Imaging Optics - Lecture 2 – 5 March 2021

Diffraction

Outline

1. Intensity
2. Phase
3. Near field
4. Plane wave
5. Gaussian beam
6. Periodic patterns; gratings
7. Non-diffracting beams

Last time we used the paraxial approximation to derive the Beam Propagation Equation (BPE) starting from the wave equation:

$$\frac{\partial A}{\partial z} = -\frac{j}{2k}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)$$

where A is the slow varying envelope of one of the transverse components of the electric field.

$$E_x(x,y,z,t) = A(x,y,z)e^{j(\omega t - kz)} \quad k = 2\pi/\lambda = \omega/v \quad v = 1/\sqrt{\mu\varepsilon}$$

where v is the speed of light. $A = |A|e^{-j\varphi}$ is the complex amplitude of the slowly varying envelope. The actual electric field is

$$\mathrm{Re}\{E_x\} = |A|\cos[\omega t - kz - \varphi]$$

In free space the speed of light is v=3x10^8 m/sec. For visible light, the wavelength is around λ=0.5 μm. This means the temporal frequency of light at this wavelength is ν=0.6x10^{15} Hz.

<u>Intensity</u>

At the high temporal frequencies of optical waves, it in not possible to directly measure currents generated by moving charges on a metalic conductor for example. Instead typically optical detectors work by freeing bound charges and then collecting the freed electrons (or holes) to a site where they can be measured. Therefore the measured signal is not proportonal to the field itself but rather proportianal to the optical power incident on the detector. The power per unit area carried by an optical field is called the intensity (I) and it is given by

$$I = \frac{\sqrt{\varepsilon}}{2\sqrt{\mu}}|A|^2$$

where $|A|^2 = AA^*$ with the asterisk indicating the complex conjugate. In order to calcultate the signal measured by a detector we use the BPE to calculate the complex field at the detectore plane and then we simply take the magnitude squared of the field.

Why is the intensity the magnitude square of the electric field? The power carried by an electromagnetic wave is given by the Poynting vector

$$\vec{P} = \vec{E} \times \vec{H}$$

and the intesity crossing into a unit area dS of the detector is

$$I = \vec{P} \cdot d\vec{S} = E_x H_y$$

For a plane wave propagating in z we can use Maxwell's equations to express H_y in terms of E_x:

$$\frac{\partial H_y}{\partial z} = jkH_y = j\omega E_x \Rightarrow H_y = \frac{\omega\varepsilon}{k}E_x = \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}E_x$$

Therefore in the case of a plane wave the intensity is proportinal to the electric field squared. This is in general true for any paraxial wave. In particlular we can calculate the intensity by taking the magnitude square of the slow varying envelope $A(x,y,z)$. In real notation

$$I(t) = \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}E_x^2 = \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}|A|^2\cos^2(\omega t - kz - \phi) = \frac{\sqrt{\varepsilon}}{2\sqrt{\mu}}|A|^2 + \frac{\sqrt{\varepsilon}}{2\sqrt{\mu}}|A|^2\cos\ (2\omega t - 2kz - 2\phi)$$

The second term oscillates at twice the optical frequency and its temporal variation is averaged out by the detector. Therefore the average intensity measured by the detector is only the first term which is the same as the magnitude square of the complex electric field.

Phase

It might appear that the phase is not important since we cannot directly measure it. This is not true. Phase is at least as important as amplitude for characterizing the propagation of optical waves. In order to appreciate this we plot in Figure 2.1 the diffraction pattern (x-z plot) of a square aperture. Figure 2.1b is a square pattern that is amplitude modulated (A=1 inside the square and zero outside) whereas Figure 2.1a shows the same for a phase modulated square aperture (φ=π inside and 0 outside). Clearly both apertures diffract even though with some differences. If we looked at the two apertures with our eyes in white light they would look very different: an empty square in a metallic screen versus a transparent thin film with some barely detectable thickness nonuniformity in the middle. The diffraction patterns by the laser are similar however. Why? Because the BPE says that as long as the transverse derivatives are non zero we will have changes as we propagate in z independently of whether these transverse variations are due to changes in amplitude or phase.

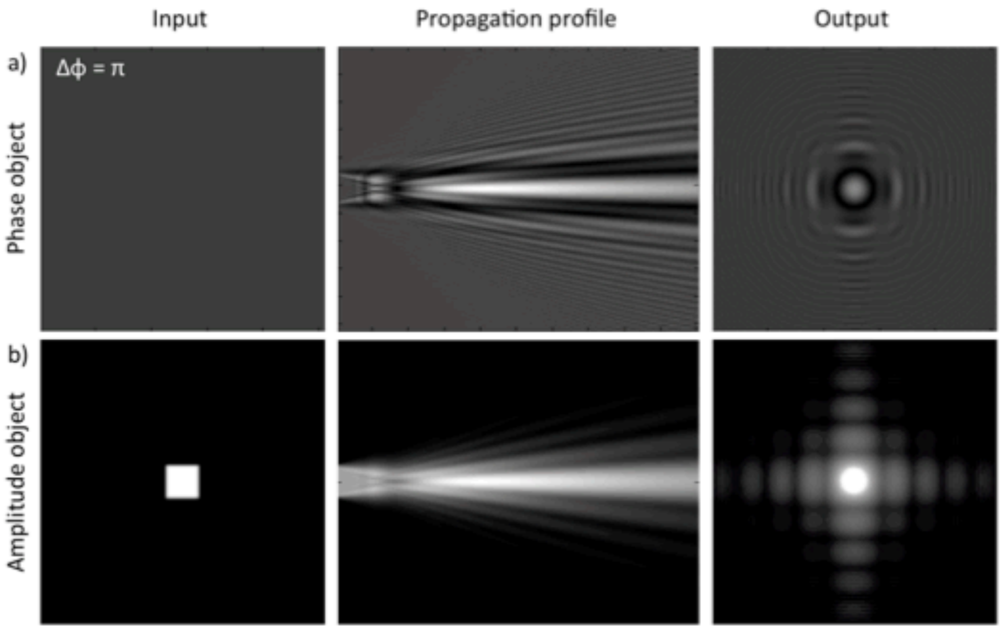

Figure 2.1

Near field

If we look_at the diffraction pattern for very small z then the field looks quite similar to what it was at z=0. We can see why that is from the BPE in the Fourier plane:

$$\tilde{A}(k_x,k_y,z) = \tilde{A}(k_x,k_y,z=0)e^{j\frac{k_x^2+k_y^2}{2k}z}$$

If $(k_x^2+k_y^2)z/2k \ll 1$ then the exponential term is approximately equal to 1. We can use this relationship to derive an estimate for the maximum z for which are still in the near field. We can, for example, calculate the mean square difference between $A(x,y,z=0)$ and $A(x,y,z)$ normalized to the input image energy as a function of z and measure how fast that rises. In this way we can experiment with the code for how fast different images get out of the near field, which means how far in z do we have to look in order to see significant diffraction effects. Figure 2.2 shows what happens with two different inputs, Sir Isaac Newton portrait and a smooth square aperture. The high spatial frequencies contained in the former yield to a mean square difference which increases faster compared to the latter's one.

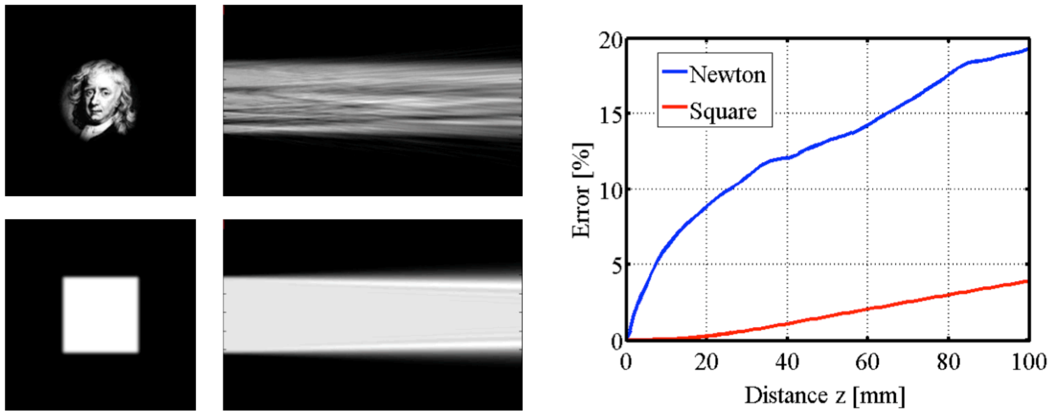

Figure 2.2

The error has been calculated as:

$$\text{error}(z) = 100 \times \frac{\iint \left| \left(|A(x,y,z)| - |A(x,y,z=0)| \right) \right|^2}{\iint |A(x,y,z=0)|^2}$$

Plane waves

If a plane wave propagates in the z direction, then A is constant and nothing changes for the slow varying envelope. For a plane wave propagating at an angle θ with respect to the z axis the electric field is

$$E_x = Ce^{j(\omega t - k_x x - k_z z)} = Ce^{j(\omega t - kz)} e^{j(kz - k_x x - k_z z)}$$

$$where \quad k_x = k\sin\theta \;\; k_z = \sqrt{k^2 - k_x^2} \simeq k - k_x^2 / 2k$$

where C is a constant that expresses the strength of the field and the paraxial approximation above is valid if θ is small . Under the paraxial approximation we can write the field as

$$E_x(x,y,z,t) = A(x,z)e^{j(\omega t - kz)} \;\; where \;\; A(x,z) = e^{-jk_x x} e^{jk_x^2 z/2k}$$

If we want to represent a plane wave in the BPM code we need to be concerned about the fact that plane waves are defined in the infinite space. Since the waves are represented in the Fourier domain in the BPM, we can select the input window to be an integral number of periods in x and y. Then the FFT automatically assumes that the input in repeated indefinitely in x and y and a plane wave can be represented in this way in the BPM code.

A better solution is to realize that in real life we don't have infinite things. In real life, we encounter find waves with planar wavefronts within a finite aperture. Then we can select an FFT window sufficiently larger than the aperture of the wave to give room to the beam to diffract. This is shown in Figure 2.3a for various values of K_x, K_y.

In order to calculate the diffraction patterns in Figure 2.3a, we set

$$A(x,y,z=0)=rect(x/a)e^{jK_x x} \quad K_x = k\sin\theta$$

where θ is the angle respect to the z axis at which the wave is directed towards. We can measure the main angle at which the wave propagates in the simulation for different θ's and compare to the value that was used to set the boundary condition (the field at z=0). Figure 2.3b is a plot of the diffracted angle measured through the simulation versus the angle inserted at the input. The two angles start deviating from one another, as the paraxial approximation is no longer satisfied.

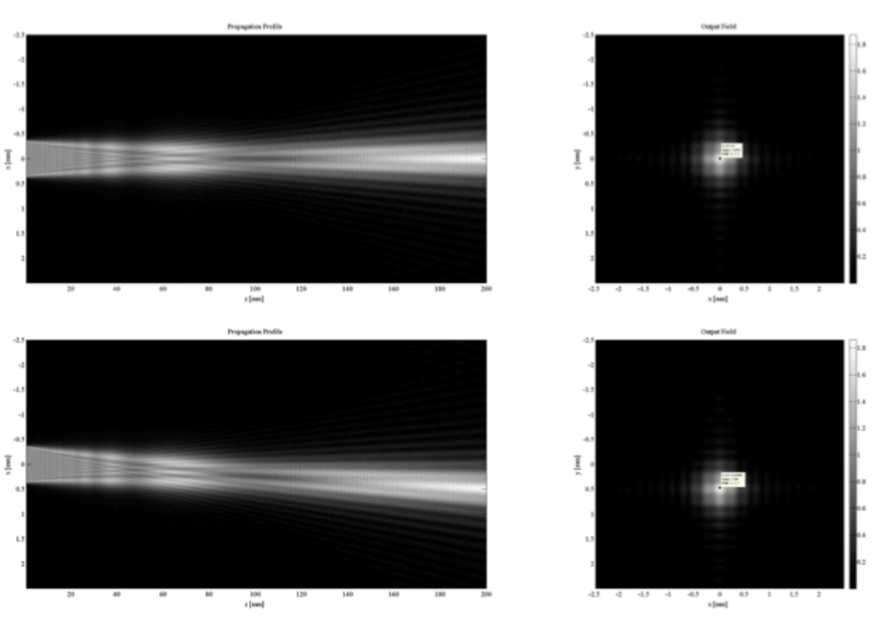

Figure 2.3a

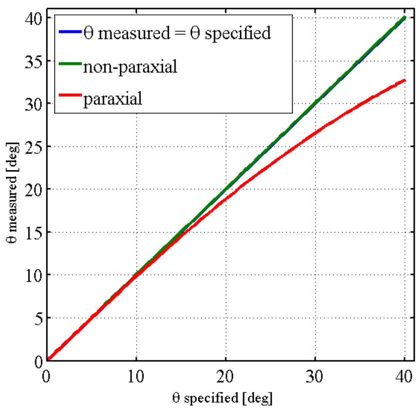

Figure 2.3b

Gaussian beams

The above example gives us the idea that finite aperture beams are easier to follow through with the BPM and closer to reality in many optical systems.. The edges of the square produce artifacts and extra scattering. We can minimize the scattering artifacts by using a **Gaussian beam.** The definition of a Gaussian beam is:

$$E(r,z) = E_0 \frac{w_0}{w(z)} \exp\left(\frac{-r^2}{w^2(z)}\right) \exp\left(-ikz - ik\frac{r^2}{2R(z)} + i\zeta(z)\right)$$

beam diameter	$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$
Rayleigh range	$z_R = \frac{\pi w_0^2}{\lambda}$
radius of curvature	$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$
divergence angle	$\theta \approx \frac{\lambda}{\pi w_0}$
Gouy phase shift	$\zeta(z) = \arctan\left(\frac{z}{z_R}\right)$

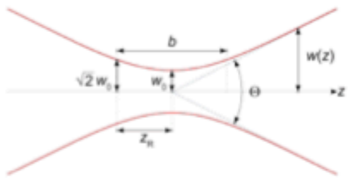

Table 2.1

In figure 2.4 we show the propagation of a Gaussian beam. It comes to a focus and back out.

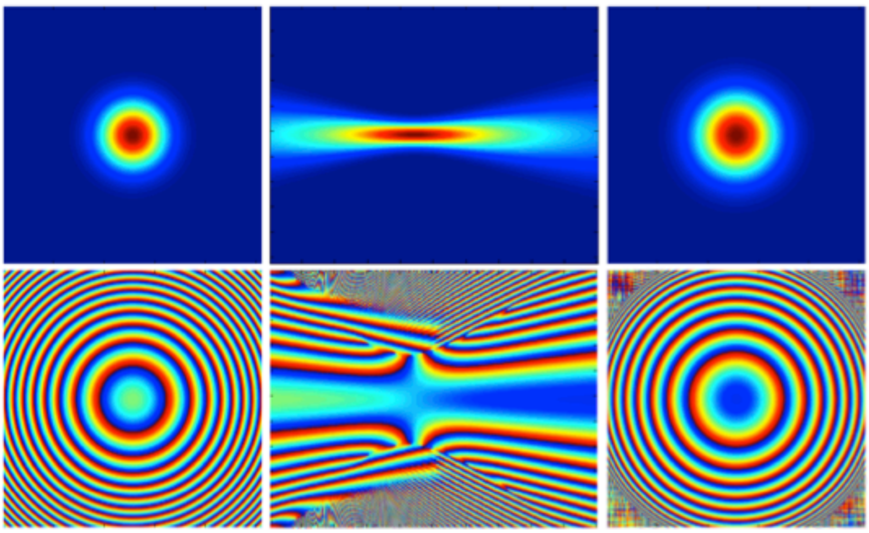

Figure 2.4

An interesting feature of a Gaussian beam is that it always stays Gaussian. We can show this using the BPE equation:

$$A(x,y,z=0)=e^{-(x^2+y^2)/a^2} \quad \Rightarrow \quad \tilde{A}(k_x,k_y,z)=FT\{e^{-(x^2+y^2)/a^2}\}e^{j(k_x^2+k_y^2)z/2k}$$

Using the FT pair $e^{-\pi x^2} \Leftrightarrow e^{-\pi u^2}$ *with* $k_x = 2\pi u$ and applying the scaling theorem you can show that the transverse shape of the amplitude of the propagating field is a Gaussian for all z and the phase is a spherical wave. Carrying out the painful algebra should give the expression for the Gaussian beam in Table 2.1 above. Notice that there is a 180 degree phase shift from z = -∞ to z= +∞. This extra phase shift in addition to the e^{jkz} phase shift is due to the detour the photons take to pass by the focus at z=0 compared to the straight through path of a place wave (that is expressed by the e^{jkz}. For large numerical aperture most of this phase shift happens right at the focus area because the detour is most abrupt there since the focusing wave has to straighten out into a collimated beam and back to a diverging wave in a small distance.

Laser cavities usually produce Gaussian beams or their close relatives as their output. The reason is the laser is usually made of a cavity where light bounces back and forth between two mirrors. In order for the cavity to be stable the beam needs to reproduce its shape after it goes through one round trip in the laser. The property of the Gaussian beam to remain a Gaussian after propagating together with the right magnification in the cavity makes the natural modes of the cavity be Gaussian.

Periodic patterns

An important class of diffraction patterns are those resulting from periodic inputs. A periodic function has the following property:

$$A(x-a,y,z=0)=A(x-a,y,z=0) \quad \Rightarrow \quad \tilde{A}(k_x,k_y,z=0)=\tilde{A}(k_x,k_y,z=0)e^{jk_x x}$$

This can only be satisfied when $a = 2\pi m / k_x$ where m is an integer and the spectrum is zero everywhere else. In other words periodic functions have discrete set of frequencies instead of a continuous spectrum. They have a Fourier series. The simplest example is a simple cosine.

$$A(x,y,z=0)=\frac{1}{2}+\frac{1}{2}\cos(K_x x)$$

The cosine is periodic with period $a = 2\pi / K_x$. The diffraction pattern of such a "grating" is shown in Figure 2.5.

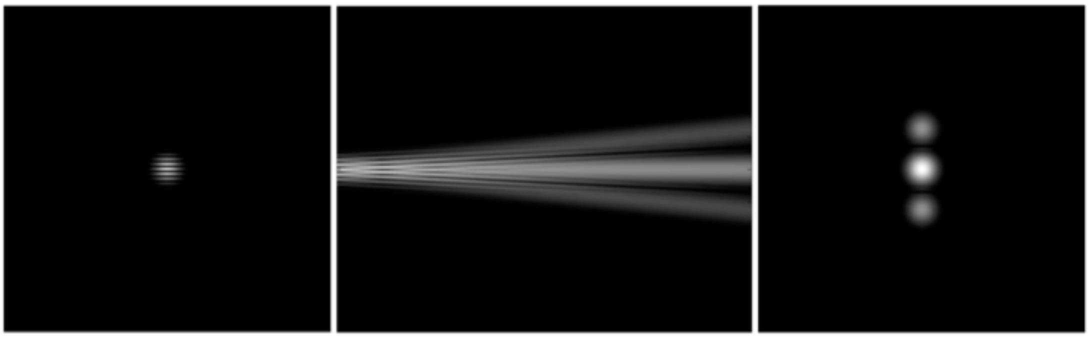

Figure 2.5

We observe 3 separate diffraction orders, one for each of the 3 frequencies that make the Fourier transform of the cosine. The idea is very similar to the idea of a plane wave as an input that we discussed earlier, except each plane wave component is now launched by the corresponding spatial frequency in the input wavefront.

We can also have period waves with phase modulation of the input frequency:

$$A(x,y,z=0)=e^{jm\sin(K_x x)}$$

The diffraction pattern of this phase sinusoidal grating is shown in Figure 2.6.

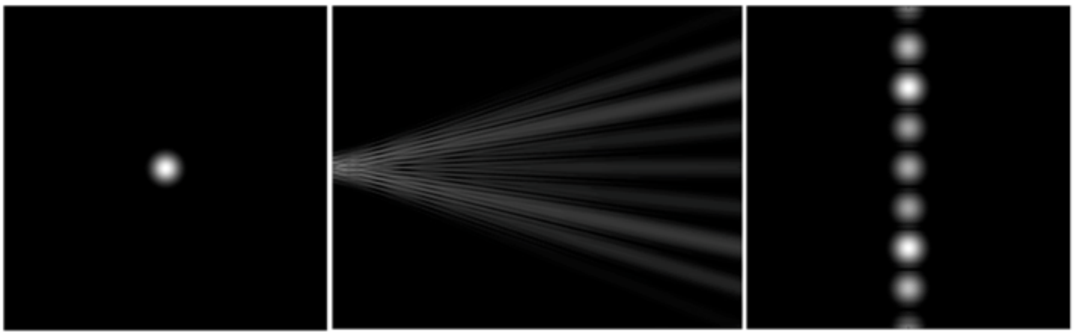

Figure 2.6

The pattern is periodic with period $a = 2\pi / K_x$ but there are more orders than in the simple sinusoidal amplitude grating. We can understand that by taking the Taylor series of the phase grating and obtain terms proportional to $\cos^n$ where n is the order in the Taylor expansion. The formula for the series expansion of the sinusoidal phase grating is:

$$A(x,y,z=0)=e^{jm\sin(K_x x)}=\sum_n J_n\left(\frac{m}{2}\right)e^{jns K_x x}$$

Other types of Gratings are square wave grating where each period is partially on (value A=1) and partially off (value A=0). The diffraction pattern of the square wave grating is shown in Fig. 2.7.

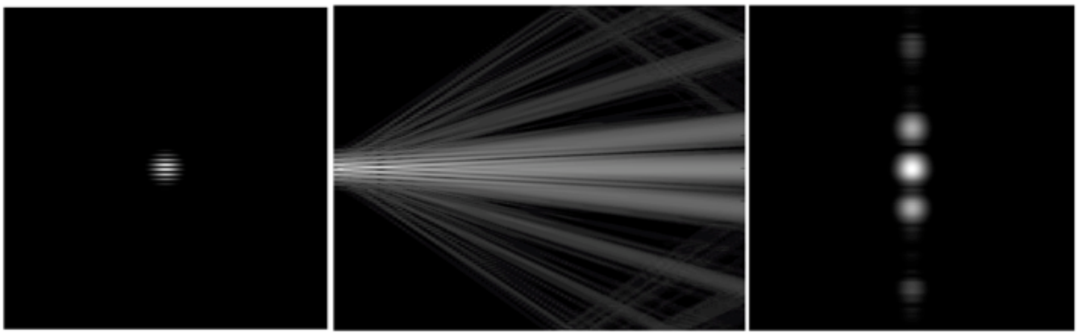

Figure 2.7

It has more than 3 orders. We can analytically predict the strength of each of the orders by calculating the Fourier transform of the square wave (the Fourier series actually). For the square wave grating we have

$$A(x,y,z=0)=\sum_{n} rect(\frac{x-na}{a/2})=\sum_{n} g_n e^{jnK_x x}$$

where $g_n = \int_{-a/2}^{+a/2} rect(\frac{x}{a/2}) e^{-j\frac{2n\pi x}{a}} dx$

We can also look at a phase square grating. Figure 2.8.

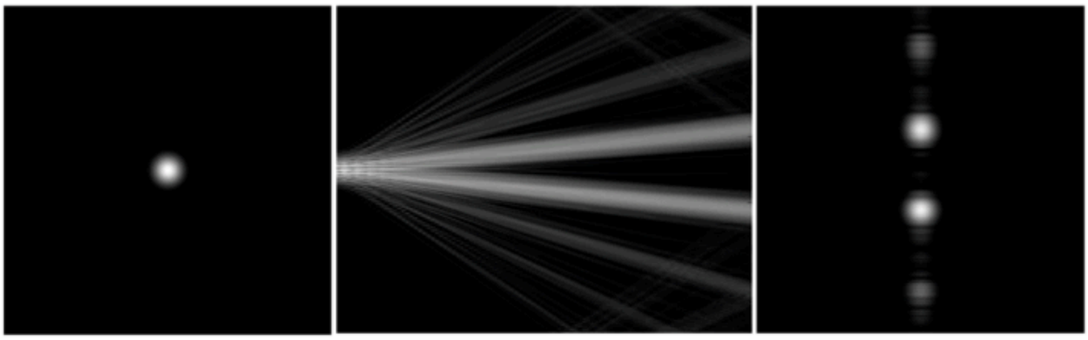

Figure 2.8

It has its own orders. Notice it does not have a zero order. Why?

The diffraction pattern of the periodic pattern in Figure 2.9 consists of a repetition of the picture of a face (Newton's face).

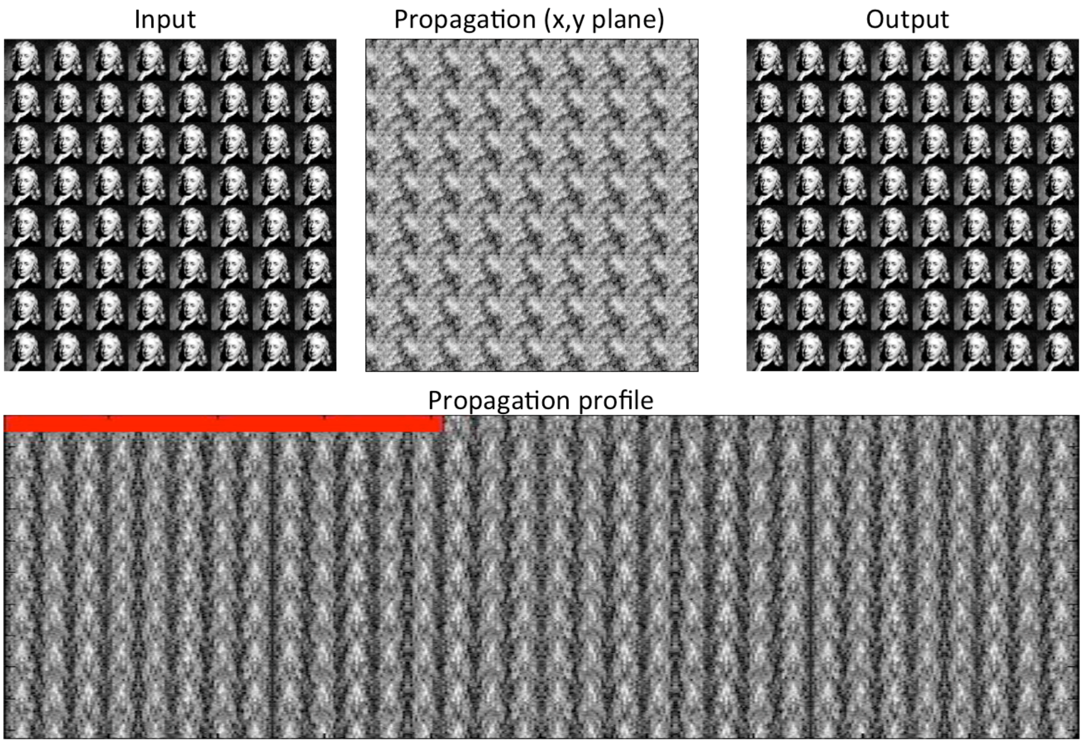

Figure 2.9

Notice that the diffraction pattern remains periodic. As we propagate in z something remarkable happens at some point. The faces reappear. As if an imaging lens had undone the smearing due to diffraction. Why does this happen? We can understand this better by considering the diffraction of a simpler periodic function: the cosine (figure 2.10).

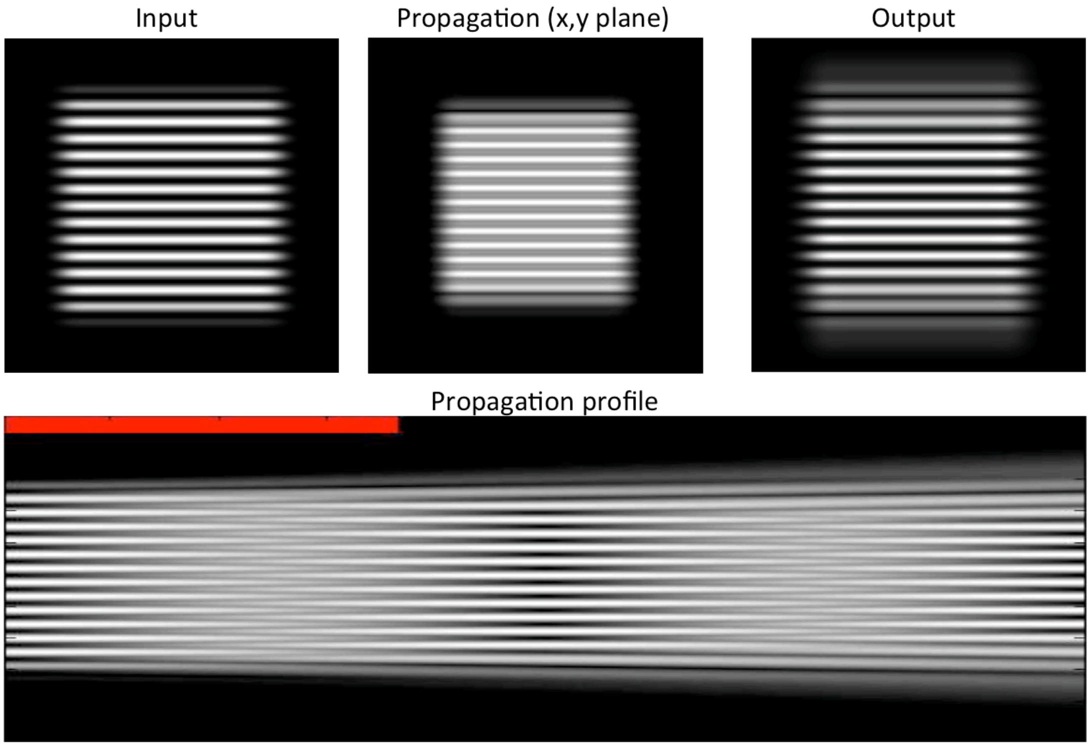

Figure 2.10

We see again that the grating replicates itself after a while and it does this multiple times. This self-duplication through diffraction process is called the **Talbot effect** and it can be explained by looking at the BPE in the Fourier plane:

$$\tilde{A}(k_x,k_y,z)=\tilde{A}(k_x,k_y,z=0)e^{j\frac{k_x^2+k_x^2}{2k}z}$$

When the input is periodic the Fourier transform is non-zero only at $k_x=\frac{2n\pi}{a}$. If we evaluate the quadratic phase term at these discrete frequency locations we obtain $e^{j\frac{k_x^2+k_y^2}{2k}z}=e^{j\frac{(2n\pi)^2}{2ka^2}z}$. This becomes 1 at z=2a^2/λ and therefore the input is reproduced at these distances.

It becomes a bit more complex for periodic patterns with more orders but the same basic idea holds: We observe Talbot planes at distances z where the quadratic term equal to 1 for all the discrete frequencies of the periodic pattern. Notice that periodic patterns in practice cannot be truly periodic because of the finite aperture of the system and effects such as the Talbot effect which relies on the periodicity eventually (for large z) do not persist due to the finite aperture.

Non-diffracting beams: If we take almost any input pattern and allow it to propagate in z (diffract) then we will see a blurring and a general expansion or spreading of the beam. We see this in the square aperture or the picture of

Newton's phase. There are some "special" patterns however that seem to defy diffraction. Consider the diffraction of the noise-like pattern in Figure 2.11.

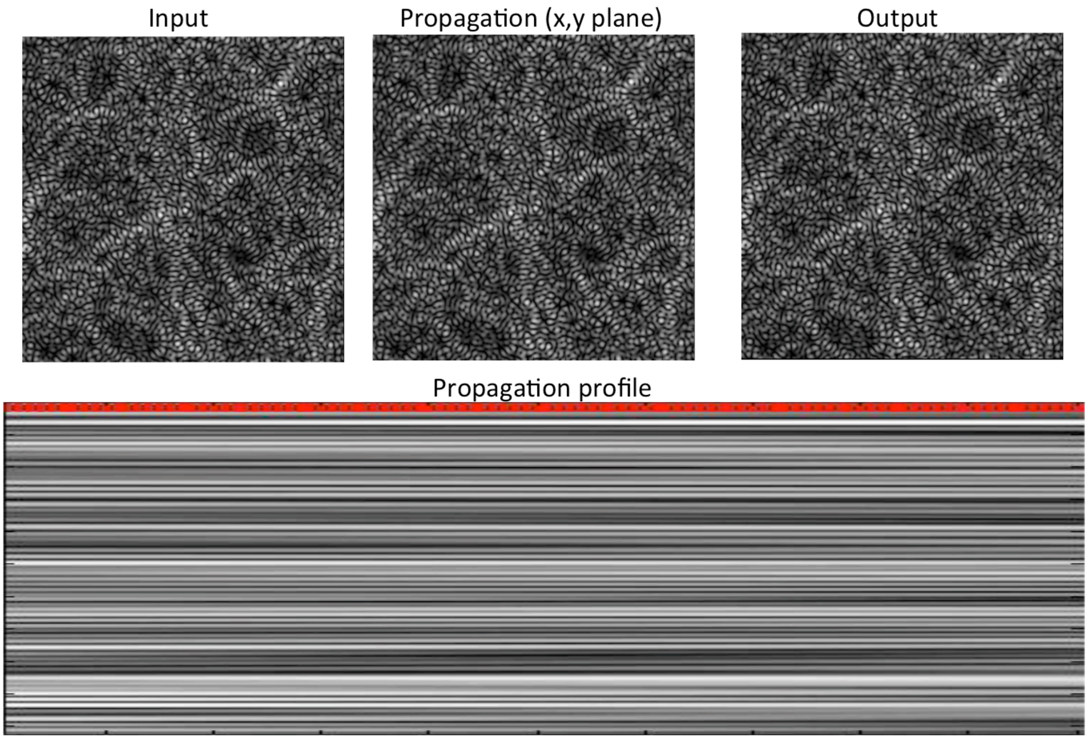

Figure 2.11

It is clear from the x-z plot that this pattern stays the same for quite a while. What is special about this pattern? If we take the 2D Fourier transform of this pattern we get a circle. The Fourier transform is non-zero only along a locus of frequency values satisfying:

$$k_x^2 + k_y^2 = \text{constant}$$

Looking at the BPE in the Fourier domain it becomes clear why this works. The $e^{j\frac{k_x^2+k_x^2}{2k}z}$ term is just a constant phase delay and therefore the patterns does not change. In Figure 2.11 a random phase pattern was multiplied by the circle and therefore the input pattern appeared random or like speckle. If we start with a clean circle and take its Fourier transform we end up with the pattern shown in Figure 2.12.

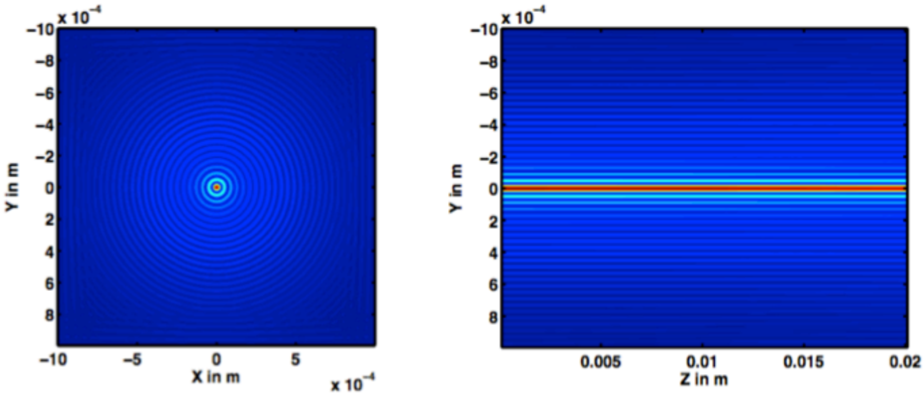

Figure 2.12

This beam is called a Bessel beam because it is a Bessel function in r, the radial direction. The field in z=0 is defined as:

$$A(x,y,z=0)=J_0(k_T\rho)$$

where $k_T^2 = k_x^2 + k_y^2$ and $\rho^2 = x^2 + y^2$.

Its Fourier transform is given by:

$$\tilde{A}(k_x,k_y)=\frac{1}{2\pi k}\delta(k-k_T)$$

If we now use the pattern $A(x,y,z=0)=J_0(k_T\rho)$ as the initial condition we get the x-z plot shown also in Figure 2.12. This is a pencil beam that propagates without diverging.