Mathematics of Data: From Theory to Computation

Prof. Volkan Cevher
volkan.cevher@epfl.ch

Lecture 13: Primal-dual optimization I

Laboratory for Information and Inference Systems (LIONS)
École Polytechnique Fédérale de Lausanne (EPFL)

EE-556 (Fall 2023)

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General nonsmooth problems

◦ We will show that the restricted template captures the familiar composite minimization:

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}).$$

- f, g are convex, nonsmooth functions; and $\mathbf{A}$ is a linear operator.

Examples

- $g(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1$ or $g(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.
- $g(\mathbf{A}\mathbf{x}) = \delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x})$, where $\delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}. \end{cases}$

Observations: ◦ The indicator example covers constrained problems, such as $\min_{\mathbf{x}\in\mathcal{X}}\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

◦ We need a tool, called Fenchel conjugation, to reveal the underlying minimax problem.

Conjugation of functions

○ Idea: Represent a convex function in $\max$-form:

Definition

Let $\mathcal{Q}$ be a Euclidean space and $\mathcal{Q}^*$ be its dual space. Given a proper, closed and convex function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : \mathcal{Q}^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \text{dom}(f)} \left\{ \mathbf{y}^T\mathbf{x} - f(\mathbf{x}) \right\}$$

is called the Fenchel conjugate (or conjugate) of f.

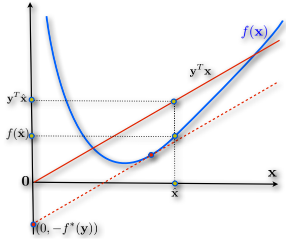

Figure: The conjugate function $f^*(\mathbf{y})$ is the maximum gap between the linear function $\mathbf{x}^T\mathbf{y}$ (red line) and $f(\mathbf{x})$.

Observations:
- ○ $\mathbf{y}$: slope of the hyperplane
- ○ $-f^*(\mathbf{y})$: intercept of the hyperplane

Conjugation of functions

Definition

Given a proper, closed and convex function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \to \mathbb{R} \cup \{+\infty\}$ such that

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Properties

- f^* is a convex and lower semicontinuous function by construction as the supremum of affine functions of $\mathbf{y}$.
- The conjugate of the conjugate of a convex function f is the same function f; i.e., $f^{**} = f$ for $f \in \mathcal{F}(\mathcal{Q})$.
- The conjugate of the conjugate of a non-convex function f is its lower convex envelope when $\mathcal{Q}$ is compact:
  - $f^{**}(\mathbf{x}) = \sup\{g(\mathbf{x}) : g \text{ is convex and } g \leq f, \forall \mathbf{x} \in \mathcal{Q}\}$.
- For closed convex f, μ-strong convexity w.r.t. $\|\cdot\|$ is equivalent to $\frac{1}{\mu}$ smoothness of f^* w.r.t. $\|\cdot\|_*$.
  - Recall dual norm: $\|\mathbf{y}\|_* = \sup_{\mathbf{x}}\{\langle \mathbf{x}, \mathbf{y} \rangle : \|\mathbf{x}\| \leq 1\}$.
  - See for example Theorem 3 in [12].

Examples

ℓ_2-norm-squared

$f(\mathbf{x}) = \frac{\lambda}{2}\|\mathbf{x}\|^2 \Rightarrow f^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x} \rangle - \frac{\lambda}{2}\|\mathbf{x}\|^2.$

∘ Take the derivative and equate to 0: $0 = \mathbf{y} - \lambda \mathbf{x} \iff \mathbf{x}^\star = \frac{1}{\lambda}\mathbf{y} \iff f^*(\mathbf{y}) = \frac{1}{\lambda}\|\mathbf{y}\|^2 - \frac{1}{2\lambda}\|\mathbf{y}\|^2 = \frac{1}{2\lambda}\|\mathbf{y}\|^2.$

ℓ_1-norm

$f(\mathbf{x}) = \lambda\|\mathbf{x}\|_1 \Rightarrow f^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x} \rangle - \lambda\|\mathbf{x}\|_1.$

∘ By definition of the ℓ_1-norm: $f^*(\mathbf{y}) = \max_{\mathbf{x}} \sum_{i=1}^n y_i x_i - \lambda|x_i| = \max_{\mathbf{x}} \sum_{i=1}^n y_i \mathrm{sign}(x_i)|x_i| - \lambda|x_i|.$

∘ By inspection:

- If all $|y_i| \leq \lambda$, then $\forall i, (y_i \mathrm{sign}(x_i) - \lambda)|x_i| \leq 0$. Taking $\mathbf{x} = 0$ gives the maximum value: $f^*(\mathbf{y}) = 0$.
- If for at least one $i, |y_i| > \lambda, (y_i \mathrm{sign}(x_i) - \lambda)|x_i| \to +\infty$ as $|x_i| \to +\infty$.

∘ $f^*(\mathbf{y}) = \delta_{\mathbf{y}:\|\cdot\|_\infty \leq \lambda}(\mathbf{y}) = \begin{cases} 0, & \text{if } \|\mathbf{y}\|_\infty \leq \lambda \\ +\infty, & \text{if } \|\mathbf{y}\|_\infty > \lambda \end{cases}$

Remark: ∘ See advanced material at the end for non-convex examples, such as $f(\mathbf{x}) = \|\mathbf{x}\|_0$.

General nonsmooth problems

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x})}$$

- By Fenchel-conjugation, we have $g(\mathbf{A}\mathbf{x}) = \max_{\mathbf{y}} \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})$, where g^* is the conjugate of g.
- Min-max formulation:

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}} \{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})\}}$$

An example with linear constraints

- If $g(\mathbf{A}\mathbf{x}) = \delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}, \end{cases}$

$$\Rightarrow g^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x}\rangle - \delta_{\{\mathbf{b}\}}(\mathbf{x}) = \max_{\mathbf{x}:\mathbf{x}=\mathbf{b}} \langle \mathbf{y}, \mathbf{x}\rangle = \langle \mathbf{y}, \mathbf{b}\rangle.$$

- We reach the minimax formulation (or the so-called "Lagrangian") via conjugation:

$$\min_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\} = \min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}} \max_{\mathbf{y}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x} - \mathbf{b}, \mathbf{y}\rangle.$$

A special case in minimax optimization

Bilinear min-max template

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - h(\mathbf{y}),$$

where $\mathcal{X} \subseteq R^p$ and $\mathcal{Y} \subseteq \mathbb{R}^n$.

- $f : \mathcal{X} \to \mathbb{R}$ is convex.
- $h : \mathcal{Y} \to \mathbb{R}$ is convex.

Example: Sparse recovery

An example from sparseland $\mathbf{b} = \mathbf{A}\mathbf{x}^\natural + \mathbf{w}$: constrained formulation

The basis pursuit denoising (BPDN) formulation is given by

$$\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \{\|\,\mathbf{x}\,\|_1 : \|\,\mathbf{A}\mathbf{x} - \mathbf{b}\,\|_2 \leq \|\,\mathbf{w}\,\|_2, \|\mathbf{x}\|_\infty \leq 1\}\,. \tag{BPDN}$$

A **primal problem** prototype

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K}\ \mathbf{x} \in \mathcal{X} \right\},$$

The above template captures BPDN formulation with

- $f(\mathbf{x}) = \|\mathbf{x}\|_1$.
- $\mathcal{K} = \{\|\mathbf{u}\| \in \mathbb{R}^n : \|\mathbf{u}\| \leq \|\mathbf{w}\|_2\}$.
- $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\|_\infty \leq 1\}$.

An alternative formulation

A **primal problem** prototype

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x}-\mathbf{b}\in\mathcal{K},\ \mathbf{x}\in\mathcal{X}\right\}, \tag{1}$$

- f is a proper, closed and convex function
- $\mathcal{X}$ and $\mathcal{K}$ are nonempty, closed convex sets
- $\mathbf{A}\in\mathbb{R}^{n\times p}$ and $\mathbf{b}\in\mathbb{R}^n$ are known
- An optimal solution $\mathbf{x}^\star$ to (1) satisfies $f(\mathbf{x}^\star)=f^\star$, $\mathbf{A}\mathbf{x}^\star-\mathbf{b}\in\mathcal{K}$ and $\mathbf{x}^\star\in\mathcal{X}$

A simplified template without loss of generality

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x}=\mathbf{b}\right\}, \tag{2}$$

- f is a proper, closed and convex function
- $\mathbf{A}\in\mathbb{R}^{n\times p}$ and $\mathbf{b}\in\mathbb{R}^n$ are known
- An optimal solution $\mathbf{x}^\star$ to (2) satisfies $f(\mathbf{x}^\star)=f^\star$, $\mathbf{A}\mathbf{x}^\star=\mathbf{b}$

Reformulation between templates

A primal problem template

$$\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K}, \mathbf{x} \in \mathcal{X} \right\}.$$

First step: Let $\mathbf{r}_1 = \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{r}_2 = \mathbf{x} \in \mathbb{R}^p$.

$$\min_{\mathbf{x},\mathbf{r}_1,\mathbf{r}_2} \left\{ f(\mathbf{x}) : \mathbf{r}_1 \in \mathcal{K}, \mathbf{r}_2 \in \mathcal{X}, \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{r}_1, \mathbf{x} = \mathbf{r}_2 \right\}.$$

- Define $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \in \mathbb{R}^{2p+n}$, $\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times p} \\ \mathbf{I}_{p\times p} & \mathbf{0}_{p\times n} & -\mathbf{I}_{p\times p} \end{bmatrix}$, $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, $\bar{f}(\mathbf{z}) = f(\mathbf{x}) + \delta_{\mathcal{K}}(\mathbf{r}_1) + \delta_{\mathcal{X}}(\mathbf{r}_2)$,
  where $\delta_{\mathcal{X}}(\mathbf{x}) = 0$, if $\mathbf{x} \in \mathcal{X}$, and $\delta_{\mathcal{X}}(\mathbf{x}) = +\infty$, o/w.

The simplified template

$$\min_{\mathbf{z}\in\mathbb{R}^{2p+n}} \left\{ \bar{f}(\mathbf{z}) : \bar{\mathbf{A}}\mathbf{z} = \bar{\mathbf{b}} \right\}.$$

From constrained formulation back to minimax

A general template

$$\min_{\mathbf{x}\in\mathbb{R}^p} \{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

Other examples:

- Standard convex optimization formulations: *linear programming, convex quadratic programming, second order cone programming, semidefinite programming and geometric programming*.
- Reformulations of existing unconstrained problems via **convex splitting**: *composite convex minimization, consensus optimization*, ...

Formulating as min-max

$$\max_{\mathbf{y}\in\mathbb{R}^n} \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}. \end{cases}$$

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} \left\{f(\mathbf{x})\colon \mathbf{A}\mathbf{x} = \mathbf{b}\right\} = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \left\{\Phi(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\}}$$

Dual problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) \colon \mathbf{A}\mathbf{x} = \mathbf{b} \right\} = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \left\{ \Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle \right\}}$$

∘ We define the dual problem

$$\max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) := \max_{\mathbf{y}\in\mathbb{R}^n} \{ \underbrace{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle }_{d(\mathbf{y})} \}.$$

Concavity of dual problem

Even if $f(\mathbf{x})$ is not convex, $d(\mathbf{y})$ is concave:

- For each $\mathbf{x}$, $d(\mathbf{y})$ is linear; i.e., it is both convex and concave.
- Pointwise minimum of concave functions is still concave.

Remark: ∘ If we can exchange $\min$ and $\max$, we obtain a **concave** maximization problem.

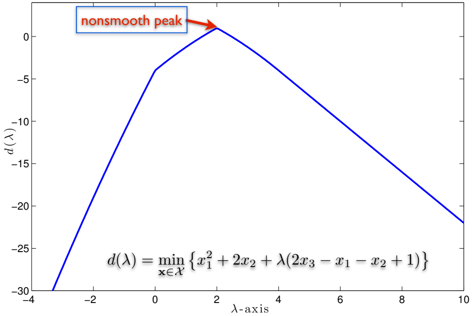
Example: Nonsmoothness of the dual function

∘ Consider a constrained convex problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^3} \quad & \left\{ f(\mathbf{x}) := x_1^2 + 2x_2 \right\}, \\ \text{s.t.} \quad & 2x_3 - x_1 - x_2 = 1, \\ & \mathbf{x} \in \mathcal{X} := [-2,2] \times [-2,2] \times [0,2]. \end{aligned}$$

∘ The **dual function** is concave and nonsmooth as written and then illustrated below.

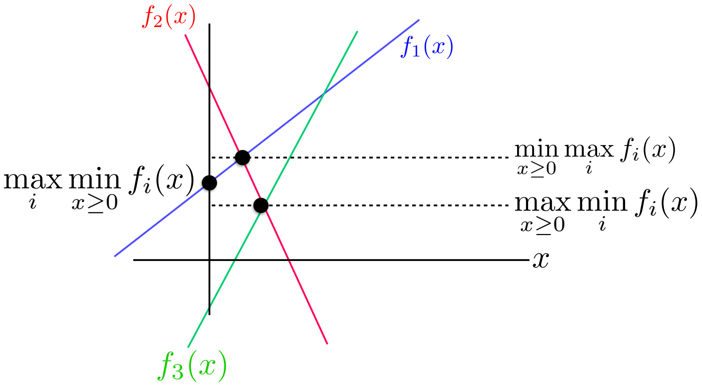
$$d(\lambda) := \min_{\mathbf{x}\in\mathcal{X}} \left\{ x_1^2 + 2x_2 + \lambda(2x_3 - x_1 - x_2 - 1) \right\}$$



Exchanging $\min$ and $\max$: A dangerous proposal

∘ Weak duality:

$$\underbrace{\max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y})}_{\text{Dual problem}} =: \boxed{\max_{\mathbf{y}\in\mathbb{R}^n}\min_{\mathbf{x}\in\mathbb{R}^p} \Phi(\mathbf{x},\mathbf{y}) \le \min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n} \Phi(\mathbf{x},\mathbf{y})} = \underbrace{\min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x}=\mathbf{b}\right\}}_{\text{Primal problem}} = \begin{cases} f^\star, \text{ if } \mathbf{A}\mathbf{x}=\mathbf{b} \\ +\infty, \text{ if } \mathbf{A}\mathbf{x}\neq\mathbf{b}\end{cases}$$



A proof of weak duality

$$\boxed{f^\star := \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\right\} = \min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n}\left\{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\}}$$

∘ Since $\mathbf{A}\mathbf{x}^\star = \mathbf{b}$, it holds for any $\mathbf{y}$

$$\begin{aligned}\Phi(\mathbf{x}^\star,\mathbf{y}) = f^\star &= f(\mathbf{x}^\star) + \langle \mathbf{y}, \mathbf{A}\mathbf{x}^\star - \mathbf{b}\rangle \\ &\geq \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\} \\ &= \min_{\mathbf{x}\in\mathbb{R}^p}\Phi(\mathbf{x},\mathbf{y}).\end{aligned}$$

∘ Take maximum of both sides in $\mathbf{y}$ and note that $f^\star$ is independent of $\mathbf{y}$:

$$f^\star = \min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n}\Phi(\mathbf{x},\mathbf{y}) \geq \max_{\mathbf{y}\in\mathbb{R}^n}\min_{\mathbf{x}\in\mathbb{R}^p}\Phi(\mathbf{x},\mathbf{y}) =: \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = d^\star.$$

Strong duality and saddle points

Strong duality

$$f^\star = f(\mathbf{x}^\star) = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \Phi(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}\in\mathbb{R}^p} \Phi(\mathbf{x},\mathbf{y}) =: \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = d^\star.$$

Under strong duality and assuming existence of $\mathbf{x}^\star$, $\Phi(\mathbf{x},\mathbf{y})$ has a saddle point. We have primal and dual optimal values coincide, i.e., $f^\star = d^\star$.

Strong duality and saddle points

Strong duality

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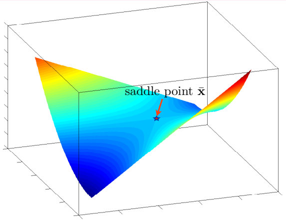
Under strong duality and assuming existence of $\mathbf{x}^\star$, $\Phi(\mathbf{x},\mathbf{y})$ has a saddle point. We have primal and dual optimal values coincide, i.e., $f^\star = d^\star$.

Recall saddle point / LNE

A point $(\mathbf{x}^\star, \mathbf{y}^\star) \in \mathbb{R}^p \times \mathbb{R}^n$ is called a saddle point of Φ if

$$\Phi(\mathbf{x}^\star,\mathbf{y}) \leq \Phi(\mathbf{x}^\star,\mathbf{y}^\star) \leq \Phi(\mathbf{x},\mathbf{y}^\star),\ \forall \mathbf{x} \in \mathbb{R}^p,\ \mathbf{y} \in \mathbb{R}^n.$$



Toy example: Strong duality

Primal problem

- Consider the following primal minimization problem: $\min_{\mathbf{x}} P(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1$
- Using conjugation and strong duality

$$
\begin{aligned}
P(\mathbf{x}^\star) = \min_{\mathbf{x}} P(\mathbf{x}) &= \min_{\mathbf{x}} \max_{\mathbf{y}} f(\mathbf{x}) + \langle \mathbf{x}, \mathbf{y} \rangle - g^*(\mathbf{y}), && \text{by conjugation} \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) + \min_{\mathbf{x}} f(\mathbf{x}) + \langle \mathbf{x}, \mathbf{y} \rangle, && \text{by changing min-max} \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) - \max_{\mathbf{x}} \langle \mathbf{x}, -\mathbf{y} \rangle - f(\mathbf{x}), && \text{by } \min f = -\max -f \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) - f^*(-\mathbf{y}), && \text{by conjugation.}
\end{aligned}
$$

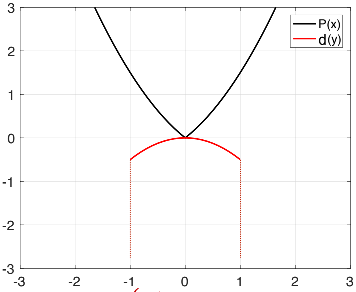
Dual problem

- Dual problem: $d^\star = \max_{\mathbf{y}} d(\mathbf{y}) = -g^*(\mathbf{y}) - f^*(-\mathbf{y})$
- Recall $f^*(-\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|^2$ and $g^*(\mathbf{y}) = \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \leq 1}(\mathbf{y})$.

Toy example: Strong duality

$$\boxed{\text{Primal problem: } \min_{\mathbf{x}} P(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1}$$

$$\boxed{\text{Dual problem: } \max_{\mathbf{y}} -\frac{1}{2}\|\mathbf{y}\|^2 - \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \leq 1}(\mathbf{y})}$$



$$d(\mathbf{y}) = \begin{cases} -\frac{1}{2}\|\mathbf{y}\|^2, & \text{if } \|\mathbf{y}\|_\infty \leq 1 \\ -\infty, & \text{if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$

Back to convex-concave: Necessary and sufficient condition for strong duality

- Existence of a saddle point is not automatic even in convex-concave setting!
- Recall the minimax template:

$$\min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \left\{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\}$$

Theorem (Necessary and sufficient optimality condition)

Under the Slater's condition: $\mathrm{relint}(\mathrm{dom}\, f) \cap \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset$, *strong duality holds, where the primal and dual problems are given by*

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{cases} \quad \textit{and} \quad d^\star := \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}).$$

Remarks:

- By definition of $f^\star$ and $d^\star$, we always have $\boxed{d^\star \leq f^\star}$ (**weak duality**).
- If a primal solution exists and the Slater's condition holds, we have $\boxed{d^\star = f^\star}$ (**strong duality**).

Slater's qualification condition

◦ Denote $\mathrm{relint}(\mathrm{dom}\, f)$ the relative interior of the domain.

◦ The Slater condition requires

$$\boxed{\mathrm{relint}(\mathrm{dom}\, f) \cap \{\mathbf{x} \;:\; \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset.} \tag{3}$$

Special cases

- If $\mathrm{dom}\, f = \mathbb{R}^p$, then (3) $\Leftrightarrow$ $\boxed{\exists \bar{\mathbf{x}} \;:\; \mathbf{A}\bar{\mathbf{x}} = \mathbf{b}}$.
- If $\mathrm{dom}\, f = \mathbb{R}^p$ and instead of $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have the feasible set $\{\mathbf{x} : h(\mathbf{x}) \leq 0\}$, where h is $\mathbb{R}^p \to R^q$ is convex, then

$$(3) \Leftrightarrow \boxed{\exists \bar{\mathbf{x}} \;:\; h(\bar{\mathbf{x}}) < 0.}$$

Example: Slater's condition

Example

Let us consider solving $\min_{\mathbf{x}\in\mathcal{D}_\alpha} f(\mathbf{x})$ and so the feasible set is $\mathcal{D}_\alpha := \mathcal{X} \cap \mathcal{A}_\alpha$, where

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \leq 1\}, \ \mathcal{A}_\alpha := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_1 + x_2 = \alpha\},$$

where $\alpha \in \mathbb{R}$.

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Let us consider solving $\min_{\mathbf{x}\in\mathcal{D}_\alpha} f(\mathbf{x})$ and so the feasible set is $\mathcal{D}_\alpha := \mathcal{X} \cap \mathcal{A}_\alpha$, where

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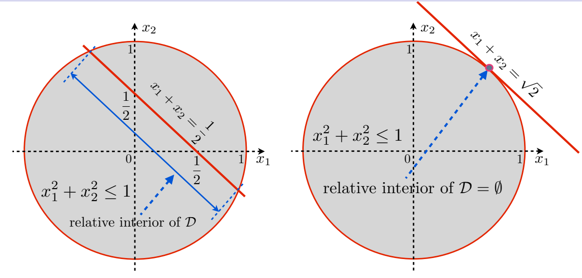
where $\alpha \in \mathbb{R}$.

Two cases where Slater's condition holds and does not hold



$\mathcal{D}_{1/2}$ satisfies Slater's condition – $\mathcal{D}_{\sqrt{2}}$-does not satisfy Slater's condition

Performance of optimization algorithms

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}, \right\}, \qquad \text{(Affine-Constrained)}$$

Exact vs. approximate solutions

- Computing an **exact solution** $\mathbf{x}^\star$ to (Affine-Constrained) is **impracticable**
- Algorithms seek $\mathbf{x}^\star_\epsilon$ that approximates $\mathbf{x}^\star$ up to ϵ in some sense

A performance metric: Time-to-reach ϵ

`time-to-reach` ϵ `= number of iterations to reach` ϵ $\times$ `per iteration time`

A key issue: Number of iterations to reach ϵ

The notion of ϵ-accuracy is elusive in constrained optimization!

Numerical ϵ-accuracy

- **Unconstrained case:** All iterates are feasible (no advantage from infeasibility)!

$$f(\mathbf{x}^\star_\epsilon) - f^\star \leq \epsilon$$

$$\boxed{f^\star = \min_{\mathbf{x} \in \mathbb{R}^p} f(\mathbf{x})}$$

- **Constrained case:** We need to also measure the infeasibility of the iterates!

$$f^\star - f(\mathbf{x}^\star_\epsilon) \leq \epsilon \quad !!!$$

$$\boxed{f^\star = \min_{\mathbf{x} \in \mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \right\}} \tag{4}$$

Our definition of ϵ-accurate solutions [16]

Given a numerical tolerance $\epsilon \geq 0$, a point $\mathbf{x}^\star_\epsilon \in \mathbb{R}^p$ is called an ϵ-solution of (4) if

$$\begin{cases} f(\mathbf{x}^\star_\epsilon) - f^\star & \leq \epsilon \text{ (objective residual)}, \\ \|\mathbf{A}\mathbf{x}^\star_\epsilon - \mathbf{b}\| & \leq \epsilon \text{ (feasibility gap)}, \end{cases}$$

- When $\mathbf{x}^\star$ is unique, we can also obtain $\|\mathbf{x}^\star_\epsilon - \mathbf{x}^\star\| \leq \epsilon$ (iterate residual).

Numerical ϵ-accuracy

Constrained problems

Given a numerical tolerance $\epsilon \geq 0$, a point $\mathbf{x}^\star_\epsilon \in \mathbb{R}^p$ is called an ϵ-solution of (4) if

$$\begin{cases} f(\mathbf{x}^\star_\epsilon) - f^\star & \leq \epsilon \text{ (objective residual)}, \\ \|\mathbf{A}\mathbf{x}^\star_\epsilon - \mathbf{b}\| & \leq \epsilon \text{ (feasibility gap)}, \end{cases}$$

- When $\mathbf{x}^\star$ is unique, we can also obtain $\|\mathbf{x}^\star_\epsilon - \mathbf{x}^\star\| \leq \epsilon$ (iterate residual).

General minimax problems

Since duality gap is 0 at the solution, we measure the primal-dual gap

$$\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\bar{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}) \leq \epsilon. \tag{5}$$

Remarks:

- ϵ can be different for the objective, feasibility gap, or the iterate residual.
- It is easy to show $\mathrm{Gap}(\mathbf{x}, \mathbf{y}) \geq 0$ and $\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 0$ iff $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a saddle point.

Primal-dual gap function for nonsmooth minimization

$$\boxed{\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \underbrace{f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})}_{\Phi(\mathbf{x},\mathbf{y})} = \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})}$$

- Primal problem: $\min_{\mathbf{x}\in\mathcal{X}} P(\mathbf{x})$ where

$$P(\mathbf{x}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}).$$

- Dual problem: $\max_{\mathbf{y}\in\mathcal{Y}} d(\mathbf{y})$ where

$$d(\mathbf{y}) = \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x},\mathbf{y}).$$

- The primal-dual gap, i.e., $\mathrm{Gap}(\bar{\mathbf{x}},\bar{\mathbf{y}})$, is literally $(\text{primal value at } \bar{\mathbf{x}}) - (\text{dual value at } \bar{\mathbf{y}})$:

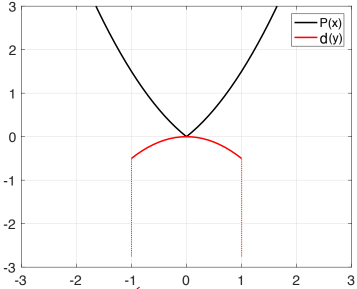
$$\mathrm{Gap}(\bar{\mathbf{x}},\bar{\mathbf{y}}) = P(\bar{\mathbf{x}}) - d(\bar{\mathbf{y}}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\bar{\mathbf{x}},\mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x},\bar{\mathbf{y}}).$$

Toy example for nonnegativity of gap

- $P(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1$
- $d(\mathbf{y}) = -\frac{1}{2}\|\mathbf{y}\|^2 - \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \le 1}(\mathbf{y})$

Recall the indicator function

$$\delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \le 1}(\mathbf{y}) = \begin{cases} 0, & \text{if } \|\mathbf{y}\|_\infty \le 1 \\ +\infty, & \text{if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$



$$d(\mathbf{y}) = \begin{cases} -\frac{1}{2}\|\mathbf{y}\|^2, & \text{if } \|\mathbf{y}\|_\infty \le 1 \\ -\infty, & \text{if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$

Primal-dual gap function in the general case

$$\boxed{\min_{\mathbf{x}\in\mathcal{X}}\max_{\mathbf{y}\in\mathcal{Y}}\Phi(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in\mathcal{Y}}\min_{\mathbf{x}\in\mathcal{X}}\Phi(\mathbf{x},\mathbf{y})}$$

◦ Saddle point $(\mathbf{x}^\star, \mathbf{y}^\star)$ is such that $\forall \mathbf{x} \in \mathbb{R}^p$, $\forall \mathbf{y} \in \mathbb{R}^n$:

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \overset{(*)}{\leq} \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \overset{(**)}{\leq} \Phi(\mathbf{x}, \mathbf{y}^\star).$$

◦ Nonnegativity of Gap:

$$\begin{aligned}
\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) &= \max_{\mathbf{y}\in\mathcal{X}} \Phi(\bar{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}) \\
&\geq \Phi(\bar{\mathbf{x}}, \mathbf{y}^\star) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), && \text{by the definition of maximization} \\
&\geq \Phi(\mathbf{x}^\star, \mathbf{y}^\star) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), && \text{by the inequality } (**) \\
&\geq \Phi(\mathbf{x}^\star, \bar{\mathbf{y}}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), && \text{by the inequality } (*) \\
&\geq 0, && \text{by the definition of minimization.}
\end{aligned}$$

◦ If $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\mathbf{x}^\star, \mathbf{y}^\star)$, then all the inequalities will be equalities and $\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 0$.

Optimality conditions for minimax

Saddle point

We say $(\mathbf{x}^\star, \mathbf{y}^\star)$ is a primal-dual solution corresponding to primal and dual problems

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{cases} \quad \text{and} \quad d^\star := \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = \max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}).$$

if it is a saddle point of $\Phi(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle$:

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \le \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \le \Phi(\mathbf{x}, \mathbf{y}^\star), \ \forall \mathbf{x} \in \mathbb{R}^p, \ \mathbf{y} \in \mathbb{R}^n.$$

Karush-Khun-Tucker (KKT) conditions

Under our assumptions, an equivalent characterization of $(\mathbf{x}^\star, \mathbf{y}^\star)$ is via the KKT conditions of the problem

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b},$$

which reads

$$\begin{cases} 0 & \in \partial_{\mathbf{x}} \Phi(\mathbf{x}^\star, \mathbf{y}^\star) & = \mathbf{A}^T \mathbf{y}^\star + \partial f(\mathbf{x}^\star), \\ 0 & = \nabla_{\mathbf{y}} \Phi(\mathbf{x}^\star, \lambda^\star) & = \mathbf{A}\mathbf{x}^\star - \mathbf{b}. \end{cases}$$

Primal approach: The Penalty Method

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x}=\mathbf{b}\right\}}$$

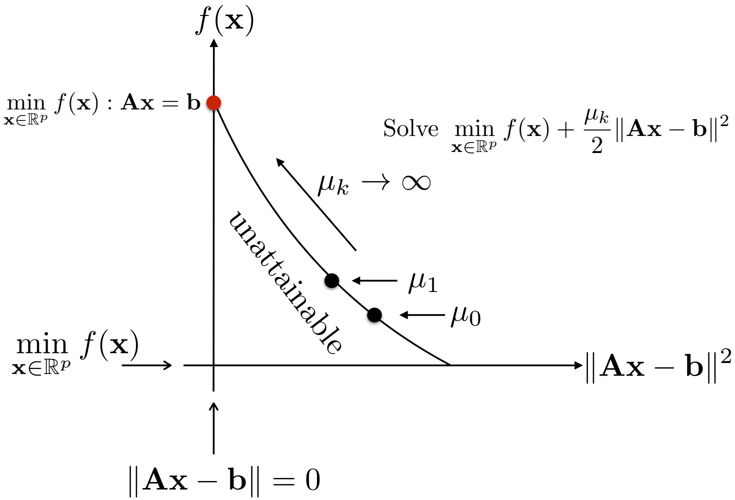
Penalty methods

- Convert constrained problem (difficult) to unconstrained (easy).
- Penalized function with penalty parameter $\mu > 0$:

$$F_\mu(\mathbf{x}) := \left\{f(\mathbf{x}) + \frac{\mu}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2\right\} \quad \overset{\mu\to\infty}{\Longleftrightarrow} \quad \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x}=\mathbf{b}\right\}.$$

- **Observations**:
  - Minimize a weighted combination of $f(\mathbf{x})$ and $\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2$ at the same time.
  - μ determines the weight of $\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2$.
  - As $\mu\to\infty$, we enforce $\mathbf{A}\mathbf{x}=\mathbf{b}$.
  - Other functions than the quadratic $\frac{1}{2}\|\cdot\|^2$ are also possible e.g., exact nonsmooth penalty functions:
    - $\mu\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2$ or $\mu\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1$
    - They work with finite μ, but they are difficult to solve [13, Section 17.2], [4]

Quadratic penalty: Intuition



Quadratic penalty: Conceptual algorithm

Quadratic penalty method (QP):
1. Choose $\mathbf{x}_0 \in \mathbb{R}^p$ and $\mu_0 > 0$.
2. For $k = 0, 1, \cdots,$ perform:
2.a. $\mathbf{x}_k := \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) + \frac{\mu_k}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 \right\}$.
2.b. Update $\mu_{k+1} > \mu_k$.

Theorem [13, Theorem 17.1]

Assume that f is smooth and $\mu_k \to \infty$. Then, every limit point $\bar{\mathbf{x}}$ of the sequence $\{\mathbf{x}_k\}$ is a solution of the constrained problem

$$\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) \colon \mathbf{A}\mathbf{x} = \mathbf{b} \right\}.$$

Limitation

- The minimization problems of step 2.a. of the algorithm become ill-conditioned as $\mu_k \to \infty$.
- Common improvements:
  - Solve the subproblem inexactly, *i.e.*, up to ϵ accuracy.
  - Linearization to simplify subproblems (up next).

Quadratic penalty: Linearization

Generalized quadratic penalty method:
1. Choose $\mathbf{x}_0 \in \mathbb{R}^p$, $\mu_0 > 0$ and positive semidefinite matrix $\mathbf{Q}_k$.
2. For $k = 0, 1, \cdots,$ perform:
2.a. $\mathbf{x}_k := \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) + \frac{\mu_k}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \frac{1}{2}\|\mathbf{x} - \mathbf{x}_{k-1}\|^2_{\mathbf{Q}_k} \right\}$.
2.b. Update $\mu_{k+1} > \mu_k$.

Ideas

- Minimize a majorizer of $F_\mu(\mathbf{x})$, parametrized by $\mathbf{Q}_k$ in step 2.a..
- $\mathbf{Q}_k = \mathbf{0}$ gives the standard QP; $\mathbf{Q}_k = \mathbf{I}$ gives strongly convex subproblems.
- $\mathbf{Q}_k = \alpha_k \mathbf{I} - \mu_k \mathbf{A}^\top \mathbf{A}$, with $\alpha_k \geq \mu_k \|\mathbf{A}\|^2$ gives

$$\mathbf{x}_k = \mathrm{prox}_{\frac{1}{\alpha_k} f}\left(\mathbf{x}_{k-1} - \frac{\mu_k}{\alpha_k}\mathbf{A}^\top(\mathbf{A}\mathbf{x}_{k-1} - \mathbf{b})\right)$$

Only one proximal operator!

and picking $\alpha_k = \mu_k\|\mathbf{A}\|^2$ gives

$$\mathbf{x}_k = \mathrm{prox}_{\frac{1}{\mu_k\|\mathbf{A}\|^2} f}\left(\mathbf{x}_{k-1} - \frac{1}{\|\mathbf{A}\|^2}\mathbf{A}^\top(\mathbf{A}\mathbf{x}_{k-1} - \mathbf{b})\right).$$

Per-iteration time: The key role of the prox-operator

Recall: Prox-operator

$$\mathrm{prox}_f(\mathbf{x}) := \arg\min_{\mathbf{z}\in\mathbb{R}^p} \left\{ f(\mathbf{z}) + \frac{1}{2}\|\mathbf{z}-\mathbf{x}\|^2 \right\}.$$

Key properties:

- single valued & non-expansive since f is a proper convex function.
- distributes when the primal problem has decomposable structure:

$$f(\mathbf{x}) := \sum_{i=1}^{m} f_i(\mathbf{x}_i), \quad \text{and} \quad \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_m.$$

  where $m \geq 1$ is the number of components.
- often efficient & has closed form expression. For instance, if $f(\mathbf{z}) = \|\mathbf{z}\|_1$, then the prox-operator performs coordinate-wise soft-thresholding by 1.

Quadratic penalty: Linearized methods

Linearized QP method (LQP)	Accelerated linearized QP method (ALQP)
1. Choose $\mathbf{x}_0 \in \mathbb{R}^p$, $\sigma_0 = 1$, $\mu_0 > 0$.	1. Choose $\mathbf{x}_0, \mathbf{y}_0 \in \mathbb{R}^p$, $\tau_0 = 1$, $\mu_0 > 0$.
2. For $k = 0, 1, \cdots:$	2. For $k = 0, 1, \cdots:$
2.a. $\mathbf{x}_{k+1} := \mathrm{prox}_{\frac{1}{\mu_k \Vert\mathbf{A}\Vert^2} f}\left(\mathbf{x}_k - \frac{1}{\Vert\mathbf{A}\Vert^2}\mathbf{A}^\top(\mathbf{A}\mathbf{x}_k - \mathbf{b})\right)$.	2.a. $\mathbf{x}_{k+1} := \mathrm{prox}_{\frac{1}{\mu_k \Vert\mathbf{A}\Vert^2} f}\left(\mathbf{y}_k - \frac{1}{\Vert\mathbf{A}\Vert^2}\mathbf{A}^\top(\mathbf{A}\mathbf{y}_k - \mathbf{b})\right)$.
2.b. Update σ_{k+1} s.t. $\frac{(1-\sigma_{k+1})^2}{\sigma_{k+1}} = \frac{1}{\sigma_k}$.	2.b. $\mathbf{y}_{k+1} := \mathbf{x}_{k+1} + \frac{\tau_{k+1}(1-\tau_k)}{\tau_k}(\mathbf{x}_{k+1} - \mathbf{x}_k)$.
2.c. Update $\mu_{k+1} = \sqrt{\sigma_{k+1}}$.	2.c. Update $\mu_{k+1} = \mu_k(1 + \tau_{k+1})$.
	2.d. Update $\tau_{k+1} \in (0, 1)$ as the unique positive root of $\tau^3 + \tau^2 + \tau_k^2\tau - \tau_k^2 = 0$.

Theorem (Convergence [17])

- ***LQP***:

$$
\begin{aligned}
|f(\mathbf{x}_k) - f(\mathbf{x}^\star)| &\leq \mathcal{O}\left(\mu_0 k^{-1/2} + \mu_0^{-1} k^{-1/2}\right) \\
\Vert\mathbf{A}\mathbf{x}_k - \mathbf{b}\Vert &\leq \mathcal{O}\left(\mu_0^{-1} k^{-1/2}\right)
\end{aligned}
$$

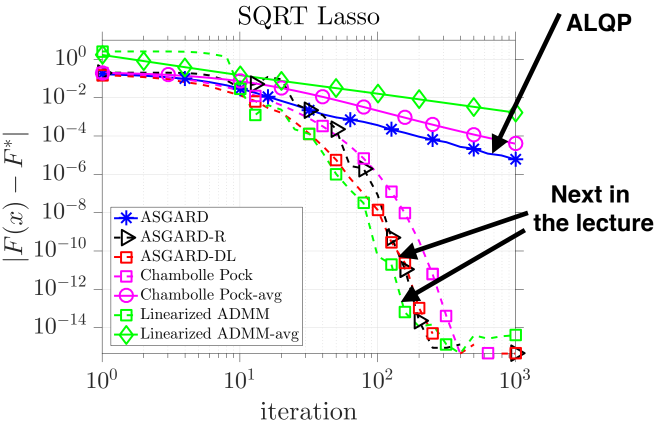
- ***ALQP***:

$$
\begin{aligned}
|f(\mathbf{x}_k) - f(\mathbf{x}^\star)| &\leq \mathcal{O}\left(\mu_0 k^{-1} + \mu_0^{-1} k^{-1}\right) \\
\Vert\mathbf{A}\mathbf{x}_k - \mathbf{b}\Vert &\leq \mathcal{O}\left(\mu_0^{-1} k^{-1}\right)
\end{aligned}
$$

In practice: poor (worst case) performance

- A nonsmooth problem: SQRT Lasso

$$\min_{\mathbf{x}\in\mathbb{R}^p} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 + \lambda\|\mathbf{x}\|_1.$$



Wrap up!

- Try to finish Homework #2...

A *convex* proto-problem for *structured* sparsity

A combinatorial approach for estimating $\mathbf{x}^\natural$ from $\mathbf{b} = \mathbf{A}\mathbf{x}^\natural + \mathbf{w}$

We may consider the sparsest estimator or its surrogate with a valid sparsity pattern:

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ \| \mathbf{x} \|_{\boldsymbol{s}} : \| \mathbf{b} - \mathbf{A}\mathbf{x} \|_2 \leq \kappa, \|\mathbf{x}\|_\infty \leq 1 \right\} \qquad (\mathcal{P}_s)$$

with some $\kappa \geq 0$. If $\kappa = \| \mathbf{w} \|_2$, then the structured sparse $\mathbf{x}^\natural$ is a feasible solution.

Sparsity and structure together [6]

Given some weights $\boldsymbol{d} \in \mathbb{R}^d, \boldsymbol{e} \in \mathbb{R}^p$ and an integer input $\boldsymbol{c} \in \mathbb{Z}^l$, we define

$$\|\mathbf{x}\|_{\boldsymbol{s}} := \min_{\boldsymbol{\omega}} \{ \boldsymbol{d}^T\boldsymbol{\omega} + \boldsymbol{e}^T\boldsymbol{s} : \boldsymbol{M} \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{s} \end{bmatrix} \leq \boldsymbol{c}, \mathbb{1}_{\mathrm{supp}(\mathbf{x})} = \boldsymbol{s}, \boldsymbol{\omega} \in \{0,1\}^d \}$$

for all feasible $\mathbf{x}$, ∞ otherwise. The parameter $\boldsymbol{\omega}$ is useful for latent modeling.

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for all feasible $\mathbf{x}$, ∞ otherwise. The parameter $\boldsymbol{\omega}$ is useful for latent modeling.

A convex candidate solution for $\mathbf{b} = \mathbf{A}\mathbf{x}^\natural + \mathbf{w}$

We use the convex estimator based on the tightest convex relaxation of $\|\mathbf{x}\|_{\boldsymbol{s}}$:
$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathrm{dom}(\|\cdot\|_{\boldsymbol{s}})} \{ \|\mathbf{x}\|_{\boldsymbol{s}}^{**} : \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \leq \kappa \}$ with some $\kappa \geq 0$, $\mathrm{dom}(\|\cdot\|_{\boldsymbol{s}}) := \{\mathbf{x} : \|\mathbf{x}\|_{\boldsymbol{s}} < \infty\}$.

Tractability & tightness of biconjugation

Proposition (Hardness of conjugation)

Let $F(s) : 2^{\mathfrak{P}} \to \mathbb{R} \cup \{+\infty\}$ be a set function defined on the support $s = supp(\mathbf{x})$. Conjugate of F over the unit infinity ball $\|\mathbf{x}\|_\infty \leq 1$ is given by

$$g^*(\mathbf{y}) = \sup_{\boldsymbol{s} \in \{0,1\}^p} |\mathbf{y}|^T \boldsymbol{s} - F(\boldsymbol{s}).$$

Observations:

- $F(\boldsymbol{s})$ is general set function

  Computation: NP-Hard

- $F(\boldsymbol{s}) = \|\mathbf{x}\|_{\boldsymbol{s}}$

  Computation: Integer Linear Program (ILP) in general. However, if
  - $\boldsymbol{M}$ is Totally Unimodular TU
  - $(\boldsymbol{M}, \boldsymbol{c})$ is Total Dual Integral TDI

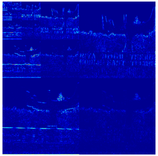
  then tight convex relaxations with a linear program (LP, which is "usually" tractable)

  Otherwise, relax to LP anyway!

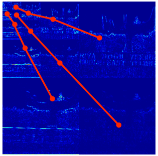
- $F(\boldsymbol{s})$ is submodular

  Computation: Polynomial-time

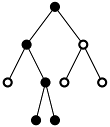
Tree sparsity [11, 5, 3, 18]



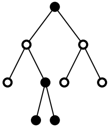
Wavelet coefficients



Wavelet tree



Valid selection of nodes



Invalid selection of nodes

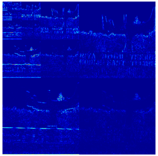
Structure: *We seek the sparsest signal with a rooted connected subtree support.*

Linear description: A valid support satisfy $s_{\text{parent}} \geq s_{\text{child}}$ over tree $\mathcal{T}$

$$\boxed{\boldsymbol{T}\mathbb{1}_{\text{supp}(\mathbf{x})} := \boldsymbol{T}\boldsymbol{s} \geq 0}$$

where $\boldsymbol{T}$ is the directed edge-node incidence matrix, which is TU.

Tree sparsity [11, 5, 3, 18]



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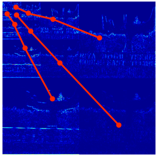

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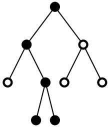

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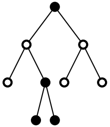

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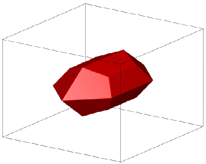
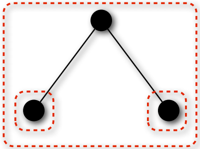
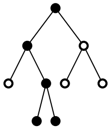
Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \min_{\boldsymbol{s}\in[0,1]^p}\{\mathbb{1}^T\boldsymbol{s} : \boldsymbol{T}\boldsymbol{s} \geq 0, |\mathbf{x}| \leq \boldsymbol{s}\}$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Tree sparsity [11, 5, 3, 18]





$\mathfrak{G}_H = \{\{1,2,3\},\{2\},\{3\}\}$



valid selection of nodes

Structure: *We seek the sparsest signal with a rooted connected subtree support.*

Linear description: A valid support satisfy $s_{\text{parent}} \geq s_{\text{child}}$ over tree $\mathcal{T}$

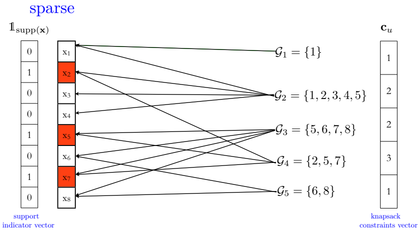
$$\boxed{\boldsymbol{T}\mathbb{1}_{\text{supp}(\mathbf{x})} := \boldsymbol{T}\boldsymbol{s} \geq 0}$$

where $\boldsymbol{T}$ is the directed edge-node incidence matrix, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \min_{\boldsymbol{s}\in[0,1]^p}\{\mathbb{1}^T\boldsymbol{s} : \boldsymbol{T}\boldsymbol{s} \geq 0, |\mathbf{x}| \leq \boldsymbol{s}\} \overset{\star}{=} \sum_{\mathcal{G}\in\mathfrak{G}_H} \|x_{\mathcal{G}}\|_\infty$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

The set $\mathcal{G} \in \mathfrak{G}_H$ are defined as each node and all its descendants.

Group knapsack sparsity [20, 8, 7]



Structure: *We seek the sparsest signal with group allocation constraints.*

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

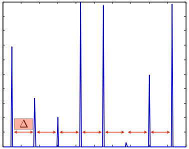
$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Remark: We can also budget a lowerbound $\boldsymbol{c}_\ell \leq \mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u$.

Group knapsack sparsity [20, 8, 7]



$$\mathfrak{B}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & & & & \\ 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_{(p-\Delta+1)\times p}$$

Structure: *We seek the sparsest signal with group allocation constraints.*

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

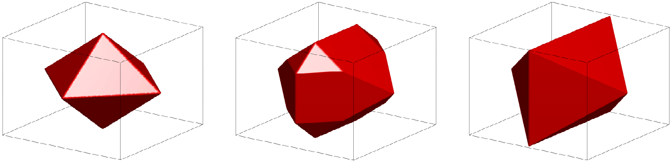
When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Remark: We can also budget a lowerbound $\boldsymbol{c}_\ell \leq \mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u$.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$

For the neuronal spike example, we have $\boldsymbol{c}_u = \mathbb{1}$.

Group knapsack sparsity [20, 8, 7]



(left) $\|\mathbf{x}\|_s^{**} \leq 1$ (middle) $\|\mathbf{x}\|_s^{**} \leq 1.5$ (right) $\|\mathbf{x}\|_s^{**} \leq 2$ for $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$

Structure: *We seek the sparsest signal with group allocation constraints.*

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Remark: We can also budget a lowerbound $\boldsymbol{c}_\ell \leq \mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u$.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$

For the neuronal spike example, we have $\boldsymbol{c}_u = \mathbb{1}$.

Group knapsack sparsity example: A stylized spike train

- Basis pursuit (BP): $\|\mathbf{x}\|_1$
- TU-relax (TU):

$$\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$$

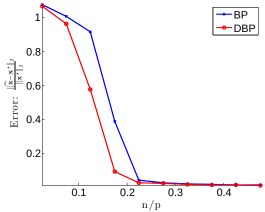
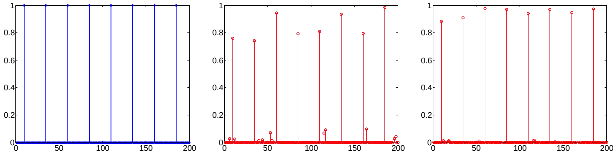


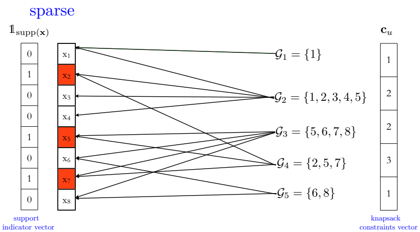
Figure: *Recovery for* $n = 0.18p$.



$\mathbf{x}^\natural$ | $\mathbf{x}^{\text{BP}}$ solution | $\mathbf{x}^{\text{TU}}$ solution

relative errors: $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{BP}}\|_2}{\|\mathbf{x}^\natural\|_2} = .200$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{TU}}\|_2}{\|\mathbf{x}^\natural\|_2} = .067$

Group knapsack sparsity: A simple variation



Structure: *We seek the signal with the minimal overall group allocation.*

$$\text{Objective: } \mathbb{1}^T \boldsymbol{s} \to \|\mathbf{x}\|_{\boldsymbol{\omega}} = \begin{cases} \min_{\omega \in \mathbb{Z}_{++}} \omega & \text{if } \mathbf{x} \in [-1, 1]^p, \mathfrak{B}^T \boldsymbol{s} \leq \omega \mathbb{1}, \\ \infty & \text{otherwise} \end{cases}$$

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \omega \mathbb{1}}$$

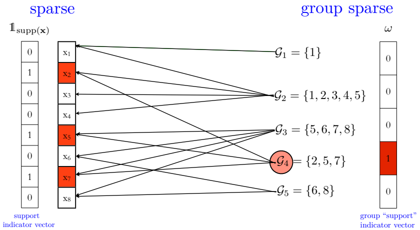
where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Biconjugate: $$\|\mathbf{x}\|_s^{**} = \begin{cases} \max_{\mathcal{G} \in \mathfrak{G}} \|\mathbf{x}^{\mathcal{G}}\|_1 & \text{if } \mathbf{x} \in [-1, 1]^p, \\ \infty & \text{otherwise} \end{cases}$$

Remark: The regularizer is known as *exclusive Lasso* [20, 15].

Group cover sparsity: Minimal group cover [2, 14, 9]



Structure: *We seek the signal covered by a minimal number of groups.*

$$\text{Objective: } \mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Group cover sparsity: Minimal group cover [2, 14, 9]

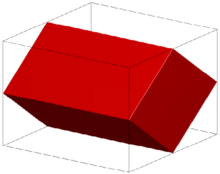


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal covered by a minimal number of groups.*

$$\text{Objective: } \mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M}\{\boldsymbol{d}^T\boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

Group cover sparsity: Minimal group cover [2, 14, 9]

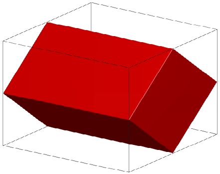


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal covered by a minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T \boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

$$\stackrel{\star}{=} \min_{\mathbf{v}_i \in \mathbb{R}^p} \{\sum_{i=1}^M d_i \|\mathbf{v}_i\|_\infty : \mathbf{x} = \sum_{i=1}^M \mathbf{v}_i, \forall \mathrm{supp}(\mathbf{v}_i) \subseteq \mathcal{G}_i\},$$

Group cover sparsity: Minimal group cover [2, 14, 9]

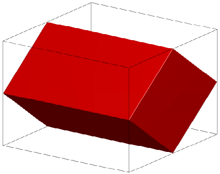


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal covered by a minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

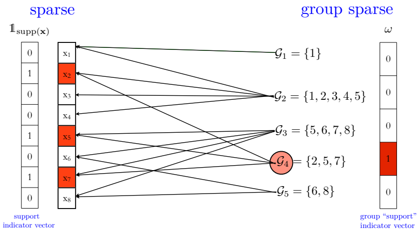
When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M} \{\boldsymbol{d}^T\boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

$$\stackrel{\star}{=} \min_{\mathbf{v}_i \in \mathbb{R}^p} \{\textstyle\sum_{i=1}^M d_i \|\mathbf{v}_i\|_\infty : \mathbf{x} = \sum_{i=1}^M \mathbf{v}_i, \forall \mathrm{supp}(\mathbf{v}_i) \subseteq \mathcal{G}_i\},$$

Remark: Weights $\boldsymbol{d}$ can depend on the sparsity within each groups (not TU) [6].

Budgeted group cover sparsity



Structure: *We seek the sparsest signal covered by G groups.*

$$\text{Objective: } \boldsymbol{d}^T\boldsymbol{\omega} \to \mathbb{1}^T\boldsymbol{s}$$

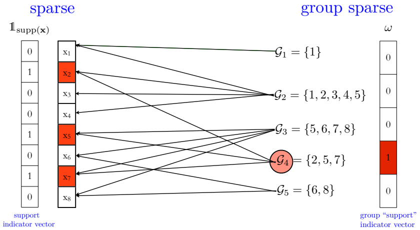
Linear description: At least one of the G selected groups cover each sparse coefficient.

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}, \mathbb{1}^T\boldsymbol{\omega} \leq G}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\begin{bmatrix}\mathfrak{B}\\ \mathbb{1}\end{bmatrix}$ is an interval matrix, it is TU.

Budgeted group cover sparsity



Structure: *We seek the sparsest signal covered by G groups.*

$$\text{Objective: } \boldsymbol{d}^T\boldsymbol{\omega} \to \mathbb{1}^T\boldsymbol{s}$$

Linear description: At least one of the G selected groups cover each sparse coefficient.

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}, \mathbb{1}^T\boldsymbol{\omega} \leq G}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\begin{bmatrix} \mathfrak{B} \\ \mathbb{1} \end{bmatrix}$ is an interval matrix, it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M}\{\|\mathbf{x}\|_1 : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|, \mathbb{1}^T\boldsymbol{\omega} \leq G\}$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Budgeted group cover example: Interval overlapping groups

- Basis pursuit (BP): $\|\mathbf{x}\|_1$
- Sparse group Lasso (SGL_q):

$$(1-\alpha)\sum_{\mathcal{G}\in\mathfrak{G}} \sqrt{|\mathcal{G}|}\|\mathbf{x}^{\mathcal{G}}\|_q + \alpha\|\mathbf{x}^{\mathcal{G}}\|_1$$

- TU-relax (TU):

$$\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M} \{\|\mathbf{x}\|_1 : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|, \mathbb{1}^T\boldsymbol{\omega} \leq G\}$$

for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise.

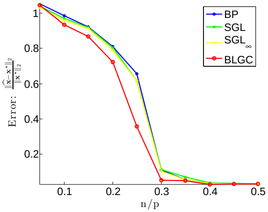
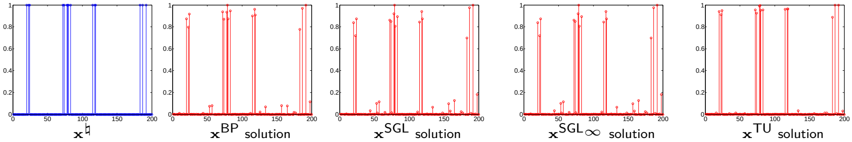
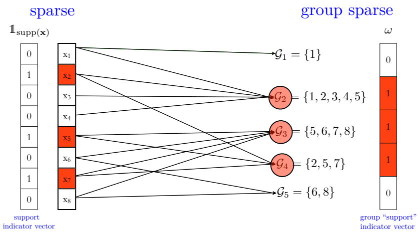


Figure: *Recovery for $n = 0.25p, s = 15, p = 200, G = 5$ out of $M = 29$ groups.*



relative errors: $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{BP}}\|_2}{\|\mathbf{x}^\natural\|_2} = .128$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{SGL}}\|_2}{\|\mathbf{x}^\natural\|_2} = .181$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{SGL}_\infty}\|_2}{\|\mathbf{x}^\natural\|_2} = .085$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{TU}}\|_2}{\|\mathbf{x}^\natural\|_2} = .058$

Group intersection sparsity [10, 19, 1]



Structure: *We seek the signal intersecting with minimal number of groups.*

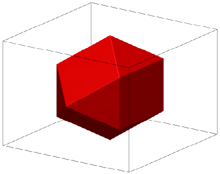
Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Group intersection sparsity [10, 19, 1]



$\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$
(left) intersection (right) cover.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

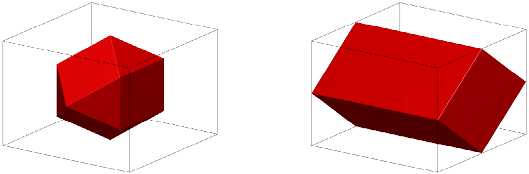
Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T \boldsymbol{\omega} : \boldsymbol{H}_k |\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\}$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Group intersection sparsity [10, 19, 1]



$\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$
(left) intersection (right) cover.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$ (*submodular*)

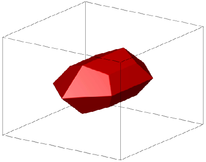
Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T \boldsymbol{\omega} : \boldsymbol{H}_k |\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\} \stackrel{\star}{=} \sum_{\mathcal{G} \in \mathfrak{G}} \|x_{\mathcal{G}}\|_\infty$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Group intersection sparsity [10, 19, 1]



$\mathfrak{G} = \{\{1,2,3\},\{2\},\{3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$ (*submodular*)

Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T \boldsymbol{\omega} : \boldsymbol{H}_k |\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\} \overset{\star}{=} \sum_{\mathcal{G} \in \mathfrak{G}} \|x_{\mathcal{G}}\|_\infty$ for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Remark: For hierarchical $\mathfrak{G}_H$, group intersection and tree sparsity models coincide.

Beyond linear costs: Graph dispersiveness

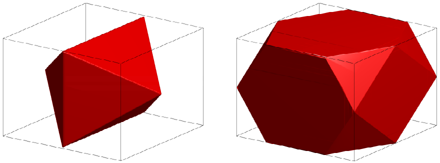


Figure: (left) $\|\mathbf{x}\|_s^{**} = 0$ (right) $\|\mathbf{x}\|_s^{**} \leq 1$ for $\mathcal{E} = \{\{1,2\},\{2,3\}\}$ (chain graph)

Structure: *We seek a signal dispersive over a given graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \sum_{(i,j)\in\mathcal{E}} s_i s_j$ (non-linear, supermodular function)

Linearization:

$$\boxed{\|\mathbf{x}\|_s = \min_{\mathbf{z}\in\{0,1\}^{|\mathcal{E}|}} \{\textstyle\sum_{(i,j)\in\mathcal{E}} z_{ij} : z_{ij} \geq s_i + s_j - 1\}}$$

When edge-node incidence matrix of $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ is TU (e.g., bipartite graphs), it is TU.

Beyond linear costs: Graph dispersiveness

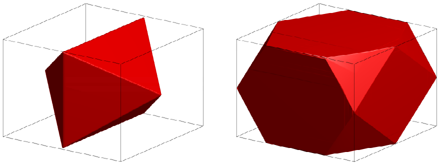


Figure: (left) $\|\mathbf{x}\|_s^{**} = 0$ (right) $\|\mathbf{x}\|_s^{**} \leq 1$ for $\mathcal{E} = \{\{1,2\},\{2,3\}\}$ (chain graph)

Structure: *We seek a signal dispersive over a given graph $\mathcal{G}(\mathfrak{P},\mathcal{E})$*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \sum_{(i,j)\in\mathcal{E}} s_i s_j$ (non-linear, supermodular function)

Linearization:

$$\boxed{\|\mathbf{x}\|_s = \min_{\mathbf{z}\in\{0,1\}^{|\mathcal{E}|}} \{\textstyle\sum_{(i,j)\in\mathcal{E}} z_{ij} : z_{ij} \geq s_i + s_j - 1\}}$$

When edge-node incidence matrix of $\mathcal{G}(\mathfrak{P},\mathcal{E})$ is TU (e.g., bipartite graphs), it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \sum_{(i,j)\in\mathcal{E}} (|x_i| + |x_j| - 1)_+$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise.

References I

[1] Francis Bach.
Structured sparsity-inducing norms through submodular functions.
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