1$$

- A graph atom is the translation of the kernel to vertex *n*

$$T_n g = \sqrt{N}(g * \delta_n) = \sqrt{N} \sum_{\ell=0}^{N-1} \sum_{k=0}^{K} \alpha_k \lambda_\ell^k \chi_\ell^*(n) \chi_\ell = \sqrt{N} \sum_{k=0}^{K} \alpha_k (\mathcal{L}^k)_n$$

$\mathcal{L}$: normalized Laplacian, χ_ℓ : eigenvector

Dictionary Structure

- A parametric graph dictionary $\mathcal{D} = [\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_S]$ is a concatenation of S subdictionaries

- Each subdictionary is built on a specific kernel

$$\mathcal{D}_s = \widehat{g_s}(\mathcal{L}) = \chi \left(\sum_{k=0}^{K} \alpha_{sk} \Lambda^k \right) \chi^T = \sum_{k=0}^{K} \alpha_{sk} \mathcal{L}^k$$

  - Each atom (column of $\mathcal{D}_s$) corresponds to a K-hop localized pattern centered on a node of the graph, i.e.,

$$\frac{1}{\sqrt{N}} T_n g_s$$

Dictionary Learning Problem

- Learning consists in computing $\{\alpha_{sk}\}_{s=1,2,\dots,S;\ k=1,2,\dots,K}$
- Given a set of training signals $Y = [y_1, y_2, ..., y_M] \in \mathbb{R}^{N \times M}$ on the graph $\mathcal{G}$, solve

$$\underset{\alpha \in \mathbb{R}^{(K+1)S},\ X \in \mathbb{R}^{SN \times M}}{\operatorname{argmin}} \quad \{||Y - \mathcal{D}X||_F^2 + \mu\|\alpha\|_2^2\}$$

$$\text{subject to} \quad \|x_m\|_0 \leq T_0, \quad \forall m \in \{1, ..., M\},$$

$$\mathcal{D}_s = \sum_{k=0}^{K} \alpha_{sk} \mathcal{L}^k, \quad \forall s \in \{1, 2, ..., S\}$$

$$0 \preceq \mathcal{D}_s \preceq c, \quad \forall s \in \{1, 2, ..., S\}$$

$$(c - \epsilon_1)I \preceq \sum_{s=1}^{S} \mathcal{D}_s \preceq (c + \epsilon_2)I,$$

The spectral constraints guarantee that:
1. The learned kernels cover the whole spectrum
2. The dictionary is a frame

Alternating optimisation

Algorithm 1 Parametric Dictionary Learning on Graphs

Input: Signal set Y, initial dictionary $\mathcal{D}^{(0)}$, target signal sparsity T_0, polynomial degree K, number of subdictionaries S, number of iterations $iter$
Output: Sparse signal representations X, polynomial coefficients α
Initialization: $\mathcal{D} = \mathcal{D}^{(0)}$
for $i = 1, 2, ..., iter$ **do:**
    Sparse Approximation Step:
        (a) Scale each atom in $\mathcal{D}$ to a unit norm
        (b) Update X using Sparse Coding
        (c) Rescale X, $\mathcal{D}$ to recover the polynomial structure
    Dictionary Update Step:
        Compute the polynomial coefficients α and update the dictionary
end for

Properties of the dictionary

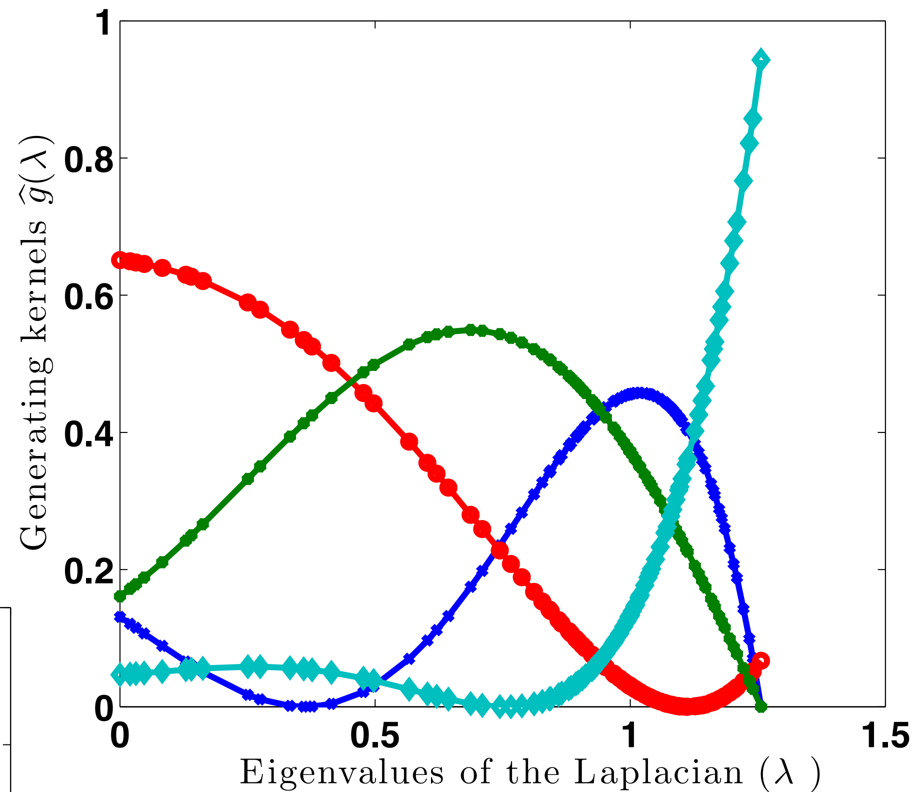
- By construction, the dictionary is a frame
- The coherence depends on the graph

$$\phi \leq \max_{n \neq n', s, s'} \frac{\nu(\sum_{\ell=0}^{N-1} |\widehat{g_s}(\ell)\widehat{g_{s'}}(\ell)|^2)^{1/2} \|deg\|^2}{|\widehat{g_s}(\lambda_0)||\widehat{g'_s}(\lambda_0)|\sqrt{deg_n}\sqrt{deg_{n'}}}$$

- Parametric structure: easy dictionary description
  - it has only $(K+1)S$ parameters
- Polynomial form: efficient implementation, esp. when the graph is sparse
  - Both forward and adjoint operators can be efficiently applied

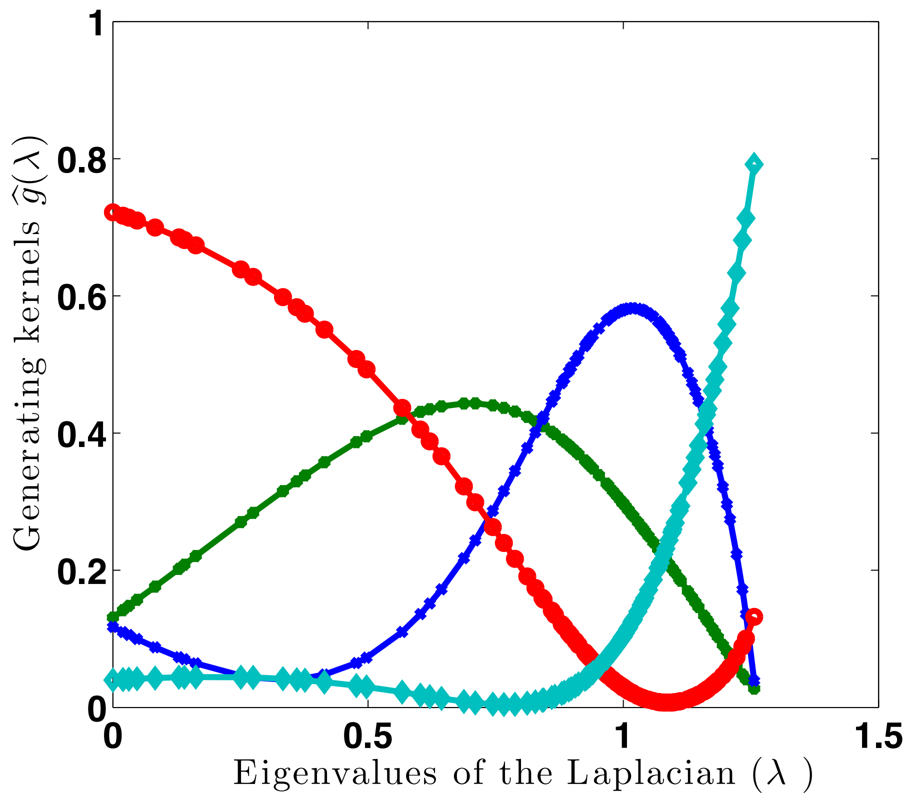
$$\mathcal{D}^T y = \sum_{s=1}^{S} \sum_{k=0}^{K} \alpha_{sk} \mathcal{L}^k y \qquad \mathcal{D}\mathcal{D}^T y = \sum_{s=1}^{S} \widehat{g_s}^2(\mathcal{L}) y$$

Recovery on synthetic data

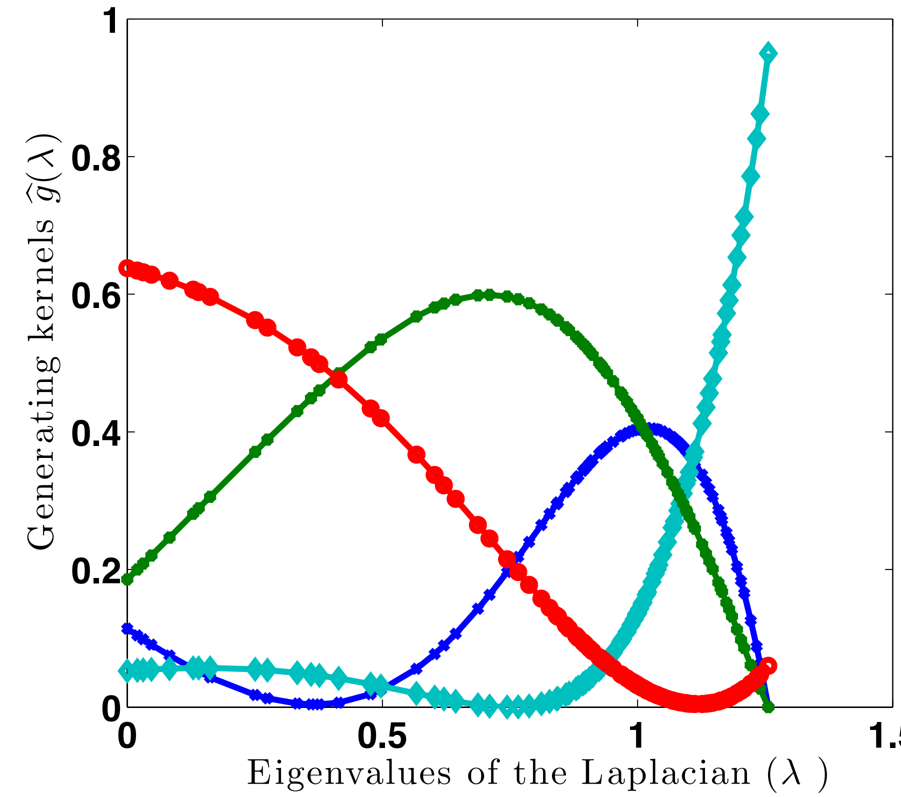


Original polynomial kernels

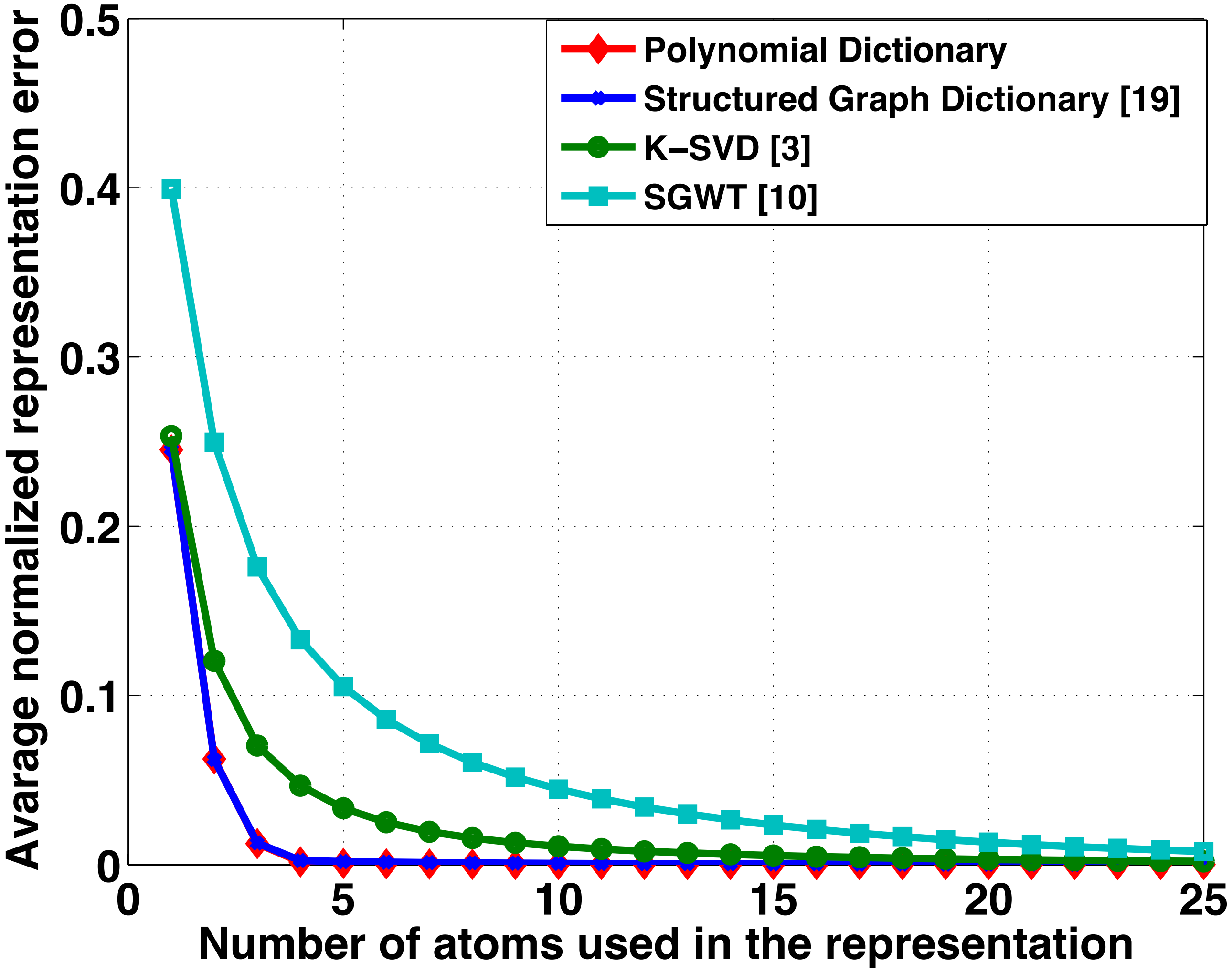
Learned kernels (M=400)



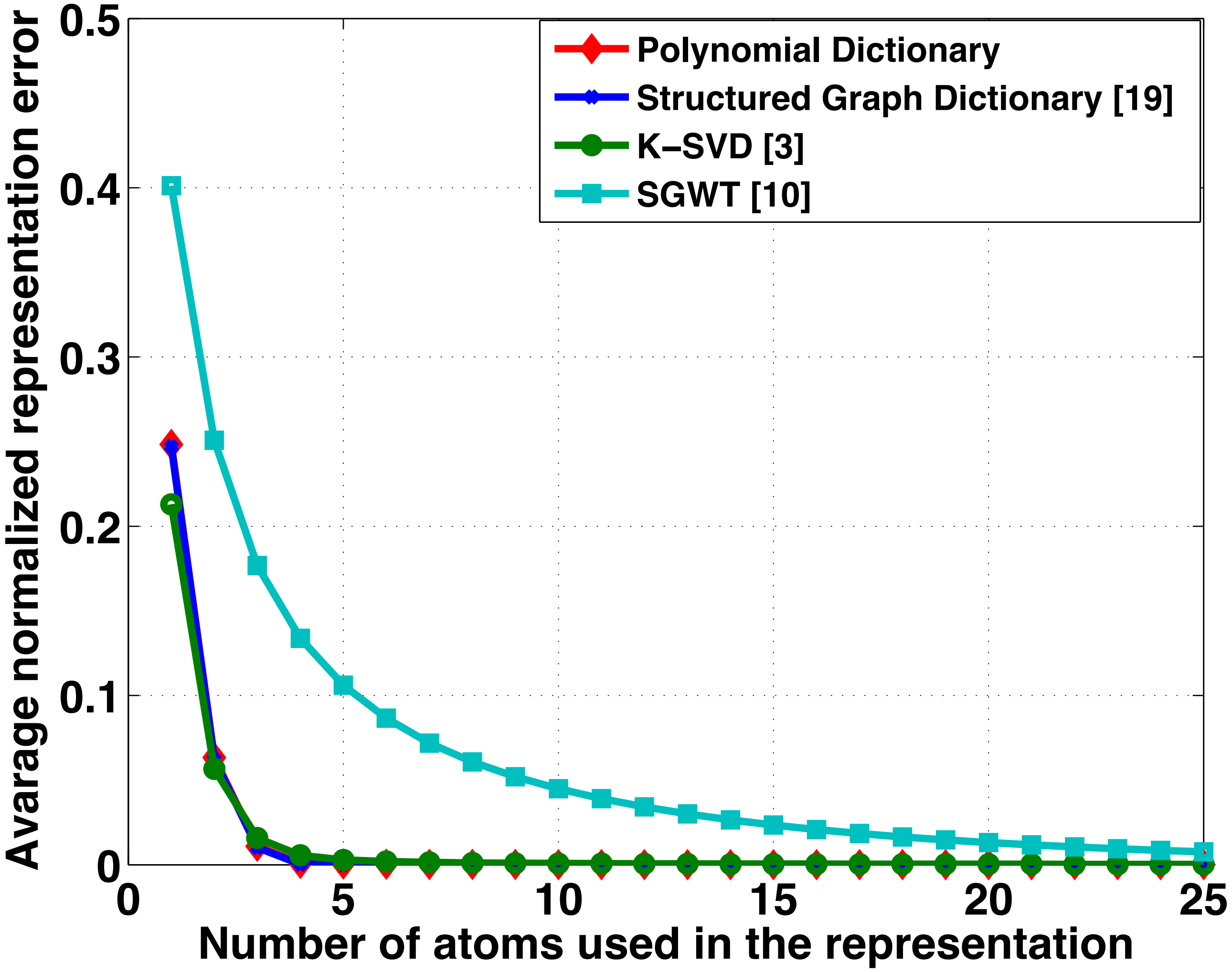
Learned kernels (M=2000)



Approximation on synthetic data



M=400

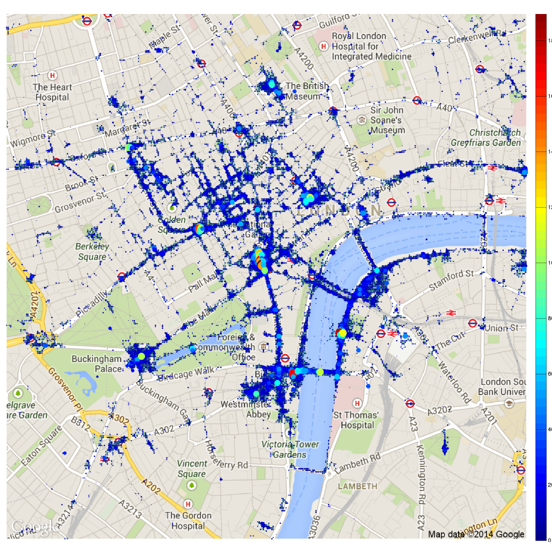


M=2000

Real World Datasets

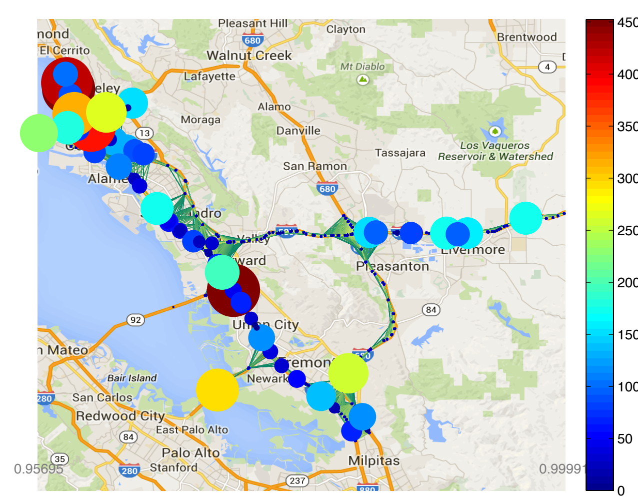
Flickr

- **Nodes:** 245 vertices in the Trafalgar Square (London), each representing a geographical area
- **Edges**: Assign edge when distance < 30m
- **Graph signals:** Daily number of distinct users that took photos between Jan. 2010 and June 2012



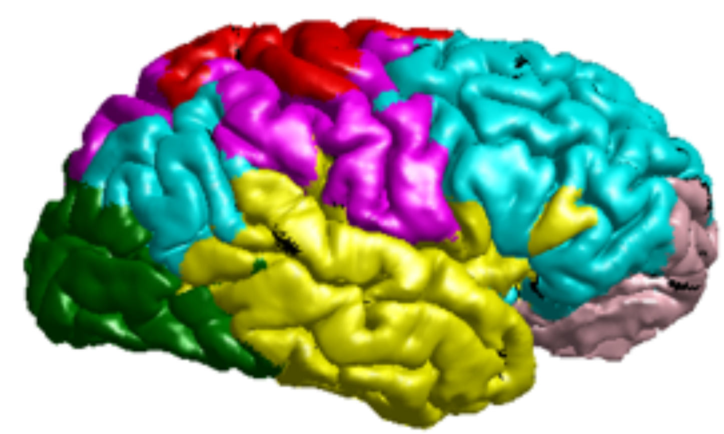
Traffic

- **Nodes:** 439 detector stations in Alameda County, CA
- **Edges**: Assign edge when distance < 13km
- **Graph signals:** Daily number of bottlenecks (in minutes) between Jan. 2007 to May. 2013

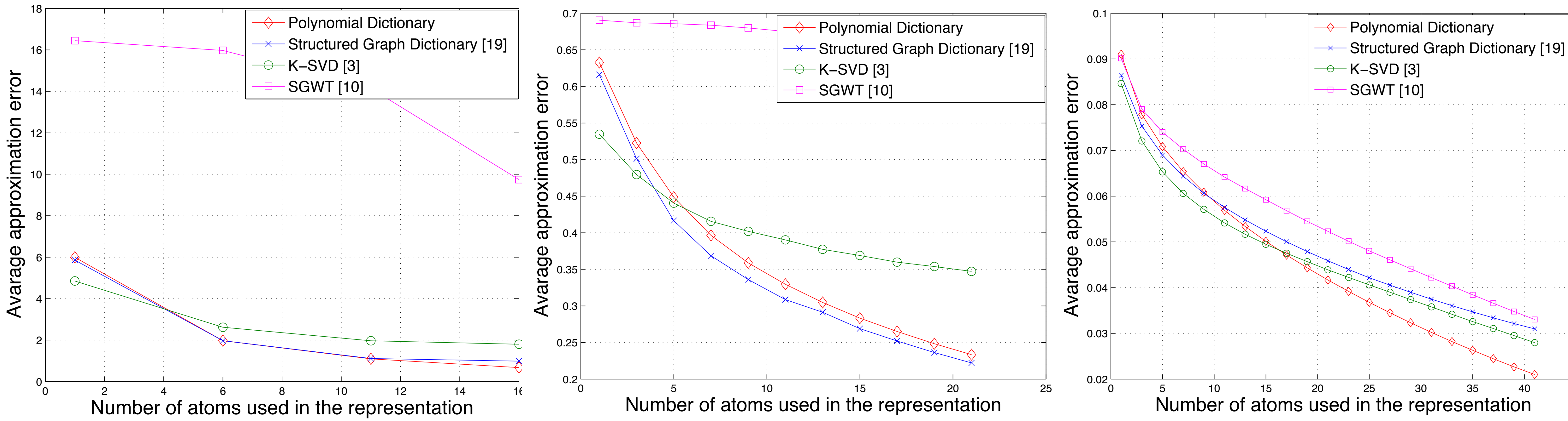


Brain

- **Nodes:** 88 brain regions of contiguous voxels
- **Edges**: Assign edge if anatomical distance < 40 mm
- **Graph signals:** fMRI signals acquired on five subjects, in different states, 1290 signals per subject



Approximation performance

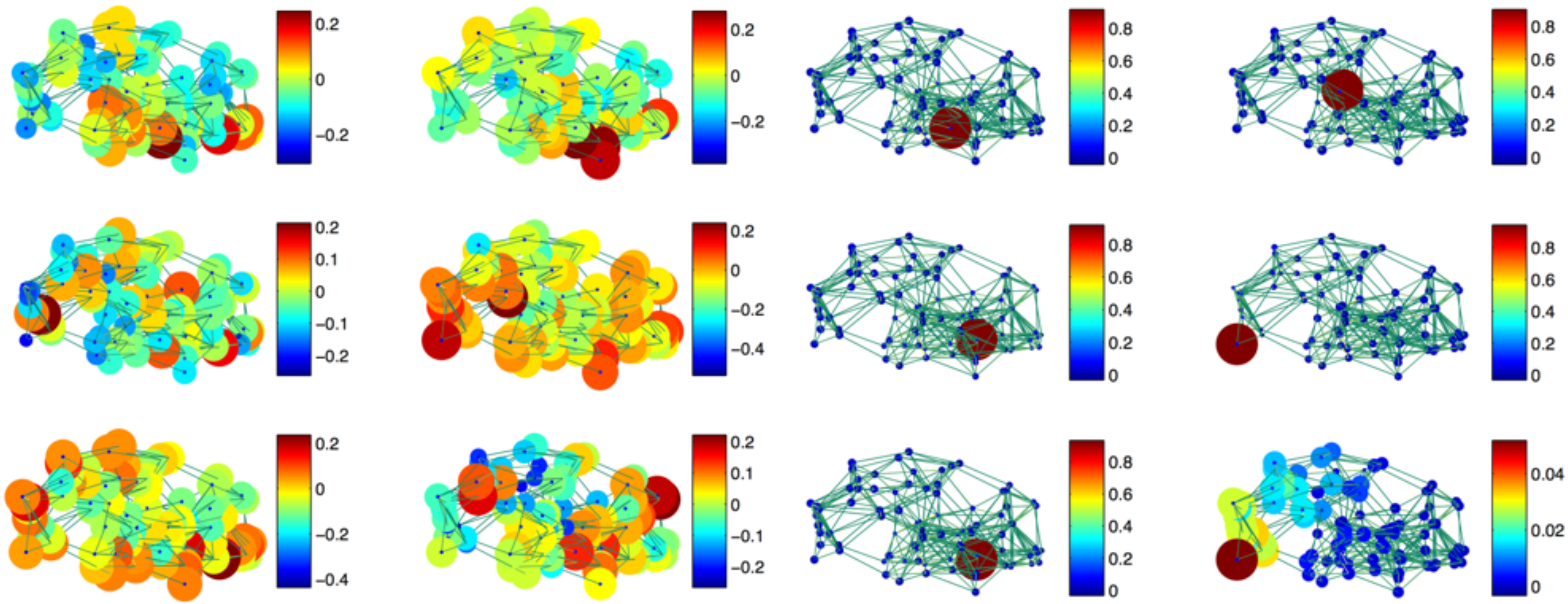


(a) Flickr dataset **(b) Traffic dataset** **(c) Brain dataset**

- As the sparsity level increases, the localisation property becomes beneficial
- The polynomial dictionary is able to learn local patterns in areas of the graph that do not show up in the training signals

Examples of Learned Atoms

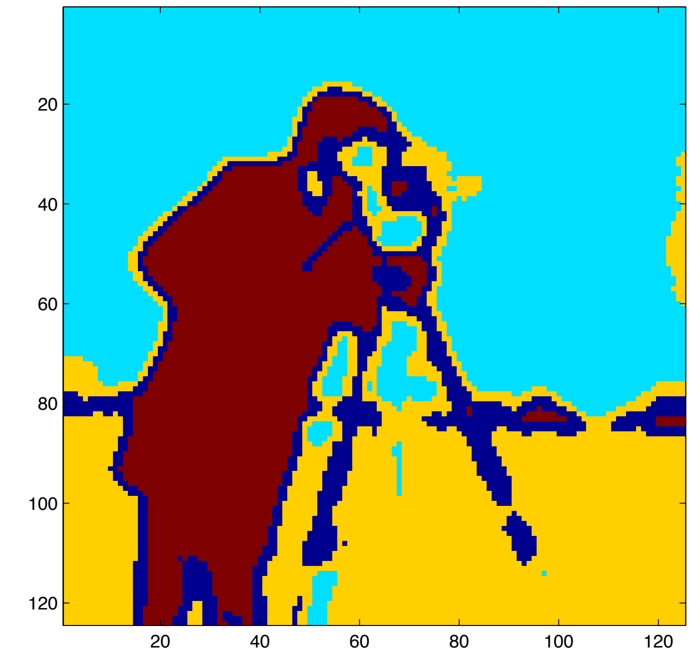
- Most common atoms in OMP expansions

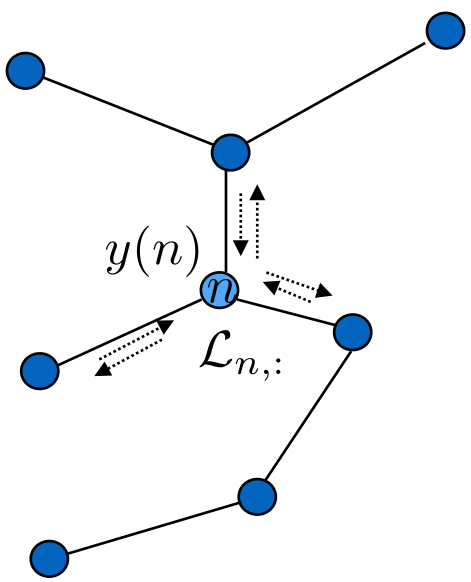


K-SVD Dictionary **Polynomial Graph Dictionary**

Applications of graph dictionaries

- Graph dictionaries apply to many sparse problems
  - sparsity prior on graphs
  - helpful when smooth priors are insufficient
- Graph dictionaries also define features on graphs
  - learning or clustering applications
- By construction, spectral graph dictionaries lead to effective implementations
  - distributed processing applications in networks

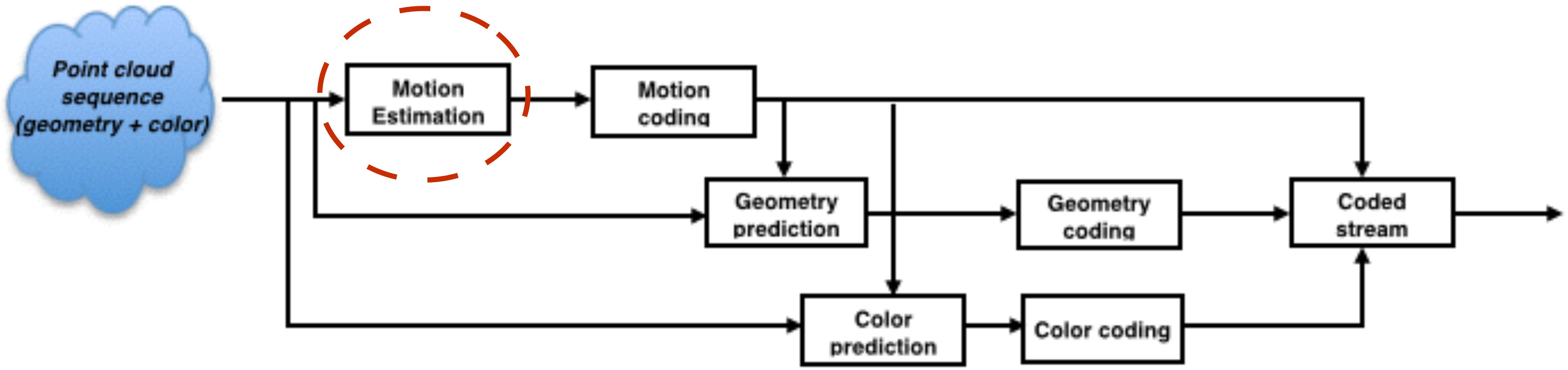




3D Point Cloud Sequences

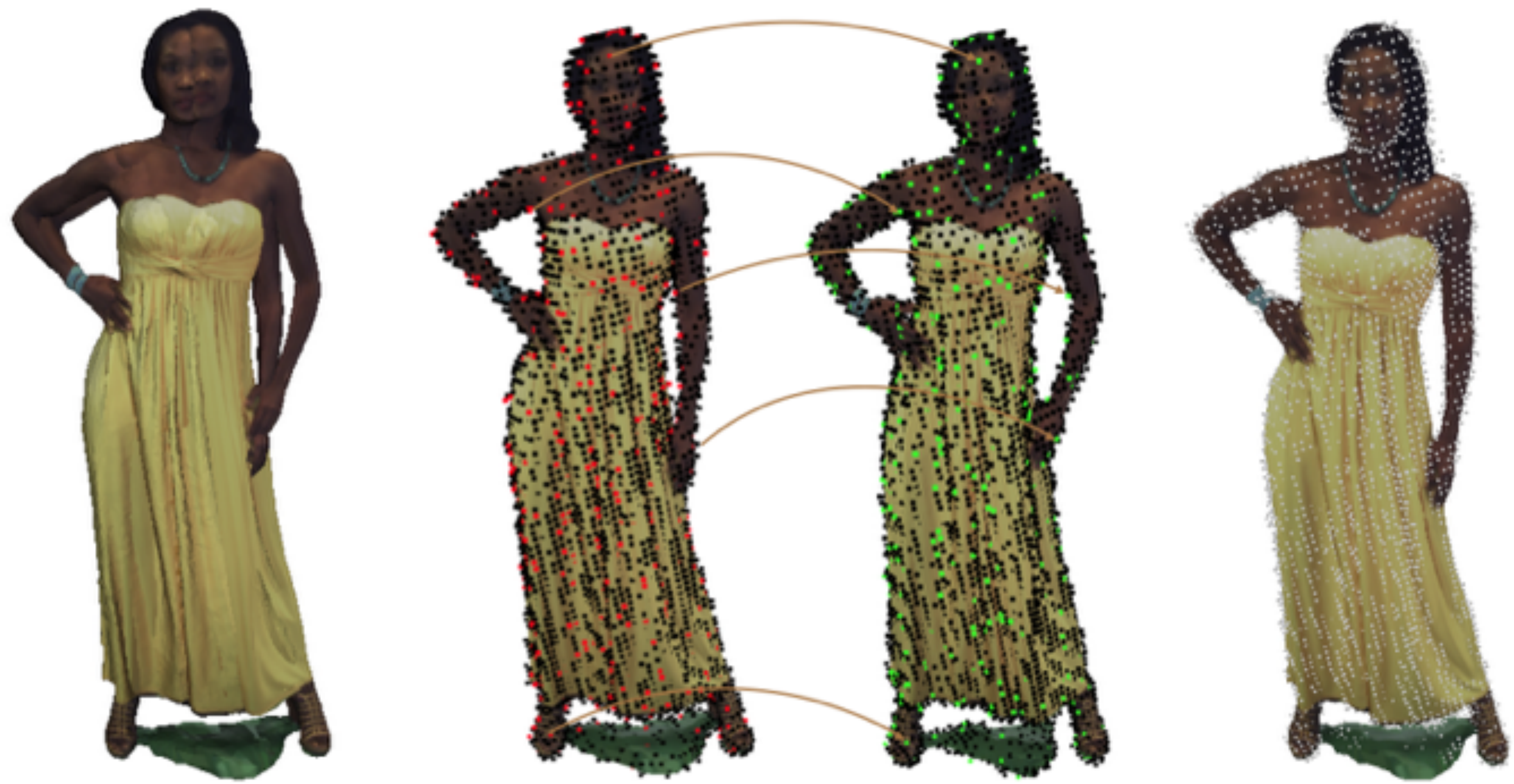
- No explicit spatio-temporal geometry structure
  - Frames have different number of points
  - No association between points over time
- Graph localised features can be used to match frames

Graph-based Motion Estimation



Graph SP representation used for motion estimation and compensation, and eventually predictive coding

Motion Compensation - Example



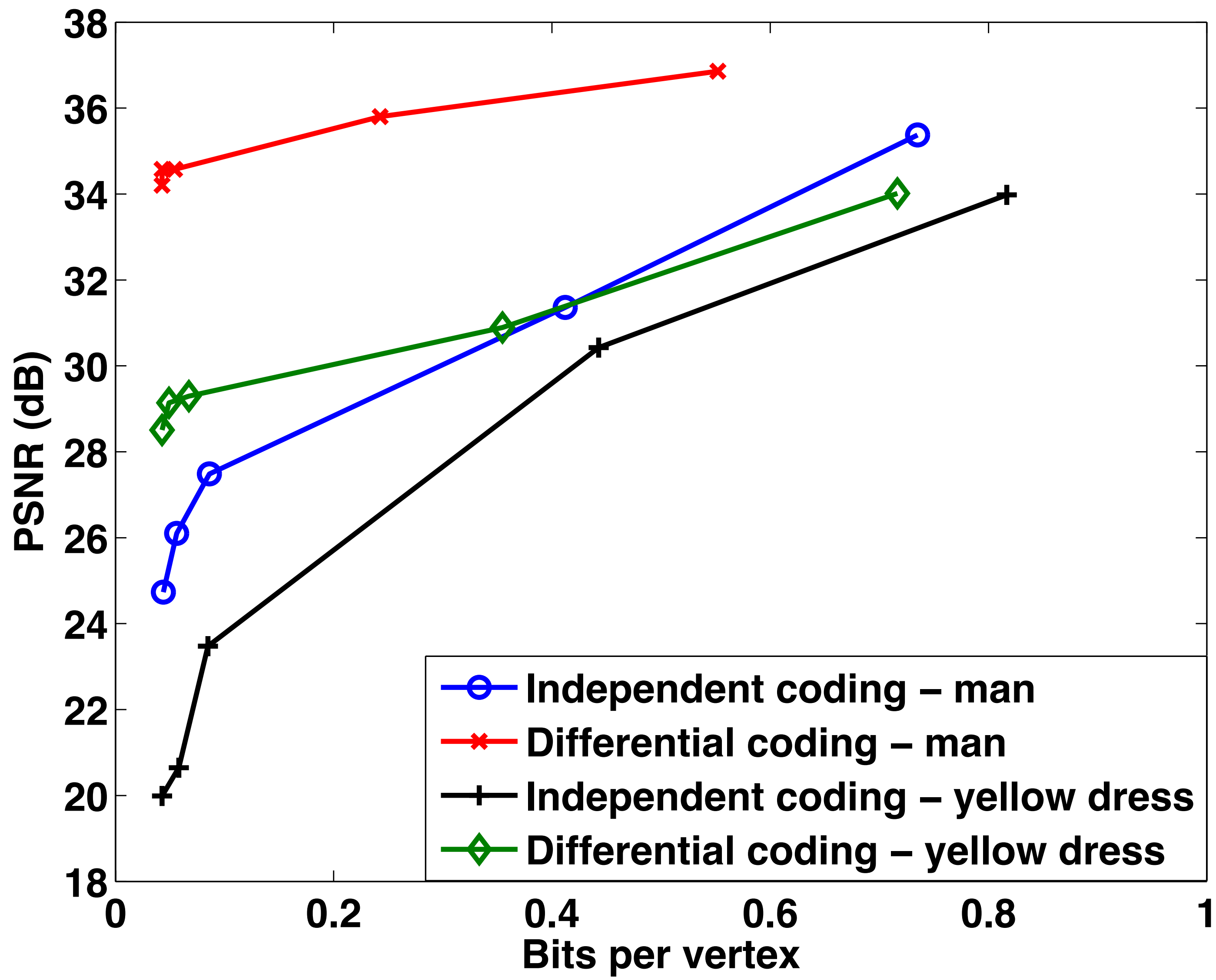
(a) reference + target frame

(b) sparse correspondences between frames

(c) motion compensated reference frame + target frame

- The sparse set of matching vertices are accurate and well-distributed in space

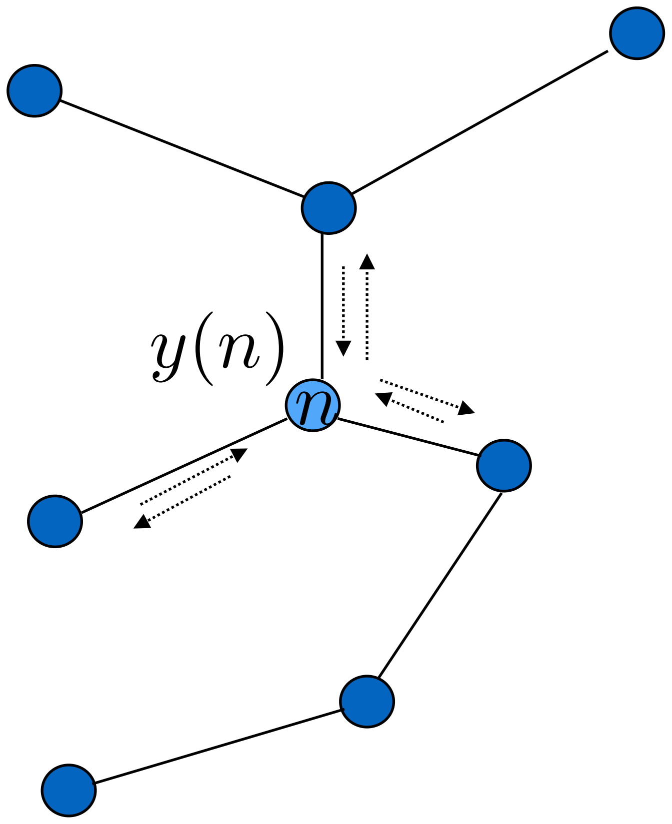
3D Color Compression Results



Compression of 10 frames in each sequence using predictive coding [Zhang:2014]

Distributed processing

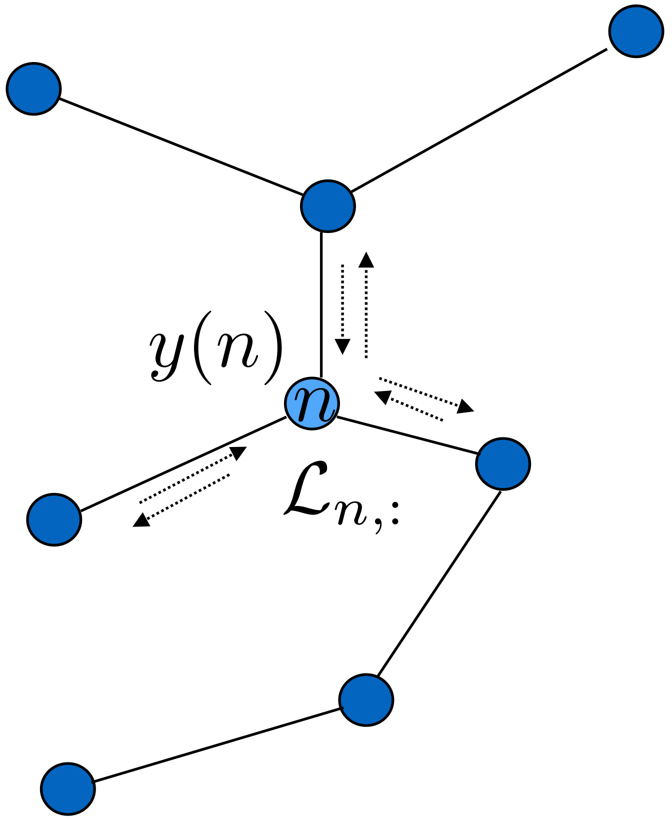
- Graph signal: function on a network
  - e.g., measurement in a wireless sensor network
- Signal processing tasks
  - denoising, reconstruction, inference
- Communication constraints
  - centralised processing is not possible
  - no node fully knows the signal
  - only local communication



Processing on graphs: denoising

- Denoising (LASSO) problem

$$x^* = \min_x \|y - \mathcal{D}x\|_2^2 + \kappa\ \|x\|_1$$



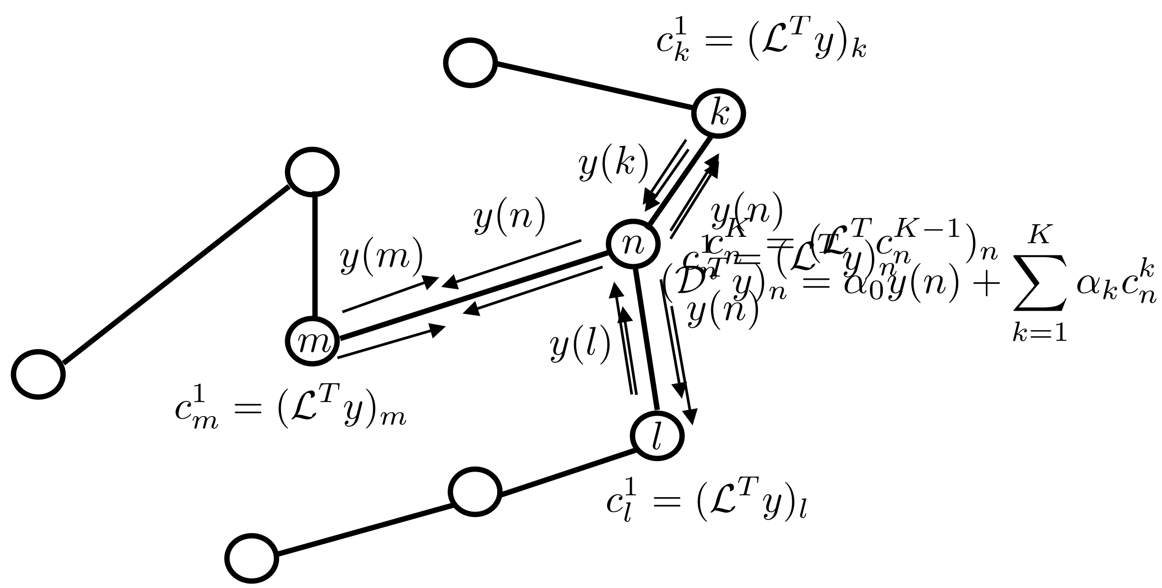
- Iterative soft thresholding solution

$$\mathcal{S}_{\kappa\tau} = \begin{cases} 0 & \text{if } |z| \leq \mu\tau \\ z - \text{sgn}(z)\kappa\tau & \text{otherwise} \end{cases}$$

$$x^t = \mathcal{S}_{\kappa\tau}\left(x^{(t-1)} + 2\tau\mathcal{D}^T\ \left(y - \mathcal{D}x^{(t-1)}\right)\right),\ t = 1, 2, \ldots$$

- Distributed solution feasible as the dictionary-based operators can be distributed!

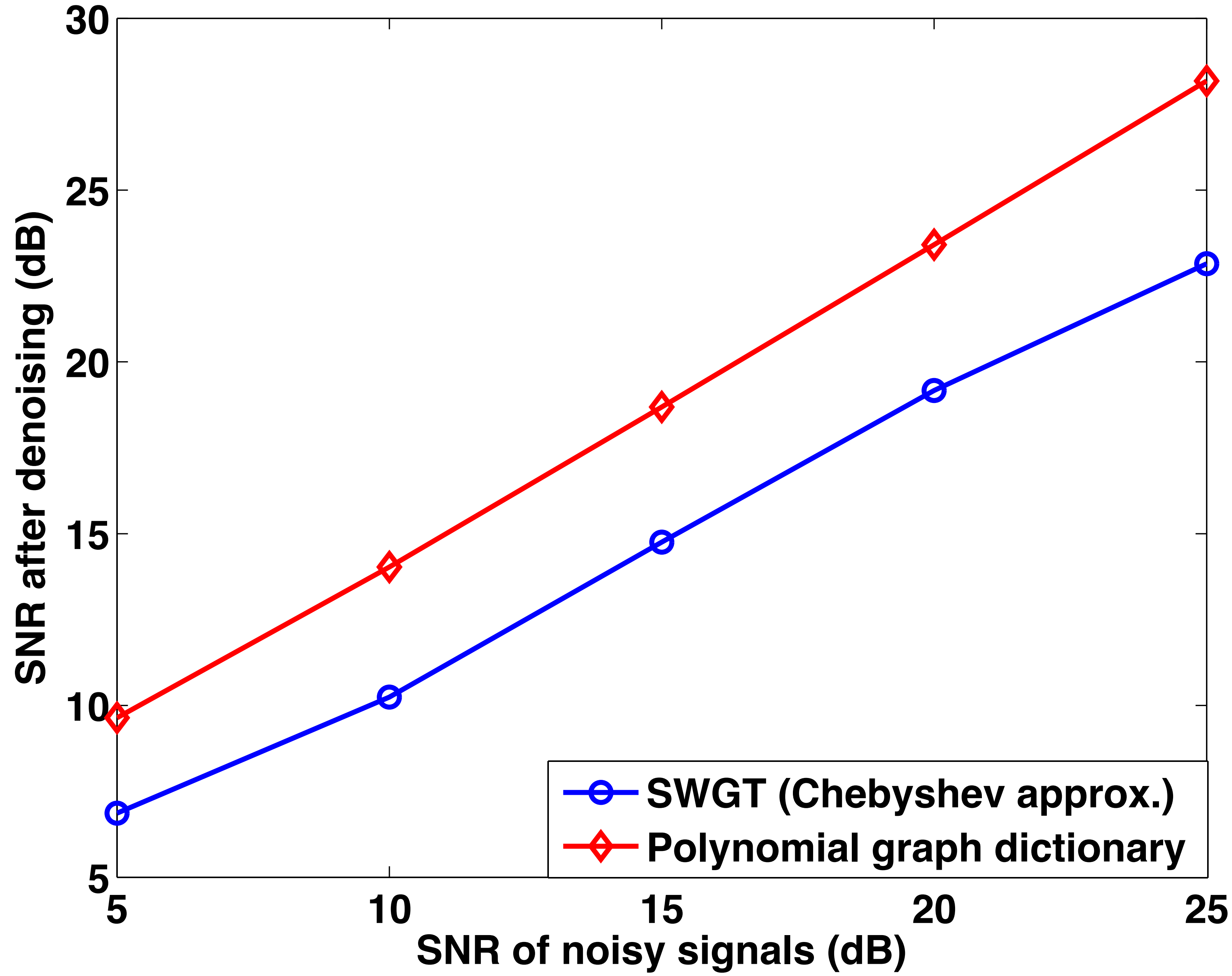
Illustration: adjoint operator



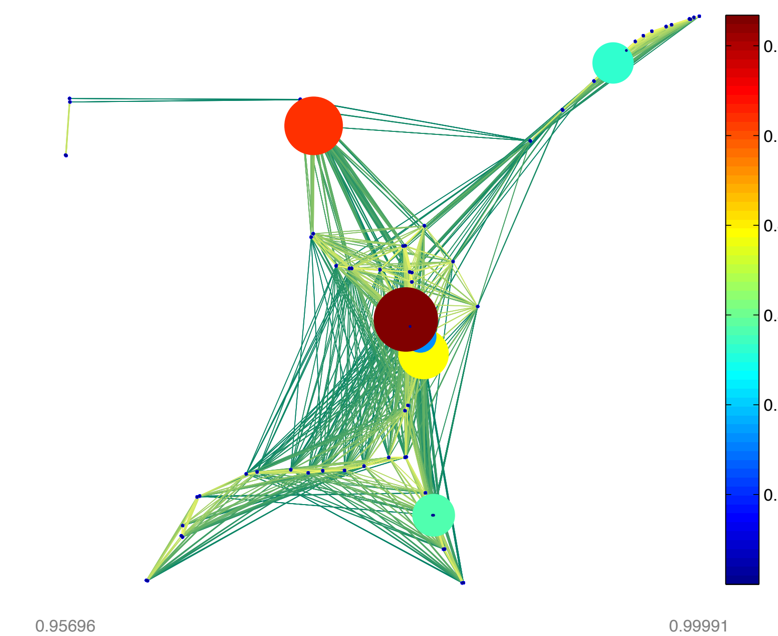
Distributed computation of $\mathcal{D}^T y$

Similar operations for the computation of $\mathcal{D}x$ and $\mathcal{D}\mathcal{D}^T y$

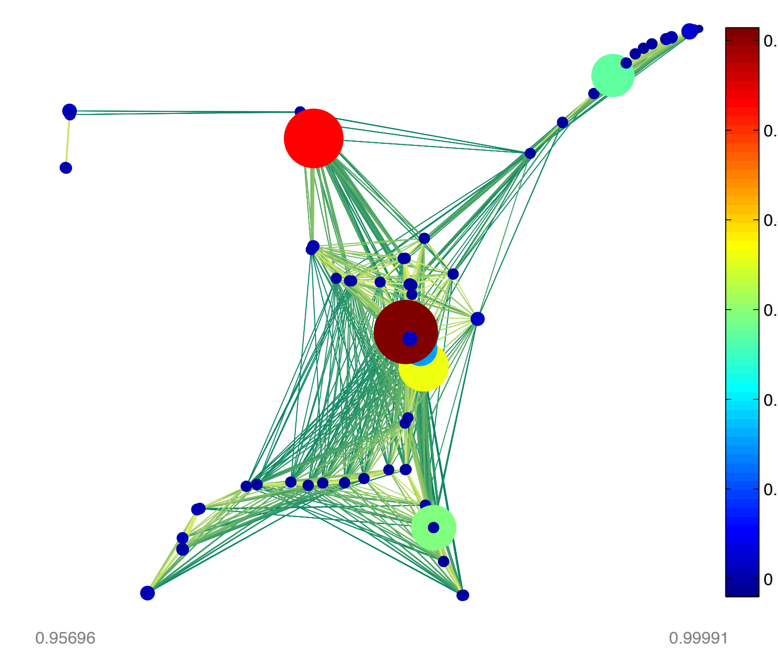
Denoising experiments



Distributed denoising with 100 ISTA iterations

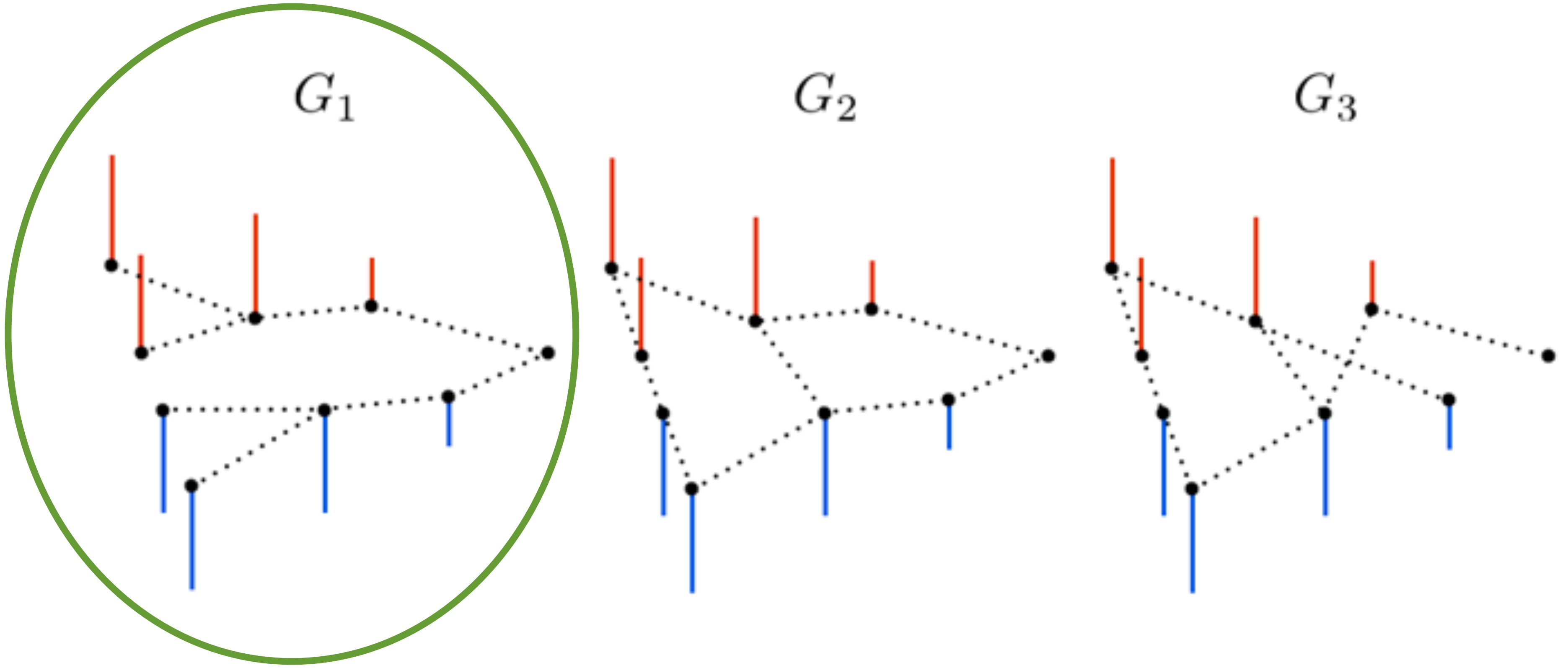


Clean traffic bottleneck signal



Denoised traffic bottleneck signal [24 dB]

The graph learning problem



Challenge:

- How to define models of relationships between signals and graph?
- How to learn graphs that enforce desirable properties of graph signals?

Factor analysis framework

- Graph signal observations could be represented explained by unobserved latent variables

$$x = \chi h + u_x + \epsilon$$

- x: observation
- χ: representation matrix (Eigenvector matrix of graph Laplacian)
- h: latent variable
- u_x: mean
- ϵ: Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2 I)$

Statistical model

$$x = \chi h + u_x + \epsilon$$

- Gaussian prior on latent variable $h \sim \mathcal{N}(0, \Lambda^\dagger)$

  with $\Lambda^\dagger$ the pseudo-inverse of the eigenvalue matrix Λ of the graph Laplacian L

- Probabilities given as

$$p(x|h) \sim \mathcal{N}(\chi h + u_x, \sigma_\epsilon^2 I_n),$$

$$p(x) \sim \mathcal{N}(u_x, L^\dagger + \sigma_\epsilon^2 I_n) \quad \text{with} \quad L^\dagger = \chi \Lambda^\dagger \chi^T$$

- χ : eigenvector of covariance of x

$$L^\dagger + \sigma_\epsilon^2 I_n = \chi(\Lambda^\dagger + \sigma_\epsilon^2 I_n)\chi^T$$

'Classical' regularisation problem

If $u_x = 0$

$$h_{\mathrm{MAP}}(x) := \arg\max_h p(h|x) = \arg\max_h p(x|h)p(h)$$

$$h_{\mathrm{MAP}}(x) = \arg\min_h \left(-\log\, p_E(x-\chi h) - \log\, p_H(h)\right)$$

$$h_{\mathrm{MAP}}(x) = \arg\min_h \left(-\log\, e^{-(x-\chi h)^T(x-\chi h)} - \alpha\, \log\, e^{-h^T \Lambda h}\right)$$

$$h_{\mathrm{MAP}}(x) = \arg\min_h ||x-\chi h||_2^2 + \alpha\, h^T \Lambda h$$

smoothness term

$$h^T \Lambda h = (\chi^T x)^T \Lambda \chi^T x = x^T \chi \Lambda \chi^T x = x^T L x$$

Graph learning problem

- When the graph is unknown,

$$\min_{\chi,\Lambda,h} ||x - \chi h||_2^2 + \alpha\ h^T \Lambda h$$

$$\min_{L,y} ||x - y||_2^2 + \alpha\ y^T L y.$$

$$\begin{aligned} \min_{L \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{n \times p}} \quad & ||X - Y||_F^2 + \alpha\ \mathrm{tr}(Y^T L Y) + \beta ||L||_F^2, \\ \text{s.t.} \quad & \mathrm{tr}(L) = n, \\ & L_{ij} = L_{ji} \leq 0,\ i \neq j, \\ & L \cdot \mathbf{1} = \mathbf{0}, \end{aligned}$$

Graph learning algorithm

Alternating optimisation

$$\text{Step 1:} \quad \arg\min_{L} \alpha \, tr(Y^T L Y) + \beta ||L||_F^2$$

$$\text{s.t.} \quad tr(L) = n, \quad L = L^T, \quad off(L) \leq 0, \quad L\mathbf{1} = \mathbf{0}$$

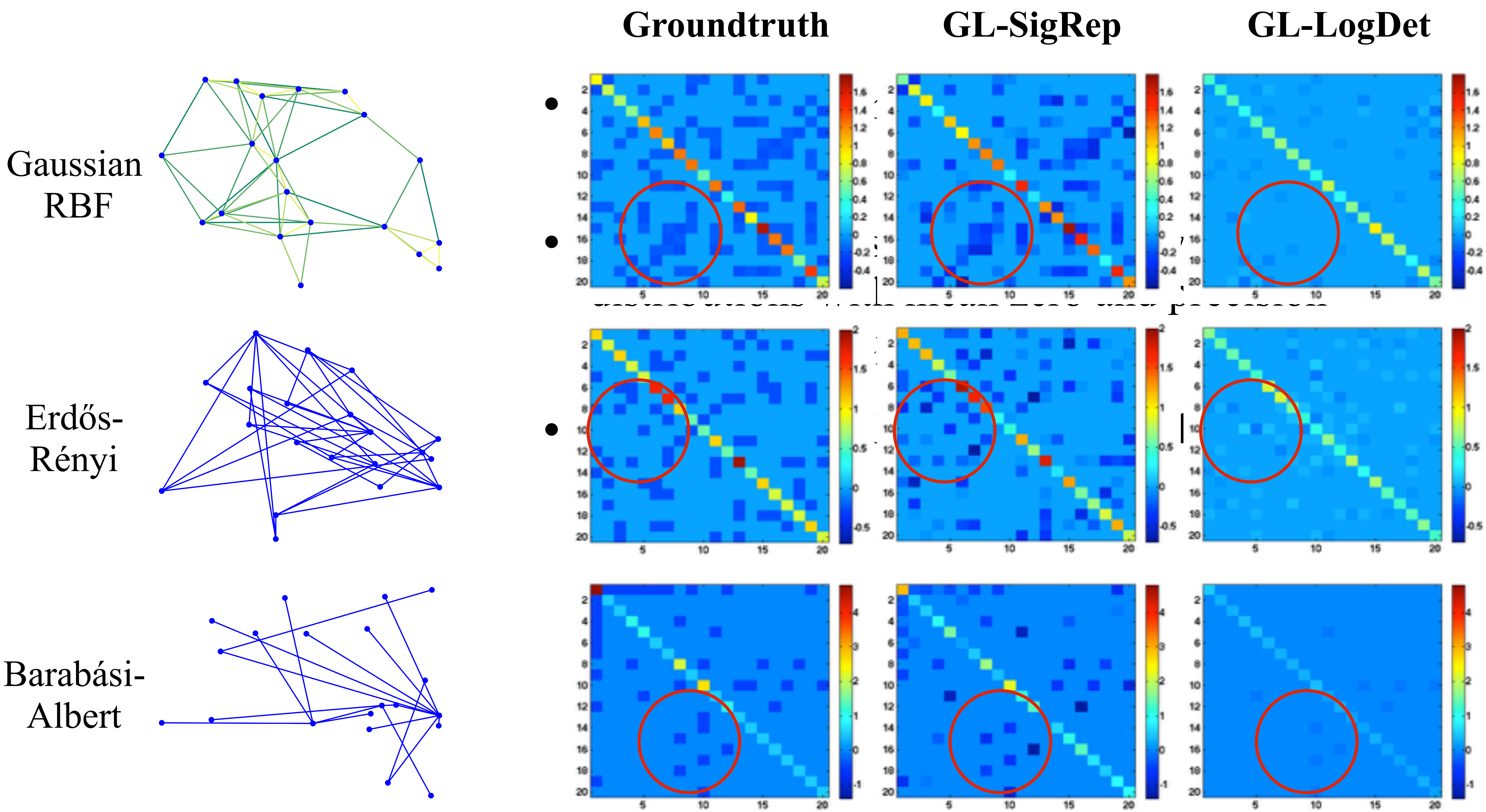
$$\text{Step 2:} \quad \arg\min_{Y} ||X - Y||_F^2 + \alpha \, tr(Y^T L Y)$$

Both steps are convex optimization problems :)

Algorithm 6 Graph Learning for Smooth Signal Representation (**GL-SigRep**)

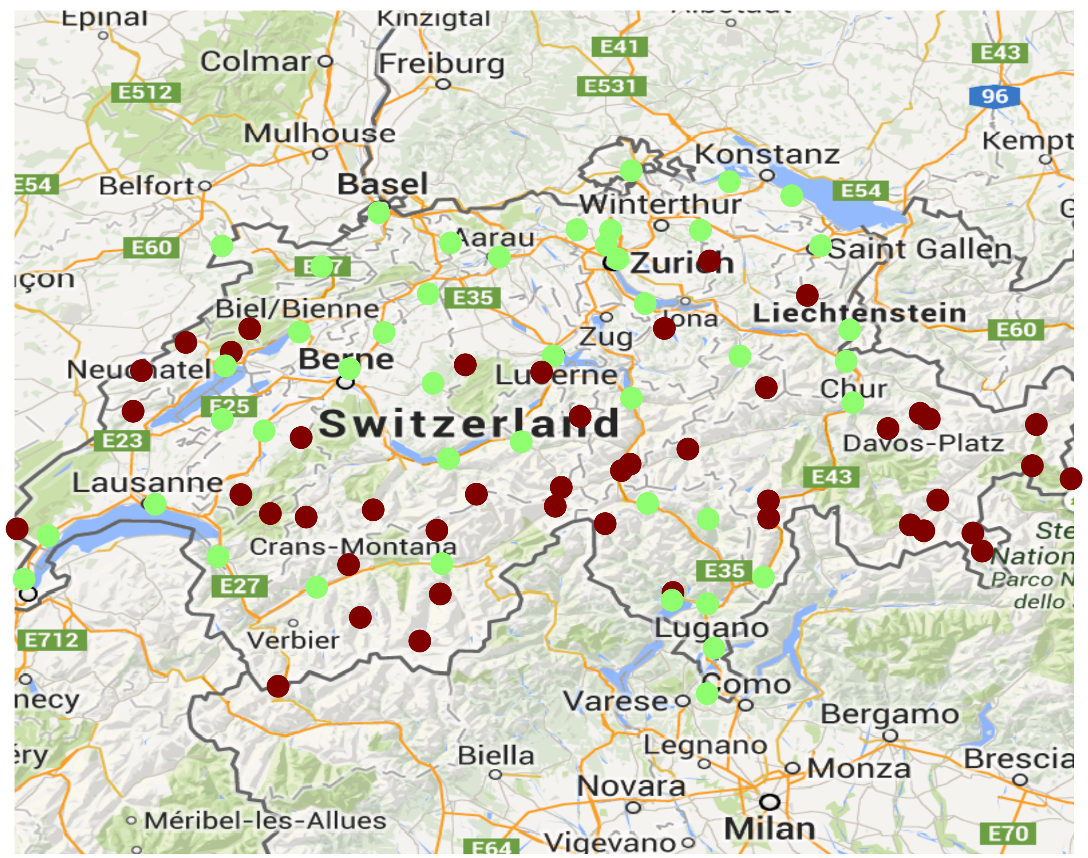
1: **Input:** Input signal X, number of iterations $iter$, α, β
2: **Output:** Output signal Y, graph Laplacian L
3: **Initialization:** $Y = X$
4: **for** $t = 1, 2, ..., iter$ **do:**
5: **Step to update Graph Laplacian** L**:**
6: Solve the optimization problem of Eq. (6.24) to update L.
7: **Step to update** Y**:**
8: Solve the optimization problem of Eq. (6.25) to update Y.
9: **end for**
10: $L = L^{iter}, Y = Y^{iter}$.

Results: Synthetic data



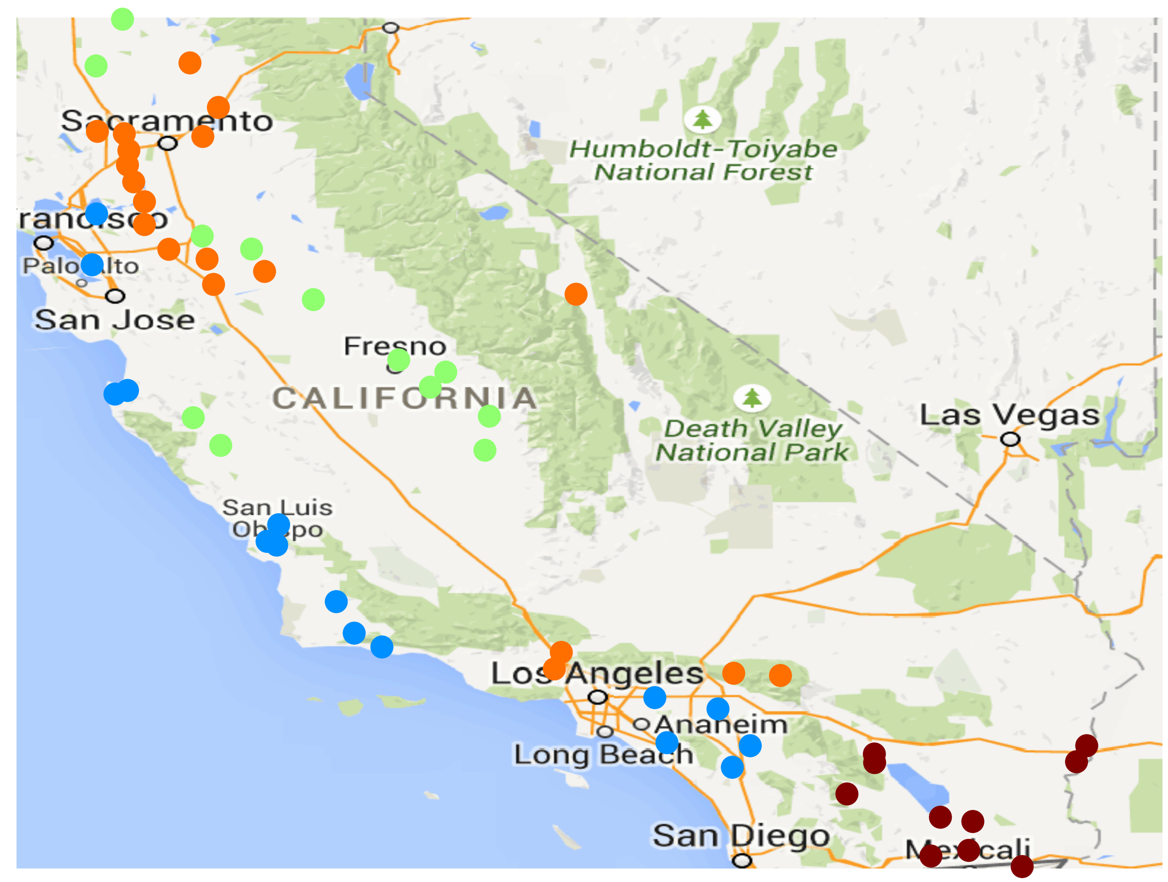
Analysis of meteorological data

- The learned graph can be used for partitioning entities into several clusters, e.g., [Ng, NIPS'01]



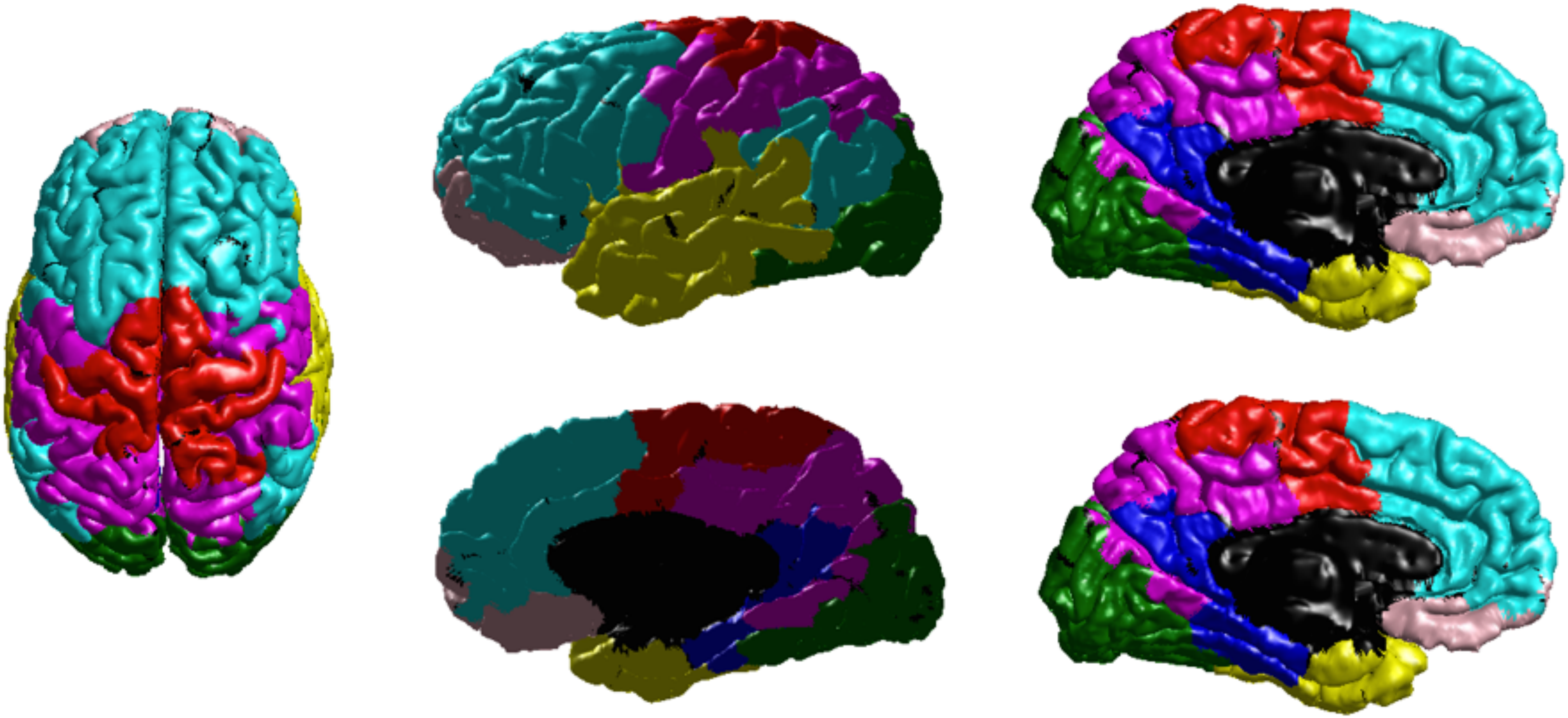
Mountain regions (Jura, Alps)

Flat regions and valleys



4 ETo zones

Learning brain networks



Summary

- Graph Signal Processing: joint consideration of the signal and the structure

- Structured adaptive representations lead to computationally effective operators

- In general, the graph is not known!

Many open challenges :)

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