Multi-legs, Superfluids & Semiclassics

Riccardo Rattazzi, EPFL

with
Gil Badel, Gabriel Cuomo, Alexander Monin, arXiv:1909.01269, arXiv:1911.08505

Quantum Mechanics is an astonishing fact

But perhaps more astonishing is Classical Physics

and

how easily Quantum Mechanics disappears behind it

REVIEWS OF

MODERN PHYSICS

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Space-Time Approach to Non-Relativistic Quantum Mechanics

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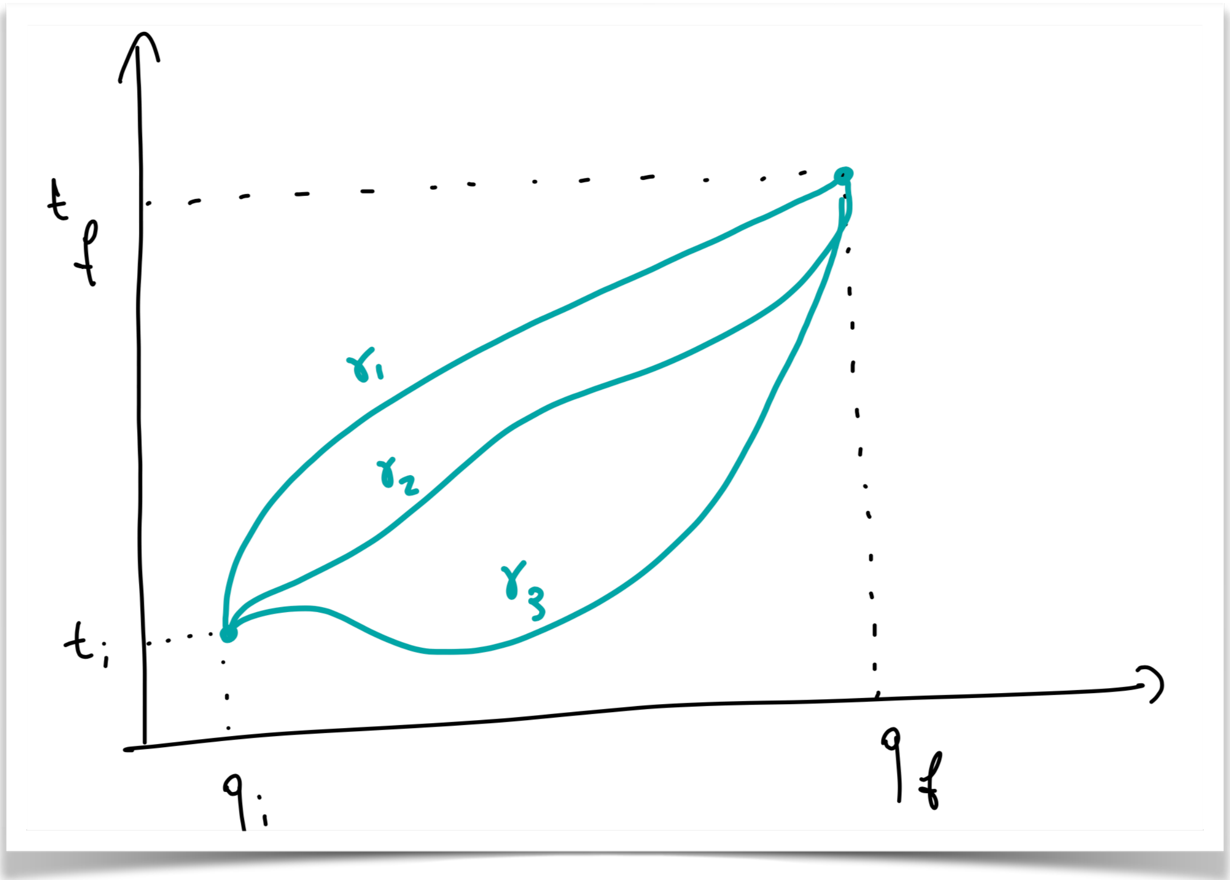
Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

A crucial perspective is offered by the path integral



$$\mathcal{A}(q_i \to q_f) = \int D\gamma\, e^{i\frac{S[\gamma]}{\hbar}}$$

Classical regime: $\mathcal{A}$ dominated by saddle point $\gamma_{c\ell}$

$$\mathcal{A} = e^{i[\frac{S(\gamma_{c\ell})}{\hbar} + \Gamma_1(\gamma_{c\ell}) + \hbar\Gamma_2(\gamma_{c\ell}) + \dots]}$$

Normally this corresponds to $S(\gamma_{c\ell}) \gg \hbar$

Could Lagrange have fancied all this?

Classical Mechanics ⟷ $\delta S = 0$

- remarkable and mysterious:
  where is the necessity of a functional of which we only use the stationary trajectories?

- perhaps Lagrange could have tried promoting his own invention by using all trajectories

- ... and he would have had to introduce a unit for S ...

{Weak vs Strong} & {Classical vs Quantum}

▲ Weak coupling: loop expansion around leading trajectory $\gamma_{c\ell}$

$$\langle \mathrm{f}, t_\mathrm{f} | \mathrm{i}, t_\mathrm{i} \rangle \;=\; e^{i[S_0 + S_1 + S_2 + \ldots]}$$

can further distinguish semi-classical and quantum observables

$\delta_q O \ll O(\gamma_{c\ell})$ semi-class — Ex: every day life

$\langle O \rangle = O(\gamma_{c\ell}) + \delta_q O$

$\delta_q O \gtrsim O(\gamma_{c\ell})$ quantum — Ex: $\langle \phi\phi\phi\phi \rangle$ around $\phi(\gamma_{c\ell}) = 0$

▲ Strong coupling: PI cannot be described by leading trajectory

Common practice: few legs in weakly coupled QFT

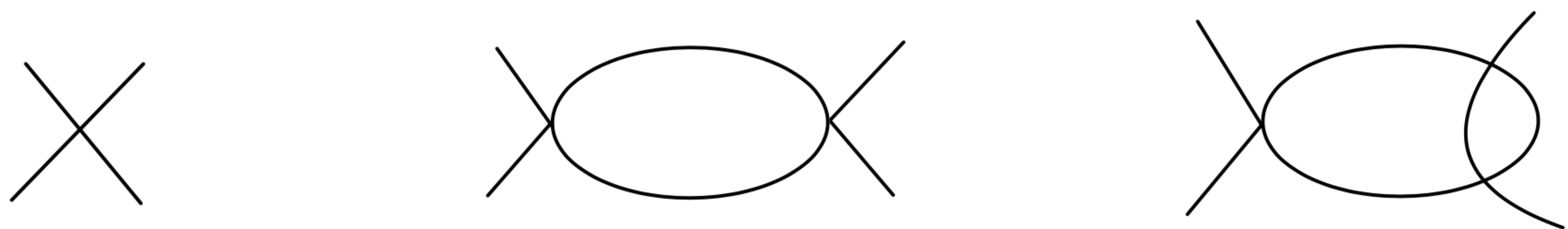
= small fluctuations around vacuum solution $\Phi_a = 0$

But, when the number of legs grows, expansion breaks down

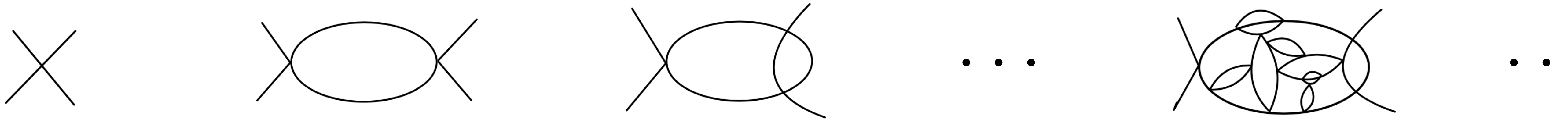
How do we describe physics in this regime?

Few Legs

Ex: $$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi + \lambda \phi^4$$



λ $\qquad$ $\frac{\lambda^2}{16\pi^2}$ $\qquad$ $\frac{\lambda^3}{(16\pi^2)^2}$



λ $\quad$ $\frac{\lambda^2}{16\pi^2}$ $\quad$ $\frac{\lambda^3}{(16\pi^2)^2}$ $\quad$ $n!\left(\frac{\lambda}{16\pi^2}\right)^n$

perturbative expansion diverges beyond $\quad n \sim \frac{16\pi^2}{\lambda}$

maximal precision $\quad e^{-c\frac{16\pi^2}{\lambda}}$

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Divergence of Perturbation Theory in Quantum Electrodynamics

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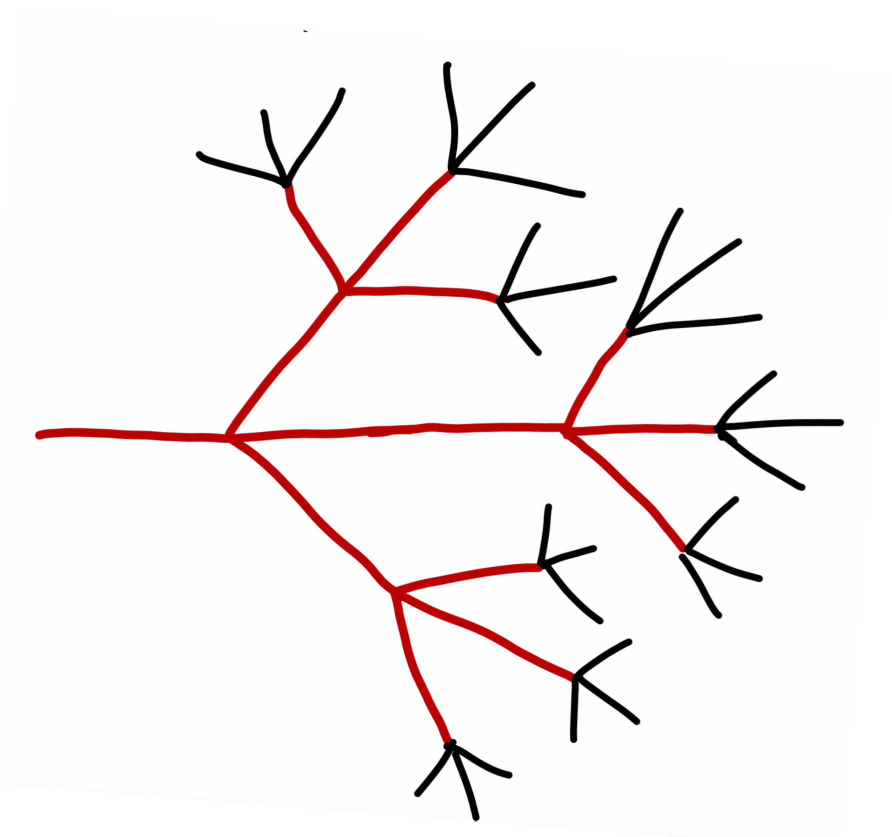
(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

n-legs in ϕ^4

see old review by Rubakov, arXiv:9511236, 1995



$$A_{1\to n} \propto n! \left(\frac{\lambda}{8m^2}\right)^{\frac{n-1}{2}}$$

$$\sigma_{1\to n} \propto n! \left(\frac{\lambda}{8}\right)^{n-1} \left(\frac{\epsilon}{m}\right)^{\frac{3n-1}{2}} \qquad \epsilon = \frac{E-nm}{n}$$

$$A_{loop} = A_{tree}\left(1 + B\lambda n^2 + C\lambda^2 n^4 + \dots\right)$$

a mess ?!

Indeed .. but not completely

all large effects can be proven to resum into

$$\sigma_{1\to n} \sim e^{nF(\lambda n,\epsilon)}$$

Libanov, Rubakov, Son, Troitsky 1994

Exponential form suggest existence of non-trivial semiclassical trajectory describing the process

Something indeed proven by Son, back in the 90's...still a reasonable mess to work out quantitatively

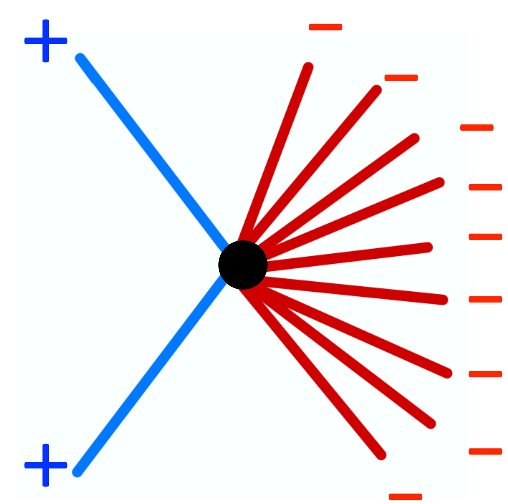
see Khoze, Reiness, 1810.01722

A "toy" problem

Complex scalar with U(1) symmetry $$\mathcal{L} = \partial_\mu \overline{\phi}\, \partial^\mu \phi + \frac{\lambda}{4}(\overline{\phi}\phi)^2$$

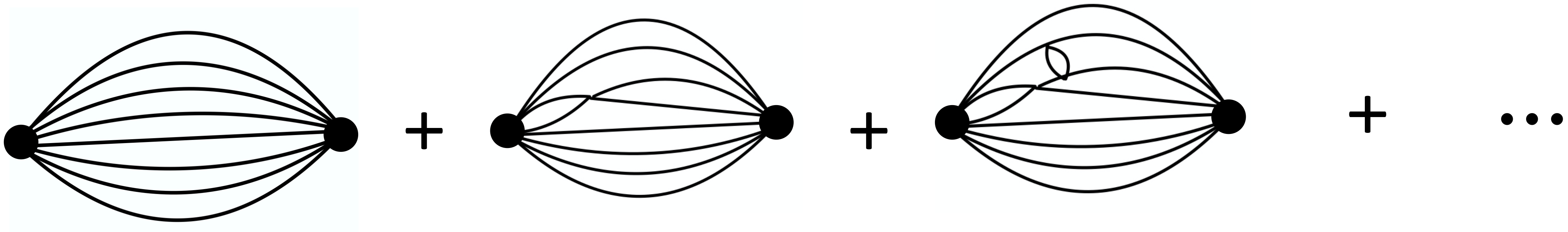
Scaling dimension of *charge-n operator* ϕ^n

Ex.1 $$\Delta\mathcal{L} = g\,\phi^n + g^*\,\bar{\phi}^n$$



$$\sigma = g^2 E^{2n-6}\,[\,1 + \text{loops}\,] \sim g^2 E^{2(n+\gamma_n)-6}$$

Ex.2 $\langle 0|\, \phi^n(x)\, \bar{\phi}^n(0)\, |0\rangle$



$$= \frac{n!}{(x^2)^{n+\gamma_n(\lambda)}} \left[\, 1 + a_2(\lambda) \ln^2 x^2 \mu^2 + \ldots \,\right]$$

$$\propto \beta(\lambda) \xrightarrow{\text{fixed point}} \beta(\lambda_*) = 0$$

$$\Longrightarrow \frac{c}{(x^2)^{n+\gamma_n(\lambda_*)}} \equiv \frac{c}{(x^2)^{\Delta_{\phi^n}}}$$

Critical Exponents in 3.99 Dimensions*

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(Received 11 October 1971)

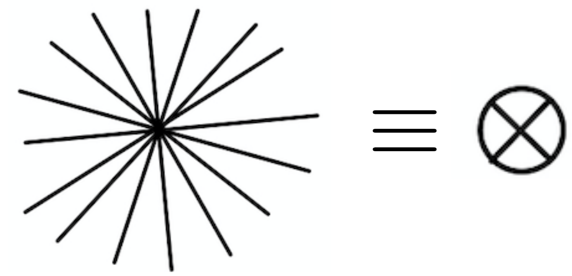
Critical exponents are calculated for dimension $d = 4 - \epsilon$ with ϵ small, using renormalization-group techniques. To order ϵ the exponent γ is $1 + \frac{1}{6}\epsilon$ for an Ising-like model and $1 + \frac{1}{5}\epsilon$ for an XY model.

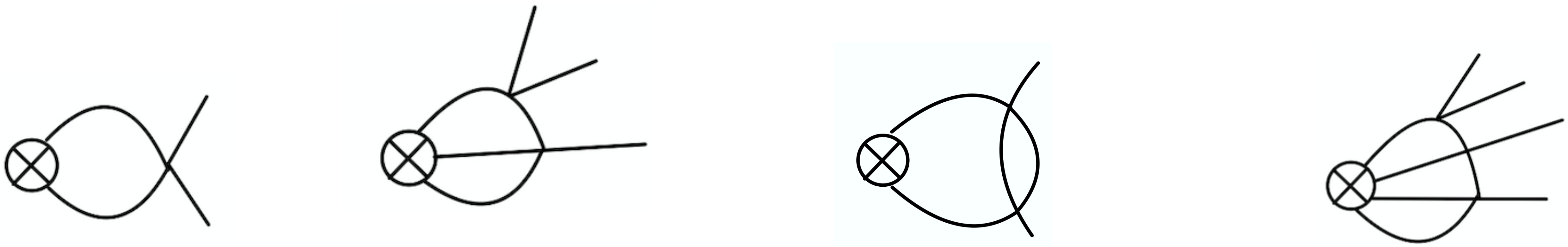
▲ fixed point in $d = 4 - \epsilon$

$$\beta_\lambda = \lambda \left[-\epsilon + 5 \frac{\lambda}{(4\pi)^2} - 15 \frac{\lambda^2}{(4\pi)^4} + O(\lambda^3) \right]$$

$$\beta(\lambda_*) = 0 \qquad \frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3\epsilon^2}{25} + O(\epsilon^3)$$

$\gamma_n(\lambda)$ with Feynman diagrams





$\lambda n(n-1)$ $\qquad$ $\lambda^2 n(n-1)(n-2)$ $\qquad$ $\lambda^2 n(n-1)$ $\qquad$ $\lambda^3 n(n-1)(n-2)(n-3)$

leading $\quad = \; n\left[\,\lambda n + (\lambda n)^2 + (\lambda n)^3 + \ldots\right]$

perturbation theory breaks down at $\quad \dfrac{\lambda n}{16\pi^2} \gtrsim 1$

series can be organized as a double expansion

$$\frac{\gamma_n}{n} \;=\; P_0(\lambda n) \;+\; \lambda P_1(\lambda n) \;+\; \lambda^2 P_2(\lambda n) \;+\; \ldots$$

similar to RG $\quad F_0(\lambda\, Log) \;+\; \lambda F_1(\lambda\, Log) \;+\; \ldots$

or to 't Hooft large-N expansion

$$\frac{\gamma_n}{n} \;=\; P_0(\lambda n) \;+\; \frac{1}{n}\bar{P}_1(\lambda n) \;+\; \frac{1}{n^2}\bar{P}_2(\lambda n) \;+\; \ldots$$

- What happens at large λn ?

- Can we compute? Yes

But expand path integral around another trajectory

Basic steps

$$\phi \to \frac{\phi}{\sqrt{\lambda}}$$

$$\int \left[\partial\bar{\phi}\partial\phi + \frac{\lambda}{4}\left(\bar{\phi}\phi\right)^2\right] \quad \longrightarrow \quad \frac{1}{\lambda}\int \left[\partial\bar{\phi}\partial\phi + \frac{1}{4}\left(\bar{\phi}\phi\right)^2\right] \equiv \frac{S}{\lambda}$$

$$\int D\phi \; \bar{\phi}^n(x_f)\,\phi^n(x_i)\; e^{-\frac{S}{\lambda}} \;=\; \int D\phi \; e^{-\frac{1}{\lambda}(S+n\lambda\ln\phi_f+n\lambda\ln\phi_i)}$$

Compute semiclassically for

$$\begin{cases} \lambda \ll 1 \\ \lambda n = \text{fixed} \end{cases}$$

saddle point $\qquad \phi_{cl} \equiv \phi_{cl}(\lambda n, x_f - x_i)$

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \int D\phi \, e^{-\frac{1}{\lambda}(S + n\lambda \ln \phi_f \phi_i)}$$

$$= e^{-\frac{1}{\lambda}S_{cl}(\lambda n, x_f - x_i) - S_1 - \lambda S_2 + \ldots}$$

$$\frac{\gamma_n}{n} = P_0(\lambda n) + \lambda P_1(\lambda n) + \ldots$$

- ‘structure’ of result manifest to all orders
- semiclassical expansion valid for all λn as long as $\lambda \ll 1$
- must match diagrammatic expansion at $\lambda n \ll 1$!

main problem: finding classical solution

$$\partial^2 \phi(x) - \frac{1}{2}\phi^2(x)\bar{\phi}(x) = -\frac{\lambda n}{\bar{\phi}(x_f)}\delta^{(d)}(x - x_f),$$
$$\partial^2 \bar{\phi}(x) - \frac{1}{2}\phi(x)\bar{\phi}^2(x) = -\frac{\lambda n}{\phi(x_i)}\delta^{(d)}(x - x_i)$$

can perform perturbation theory in λn

not straightforward to find solution at finite λn

basic difficulty

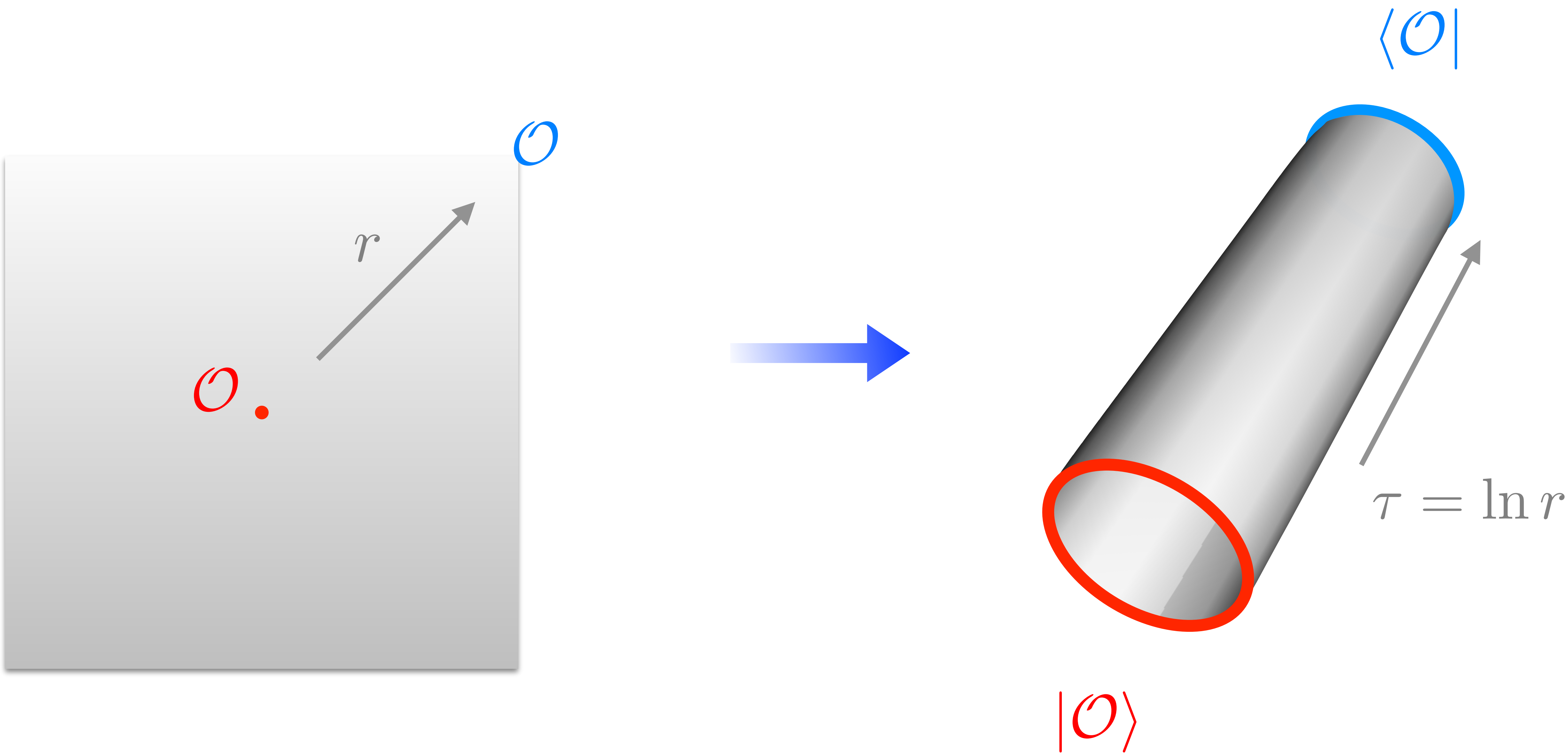
must regulate to $d = 4 - \epsilon$ where ϕ^4 not scale invariant: radial dependence non-trivial

Way out

- at fixed point $$\frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3\epsilon^2}{25} + O(\epsilon^3)$$

- Conformally map theory to cylinder $\mathbb{R} \times S_{d-1}$

Mapping to the cylinder & operator state correspondence



$$\langle \, \mathcal{O}(r) \, \mathcal{O}(0) \, \rangle = \frac{1}{r^{2\Delta}} \quad \longrightarrow \quad \langle \mathcal{O} | e^{-H\tau} | \mathcal{O} \rangle = e^{-\Delta \tau}$$

- on cylinder $D = H_{cy\ell}$ is conserved off-criticality

- expect solution to be stationary for $\tau_f \gg \tau_i$

- simple consistent ansatz at $\tau_f \gg \tau \gg \tau_i$

$$\phi_{c\ell} = \rho e^{i\chi} \qquad \rho = \text{const} \qquad \chi = -i\mu\tau$$

this corresponds to a homogeneous superfluid boundary eom $$2\rho^2\mu = \frac{\lambda n}{S_{d-1}}$$ $$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

bulk eom $$-\mu^2 + m_d^2 + \frac{1}{2}\rho^2 = 0$$ $$m_d^2 \equiv \left(\frac{d-2}{2}\right)^2$$

$$\mu(\mu^2 - m_d^2) = \frac{\lambda n}{4S_{d-1}}$$

Plug back into action and perform systematic loop expansion

$$\Delta_{\phi^n} = \frac{1}{\lambda_*}\Delta_{-1}(\lambda_* n) + \Delta_0(\lambda_* n) + \lambda_*\Delta_1(\lambda_* n) + \ldots .$$

Leading order $d \to 4$

$$\left[\begin{array}{l} \lambda = \lambda_* \propto \epsilon \to 0 \\ \lambda n \ = \ \text{fixed} \end{array} \right]$$

$$x \equiv \frac{\lambda_* n}{16\pi^2}$$

$$\frac{1}{\lambda_* n}\Delta_{-1} = \frac{3\left[9x-\sqrt{81x^2-3}\right]^{1/3}+3^{2/3}\left[9x-\sqrt{81x^2-3}\right]}{\left[\left(9x-\sqrt{81x^2-3}\right)^{2/3}+3^{1/3}\right]^2}$$

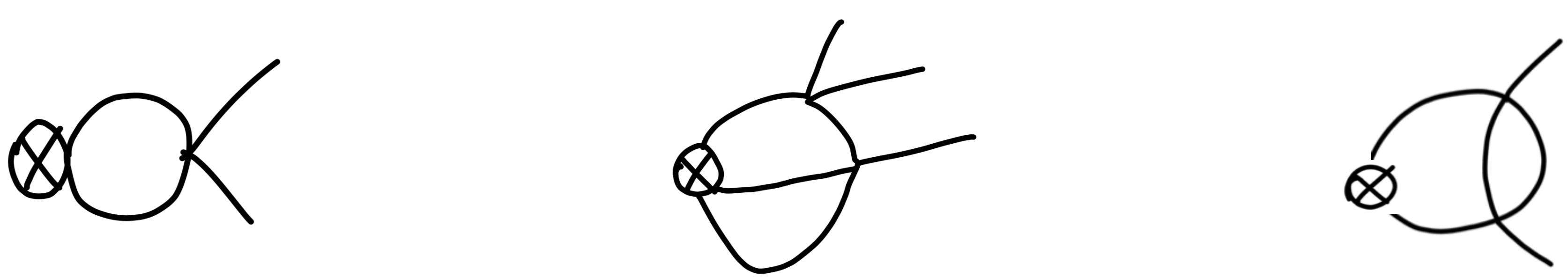
$$+\frac{9\times 3^{1/3}x\left[9x-\sqrt{81x^2-3}\right]^{2/3}}{2\left[\left(9x-\sqrt{81x^2-3}\right)^{2/3}+3^{1/3}\right]^2}$$

Supposed to resum leading powers of n at all loops!

expanding at small λn

$$\Delta_{\phi^n} = n + \frac{\lambda n^2}{32\pi^2} - \frac{\lambda^2 n^3}{512\pi^4} + \frac{\lambda^3 n^4}{4096\pi^6} + O\left(\lambda^4 n^5\right)$$

and comparing with diagrams



$$\gamma_n = \frac{\lambda n(n-1)}{32\pi^2} - \frac{\lambda^2 n^2(n-1)}{512\pi^4} + \ldots$$

they happily agree

Makes one want to check subleading terms

To do so fixed point condition in $d = 4 - \epsilon$ is essential

diagrammatic computation gives

$$\gamma_n\Big|_{\text{diag}} = \epsilon\frac{n(n-1)}{10} - \epsilon^2\frac{n(n^2-4n)}{50} + O\left(\epsilon^3 n^4\right)$$

Semiclassically, subleading terms should come from Casimir energy on cylinder: the explicit computation confirms

Large charge limit for $d \to 4$

$$\left[\begin{array}{l} \lambda = \lambda_* \propto \epsilon \to 0 \\ \lambda n \gg 1 \end{array}\right]$$

$$\Delta_{\phi^n} = \frac{\pi^2}{\lambda}\left[\frac{3}{8}\left(\frac{\lambda n}{\pi^2}\right)^{4/3} + \left(\frac{\lambda n}{\pi^2}\right)^{2/3} - \frac{2}{3} + O\left(\left(\lambda n/\pi^2\right)^{-2/3}\right)\right]$$

Interpretation: $m_\rho^2 \sim \mu^2 \sim (\lambda n)^{2/3} \gg \frac{1}{R^2} = 1$

→ integrate out radial mode: 'pure' conformal superfluid EFT

$$\mathcal{L} \sim (\partial\chi)^4 + \mathcal{R}(\partial\chi)^2 + \mathcal{R}^2 + \ldots$$

known large charge behaviour

Hellerman, Orlando, Reffert, Watanabe '15
Monin, Pirtskhalava, RR, Seibold '16
Jafferis, Mukhametzhanov, Zhiboedov, '17

For $d = 4 - \epsilon$ quantum corrections provide proper scaling

$$\Delta_{\phi^n} = \frac{1}{\epsilon}\left[c_{-1}(\epsilon)\left(\frac{2}{5}\epsilon n\right)^{\frac{4-\epsilon}{3-\epsilon}} + c_0(\epsilon)\left(\frac{2}{5}\epsilon n\right)^{\frac{3-\epsilon}{2-\epsilon}} + \dots\right]$$

	$c_{-1}(1)$	$c_0(1)$
Monte-Carlo	0.337(3)	0.27(4)
ε-expansion: LO	0.47	0.79
ε-expansion: NLO	0.42	0.04

Summary

Wilson-Fisher fixed points: simple but rich playground to get structural insight on $\lambda n \gg 1$ regime in QFT

Loop expansion for γ_{ϕ^n} non-trivially and systematically encapsulated by semiclassical supefluid configuration

Properties of nearby operators, ex $\phi^{n-2}\partial\phi\partial\phi$, described by hydrodynamic modes

$\langle\phi^{n_1}\dots\phi^{n_p}\rangle$ can be studied by extension of method, providing dynamical information, akin to amplitudes

...but it would be nice to get back to the SM...
someone surely will before the next Big Collider turns on