# <u>**Imaging Optics** - Lecture 1 – 26 February 2021</u>

## Light Propagation

The first thing we all learned about light is that it propagates in straight lines in air. This is **rays optics.**

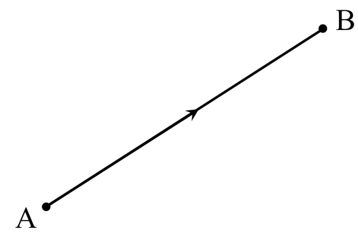



**Figure 1.1 –** Rays travel in straight lines

We can also use ray optics in inhomogeneous media. Think of a point source illuminating a dielectric interface (Figure 1.2). We can get a good idea for what is going on by using Snell's law and rays. Ray optics are in fact widely used for optical design, which is the design of optical lenses and related systems such as microscopes, telescopes, eyeglasses, spectrometers, etc. There are excellent ray tracing programs that can be used to design and optimize lenses and other optical elements. An entire course (or even a sequence of courses) on "optical design" can be offered.

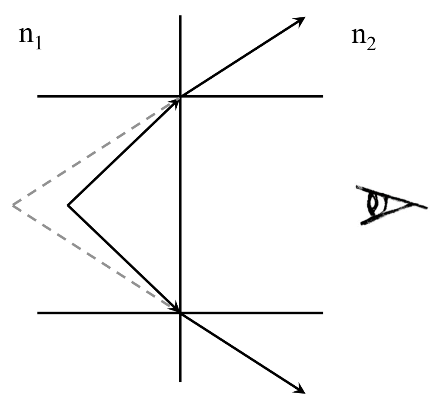



**Figure 1.2 –** Rays bend when index changes

This is not what we will do in this course. We will instead focus on wave optics and corresponding applications where the wave nature of light is essential of the problem at hand. The classic example for which ray optics fails is diffraction (Figure 1.3). Light passing through an aperture "bends" and it enters the geometric shadow area. This can be readily explained with wave optics (the Huygens principle for example). Interference is another phenomenon that can only be handled by wave optics.

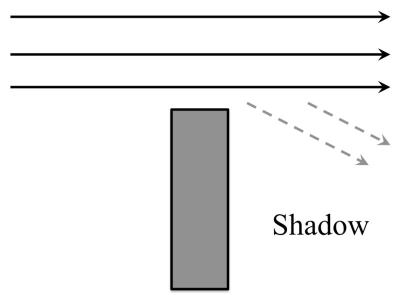



**Figure 1.3 –** Rays do not diffract

Formally, an optical ray is the vector perpendicular to the optical wavefront (Figure 1. 4). The wavefront is the locus of points in 3D space where the phase of the optical wave is the same.

Formally then

$\vec{S} = \nabla\phi$

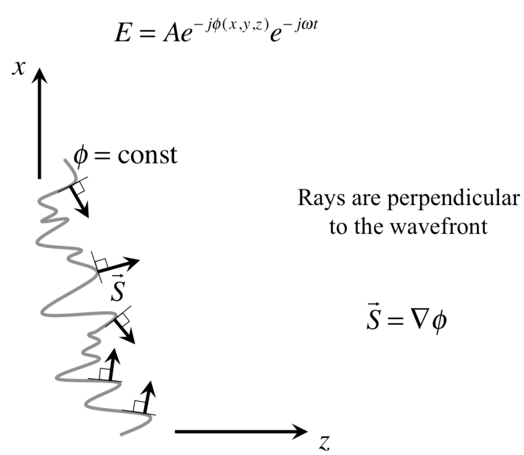



**Figure 1.4 –** Wavefronts and rays

We can follow the evolution of rays even in the presence of index variations but what we cannot do is account for the amplitude variations of the wavefront.  The aperture is an example of such amplitude variations.

Another way to understand light propagation is by recognizing that light is an electromagnetic wave and must obey Maxwell's equations. (Maxwell determined light is an electromagnetic wave when he found theoretically that the speed of electromagnetic waves is the same as the speed of light which had been previously measured).

If we start with Maxwell's equations

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

$$\nabla \cdot \varepsilon \vec{E} = \rho$$

and we are given the boundary conditions for the field we can then find the field everywhere in space. Finite element methods are often used to calculate the field everywhere. This is possible but difficult in general and certainly not intuitive. We will concentrate on a different method, the ***beam propagation method.*** We will start by solving analytically Maxwell's equations assuming we are in a region without any free charges and constant dielectric constant (homogeneous medium) and derive the simplest version of the ***beam propagation equation*** which we will then numerically simulate.

We first transform Maxwell's equations into the wave equation by taking the curl of the top equation:

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \left( \nabla \times \vec{H} \right)}{\partial t} == -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \nabla \left( \nabla \cdot \vec{E} \right) = -\nabla^2 \vec{E}$$

This gives us the wave equation

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

This is a vector equation. We can write in terms of each of the elements of the vector, for example $E_x$.

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu\varepsilon E_x$$

The solution to this is the plane wave:

$$E_x\left(x,y,z,t\right) = A\cos\left(\omega t - k_x x - k_y y - k_z z\right)$$

Or in complex notation

$$E_x(x,y,z,t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$$

Substituting this back into the wave equation

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon$$

This result holds only for a homogeneous medium where ε and μ are constants. For this important case then we have everything we need. We can expand the field in one plane into a superposition of plane waves and then we can find the field everywhere. We will pursue this route in this course and in addition we will often make the **paraxial approximation**. The idea of the paraxial approximation can be understood through the "typical" optical system in Figure 1.5.

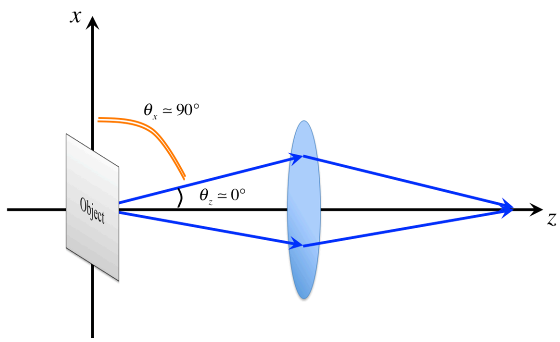



**Figure 1.5 –** Paraxial approximation

A plane wave propagates near the optical axis z since the light has to be captured by the lenses and other apertures that are in the path of the optical system. For a plane wave,

$$k_x = k\cos\theta_x$$
$$k_y = k\cos\theta_y$$
$$k_z = k\cos\theta_z$$

with the x and y angles being nearly 90 degree and $\theta_z$ almost zero degrees. If I were to plot E as a function of x or y (see Figure 1.6) it will be a very slow varying function of x and y whereas if we plot it as function of z is it will be a very rapidly varying function.For a wave propagating along the z-axis (at zero degrees angle) we can write is as

$$\boldsymbol{E(z,t) = Ae^{j\omega t}e^{-jkz}}$$

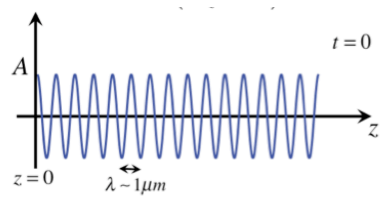
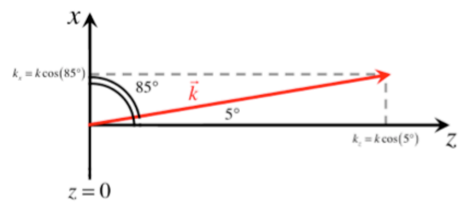



**Figure 1.6a. Plane propagating in the z-direction**

For a wave propagating at an angle θ with respect to the z –axis the plot the following equation for the field:

$$\boldsymbol{E(x,z,t) = Ae^{j\omega t}e^{-jk\sin\theta x}e^{-jk\cos\theta z}}$$

The variation of the field in z axis as compared to the x axis is plotted for $\theta=5^0$ in Figure 1.6b. In addition, the slowly varying field described by the wavevector k-$k_z$ is also presented



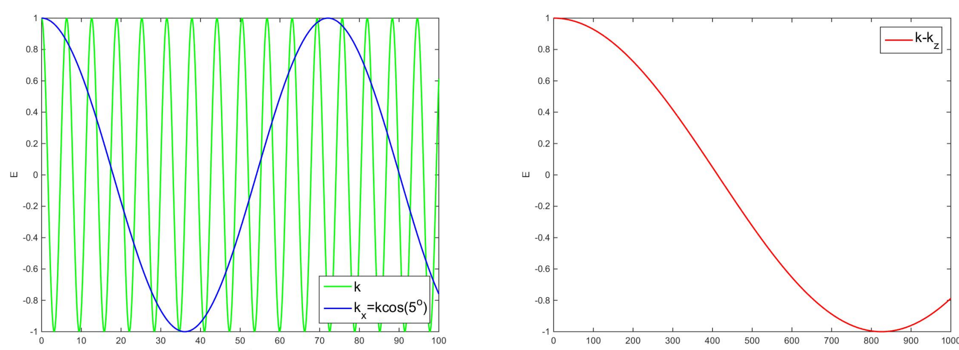




**Figure 1.6b Plane wave propagating at a an angle of $5^0$ with respect to z**

The *paraxial approximation* or equivalently the slowly varying envelope approximation gives us a way to write a more convenient or intuitive form of the wave equation:

$$E_x(x,y,z,t) = A(x,y,z)e^{j\omega t}e^{-jkz}$$

In other words the paraxial wave is almost like a plane wave which has slow variations in the x and y directions. The dependence of *A* on z is also slow because the fast variations of the field in z was already taken out by the $e^{-jkz}$ term. We call $A(x,y,z)$ the slow varying envelope. So far no approximation has been made. If we plug the slowly varying envelope of the field expression in the wave equation we obtain:

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right) e^{j\omega t} e^{-jkz} = \frac{\partial^2 A}{\partial x^2} e^{j\omega t} e^{-jkz} \qquad \text{same for y}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial A}{\partial z} e^{-jkz} - jkAe^{-jkz}\right) e^{j\omega t} = \left(\frac{\partial^2 A}{\partial z^2} - 2jk\frac{\partial A}{\partial z} - k^2 A\right) e^{j\omega t} e^{-jkz}$$

Plugging back into the wave equation and dropping the oscillatory terms that are common to all terms we obtain:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} - 2jk\frac{\partial A}{\partial z} - k^2 A = -\omega^2 \mu\varepsilon A$$

Recognizing that

$$-k^2 A = -\omega^2 \mu\varepsilon A$$

and making the approximation that the second derivative in z of the *A* envelope is negligible we obtain

$$2jk\frac{\partial A}{\partial z} \simeq \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}$$

or

$$\frac{\partial A}{\partial z} \simeq -\frac{j}{2k}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)$$

This is the beam propagation equation for a homogeneous medium such as air or free space.

At this point it may not be clear why we took a perfectly nice wave equation with its clean plane wave solution, which we could use to represent ANY wavefront and then propagate each plane wave separately, and manipulated it to the beam propagation equation. As we will see it will be more convenient for paraxial optics to analyze and numerically simulate the beam propagation equation. Moreover, as we will see in a few weeks, it is easier to extend this method inhomogeneous and nonlinear media.

To solve numerically the BPE we do not want to evaluate the derivatives. Instead we will use a Fourier transform trick, which proves computationally more efficient and numerically more stable.

In general a tilda superscript will denote the Fourier transform. If

$$\tilde{A}(k_x, k_y, z) = \int A(x, y, z) e^{-j(k_x x + k_y y)} dxdy$$

and

$$A(x,y,z)=\frac{1}{4\pi^2}\int \tilde{A}(k_x,k_y,z)e^{+j(k_xx+k_yy)}dk_xdk_y$$

then knowing the Fourier transform of a function we can calculate the Fourier transform of its second derivative by multiplying by - $k_x^2$:

$$\iint\frac{\partial^2 A(x,y,z)}{\partial x^2}e^{-j(k_xx+k_yy)}dxdy$$
$$=\frac{1}{(2\pi)^2}\iint\frac{\partial^2}{\partial x^2}[\iint\tilde{A}(k_x^{'},k_y^{'},z)e^{+j(k_x^{'}x+k_y^{'}y)}dk_x^{'}dk_y^{'}]e^{-j(k_xx+k_yy)}dxdy$$
$$=\frac{-k_x^2}{(2\pi)^2}\iint\ \iint\tilde{A}(k_x^{'},k_y^{'},z)e^{+j(k_x^{'}x+k_y^{'}y)}e^{-j(k_xx+k_yy)}dxdy\,dk_x^{'}dk_y^{'}$$
$$=\frac{-k_x^2}{(2\pi)^2}\iint\tilde{A}(k_x^{'},k_y^{'},z)\delta(k_x^{'}-k_x)dk_x^{'}dk_y^{'}=-k_x^2\tilde{A}(k_x^{'},k_y^{'},z)$$

Going back to the Beam Propagation Equation and taking the Fourier transform of both sides we have

$$\frac{\partial\tilde{A}(k_x,k_y,z)}{\partial z}=-\frac{j}{2k}\left[-k_x^2-k_y^2\right]\tilde{A}(k_x,k_y,z)=j\left[\frac{k_x^2+k_y^2}{2k}\right]\tilde{A}(k_x,k_y,z)$$

$$\tilde{A}(k_x,k_y,z)=e^{j\left[\frac{k_x^2+k_y^2}{2k}\right]z}\tilde{A}(k_x,k_y,z=0)\qquad \textit{Equation}\quad 1$$

Therefore to implement this in MATLAB we need to take the following steps:

1. Import $A(x,y,z=0)$

2. Take the FFT of A(x,y,z=0) to produce $\tilde{A}(k_x,k_y,z=0)$

3. Multiply $\tilde{A}(k_x,k_y,z=0)$ by $e^{j[\frac{k_x^2+k_y^2}{2k}]z}$. Need to be careful in this step. Need to know how to match the scale of the $k_x$'s.

4. Inverse FFT the product to get $A(x,y,z)$

Fresnel diffraction

Convolution theorem: Given a pair of 2D functions f(x,y) and h(x,y) the convolution between f and h is given by:

$$g(x,y) = \int \int f(x',y')h(x-x,y'-y)dx'dy'$$

The Fourier transform of g(x), denoted by F(u) is given by

$$G(k_x,k_y) = F(k_x,k_y)H(k_x,k_y)$$

where F and H are the Fourier transforms of f and h. The

We can apply the convolution theorem to equation 1 above and use the Fourier transform pair

$$e^{-\pi(x^2+y^2)} \Leftrightarrow e^{-\pi(u^2+v^2)} \quad where \quad k_x = 2\pi u \quad k_y = 2\pi v$$

to obtain the ***Fresnel diffraction*** formula:

$$E_x(x,y,z,t) = A(x,y,z)e^{j\omega t}e^{-jkz} = e^{j\omega t}e^{-jkz}\iint A(x',y',z=0)e^{-\frac{j\pi[(x-x')^2+(y-y')^2}{\lambda z}}dx'dy'$$

A spherical wave can be written as

$$E_x(x,y,z,t) = j\frac{A}{\lambda r}e^{j\omega t}e^{-jkr} \approx j\frac{A}{\lambda r}e^{j\omega t}e^{-jkz}e^{-j\frac{\pi}{\lambda z}[(x-x')^2+(y-y')^2]}$$

Comparing the expression for the Fresnel diffraction with the spherical wave we recognize that the Fresnel diffraction is a mathematical expression of the Huygen's principle where each point in a wavefront acts as a secondary point source emanating a spherical wave. The total field at the observation plane is a superposition of the spherical waves.

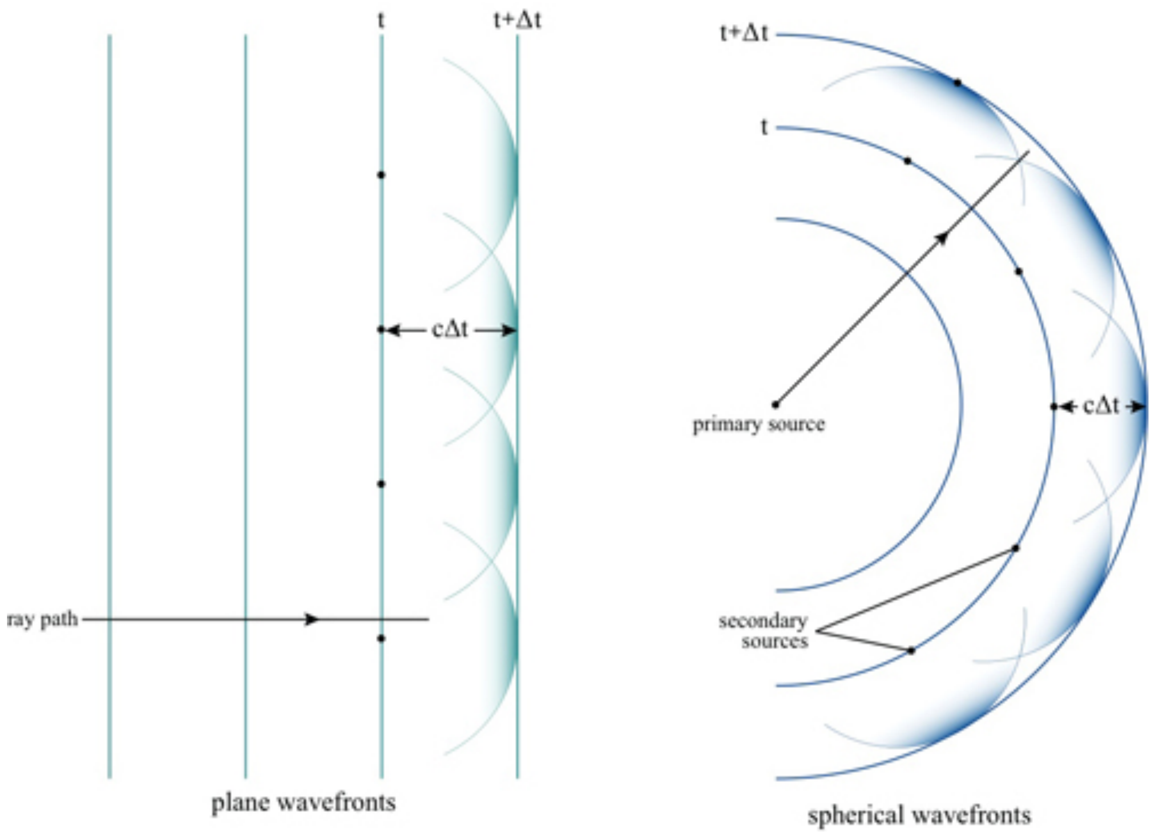



**Figure 1.7 Depiction of Huygens's principle**