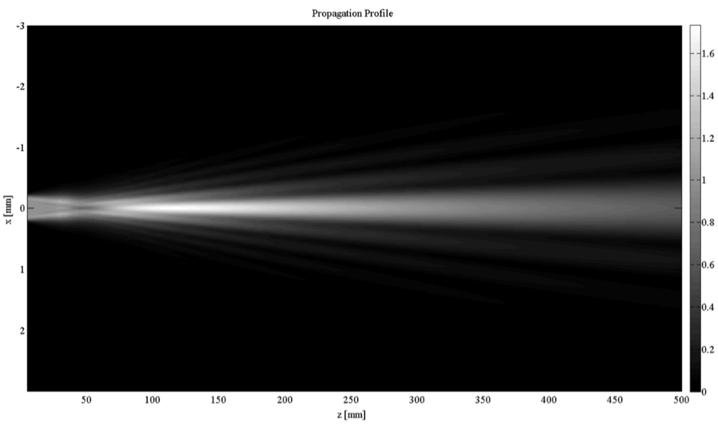
Diffraction

Outline

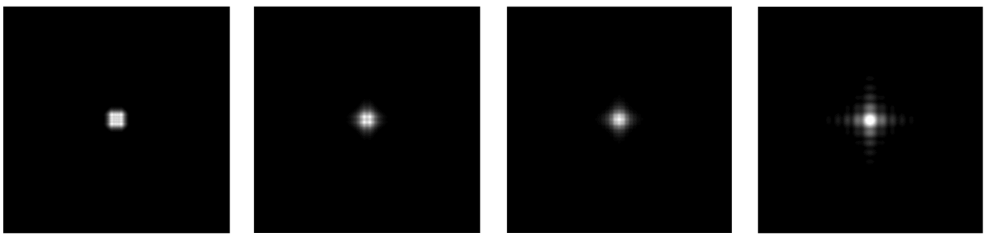
1. Near field/far field
2. Fourier transforms and linear systems
3. Huygens, spherical waves, Fresnel diffraction
4. Fraunhoffer diffraction
5. Angular spectrum and non-paraxial propagation
6. Evanescent waves
7. Wavelength dependence of diffraction

<u>Near field-Far field</u>: Let's start by propagating a square aperture to the far field using the code. Very close to the aperture we have the pattern being same as the input. This is the near field.

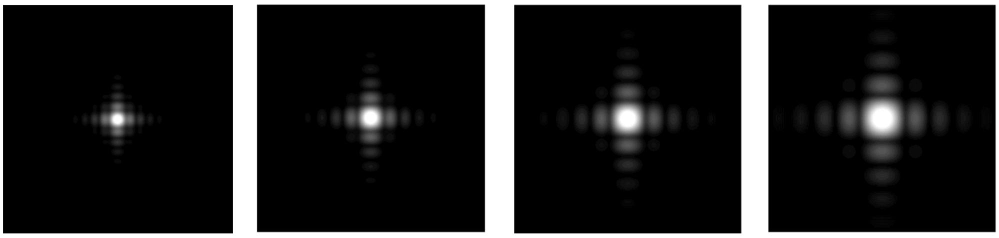
After a while the diffraction pattern of the square breaks up in multiple lobes that keep evolving (Figure 3.1a) and eventually the pattern stabilizes (Figure 3.1b) maintaining its shape but growing as z increases (Figure 3.1c).



(a)



(b) z=20mm z=60mm z=80mm z=200mm



(c) z=200mm z=300mm z=400mm z=500mm

Figure 3.1

By definition we are in the far field when the amplitude of the diffraction pattern no longer changes its shape as z increases but it gets larger. The phase of the pattern is quadratic. When viewed over a small fixed region in x-y the quadratic curvature becomes increasingly slower finally at large z the wavefront becoming practically planar. If we look at the entire x-y plane for large z, the magnitude of the field becomes a sync function, which is the magnitude of the Fourier transform the square: This is generally true: ***The magnitude of the far field is the magnitude of the Fourier transform of the near field.***

To get a better idea for why this is let's look at the diffraction pattern of a square aperture with a grating inside (Figure 3.2). As we propagate the diffraction pattern starts splitting into 3 distinct beams, eventually each of the 3 beams becomes a separate sync function (the FT of the square aperture).

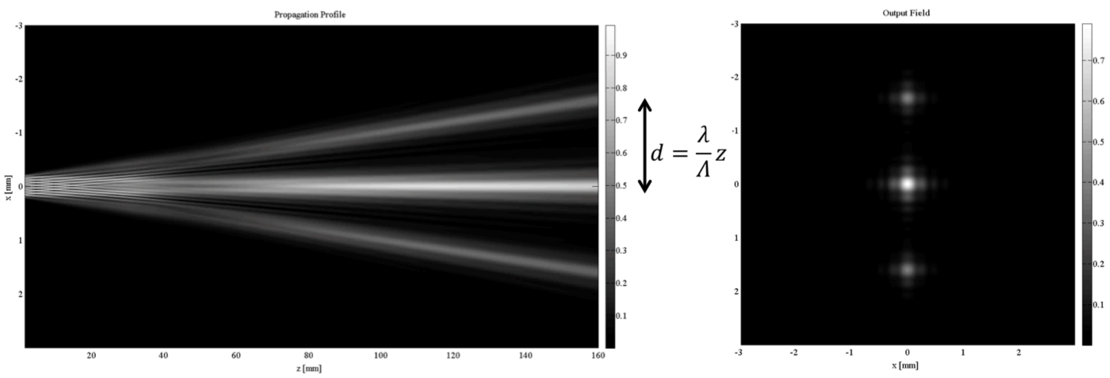


Figure 3.2

We can interpret this as each of the 3 frequency components of the input grating launching a wave in a separate direction. For a grating vector $\vec{K} = (K_x, K_y)$ the wavevector launched is a plane wave with wavevector $\vec{k} = (k_x = K_x, k_y = K_y, k_z = \sqrt{k^2 - K_x^2 - K_x^2} \quad with\ k_x = k\cos\theta_x \quad k_y = k\cos\theta_y$

As these beams propagate at different angles away from the near field they separate spatially (in x and y) from each other and eventually the field we measure at a particular location in x-y in the far field is the strength (amplitude) of the field of the wave that is launched in a particular direction.

A beam launched in a direction θ_x will arrive at position $x = \lambda z / \Lambda$. Recognizing that u=1/Λ is the spatial frequency we have a mapping from frequency in the near field to spatial coordinate (x and y) in the far field.

Returning to the BPE, in the frequency domain we have

$$\tilde{A}(k_x, k_y, z) = \tilde{A}(k_x, k_y, z=0) e^{+j\left(\frac{k_x^2 + k_y^2}{2k}\right) z}$$

In order to get back to the space domain we take the Fourier transform of the product above. This yields the convolution of $A(x,y,z=0)$ with the Fourier transform of the quadratic phase term. What we would like to do next is switch to the integral based description of diffraction (traditional Fourier Optics) instead of the differential equation description of the BPE approach. We will first do a parenthesis to review Fourier transforms and in a particular 2D Fourier transforms.

2D Fourier transforms:

The 2D FT of a function A(x,y) is

$$\tilde{A}(k_x, k_y) = \iint A(x,y) e^{-j(k_x x + k_y y)} \, dx \, dy$$

where u and v are the spatial frequencies in cycles per mm (c/mm). The amplitude of the FT measures the strength of a frequency component in A(x,y) with grating vector

$$\vec{k} = \begin{bmatrix} K_x \\ K_y \end{bmatrix} = \begin{bmatrix} 2\pi u \\ 2\pi v \end{bmatrix}, \text{where } u = \frac{1}{\Lambda_x}, v = \frac{1}{\Lambda_y}$$

$$\left|\vec{k}\right| = \frac{2\pi}{\Lambda} = \sqrt{\left(\frac{2\pi}{\Lambda_x}\right)^2 + \left(\frac{2\pi}{\Lambda_y}\right)^2}$$

We plot in Figure 3.3 $\cos(K_x x + K_y y)$ for $K_y = 0$, $K_x = 0$, and $K_x = K_y$. Notice that we can think of each of the 2D gratings as having a frequency 1/Λ and that can be rotated in any direction. The Fourier Transform is a complex function. The magnitude conveys the strength of the grating component in A(x,y) and the phase the precise position of the fringes. The inverse FT is

$$A(x,y) = \iint \tilde{A}(k_x, k_y) e^{+j(k_x x + k_y y)} \frac{dk_x dk_y}{(2\pi)^2}$$

Simply plugging in the FT equation into the inverse FT we can verify that the inverse reproduces the original A(x,y). The inverse FT says that A(x,y) can be expressed as a linear combination of complex sinewaves.

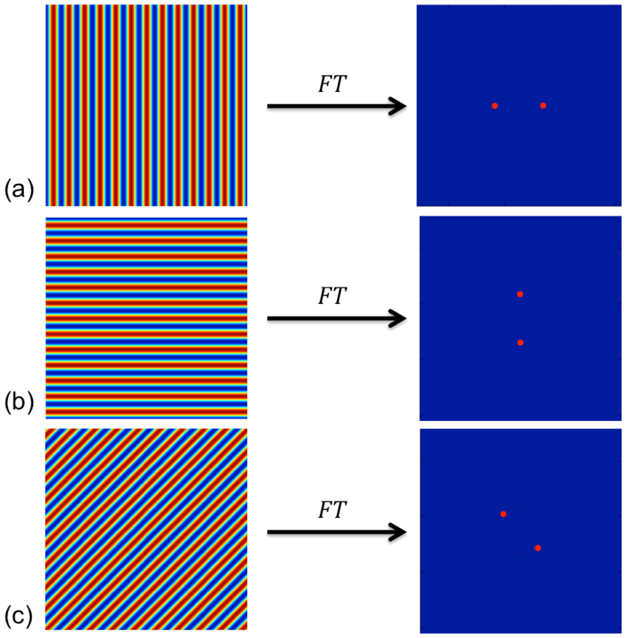


Figure 3.3

We show a number of examples of 2D FTs in Figure 3.4. Of particular interest in the 2D sync and the Airy functions.

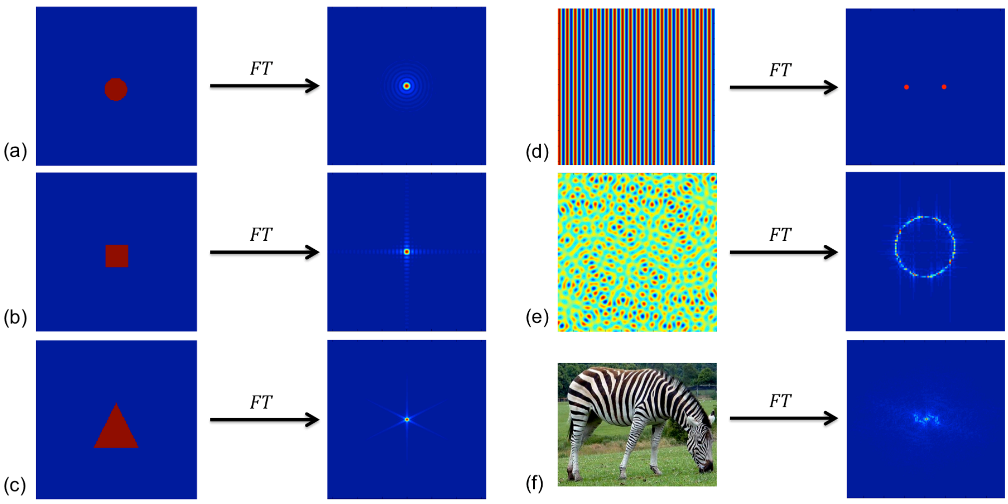


Figure 3.4

<u>Fresnel Diffraction</u>: The Fresnel diffraction formula is written below:

$$A(x',y',z) = \frac{1}{\lambda z}\iint A(x,y,z=0)e^{-j\frac{\pi}{\lambda z}\left[(x-x')^2+(y-y')^2\right]}dx\,dy$$

We can derive the Fresnel diffraction formula from the Huygens principle. Each point at the input radiates a spherical wave with amplitude and phase given by the amplitude and phase of the field at that point. The expression for the spherical wave is:

$$E_{sph}(x,y,z,t) = \frac{A_0}{j\lambda r} e^{j(\omega t - kr)}, \text{ where } r = \sqrt{x^2 + y^2 + z^2}$$

The phase kr is due to the delay of a wave disturbance propagating outward from the point source of the wave, the $1/r$ term is due to energy conservation. The area of the sphere which the wave crosses as it propagates away from the origin grows as r^2 and the power crossing the sphere at any r needs to stay constant therefore the field is 1/r and the intensity $1/r^2$. We can rewrite the spherical wave using the paraxial approximation $r = \sqrt{x^2 + y^2 + z^2} \sim z + \frac{x^2 + y^2}{2z}$ in the exponent and using $r \sim z$ in the denominator. By inspection then the Fresnel diffraction formula simply says that we calculate the wave propagation as a superposition of spherical waves launched by the wave at preceding z-planes. This is the interference point of view of the diffraction process.

<u>Alternative derivation of the Frensel Formula:</u> Sinewaves are important because they are eigenmodes of linear shift invariant systems. Specifically consider the system we have been dealing with and shown in Figure 3.5. If each point in A(x,y,z=0) is one of the inputs and A(x,y,z) at each point (x,y) at z is an output, then each output in a linear combination of the inputs since linear superposition holds in the wave equation and hence the BPE. If we say that the "weight" of the connection from point (x,y) at z=0 to a point (x',y') at z is h(x,y,x',y') then we can write the following:

$$A(x',y',z) = \iint A(x,y,z=0)h(x,y,x',y')\,dx\,dy$$

The "kernel" or impulse response *h* is a 4D function and it can be simplified by observing that a shift in x or y in the input is expected to cause the same shift in the output:

$$A(x',y',z) = \iint A(x,y,z=0)h(x-x',y-y')\,dx\,dy$$

This system is a linear shift invariant system. It is also called the convolution between A(x,y,z=o) and h(x,y). If the input to this system is a complex 2D sinewave e $^{j(K_x x+K_y y)}$ then the output is the same complex sinewave (an eigenfunction) times a constant (the eigenvalue). The only pattern that has this magical property is the sinewave. If we enter the input in the equation above we obtain:

$$\iint e^{+j(K_x x+K_y y)} h(x-x',y-y')\,dx\,dy = e^{+j(K_x x+K_y y)} \iint h(x'',y'') e^{-j(K_x x''+K_y y'')} \frac{dx''dy''}{(2\pi)^2}$$

where $x'' = x' - x \Rightarrow x = x' - x''$

This is interesting: The eigenvalue is the 2D FT of h(x,y). We expect *h* to depend on z. We know from the plane wave solution of the wave equation that

$$H(K_x,K_y) = e^{-j\sqrt{k^2-K_x^2-K_y^2}\,z} \approx e^{-jk_z}e^{+j\left(\frac{K_x^2+K_y^2}{2k}\right)z}$$

Using the FT pair for a complex Gaussian beam we can obtain once again the Fresnel diffraction formula. This is left as an exercise.

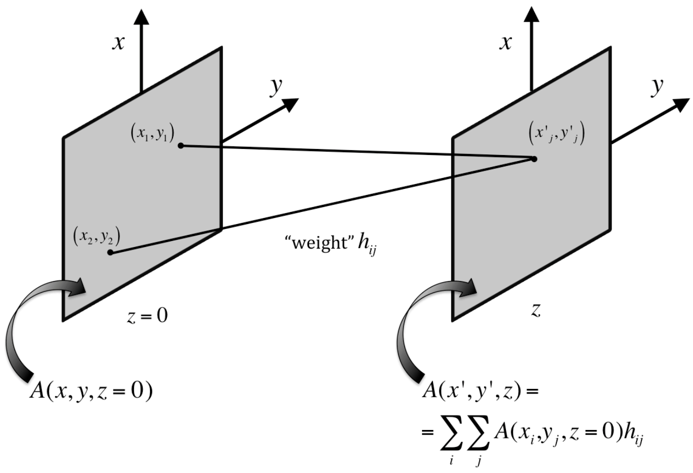


Figure 3.5

Far Field or Fraunhoffer diffraction: It is now easy to show from the Fresnel formula that the far field is the FT of the near field distribution:

$$A(x',y',z) = \frac{1}{\lambda z}\iint A(x,y,z=0)e^{-j\frac{\pi}{\lambda z}\left[(x-x')^2+(y-y')^2\right]}dx\,dy$$

In order to obtain the FT relationship the quadratic phase term $e^{j\pi x^2/\lambda z}$ must be approximately one. This gives us a condition for being in the far field:

$$\frac{x^2}{\lambda z} << 1$$

Angular Spectrum: The form of the BPE in the Fourier domain suggests suggests that each frequency component that makes up the input waveform A(x,y,z=0) picks up a quadratic phase $e^{j\frac{k_x^2+k_y^2}{2k}z}$ as it propagates a distance z. Multiplying both sides of the BPE in the Fourier domain by e^{-jkz} we can recognize that the phase term that each frequency component picks up is simply $e^{-jk_z z}$ written after making the paraxial approximation. The spectral decomposition of an input distribution in its 2D frequency components k_x *and* k_y and the launching of a plane wave with wavevector $\vec{k} = (k_x,k_y,k_z)$ with a corresponding complex amplitude $\tilde{A}(k_x, k_y, z)$ is called the *Angular Spectrum.* It is the complementary point of view of the Huygens principle of the launching of spherical weaves conveyed by the Fresnel diffraction formula.

Non-paraxial Beam propagation: A very interesting observation from the discussion in the previous paragraph is that we can calculate the propagation of the optical wave without ever having to invoke the paraxial approximation. We can simply use $e^{-jk_z z} = e^{-j\sqrt{k^2 - k_x{}^2 - k_y^2}\, z}$ and in free space this is 100% accurate. The beam propagation code contains a version that is non-paraxial. Using this code we can compare the propagation of a grating with and without the paraxial approximation for two different spatial frequencies. There is so no real advantage in using the paraxial approximation if we use a computational approach. The advantage of the paraxial version of the BPE and the Fresnel diffraction formula is that is much easier to handle analytically the propagation of classic paraxial behaviors such as Gaussian beams, Bessel beams, and Talbot planes.

Evanescent Waves: As the spatial frequency of k_x and or k_y increases then the argument of the square root becomes negative and therefore the square root imaginary. This gives us

$$e^{-jk_z z} = e^{-j\sqrt{k^2 - k_x^2 - k_y^2}\, z} = e^{-\alpha z}$$

a real exponential. Without any gain this will be a decaying exponential. The rate at which the field decays is $\alpha = \sqrt{k_x^2 - k^2}$. This quantity is zero when the period Λ of the grating is equal to the wavelength of light λ. Figure 3.6 shows the non-paraxial propagation of gratings of different periods demonstrating that for subwavelength grating periods the diffracted wave decays.

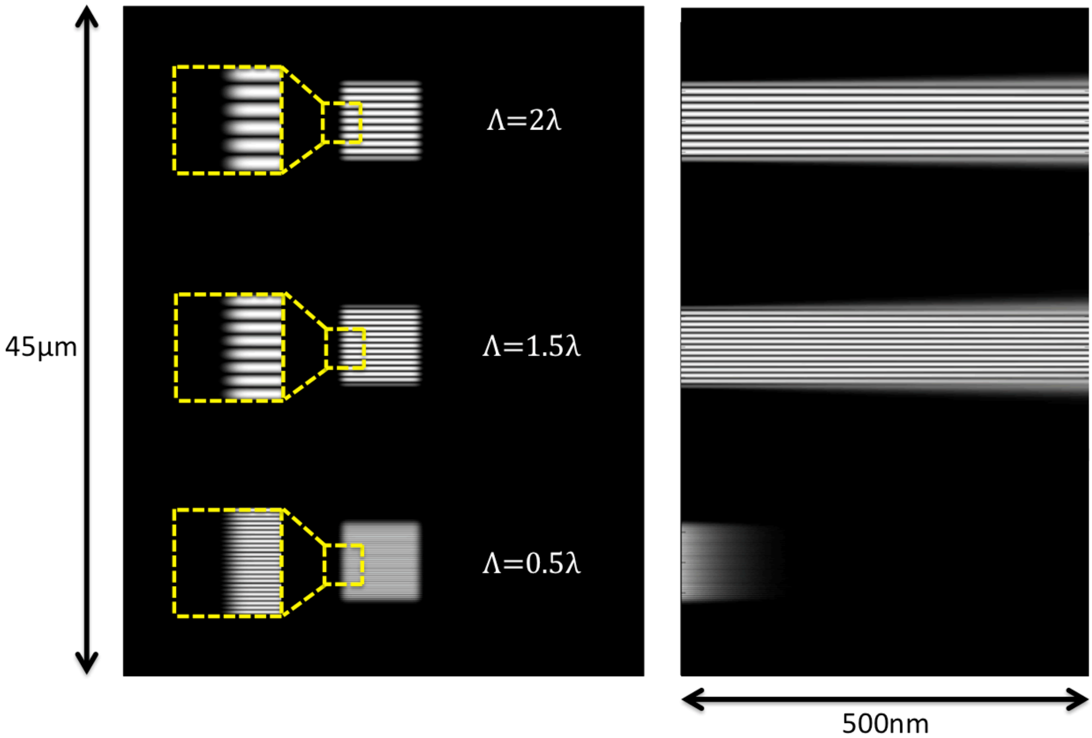
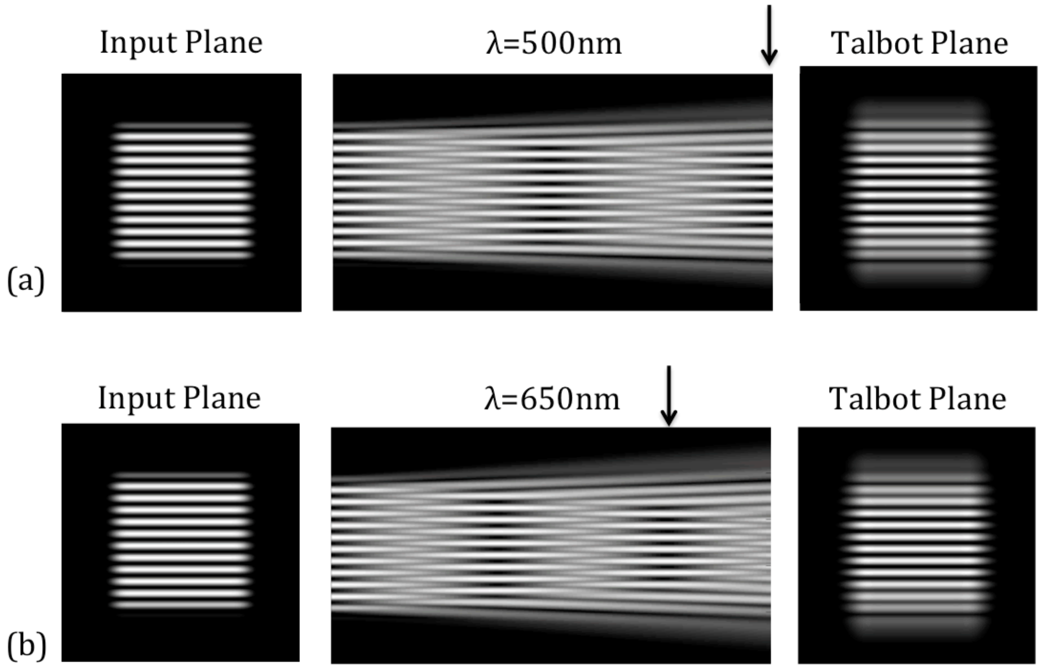


Figure 3.6

<u>**Wavelength Dependence:**</u> The diffraction equations contain the wavelength of light λ. Therefore, we expect the pattern to change as we change the wavelength of the light. Inspection of the Fresnel diffraction formula shows that λ always appears as a product with z. That means if $\lambda \rightarrow \lambda'$ then the diffracted wave at $z' = z\lambda / \lambda'$ will be the same as the one at z at wavelength λ. For example the light diffracted by a grating with $K_x = 1/\Lambda$ will diffract light at an angle $\sin(\theta) = \lambda / \Lambda$. In other words the diffraction angle will change with wavelength but the lateral displacement (in x) of the beam will be the same if we move from z to z'. Figure 3.7 shows the diffraction of a grating (including the Talbot planes) at two different wavelengths.



Input Plane
λ=500nm
Talbot Plane
(a)
Input Plane
λ=650nm
Talbot Plane
(b)

Figure 3.7