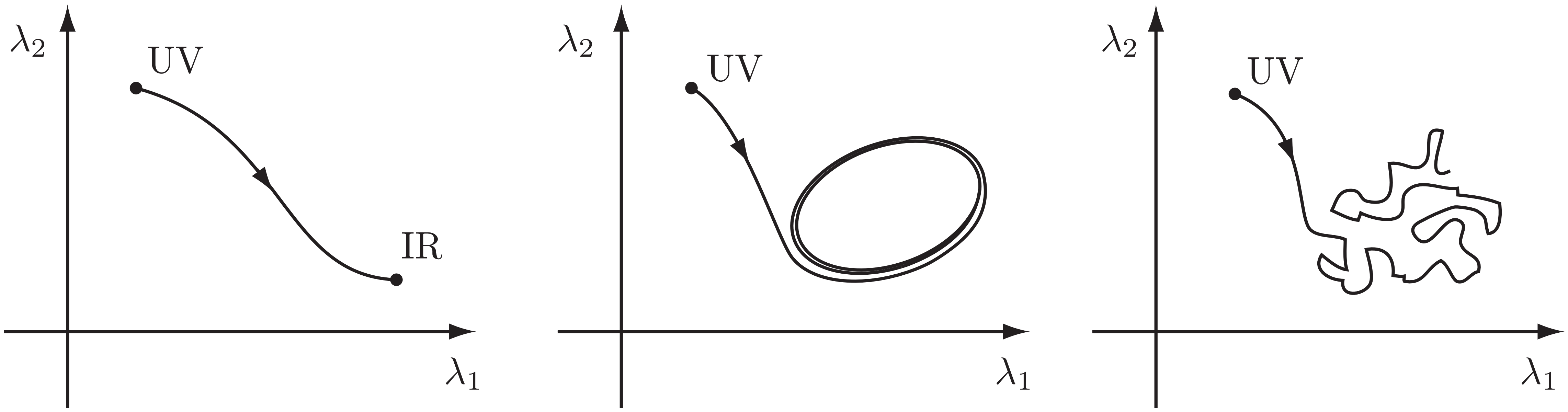
Lecture II

Constraining the structure of RG flows in 4D

conceivable RG flows



(a) (b) (c)

but all known examples asymptote to a CFT fixed point

- free (QED, massless QCD)
- strongly coupled (Supersymmetry)
- trivial (real QCD)

In particular: there are no known SFT asymptotics !

Scale invariance

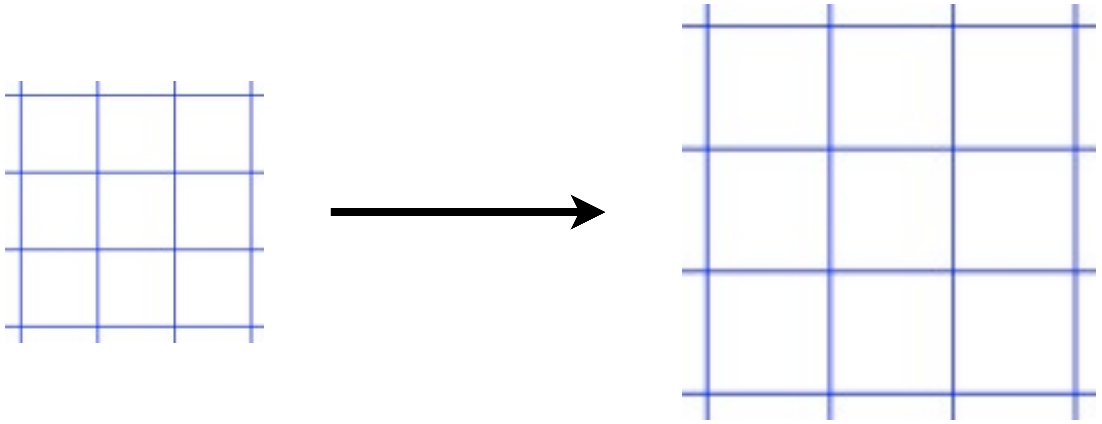
versus

Conformal Invariance

Wess 1960
Polchinski 1988

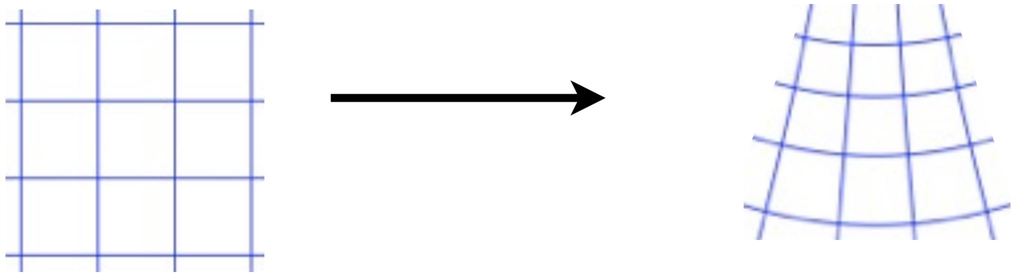
Geometric picture

dilations $x^\mu \to \tilde{x}^\mu = k x^\mu$



$$(d\tilde{x})^2 = k^2 (dx)^2$$

conformal $x^\mu \to \tilde{x}^\mu = \dfrac{x^\mu + b^\mu x^2}{1 + b^2 x^2 + 2b \cdot x}$



$$(d\tilde{x})^2 = \frac{1}{(1 + b^2 x^2 + 2b \cdot x)^2} (dx)^2$$

Conformal Group:

$x \to \tilde{x}(x)$ such that $\eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = F(x) \eta_{\mu\nu} dx^\mu dx^\nu$

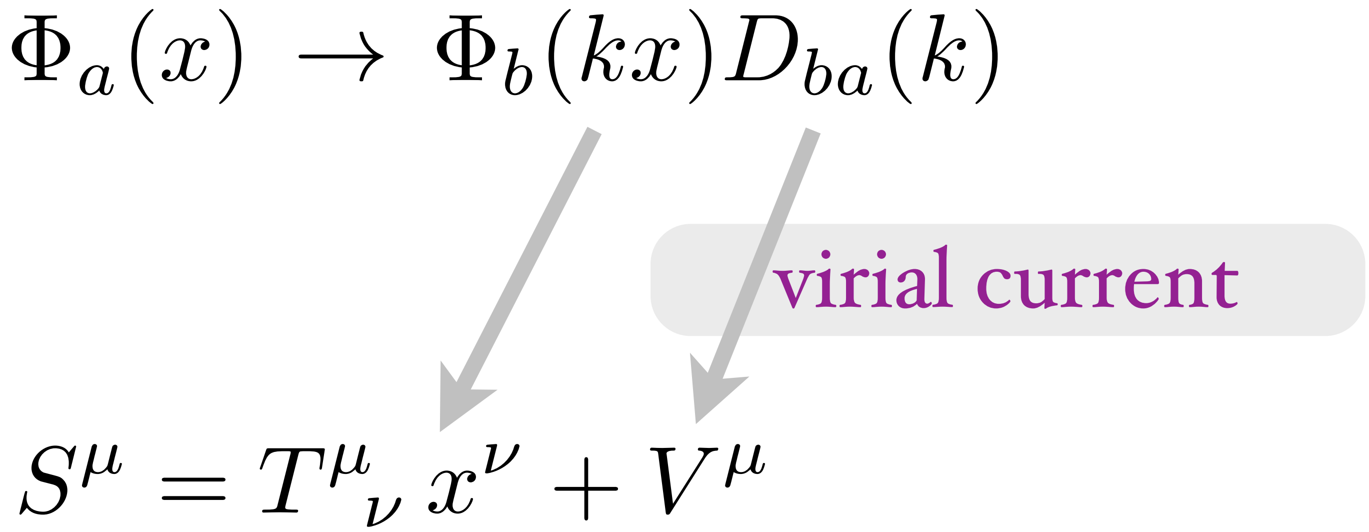
In D>2 the group is O(D,2)

- Poincaré
- dilations
- special conformal

In D=2 the symmetry algebra is generated by the infinite set of harmonic functions

In principle one could have just a scale invariant field theory (SFT) with Poincaré $\times$ dilations symmetry

Scale transformation $\Phi_a(x) \to \Phi_b(kx) D_{ba}(k)$



virial current

Noether current $S^\mu = T^\mu{}_\nu x^\nu + V^\mu$

$$\partial_\mu S^\mu = 0 \quad \Longleftrightarrow \quad T^\mu_\mu = -\partial_\mu V^\mu \neq 0$$

- If $V_\mu = \partial^\nu L_{\mu\nu}$
- improvement exists $T_{\mu\nu} \to \Theta_{\mu\nu}$ $S^\mu \to \tilde{S}^\mu = \Theta^\mu{}_\nu x^\nu$

$$\Theta^\mu{}_\mu = 0$$

$$\Theta^{\mu}{}_{\mu} = 0$$

conformal
symmetry
as a ‘bonus’

$$K^{\mu}{}_{\nu} = 2x_{\nu}x^{\rho}\Theta^{\mu}{}_{\rho} - x^2\Theta^{\mu}{}_{\nu}$$

$$\partial_{\mu}K^{\mu}{}_{\nu} = 2x_{\nu}\Theta^{\mu}{}_{\mu} = 0$$

✦ happens in all classical examples

Callan, Coleman, Jackiw 1970

- SFT examples, if any, necessarily entail quantum effects

$V_\mu \equiv$ genuine non-conserved current with scaling dimension exactly equal to 3 even including quantum effects

- Often, there simply doesn't exist a candidate for V_μ

Ex.: axial current in massless (S)QCD excluded by parity selection rule

- But in general?

Exploring the structure of QFT by turning on an external metric

- Irreversibility of CFT-to-CFT RG flows: a-theorem
- Ruling out non-CFT asymptotics in perturbation theory
- Towards a non-perturbative result

RG flow describes the change of the dynamics under a dilation

≡ change of the action under a dilation

Whenever we have some explicitly broken symmetry it proves useful to

- formally restore it by promoting couplings to sources transforming non trivially
- gauge it by adding the suitable gauge field

We shall play various related games

A. $\eta_{\mu\nu}, \quad \lambda_i \quad \longrightarrow \quad g_{\mu\nu}(x), \quad \lambda_i(x)$ + Weyl symmetry

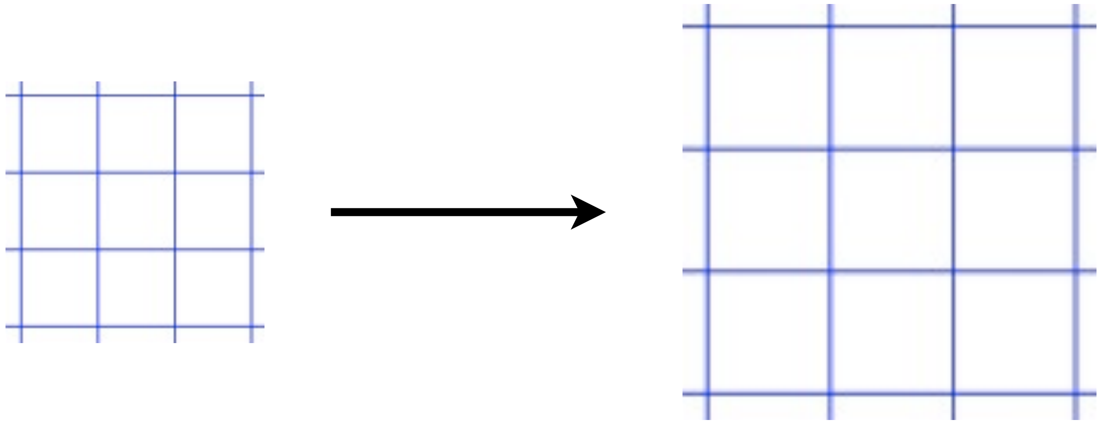
B. $\lambda_i = \text{const}$

$$\eta_{\mu\nu} \quad \longrightarrow \quad e^{-2\tau}\eta_{\mu\nu}$$

$e^{-\tau} \equiv \Omega \equiv 1 + \varphi$ background dilaton field

Geometric picture

dilations $x^\mu \to \tilde{x}^\mu = k x^\mu$



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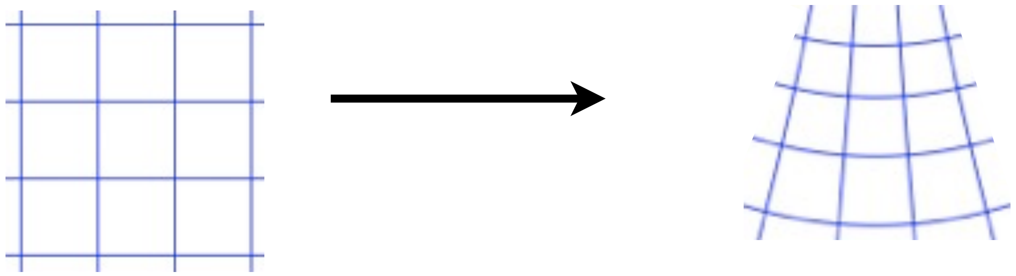

$$(d\tilde{x})^2 = \frac{1}{(1 + b^2 x^2 + 2b \cdot x)^2} (dx)^2$$

QFT in a gravitational background

Weyl Symmetry

$$\begin{aligned} g_{\mu\nu}(x) &\rightarrow e^{-2\sigma(x)} g_{\mu\nu} \\ \Phi_a(x) &\rightarrow e^{-k_a \sigma(x)} \Phi_a(x) \end{aligned}$$

O(D,2) = subgroup of Weyl x Diffs that leaves $\eta_{\mu\nu}$ invariant

$S[g, \Phi]$ Weyl invariant → $S[\eta, \Phi]$ Conformal invariant

Converse is also true (at classical level)

Ex.: free massless scalar field

$$\mathcal{L}_{flat} = \frac{1}{2}(\partial\varphi)^2$$

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{\eta_{\mu\nu}}{2}(\partial\varphi)^2$$

$$T^\mu_\mu = -(\partial\varphi)^2 \neq 0$$

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$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\varphi^2$$

$$\Theta^\mu_\mu = 0$$

improvement

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improvement

$$\mathcal{L}_{curved} = \sqrt{g}\frac{1}{2}\left[(\partial\varphi)^2 + \frac{1}{6}R\varphi^2\right] = \frac{1}{6}\sqrt{\hat{g}}R(\hat{g})$$

$$\hat{g}_{\mu\nu} \equiv \varphi^2 g_{\mu\nu}$$

$$\varphi \to e^\sigma\varphi$$

$$g_{\mu\nu} \to e^{-2\sigma}g_{\mu\nu}$$

Weyl symmetry
manifest

Weyl symm: $\int \sigma(x) \left(2g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}(x)} + k_a \Phi_a \frac{\delta S}{\delta \Phi_a(x)} \right) = 0$

$\sigma(x)$ arbitrary $\Downarrow$ Φ_a on-shell

$$T^\mu_\mu \equiv g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}(x)} = 0$$

With only global Weyl, $\Box\sigma$ = constant, we would instead deduce

$$\int T^\mu_\mu = 0 \qquad \longrightarrow \qquad T^\mu_\mu = \partial^\mu V_\mu$$

QFT in gravity background

✦ quantum effective action $e^{iW[g_{\mu\nu}]} = \int D[\Phi] e^{iS[g,\Phi]}$

- need regulation
- diff invariant
- finite by adding suitable local counterterms

In general the introduction of a regulator in curved background breaks explicitly Weyl invariance even when flat space theory is conformally invariant

$$\delta_\sigma \equiv \int d^4x\, 2\sigma(x)\, g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)}$$

In ordinary QFT $\delta_\sigma W =$ non-local

In CFT $\delta_\sigma W = \int \sigma(x)\sqrt{g}\,\mathcal{A}(x) =$ Weyl Anomaly (local!)

also written as $\langle T \rangle \equiv \langle T^\mu_\mu \rangle = \mathcal{A}(x)$

Christensen, Duff '74

The structure of the Weyl anomaly in a CFT

$\mathcal{A}(x)$ is a scalar function of the metric

in general

$$\mathcal{A}(x) \;=\; aE_4 - bR^2 - cW^2 - d\Box R$$

$$\begin{aligned} E_4 &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \\ W^2 &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2 \end{aligned}$$

$$\int \sigma(x)\sqrt{g}\left(-d\Box R + e\Lambda^2 R + f\Lambda^4\right) \;=\; \delta_\sigma \int (-1)\sqrt{g}\left(\frac{d}{12}R^2 + \frac{e}{2}\Lambda^2 R + \frac{f}{4}\Lambda^4\right)$$

the last three terms can be written as variation of local functional

⬇

they can be eliminated by a choice of counterterms

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the last three terms can be written as variation of local functional

they can be eliminated by a choice of counterterms

Wess-Zumino consistency condition

$$\delta_\sigma W \ = \ \int \sigma(x)\sqrt{g}\,\mathcal{A}(x)$$

Weyl symmetry is abelian

$$[\delta_{\sigma_2}, \delta_{\sigma_1}]\, W = \delta_{\sigma_2}\left(\int d^4x_1\sigma_1\sqrt{g}\mathcal{A}\right) - \delta_{\sigma_1}\left(\int d^4x_2\sigma_2\sqrt{g}\mathcal{A}\right) = 0$$

$$\mathcal{A}(x) \ = \ aE_4 - bR^2 - cW^2 \quad \longrightarrow \quad \boxed{\quad aE_4 - cW^2 \quad}$$

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Easy to check using

$$
\begin{aligned}
\delta\sqrt{g} &= -4\sigma\sqrt{g} \\
\delta\Box &= 2\sigma\Box - 2\nabla_\mu\sigma\nabla^\mu \\
\delta R &= 2\sigma R + 6\Box\sigma \\
\delta E_4 &= 4\sigma E_4 - 8G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma \\
\delta W^2 &= 4\sigma W^2 \\
\delta G_{\mu\nu} &= 2\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\sigma
\end{aligned}
$$

- In general CFT

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{c}{x^8} I_{\mu\nu\rho\sigma}(x)$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) T_{\gamma\delta}(y)\rangle = c\, C_{\mu\nu\rho\sigma\gamma\delta}(x,y) + a\, A_{\mu\nu\rho\sigma\gamma\delta}(x,y)$$

Stanev ’88
Osborn, Petkou ’94

- In a free field theory one has

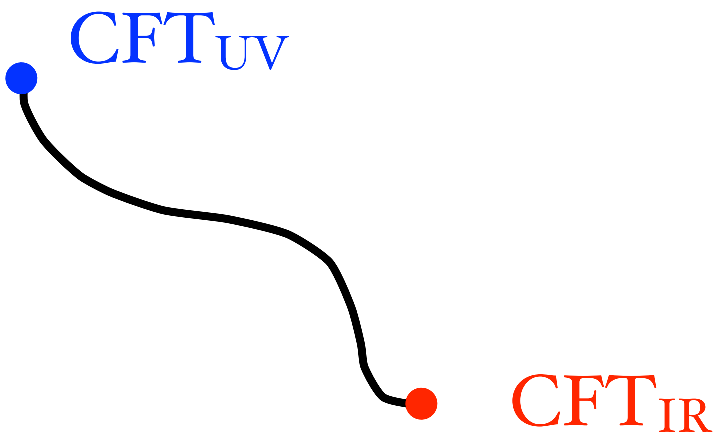
Christensen, Duff ’79

$$a = \frac{1}{5760\pi^2}\left(n_s + \frac{11}{2} n_f + 62 n_v\right)$$

$$c = \frac{1}{5760\pi^2}\left(3n_s + 9n_f + 36 n_v\right)$$

both *a* and *c* are a weighted measure of the number of degrees of freedom

Cardy's Conjecture (1988) : a decreases monotonically along the RG flow



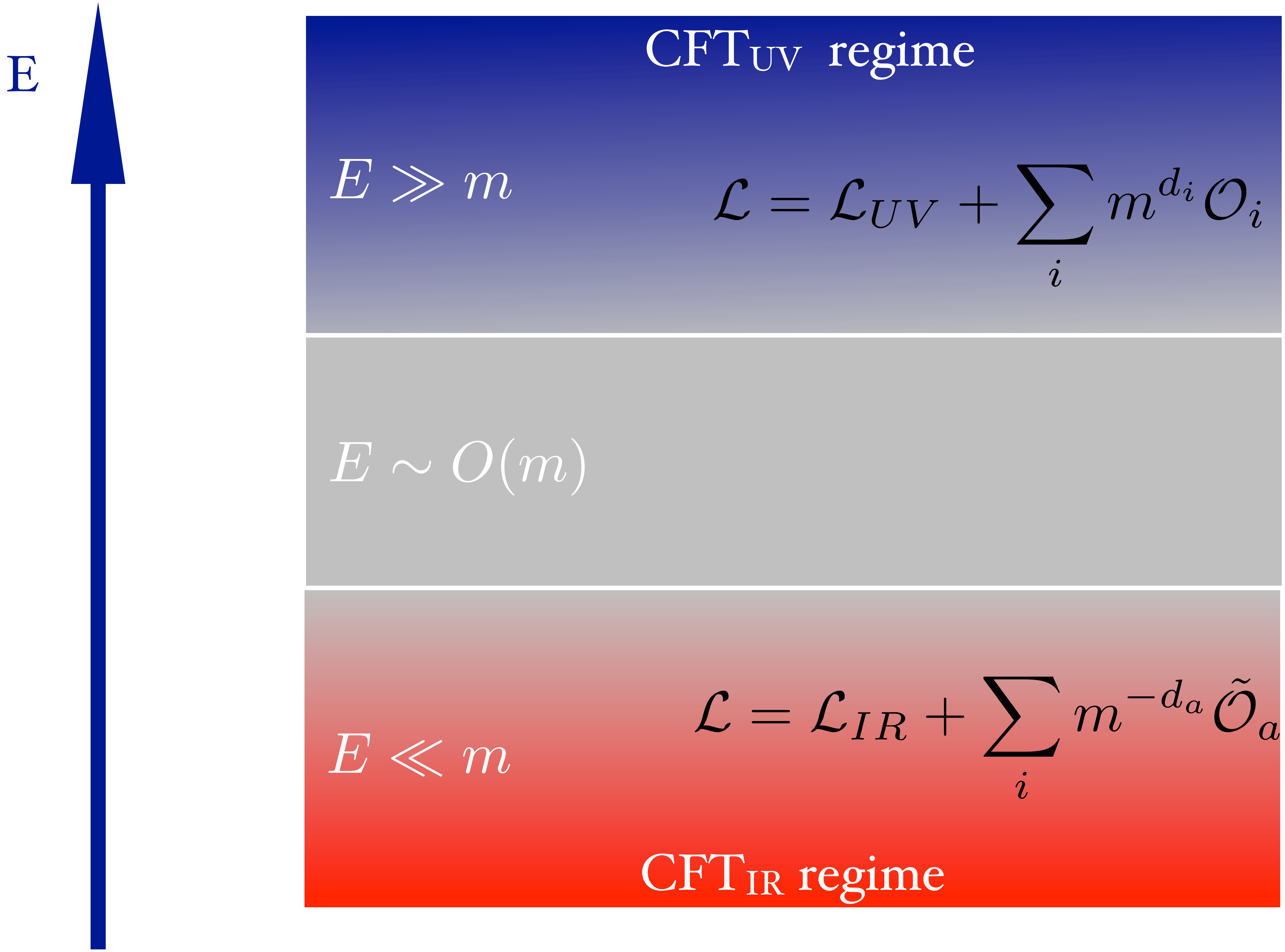
$$\boxed{a_{UV} \quad > \quad a_{IR}}$$

In 2D there was already Zamolodchikov's c-theorem (1986) stating the monotonicity of the unique coefficient in the 2D Weyl anomaly

$$\delta_\sigma W = \int d^2x \sqrt{g}\, c\, R(g) \equiv \int d^2x \sqrt{g}\, c\, E_2(g)$$

Proof of Cardy's Conjecture: the a-theorem

Komargodski and Schwimmer 2011



E

CFT$_{\mathrm{UV}}$ regime

$E \gg m$

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i m^{d_i} \mathcal{O}_i$$

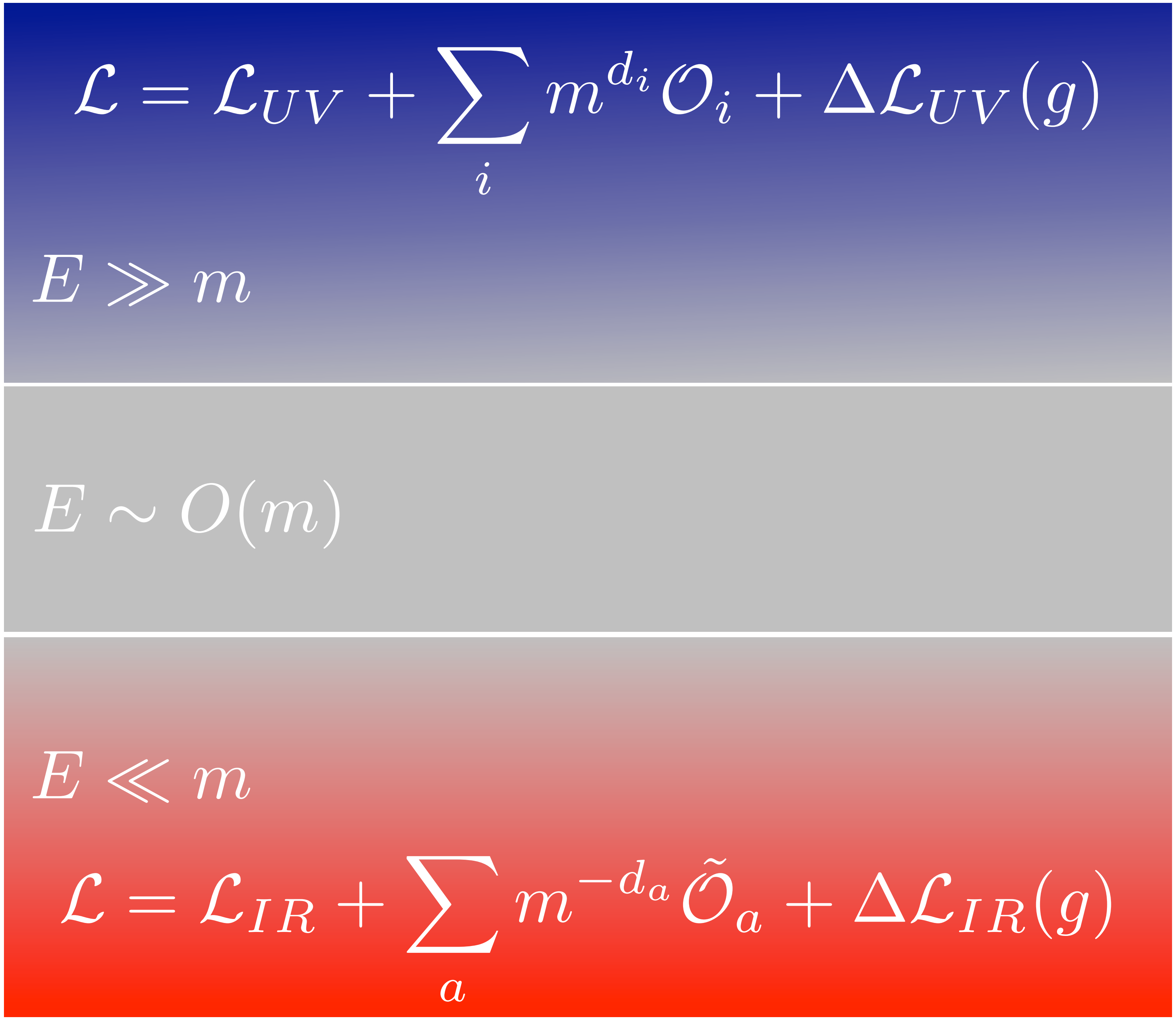
$E \sim O(m)$

$E \ll m$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_i m^{-d_a} \tilde{\mathcal{O}}_a$$

CFT$_{\mathrm{IR}}$ regime

Consider now putting this system in an extermal metric



$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i m^{d_i} \mathcal{O}_i + \Delta\mathcal{L}_{UV}(g)$$

$E \gg m$

$E \sim O(m)$

$E \ll m$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a m^{-d_a} \tilde{\mathcal{O}}_a + \Delta\mathcal{L}_{IR}(g)$$

$\Delta\mathcal{L}_{UV} \equiv$ metric (curvature) dependent counterterms needed to define a renormalized quantum action W[g]

$\Delta\mathcal{L}_{IR} \equiv$ metric dependent terms associated with positive powers of m

The general structure of $\Delta\mathcal{L}_{UV}$ and $\Delta\mathcal{L}_{IR}$ is the same:

local scalar functions of dimension ≤ 4

In the classification of these terms it is crucial to consider what happens for the case of a conformally flat background metric

$$\hat{g}_{\mu\nu} = \Omega(x)^2 \, \eta_{\mu\nu}$$

Counterterms

$d > 4$ in sensible theories

I. $\sqrt{\hat{g}} R \, \mathcal{O}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \sqrt{\hat{g}} \nabla_\mu R \, J^\mu$ $\quad \sqrt{\hat{g}} R_{\mu\nu} \, J^{\mu\nu}$

II. $\sqrt{\hat{g}}$ $\quad \sqrt{\hat{g}} R$ $\quad \sqrt{\hat{g}} R^2$ $\quad \sqrt{\hat{g}} E_4$ $\quad \sqrt{\hat{g}} W^2$

Counterterms

$d > 4$ in sensible theories

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irrelevant

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Counterterms

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Counterterms

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$$\begin{array}{ccc} \sqrt{\hat{g}} R\,\mathcal{O} & \qquad & d_{\mathcal{O}} \leq 2 \\ \downarrow & & \\ \Omega \Box \Omega\, \mathcal{O} & & \end{array}$$

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$$\begin{array}{ccccc} \sqrt{\hat{g}} & \sqrt{\hat{g}} R & \sqrt{\hat{g}} R^2 & \sqrt{\hat{g}} E_4 & \sqrt{\hat{g}} W^2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \Omega^4 & \Omega \Box \Omega & \Omega^{-2}(\Box\Omega)^2 & 0 & 0 \end{array}$$

On shell dilaton : $\Box \Omega = 0$

Counterterms

$d > 4$ in sensible theories

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$d > 4$ in sensible theories

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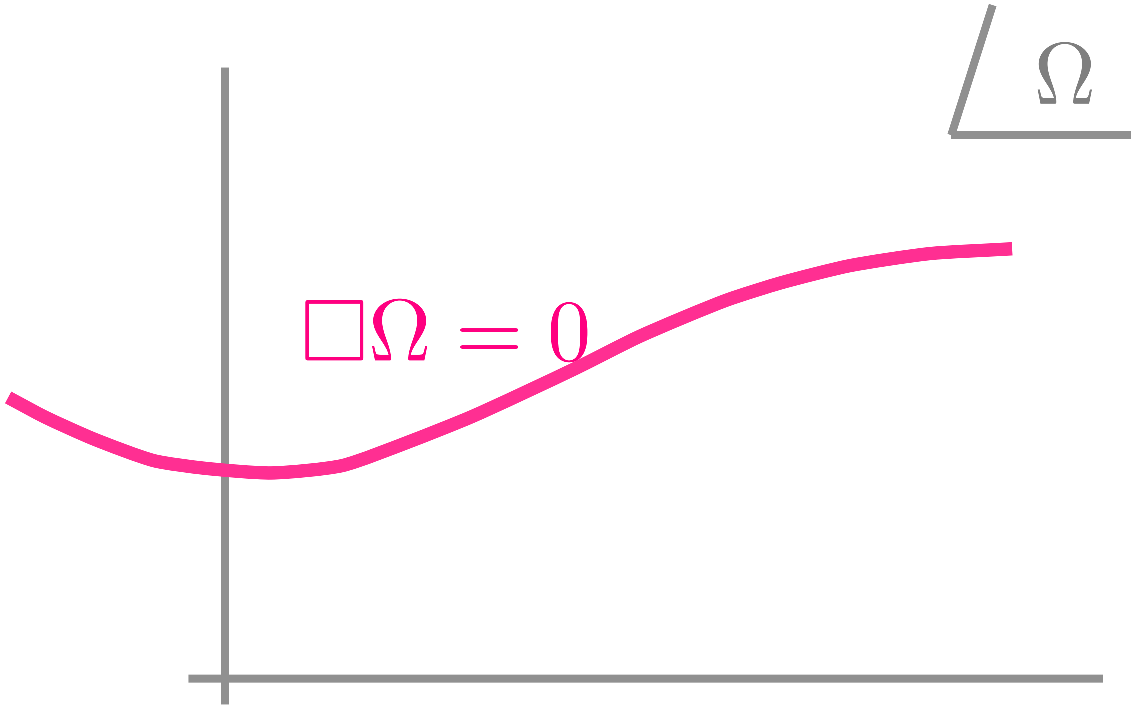
On shell dilaton : $\quad \Box\Omega = 0$

$$\boxed{\quad W\Big|_{\Omega\,:\,\Box\Omega=0} \quad = \quad W[\Omega, \lambda_{QFT}, \Lambda_{cc}]}$$

schematically



consider from here on $W[\Omega] \equiv W[\Omega]\Big|_{\Box\Omega=0}$

$$\frac{\delta}{\delta\Omega(x_1)}\cdots\frac{\delta}{\delta\Omega(x_n)}W\Big|_{\Omega=1} = \mathcal{M}(x_1,\ldots,x_n)$$

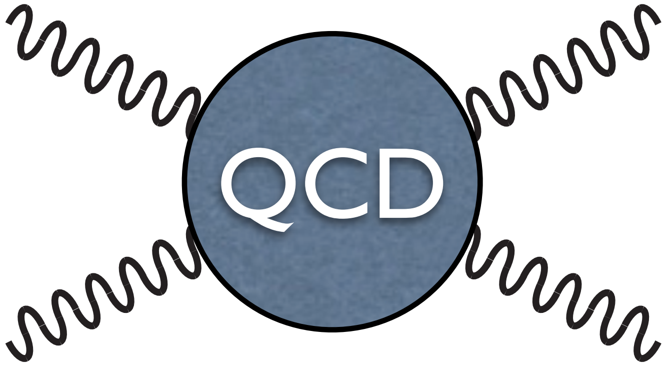
$\mathcal{M}$ can be interpreted as the n-dilaton scattering amplitude

QCD analogy

Effective QCD action in background photon field : $W[A_\mu]$

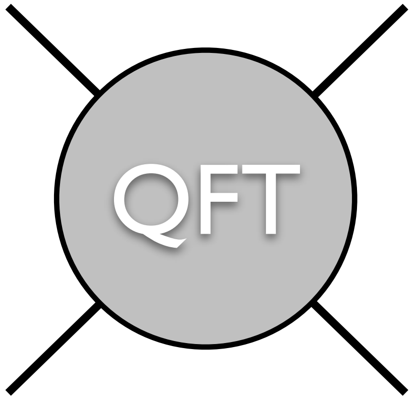
$$\frac{\delta}{\delta A_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta A_{\mu_n}(x_n)} W \quad = \quad \text{QCD mediated n-photon amplitude}$$

Ex. $$\frac{\delta}{\delta A_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta A_{\mu_4}(x_4)} W \quad =$$



light-by-light
scattering

The analogue QFT mediated dilaton-by-dilaton scattering



affords a remarkable insight into the structure of our QFT

Like for all on-shell n-point dilaton amplitudes the only renormalization needed to define this amplitude concerns a constant term associated with the cosmological constant

$$\mathcal{M} \equiv \mathcal{M}(\lambda_{QFT}, \Lambda_{cc})$$

4-point amplitude

$$\hat{g}_{\mu\nu} = \Omega(x)^2 \eta_{\mu\nu}$$

$$\frac{\delta}{\delta\Omega} = \frac{1}{\Omega}\frac{\delta}{\delta\ln\Omega}$$

$$-\frac{\delta}{\delta\ln\Omega} W = T^{\mu}_{\mu} \equiv T$$

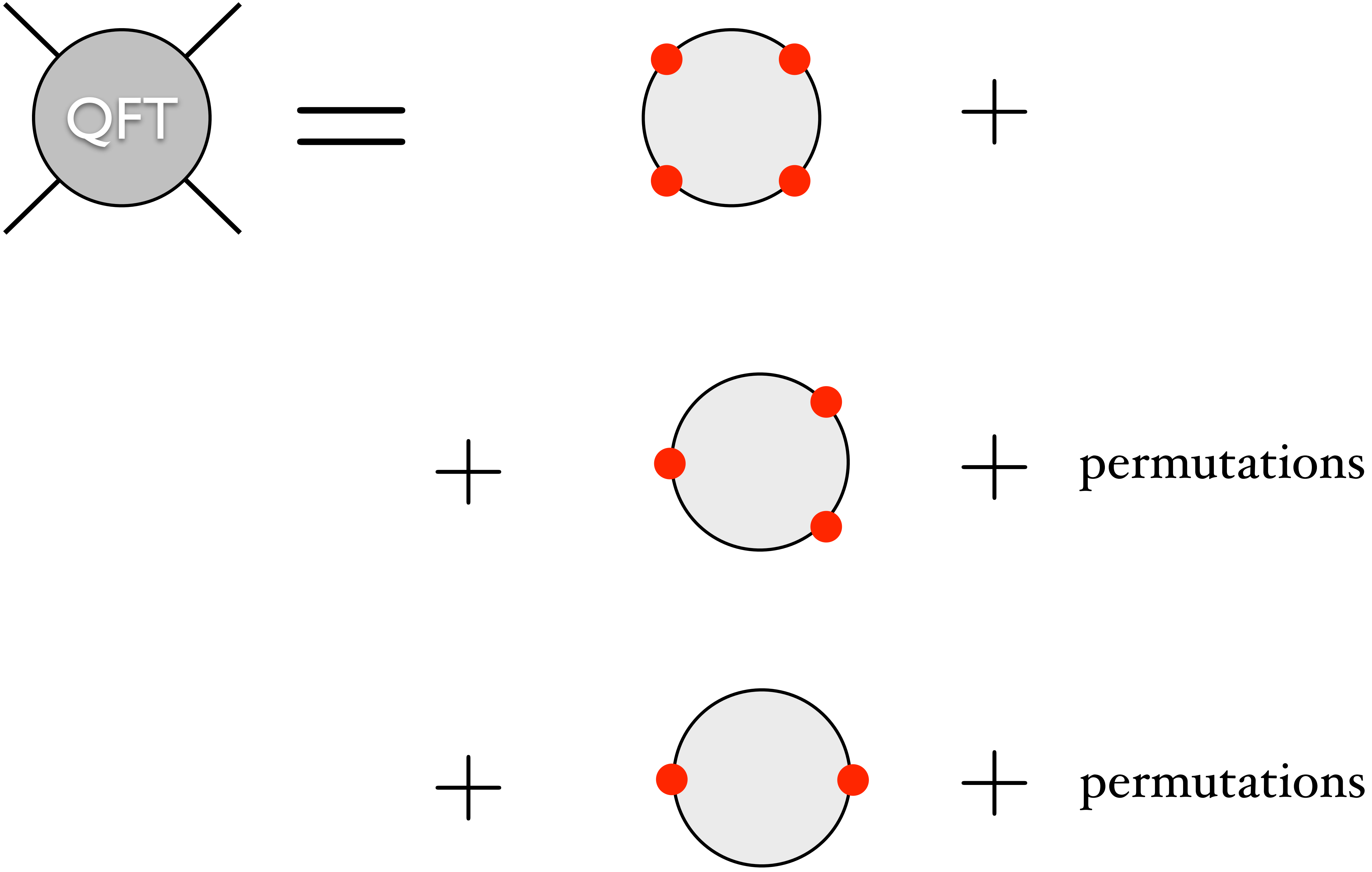
$$\mathcal{M}(p_1,\ldots,p_4) = \frac{\delta^4 W}{\delta\Omega(p_1)\cdots\delta\Omega(p_4)} = \langle T(p_1)T(p_2)T(p_3)T(p_4)\rangle + \langle T(p_1+p_2)T(p_3)T(p_4)\rangle + \text{permutations}$$

$$+\langle T(p_1+p_2)T(p_3+p_4)\rangle + \text{permutations}$$

$$+\langle T(p_1+p_2+p_3)T(p_4)\rangle + \text{permutations}$$

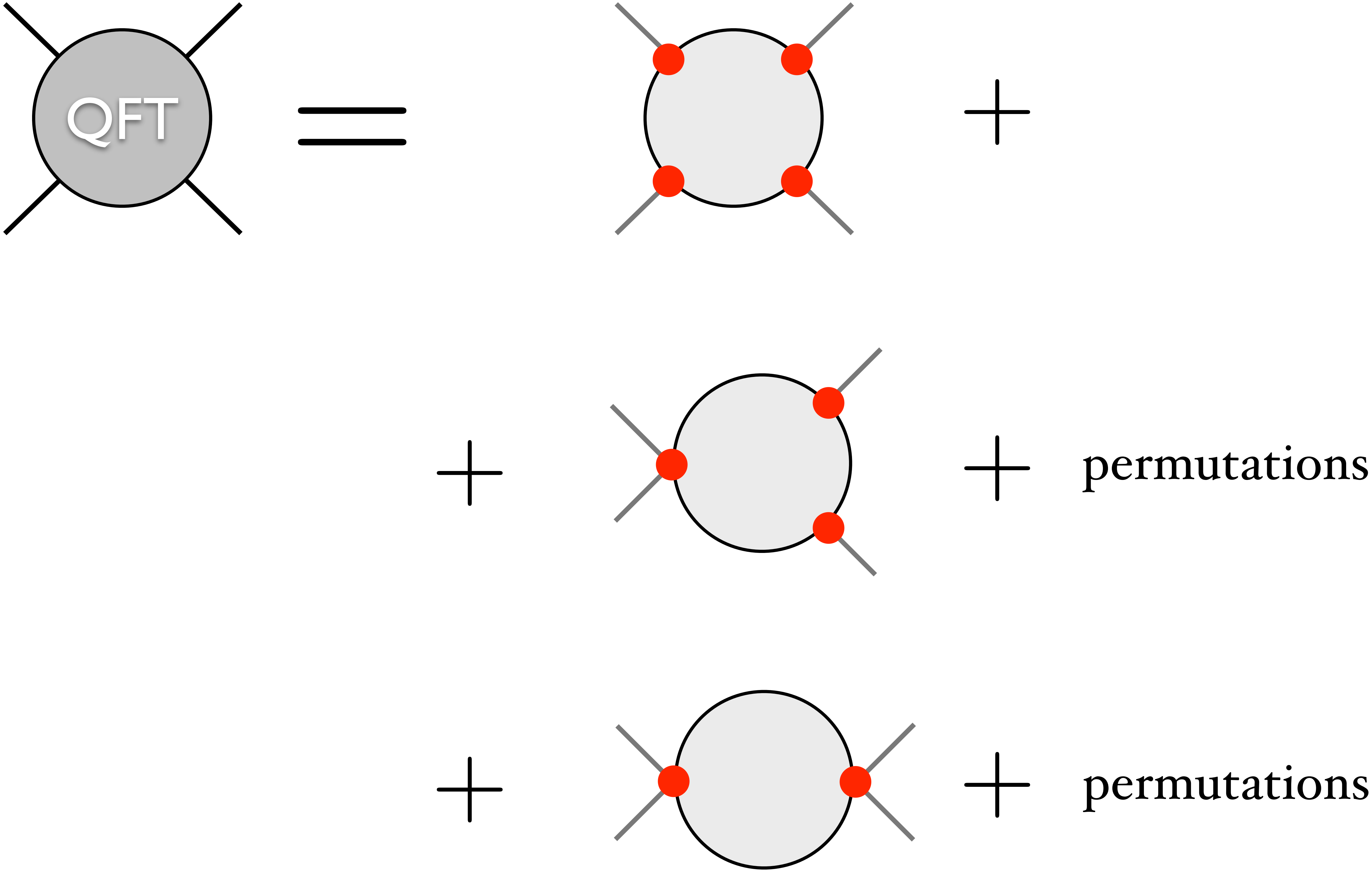
$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$$

$$\mathcal{M}(p_1,\ldots,p_4) \equiv (2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\, A(s,t)$$



permutations

permutations


QFT
=
+
+
+ permutations
+
+ permutations

In a CFT $W[\Omega]$ is local and fully determined by the Weyl anomaly up to cosmological constant term

$$\xrightarrow{\text{on shell} \quad \Omega} \qquad \Delta W = \frac{\Lambda_{cc}}{4!}\,\Omega^4$$

neglecting momentarily CC term

$$\boxed{W_{CFT}[\Omega^2 g_{\mu\nu}] = W_{CFT}[g_{\mu\nu}] - S_{\rm WZ}[g_{\mu\nu}, \Omega; a, c]}$$

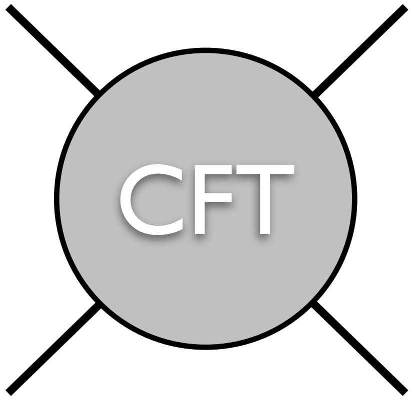
Cappelli, Coste 1988
Tomboulis 1990
Schwimmer, Theisen '11

$$S_{\rm W\mathit{Z}}[g_{\mu\nu}, \Omega; a, c] = \int d^4x\sqrt{-g}\,\Big\{a\Big[\ln\Omega\, E_4(g) - 4\big(R^{\mu\nu}(g) - \frac{1}{2}g^{\mu\nu}R(g)\big)\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega - 4\Omega^{-3}(\partial\Omega)^2\Box\Omega + 2\Omega^{-4}(\partial\Omega)^4\Big] - c\,\ln\Omega\, W^2(g)\Big\}.$$

$$g_{\mu\nu} = \eta_{\mu\nu}$$

$$\Box\Omega = 0$$

$$W_{CFT}[\Omega] \quad \longrightarrow \quad -2\,a\;\Omega^{-4}(\partial\Omega)^2(\partial\Omega)^2 \;+\; \frac{\Lambda}{4!}\,\Omega^4$$



$$\boxed{A(s,t) \;=\; -4\,a\,\left[s^2 + t^2 + (s+t)^2\right] \;+\Lambda}$$

this basic result leads to a simple proof of the a-theorem

Komargodski, Schwimmer ’11

$\mathrm{CFT}_{\mathrm{UV}}$

$\mathrm{CFT}_{\mathrm{IR}}$

CFT$_{\text{UV}}$

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i c_i m^{4-d_i} \mathcal{O}_i$$

$$d_i < 4$$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a b_a \frac{1}{m^{d_a-4}} \tilde{\mathcal{O}}_a$$

$$d_a > 4$$

CFT$_{\text{IR}}$

CFT$_{\mathrm{UV}}$

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i c_i m^{4-d_i} \mathcal{O}_i$$

$$d_i < 4$$

$$A(s,0) = -8a_{UV}\, s^2 \left[1 + \left(\frac{m}{\sqrt{s}}\right)^{\#}\right] + \Lambda_{cc}^{UV}$$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a b_a \frac{1}{m^{d_a-4}} \tilde{\mathcal{O}}_a$$

$$d_a > 4$$

$$A(s,0) = -8a_{IR}\, s^2 \left[1 + \left(\frac{\sqrt{s}}{m}\right)^{\#}\right] + \Lambda_{cc}^{IR}$$

CFT$_{\mathrm{IR}}$

Can relate UV to IR via dispersive argument

Using

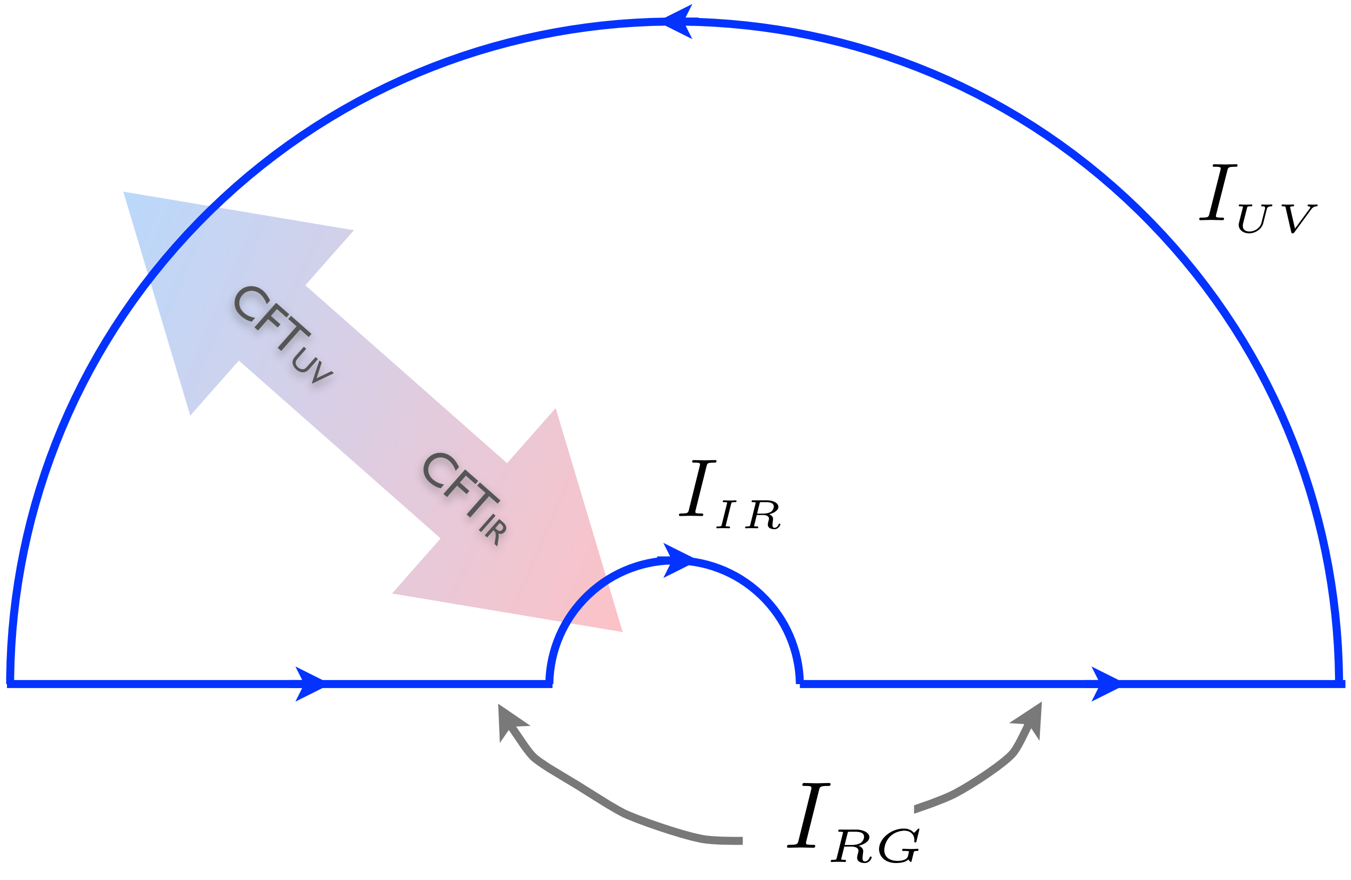
- $A(s) \equiv A(s,0)$ is analitic with cut on real s axis

- crossing $A(s) = A(-s)$ $(t=0,\ s \leftrightarrow u \quad \equiv \quad s \leftrightarrow -s)$

- 'reality' $A^*(s) = A(s^*)$

- optical theorem

$$-i[A(s+i\epsilon) - A(-s+i\epsilon)]$$
$$= \ \mathrm{Im}A(s+i\epsilon) \ = \ s\,\sigma(\Omega\Omega \to \mathrm{QFT})$$



$$\frac{1}{2\pi i}\oint_C \frac{A(s,0)}{s^3}ds = 0$$

$$I_{IR} + I_{UV} + I_{RG} = 0$$

$$I_{IR} = 4\,a_{IR}$$

$$I_{UV} = -4\,a_{UV}$$

CC term drops !

$$I_{RG} = \frac{1}{\pi}\int \frac{\mathrm{Im}\,A}{s^3} = \frac{1}{\pi}\int \frac{\sigma(\Omega\Omega \to \mathrm{QFT})}{s^2} > 0$$

$$\boxed{a_{IR} = a_{UV} - \frac{1}{4}I_{RG} < a_{UV}}$$

- $I_{RG} = a_{UV} - a_{IR}$ is nicely finite in CFT-to-CFT flows

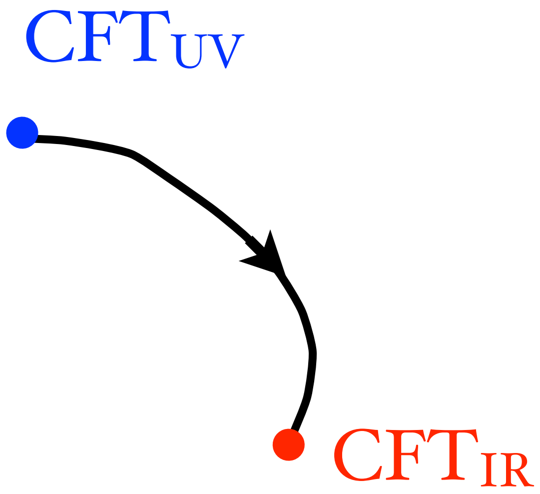
- can check directly that convergence of I_{RG} in both UV and IR corresponds to convergence of RG flow to a CFT

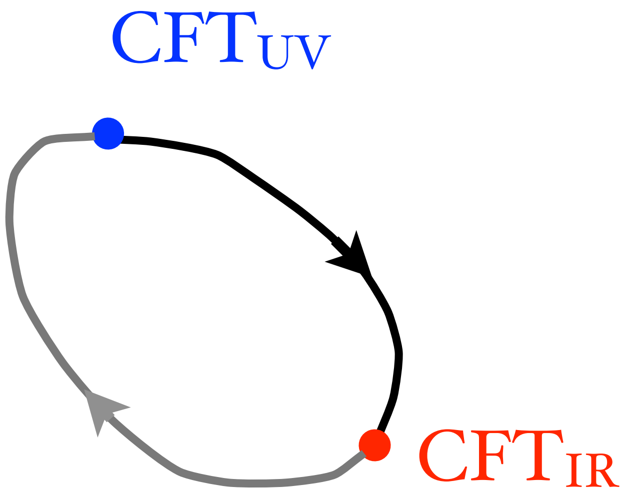
- It had to be so, since d^2A/ds^2 is finite; just a function of the renormalized QFT couplings

- Finiteness of I_{RG} → constraint on QFT asymptotics

a-theorem implies the deep notion of irreversibility of RG flow

given





is
impossible

however I do not know of insightful applications in particle physics

Does a-theorem constrains phases of N_C,N_F QCD ?

$$a \propto N_S + 11N_F + 62N_V$$

UV: quarks and gluons $a_{UV} \propto 11N_F + 62(N_C^2 - 1)$

IR: assume chiral symmetry breaking vacuum and mass gap

$N_F^2 - 1$ NG-bosons $a_{IR} \propto N_F^2 - 1$

easy to check that $a_{IR} < a_{UV}$

for any asymptotically free choice of N_F and N_C