License Information for Theory and Methods for Reinforcement Learning (EE-618)

- This work is released under a Creative Commons License with the following terms:
  - **Attribution**
    - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees must give the original authors credit.
  - **Non-Commercial**
    - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees may not use the work for commercial purposes – unless they get the licensor’s permission.
  - **Share Alike**
    - The licensor permits others to distribute derivative works only under a license identical to the one that governs the licensor’s work.
  - **Full Text of the License**
Recap - Reinforcement learning objective

- Reinforcement Learning: Sequential decision making in **unknown** environment
- Markov decision process: $M = (S, A, P, r, \mu, \gamma)$
- Stationary stochastic policy $\pi : S \rightarrow \Delta(A)$, $a_t \sim \pi(\cdot | s_t)$
- State-value function: $V^\pi(s) := \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s, \pi \right]$
- Performance objective: $\max_\pi (1 - \gamma) \sum_{s \in S} \mu(s) V^\pi(s)$

**Challenges:**
- Infer long-term consequences based on limited, noisy short-term feedback.
- Unknown dynamics - Knowledge only through sampled experience.
- Large state and actions spaces.
- Highly nonconvex objective.
Motivation

- Approximate dynamic programming
  - Attempts to find approximate fixed-point solutions to the (nonlinear) Bellman equation.
  - Pros:
    + Well-studied setting for tabular MDPs that comes with theoretical convergence guarantees.
      - See Lecture 2.
    + Deep-learning variants (e.g., DQN [19]) are powerful.
  - Cons:
    - Training can oscillate or even diverge under the simplest parameterizations or in offline settings.
      - For divergent examples for TD-learning with nonlinear parameterizations, see e.g., Ex 6.6 and 6.7 in [3].
      - For divergent example for approximate VI with linear parameterizations, see e.g., Ex. 6.11 in [3].
    - Incompatible with classical machine-learning tools that are rooted in convex optimization.
Motivation (cont’d)

- The linear programming approach (this lecture)
  - Introduces the linear programming (LP) approach, i.e., an alternative convex viewpoint that formulates the RL problem as a linear program.
  - Overviews recent scalable algorithms with theoretical guarantees rooted in the LP approach.
  - Highlights how historical key limitations have been eliminated.
Revisiting Bellman optimality equation

- Finding \( V^* \) satisfying Bellman optimality equation can be written as a feasibility problem:

\[
\min_V 0 \\
\text{s.t. } V(s) = \max_{a \in A} \left[ r(s, a) + \gamma \sum_{s' \in S} P(s'|s,a)V(s') \right], \quad \forall s \in S.
\]

- The only feasible point is \( V^* \).

- The above constraints are nonlinear in \( V \).
Relaxation of Bellman optimality condition

- The Bellman optimality equation suggests that $V^*$ is the “least feasible solution" of all $V \in \mathbb{R}^{\mid S\mid}$ satisfying

$$V(s) \geq r(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V(s'), \quad \forall s \in S, \ a \in A.$$ 

- Note that the new inequality constraint is linear in $V \implies \text{Linear Programming (LP)}$.

**Figure**: Graphical interpretation of Bellman inequality
Primal LP

Let \( \mu(s) > 0, s \in S \) be the initial distribution (or any positive weights).

\[
\min_{V} \quad (1 - \gamma) \sum_{s \in S} \mu(s)V(s)
\]

\[
\text{s.t.} \quad V(s) \geq r(s, a) + \gamma \sum_{s' \in S} P(s'|s, a)V(s'), \quad \forall s \in S, \ a \in A.
\]

Remarks:
- The optimal value function \( V^* \) is the unique solution to the above LP.
- Number of decision variables: \(|S|\), number of constraints: \(|S||A|\).
- An optimal (deterministic) policy is the associated greedy policy

\[
\pi^*(s) \in \arg \max_{a \in A} \left[ r(s, a) + \gamma \sum_{s' \in S} P(s'|s, a)V^*(s') \right]. \tag{1}
\]

- The factor \((1 - \gamma)\) in the objective ensures that the dual variables are in the simplex.
Corollary (LP Formulation and $V^*$)

$V^*$ is the unique optimal solution to the above LP formulation for any positive weights $\{\mu(s)\}$.

Proof Sketch

- First, we establish that $V^*$ is a feasible solution.
- Then, we need to show that $V^*$ minimizes the objective.
- By the monotonicity property of the Bellman operator, we get that $V \geq V^*$, for any feasible $V$.

Remark:

- The unique optimizer does not depend on the positive weights $\{\mu(s)\}$.
- Slide 21 discusses how does the choice of $\{\mu(s)\}$ affect the performance guarantees of approximate linear programming schemes.
A closer look at the primal LP

Recall: Primal LP

Let $\mu(s) > 0, s \in S$ be the initial distribution (or any positive weights).

$$\min_V (1 - \gamma) \sum_{s \in S} \mu(s) V(s)$$

s.t. $V(s) \geq r(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V(s'), \forall s \in S, a \in A.$

(P)

Observations:

- Any $V^*$ is feasible as

$$V^*(s) = TV^*(s) \geq r(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^*(s'), \forall (s, a) \in S \times A.$$ 

This implies feasibility.

- For any feasible $V$, we have $V \geq TV$. By monotonicity of the Bellman operator $T$, we have

$$V \geq TV \geq T^2 V \geq \cdots \geq T^\infty V = V^*.$$ 

This implies optimality.
Solving MDPs with LP - Dual LP formulation

**Dual LP**

\[
\begin{align*}
\text{max} & \quad \sum_{s \in S} \sum_{a \in A} r(s, a) \lambda(s, a) \\
\text{s.t.} & \quad \sum_{a \in A} \lambda(s, a) = (1 - \gamma) \mu(s) + \gamma \sum_{s' \in S, a' \in A} P(s'|s, a') \lambda(s', a'), \quad \forall \ s \in S, \\
& \quad \lambda(s, a) \geq 0, \quad \forall \ s \in S, a \in A.
\end{align*}
\]

**(D)**

**Remarks:**
- The number of decision variables: $|S||A|$.
- The number of constraints: $|S| + |S||A|$.
- The constraints implicitly implies the decision variables are in the probability simplex.
- The solution to the dual LP, $\lambda^*$, corresponds to the state-action occupancy of $\pi^*$. 

Theory and Methods for Reinforcement Learning | Prof. Niao He & Prof. Volkan Cevher, niao.he@ethz.ch & volkan.cevher@epfl.ch
A closer look at the dual LP

○ For any policy \( \pi \) and \( s_0 \sim \mu \), define the state-action visitation distribution \( \lambda^\pi(s,a) \) as

\[
\lambda^\pi(s,a) := (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s, a_t = a \mid s_0 \sim \mu, \pi)
\]

○ We can write

\[
(1 - \gamma) \mathbb{E}_{s \sim \mu} [V^\pi(s)] = (1 - \gamma) \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 \sim \mu, \pi \right]
\]

\[
= (1 - \gamma) \sum_{s \in \mathcal{S}, a \in \mathcal{A}} \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s, a_t = a \mid s_0 \sim \mu, \pi) r(s, a)
\]

\[
= \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} \lambda^\pi(s,a) r(s,a)
\]

\[\Rightarrow\text{ primal objective (P)}\]

\[\Rightarrow\text{ dual objective (D)}\]
A closer look at the dual LP (cont’d)

Recall: Dual LP

\[
\begin{align*}
\max_{\lambda} & \sum_{s \in S} \sum_{a \in A} r(s,a) \lambda(s,a) \\
\text{s.t.} & \sum_{a \in A} \lambda(s,a) = (1 - \gamma) \mu(s) + \gamma \sum_{s' \in S, a' \in A} P(s|s',a') \lambda(s',a'), \quad \forall \ s \in S, \\
& \lambda(s,a) \geq 0, \quad \forall \ s \in S, a \in A.
\end{align*}
\]

Observations:

- Easy to verify that $\lambda^\pi(s,a)$ satisfies the constraints in the dual LP.
- By Markov property, we have (see supplementary material for details)

\[
\lambda^\pi(s,a) = (1 - \gamma) \mu(s) \pi(a|s) + \gamma \sum_{s',a'} \pi(a|s) P(s'|s,a') \lambda^\pi(s',a').
\]

Summing over $a$ implies feasibility.
A closer look at the dual LP (cont’d)

Dual LP

\[
\begin{align*}
\max_{\lambda} & \quad \sum_{s \in S} \sum_{a \in A} r(s, a) \lambda(s, a) \\
\text{s.t.} & \quad \sum_{a \in A} \lambda(s, a) = (1 - \gamma) \mu(s) + \gamma \sum_{s' \in S, a' \in A} P(s'|s, a') \lambda(s', a'), \quad \forall s \in S, \\
& \quad \lambda(s, a) \geq 0, \quad \forall s \in S, a \in A.
\end{align*}
\]

(D)

Observations:

- For any \( \lambda \) feasible to the dual LP, we can define a policy

\[
\pi_\lambda(a \mid s) = \frac{\lambda(s, a)}{\sum_{a \in A} \lambda(s, a)}.
\]

It then holds \( \lambda \pi_\lambda = \lambda \).

- Note that \( \lambda^*(s, a) = \lambda \pi^*(s, a) \) and \( \pi^*(a \mid s) = \frac{\lambda^*(s, a)}{\sum_{a \in A} \lambda^*(s, a)} \). (self-check)

- Optimal policy does not depend on \( \mu \). (LP sensitivity analysis)
Finding the optimal policy

◦ Primal LP approach:
  ▶ Solve primal LP to obtain for the optimal value function $V^*$
  ▶ Then construct the optimal policy (deterministic) through the greedy policy
    
    $$
    \pi^*(s) \in \arg \max_{a \in A} \left[ r(s, a) + \gamma \sum_{s' \in S} P(s' | s, a)V^*(s') \right].
    $$

◦ Dual LP approach:
  ▶ Solve the dual LP to obtain the optimal state-action occupancy $\lambda^*$
  ▶ Then construct the optimal policy (randomized) by
    
    $$
    \pi^*(a | s) = \frac{\lambda^*(s, a)}{\sum_{a \in A} \lambda^*(s, a)}.
    $$

◦ Reference: See [29] (Section 6.9)
Linear Programming - Summary

Primal LP:

\[ \min_{V \in \mathbb{R}^{|S|}} (1 - \gamma)\langle \mu, V \rangle \]
\[ \text{s.t. } EV \geq r + \gamma PV. \quad (P) \]

- Primal LP over value functions
- \(|S|\) decision variables and \(|S||A|\) constraints
- \(\forall V\) primal feasible \(\Rightarrow V^* \leq V\)
- Optimal value function \(V^*\) is the optimizer
- Optimal policy is the associated greedy policy

Dual LP

\[ \max_{\lambda \in \mathbb{R}^{|S||A|}} \langle \lambda, r \rangle \]
\[ \text{s.t. } E^\top \lambda = (1 - \gamma)\mu + \gamma P^\top \lambda, \quad \lambda \geq 0. \quad (D) \]

- Dual LP over occupancy measures
- \(|S||A|\) variables and \(|S| + |S||A|\) constraints
- \(\forall \) policy \(\pi\), the induced \(\lambda^\pi\) is dual feasible
- \(\forall \) feasible \(\lambda \Rightarrow \pi_\lambda\) has occupancy measure \(\lambda\)
- We have \(\lambda^* = \lambda^{\pi^*}\) and \(\pi^* = \pi_{\lambda^*}\)
Dynamic programming vs Linear programming (exact solutions)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Component</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value Iteration (VI)</td>
<td>Bellman Optimality Operator $T$</td>
<td>$V^*$ (control)</td>
</tr>
<tr>
<td>Policy Iteration (PI)</td>
<td>(Multiple) Bellman Operator $T^\pi +$ Greedy Policy</td>
<td>$\pi^*$ (control)</td>
</tr>
<tr>
<td>Linear Programming (LP)</td>
<td>LP solver (Simplex, Interior Point Method)</td>
<td>$V^<em>, \pi^</em>$ (control)</td>
</tr>
</tbody>
</table>

Dynamic Programming:
- Simple iterative updates.
- Polynomial complexity in $|S|$ and $|A|$.
- Works better for small problems.

Linear Programming:
- Rich library of fast LP solvers.
- Polynomial complexity in $|S|$ and $|A|$.
- Works better for large problems.
The LP approach - Pros and Cons

○ Why is this useful?
  ▶ Defining optimality is simple: no value functions, no fixed-point equations, just the numerical objective.
  ▶ Easily comprehensible with an optimization background.
  ▶ A disciplined convex optimization template with a rich set of algorithms.

○ End User License Agreement:
  ▶ Need to ensure $\sum_{a \in A} \lambda(s, a) > 0$ to extract a policy.
  ▶ Number of variables is large.
  ▶ Intractable number of constraints.
  ▶ Constraints may be not satisfied when working with function approximators.
Beyond exact solutions - A bit of history of approximate linear programming (ALP)

- [Manne 1960] [18]
  - Formulated the primal LP over value functions and showed equivalence to Bellman equations.

  - Studied the LP approach to MDPs with continuous state and action spaces.
  - The corresponding LPs are infinite-dimensional.

- [Schweitzer & Seidman 1982] [33]
  - Proposed linear function approximators to reduce the number of decision variables
  - Proposed a relaxation to reduce the number of constraints.

- [De Farias & Van Roy 2003, 2004] [7, 8]
  - Analyzed the reduction [Schweitzer & Seidman 1982] [33].
  - Inspired some follow-up work in RL [Petrik et al. 2009,2010] [27, 26], [Desai et al. 2012] [9], [Abbasi-Yadkori et al. 2014] [1], [Lakshminarayanan et al. 2018] [16].

Theory and Methods for Reinforcement Learning | Prof. Niao He & Prof. Volkan Cevher, niao.he@ethz.ch & volkan.cevher@epfl.ch
Prior works in ALP - Linear function approximation

Large-scale MDPs $\Rightarrow$ Large-scale optimization

- Reduce the number of decision variables by projecting onto a lower-dimensional subspace.
  - Let $\phi_1, \ldots, \phi_k : S \rightarrow \mathbb{R}$ be $k$ basis functions (or features).
  - $\Phi := [\phi_1 \ldots \phi_k] \in \mathbb{R}^{|S| \times k}$ is the corresponding feature matrix.
  - The (ALP) is obtained by adding the linear constraint $V = \Phi \theta = \sum_{i=1}^{k} \theta_i \phi_i$ to the original primal LP (P).

Approximate linear program [Schweitzer & Seidman 1982] [33]

$$\min_{\theta \in \mathbb{R}^k} (1 - \gamma) \sum_{s \in S} \mu(s)(\Phi \theta)(s)$$

s.t. $(\Phi \theta)(s) \geq r(s,a) + \gamma \sum_{s' \in S} P(s'|s,a)(\Phi \theta)(s'), \ \forall \ s \in S, \ a \in A.$
Prior works in ALP - Linear function approximation (cont’d)

Assumptions:  
- The set \( \{\phi_1, \ldots, \phi_k\} \) is linearly independent.
- \( 1 \in \text{span}\left(\{\phi_1, \ldots, \phi_k\}\right) := \{\Phi \theta \mid \theta \in \mathbb{R}^k\} \). This ensures that (ALP) is feasible \([7]\).
- The values \( \sum_{s' \in S} P(s'|s,a)\phi_i(s') \) and \( \mu^\top \phi_i, \ i = 1, \ldots, k \), can be accessed in \( \mathcal{O}(1) \) time.

Quality of the approximate solution (Th.2 in [De Farias & Van Roy 2003] \([7]\))

\[
\|V^* - V^*_{\text{ALP}}\|_{1, \mu} \leq \frac{2}{1 - \gamma} \min_{\theta} \|V^* - \Phi \theta\|_\infty.
\]

\( \varepsilon_{\text{approx}} \): approximation error

Notation:  
- \( \theta^*_{\text{ALP}} \) is optimal to (ALP) and \( V^*_{\text{ALP}} = \Phi \theta^*_{\text{ALP}} \) is the approximate value function.
- \( \|V\|_{1, \mu} := \sum_{s \in S} \mu(s)|V(s)| \) is the \( \mu \)-weighted \( \ell_1 \)-norm, where \( \mu > 0 \).
- \( \Phi \theta^* \) is the \( \| \cdot \|_\infty \)-norm projection of \( V^* \) to the subspace \( V = \Phi \theta \).
- \( \varepsilon_{\text{approx}} := \min_{\theta} \|V^* - \Phi \theta\|_\infty = \|V^* - \Phi \theta^*\|_\infty \) is called the approximation error.
Prior works in ALP - Linear function approximation (cont’d)

Quality of the approximate solution

\[
\|V^* - V_{ALP}^*\|_{1,\mu} \leq \frac{2}{1 - \gamma} \varepsilon_{\text{approx}}.
\]

Remarks:

- \(\varepsilon_{\text{approx}} = \min_{\theta} \|V^* - \Phi \theta\|_{\infty}\) captures the approximation power of the feature map.

- If \(V^* \in \text{span}(\phi_1, \ldots, \phi_k)\), then \(V^* = \Phi \theta_{ALP}^*\).

- In general, \(\|V^* - V_{ALP}^*\|_{1,\mu} = O(\varepsilon_{\text{approx}})\).

- Focus on finding a good basis, leaving the search of the “right” weights to an LP solver.

Figure: Graphical interpretation of ALP [7]
Prior works in ALP - Constraint sampling

- Reduce the number of constraints by constraint sampling.
  - \((x, a)\) is treated as an uncertainty parameter.
  - \(S \times A\) is the uncertainty space.
  - \(P\) is a probability distribution on \(S \times A\).
  - \(\{(s_i, a_i)\}_{i=1}^{N}\) i.i.d. samples on \((S \times A, P)\).
  - \(\mathcal{N} \subseteq \mathbb{R}^k\) is a bounding set.
  - The relaxed LP (RLP) is obtained from (ALP) by restricting \(\theta \in \mathcal{N}\) with \(N\) sampled constraints.

Relaxed linear program \cite{De_Farias_Van_Roy_2001} \cite{8}

\[
\begin{align*}
\min_{\theta \in \mathcal{N}} \quad & (1 - \gamma) \sum_{s \in \mathcal{S}} \mu(s) (\Phi \theta)(s) \\
\text{s.t.} \quad & (\Phi \theta)(s_i) \geq r(s_i, a_i) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s_i, a_i)(\Phi \theta)(s'), \quad \forall \ i = 1, \ldots, N.
\end{align*}
\]
Prior works in ALP - Constraint sampling (cont’d)

Assumptions:

○ The set $\mathcal{N} \subset \mathbb{R}^k$ is compact, i.e., bounded and closed.

○ The optimal solution $\theta^*_{\text{ALP}}$ to (ALP) is in $\mathcal{N}$.

○ The sampling probability distribution is $P \propto \lambda \pi^*$, i.e., the state-action visitation distribution induced by an optimal policy $\pi^*$.

How many samples give a good solution (Th.3.1 in [De Farias & Van Roy 2004] [8])

Let $\varepsilon, \delta \in (0, 1)$. If $N \geq \tilde{O}\left(\frac{4k \log(\frac{1}{\delta})}{(1-\gamma)\varepsilon} \sup_{\theta \in \mathcal{N}} \|V^* - \Phi \theta\|_\infty \right)$, then with probability at least $1 - \delta$, we have

$$\|V^* - V_{\text{RLP}}\|_{1,\mu} \leq \|V^* - V^*_{\text{ALP}}\|_{1,\mu} + \varepsilon \|V^*\|_{1,\mu},$$

where the probability is taken over the random sampling of constraints.

Notation:

○ $\theta^*_{\text{RLP}}$ is optimal to (RLP) and $V^*_{\text{RLP}} = \Phi \theta^*_{\text{RLP}}$ is the approximate value function.

○ $\varepsilon \in (0, 1)$ is the desired approximation accuracy.

○ $\delta \in (0, 1)$ is the desired confidence level.
Prior works in ALP - Constraint sampling (cont’d)

Remarks:

○ (RLP) is a relaxation of (ALP).

○ The constraint $\theta \in \mathcal{N}$ ensures that the optimal value of (RLP) is bounded.

○ The relaxed linear program (RLP) is random.

○ $\theta^*_{RLP}$ and $V^*_{RLP} = \Phi \theta^*_{RLP}$ are random variables.

○ A lower bound on the number of samples needed to achieve an $\varepsilon$-accurate solution with probability at least $1 - \delta$, is called the sample complexity of the problem.

○ The sample complexity bound depends on the choice of the bounding set $\mathcal{N}$.

○ The sample complexity bound requires access to samples from the optimal state-action visitation distribution (which is not known a priori).
Common theme of all prior ALP works

- Reduce the number of decision variables by projecting on a low-dimensional subspace.
- Reduce the number of constraints (e.g., by constraint sampling).
- Solve the resulted LP with generic solver.
- Analyze the quality of the approximate solution.
- Either scale badly with the size of the state-action spaces or
- Require access to samples from a distribution that depends on the optimal policy.
- Require knowledge of dynamics or access to a simulator.
- Focus mainly on the approximation of the optimal value function but not so much on extracting a nearly optimal policy.

Is this the best we can do?
Some notation: towards an unconstrained problem.

- We will write an equivalent unconstrained problem.
- To simplify the notation, we need to introduce a couple of operators:
  - $E : \mathbb{R}^{S \times A} \to \mathbb{R}^S$ such that $(EV)(s, a) = V(s)$.
  - $P : \mathbb{R}^{S \times A} \to \mathbb{R}^S$ such that $(PV)(s, a) = \sum_{s'} P(s'|s, a)V(s')$.
- Their adjoints are given by
  - $E^T : \mathbb{R}^S \to \mathbb{R}^{S \times A}$ such that $(E^T \lambda)(s) = \sum_a \lambda(s, a)$.
  - $P^T : \mathbb{R}^S \to \mathbb{R}^{S \times A}$ such that $(P^T \lambda)(s') = \sum_{s, a} P(s'|s, a)\lambda(s, a)$.
Towards the Lagrangian

- Instead of working solely with the primal or dual LP formulation, we work with an object between them.
- Introducing the Lagrangian multipliers vector $\lambda \in \mathbb{R}^{|S||A|}$, we can write the Lagrangian as follows:

Primal LP:

$$\min_{V \in \mathbb{R}^{|S|}} \ (1 - \gamma) \langle \mu, V \rangle \quad \text{(P)}$$

subject to:

$$EV \geq r + \gamma PV.$$ 

Dual LP

$$\max_{\lambda \in \mathbb{R}^{|S||A|}} \langle \lambda, r \rangle \quad \text{(D)}$$

subject to:

$$E^\top \lambda = (1 - \gamma) \mu + \gamma P^\top \lambda, \quad \lambda \geq 0.$$ 

Saddle point formulation

$$\min_{V} \max_{\lambda \geq 0} (1 - \gamma) \langle \mu, V \rangle + \langle \lambda, r + \gamma PV - EV \rangle. \quad \text{(Saddle-point problem)}$$
Minimax optimization

**Bilinear min-max template**

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \ f(x) + \langle Ax, y \rangle - h(y),
\]

where \( \mathcal{X} \subseteq \mathbb{R}^p \) and \( \mathcal{Y} \subseteq \mathbb{R}^n \).

- \( f : \mathcal{X} \rightarrow \mathbb{R} \) is convex.
- \( h : \mathcal{Y} \rightarrow \mathbb{R} \) is convex.

**Convex-concave min-max template**

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y),
\]

where \( \Phi(x, y) \) is convex in \( x \) and concave in \( y \).
Basic algorithms for minimax

- Given \( \min_{x \in X} \max_{y \in Y} \Phi(x, y) \), define \( V(z) = [\nabla_x \Phi(x, y), -\nabla_y \Phi(x, y)] \) with \( z = [x, y] \).

Figure: Trajectory of different algorithms for a simple bilinear game \( \min_x \max_y xy \).

- (In)Famous algorithms
  - Gradient Descent Ascent (GDA)
  - Proximal point method (PPM)
  - Extra-gradient (EG)
  - Optimistic Gradient Descent Ascent (OGDA)
  - Reflected-Forward-Backward-Splitting (RFBS)

- EG and OGDA are approximations of the PPM

- \( z^{k+1} = z^k - \eta V(z^k) \).
- \( z^{k+1} = z^k - \eta V(z^{k+1}) \).
- \( z^{k+1} = z^k - \eta V(z^k - \alpha V(z^{k-1})) \).
- \( z^{k+1} = z^k - \eta [2V(z^k) - V(z^{k-1})] \).
- \( z^{k+1} = z^k - \eta V(2z^k - z^{k-1}) \).
Primal-dual $\pi$-learning

Saddle point formulation

\[
\min_V \max_{\lambda \in \Delta_S \times A} \left( 1 - \gamma \right) \langle \mu, V \rangle + \langle \lambda, r + \gamma PV - EV \rangle. 
\]  
(Saddle-point problem)

- For known dynamics, it can be solved via primal-dual updates:
  - \( V_{k+1} = V_k - \eta \left( (\gamma P - E) \top \lambda_k + \mu \right) \).
  - \( \lambda_{k+1} \propto \lambda_k \odot e^{\eta (r + \gamma PV_k - EV_k)} \), where \( \odot \) denotes entry wise multiplication.

- Gradients are expectations under the occupancy measure iterates \( \lambda_k \) and the transition law \( P \)
  \[ \Rightarrow \] efficient stochastic implementation [Chen et al. 2018] [6], [Jin & Sidford. 2018] [13].

- State-of-the-art sample complexity for solving small MDPs.
  - \( \mathcal{O} \left( \frac{|S||A| \log \left( \frac{1}{\delta} \right)}{(1-\gamma)^4 \varepsilon^2} \right) \) samples for finding an \( \varepsilon \)-optimal policy with probability at least \( 1 - \delta \).
Scaling up

**Large-scale MDPs \Rightarrow Large-scale optimization**

- Parameterize $\lambda$ and $V$ via linear functions
  - $\lambda_\nu = \Psi_\nu$, for some feature matrix $\Psi \in \mathbb{R}^{|S|A| \times n}$
  - $V_\theta = \Phi_\theta$, for some feature matrix $\Phi \in \mathbb{R}^{|S| \times m}$

**Assumption:** The columns of $\Psi$ are probability distributions.

**Relaxed saddle point formulation**

$$\min_{\theta} \max_{\nu \in \Delta[n]} \left(1 - \gamma\right)\langle \mu, \Phi \theta \rangle + \langle \nu, \Psi^\top(r + \gamma P \Phi \theta - E \Phi \theta) \rangle$$
Scaling up (cont’d)

Relaxed saddle point formulation

\[
\min_{\theta} \max_{\nu \in \Delta_{[n]}} \left( (1 - \gamma) \langle \mu, \Phi \theta \rangle + \langle \nu, \Psi^\top (r + \gamma P \Phi \theta - E \Phi \theta) \rangle \right)
\]

- Primal-dual updates:
  - \( \theta_{k+1} = \theta_k - \eta \left( (\gamma P \Phi - E \Phi)^\top \Psi \nu_k + \Phi^\top \mu \right) \),
  - \( \nu_{k+1} \propto \nu_k \odot e^{\eta \Psi^\top (r + \gamma P \Phi \theta_k - E \Phi \theta_k)} \).

- Implementable with only sample access to the columns of \( \Psi \) and the transition law \( P \) [Chen et al. 2018] [6].
  - \( \mathcal{O} \left( \frac{nm \log(\frac{1}{\delta})}{(1-\gamma)^4 \varepsilon^2} \right) \) samples for finding an \( \varepsilon + \varepsilon_{\text{approx}} \)-optimal policy with probability at least \( 1 - \delta \).
  - \( \varepsilon_{\text{approx}} \) captures the expressivity of the approximation architecture.
Proximal point method (PPM)

- Consider the following smooth unconstrained optimization problem:
  \[ \min_{\mathbf{x} \in \mathbb{R}^p} f(\mathbf{x}) \]

**Proximal point method for convex minimization.**

For a step-size \( \tau > 0 \), PPM can be written as follows

\[
\mathbf{x}^{k+1} = \arg \min_{\mathbf{x} \in \mathbb{R}^p} \left\{ f(\mathbf{x}) + \frac{1}{2\tau} \| \mathbf{x} - \mathbf{x}^k \|^2 \right\} := \text{prox}_{\tau f}(\mathbf{x}^k) \tag{3}
\]

**Observations:**
- The optimality condition of (3) reveals a simpler PPM recursion for smooth \( f \):
  \[
  \mathbf{x}^{k+1} = \mathbf{x}^k - \tau \nabla f(\mathbf{x}^{k+1}).
  \]
- PPM is an **implicit**, non-practical algorithm since we need the point \( \mathbf{x}^{k+1} \) for its update.
- Each step of PPM can be as hard as solving the original problem.
- Convergence properties are well understood due to Rockafellar [32].
PPM and minimax optimization

PPM applied to the minimax template: \( \min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^n} \Phi(x, y) \)

Define \( z = [x, y]^\top \) and \( V(z) = [\nabla_x \Phi(x, y), -\nabla_y \Phi(x, y)]^\top \). PPM iterations with a step-size \( \tau > 0 \) is given by

\[
z^{k+1} = z^k - \tau V(z^{k+1}).
\]

**Derivation:**

- For \( \tau > 0 \), \((x^{k+1}, y^{k+1})\) is the unique solution to the saddle point problem,

\[
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^n} \Phi(x, y) + \frac{1}{2\tau} \| x - x^k \|^2 - \frac{1}{2\tau} \| y - y^k \|^2
\]

\[(4)\]

- Writing the optimality condition of the update in (4)

\[
\begin{align*}
x^{k+1} &= x^k - \tau \nabla_x \Phi(x^{k+1}, y^{k+1}), \\
y^{k+1} &= y^k + \tau \nabla_y \Phi(x^{k+1}, y^{k+1})
\end{align*}
\]

\[(5)\]

**Observation:**

- PPM is an implicit algorithm.

- For the bilinear problem, PPM is implementable!
Proximal point methods in the Bregman setup

Definition: Bregman distance

Let $\omega : \mathcal{X} \rightarrow \mathbb{R}$ be a distance generating function where $\omega$ is $1-$strongly convex w.r.t. some norm $\| \cdot \|$ on the underlying space and is continuously differentiable. The Bregman distance induced by $\omega(\cdot)$ is given by

$$D_\omega(z, z') = \omega(z) - \omega(z') - \nabla \omega(z')^T (z - z').$$

The proximal point method in the Bregman setup reads as follows:

$$x^{k+1} = \arg \min_{x \in \mathbb{R}^p} \left\{ f(x) + \frac{1}{\tau} D_\omega(x, x^k) \right\}$$

Remarks:

- Choosing the negative entropy as a generating function $\omega(x) = \langle x, \log x \rangle$, we obtain the KL divergence. Such $\omega(x)$ is $1-$strongly convex in $\| \cdot \|_1$ norm.
- This choice will allow to avoid projection in the simplex constraints and it improves the dependence on the domain dimension.
- Now, we will see PPM in action on the Lagrangian.
REPS: A success story

- REPS is widely popular in the robotics community.
- It applies proximal point to the Dual LP.
- A robot trained with REPS manages to play table tennis.

Figure: Source: Relative Entropy Policy Search [25]
Towards REPS: Proximal point on the Dual LP

- Recall: Proximal point is generally an implicit method.
- However, for a linear objective PPM can be implemented.
- Hence, we can apply proximal point updates on the Lagrangian, which is just a bilinear form.

Recall: Dual LP

\[ \lambda_k = \arg\max_{\lambda \in \Delta} \langle \lambda, r \rangle \]
\[ \text{s.t. } E^T \lambda = \gamma P^T \lambda + (1 - \gamma) \mu. \]

Remarks:
- The problem in the current form suffers from \(|S|\) many constraints.
The Lagrangian: Towards an unconstrained problem.

- The corresponding Lagrangian is:

\[
\max_{\lambda \in \Delta} \min_V \langle \lambda, r \rangle + \langle V, \gamma P^T \lambda - E^T \lambda \rangle + (1 - \gamma) \langle V, \mu \rangle.
\]

- Applying **proximal point** we obtain the following update:

\[
\lambda_k = \arg \max_{\lambda \in \Delta} \min_V \langle \lambda, r \rangle + \langle V, \gamma P^T \lambda - E^T \lambda \rangle + (1 - \gamma) \langle V, \mu \rangle - \frac{1}{\eta} D_{KL}(\lambda, \lambda_{k-1}).
\]

\[
:= f(\lambda)
\]
KKT conditions on the Lagrangian update.

**Derivation:**
- We notice by convexity of the Bregman divergence that the update is convex in $\lambda$.
- We introduce an auxiliary problem for any $V$ as follows:

$$
\lambda_k^V = \arg\max_{\lambda \in \Delta} \langle \lambda, r \rangle + \langle V, \gamma P^T \lambda - E^T \lambda \rangle + (1 - \gamma) \langle V, \mu \rangle - \frac{1}{\eta} D_{KL}(\lambda, \lambda_{k-1}).
$$
- By optimality conditions, it must hold

$$
r + \gamma PV - EV - \frac{1}{\eta} \nabla_{\lambda} D_{KL}(\lambda_k^V, \lambda_{k-1}) = 0.
$$
- Thus, $\lambda_k^V$ can be computed in closed form for any $V$

$$
\lambda_k^V(s, a) = \frac{\lambda_{k-1}(s, a)e^r(s, a)+\gamma(PV)(s, a)-(EV)(s, a)}{\sum_{s, a} \lambda_{k-1}(s, a)e^r(s, a)+\gamma(PV)(s, a)-(EV)(s, a)}. 
$$
The unconstrained problem

○ We can leverage the KKT conditions to write an unconstrained problem where the only decision variable is $V$:

$$
\min_V \langle \lambda_k^V, r \rangle + \langle V, \gamma P^T \lambda_k^V - E^T \lambda_k^V \rangle + (1 - \gamma) \langle V, \mu \rangle - \frac{1}{\eta} D_{KL}(\lambda_k^V, \lambda_{k-1}).
$$

○ With some calculus, we have the following compact form.

**Unconstrained problem (REPS)**

$$
V_k = \min_V (1 - \gamma) \langle \mu, V \rangle + \frac{1}{\eta} \log \sum_{s, a} \lambda_{k-1}(s, a) e^{r(s, a)+\gamma(PV)(s, a)-(EV)(s, a)}.
$$

Remarks:
○ The decision variable $V$ has dimension $|S|$.
○ The objective is convex and smooth with Lipschitz continuous gradient.
The REPS algorithm [25]

<table>
<thead>
<tr>
<th>Algorithm: REPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initialize $\lambda_0$ (for example uniform)</td>
</tr>
<tr>
<td>for each iteration $k = 1, \ldots, K$ do</td>
</tr>
<tr>
<td>Solve the problem</td>
</tr>
<tr>
<td>$V_k = \min_V (1 - \gamma)\langle \mu, V \rangle + \frac{1}{\eta} \log \sum_{s,a} \lambda_{k-1}(s,a)e^{r(s,a) + \gamma(PV_k)(s,a) - (EV_k)(s,a)}$</td>
</tr>
<tr>
<td>Update the occupancy measure:</td>
</tr>
<tr>
<td>$\lambda_k(s,a) \propto \lambda_{k-1}(s,a)e^{r(s,a) + \gamma(PV_k)(s,a) - (EV_k)(s,a)}$</td>
</tr>
<tr>
<td>end for</td>
</tr>
</tbody>
</table>
Sample complexity of REPS [24]

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Oracle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>REPS</td>
<td>Exact gradient</td>
<td>$O \left( \frac{</td>
</tr>
<tr>
<td>REPS</td>
<td>Stochastic Biased Gradients</td>
<td>$O \left( \frac{</td>
</tr>
</tbody>
</table>

Remarks:
- The exact gradient case achieves the best-known sample complexity, e.g., comparable to NPG (see Lecture 5)
- The sample complexity with stochastic gradients degrades.
- For the stochastic gradient case, one needs to assume that $\lambda_k(s,a) \geq \beta > 0$. It solves the exploration problem by assumption.
Off-policy reinforcement learning (aka batch reinforcement learning)

- Learn to control from a previously collected dataset.
- Important for safety-critical applications, where deploying a suboptimal policy during learning is impossible.
  - Think about drug testing.

Remarks:
- This setting is distinct from IRL, where the data is given by an “expert” policy.
- In this setting, we do have access to a reward signal from previous experience.
- We assume that the data covers the state-action space sufficiently well.
Off-policy reinforcement learning: The formalism

○ In off-policy RL, we focus on the usual objective, which is:

\[ J(\pi) = \mathbb{E}_{s \sim \mu} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s, \pi \right] . \]

○ However, we assume access only to samples from a fixed policy \( \tilde{\pi} \).

Remarks: ○ The policy \( \tilde{\pi} \) represents the policy previously used to collect the experience dataset.

○ In drug testing, \( \tilde{\pi} \) may represent the policy used by the human doctors (not necessarily optimal).
A useful subproblem: Offline policy evaluation

- We saw that often we find an optimal policy via learning the state-action value function:

\[
Q^\pi(s, a) = \mathbb{E}\left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s, a_0 = a, \pi \right].
\]

- However, we assume access only to samples from a fixed policy \(\tilde{\pi}\).

- Estimating \(Q^\pi(s, a)\) using samples from \(\tilde{\pi}\) is known as offline policy evaluation.

- Next, we derive a convex programming approach to compute \(Q^\pi(s, a)\).

**Self-study:**
- Compare to the derivation of the Primal LP to compute \(V^*\).
An offline policy evaluation (OPE) approach

**OPE via \( f \)-divergences**

Let \( g \) be the convex conjugate of an \( f \)-divergence. [21] proposes to use the following formulation via \( Q^\pi \):

\[
Q^\pi = \arg\min_Q \mathbb{E}_{\lambda^\pi} g(r - \mathcal{L}_\pi Q) + (1 - \gamma) \langle Q, c \rangle, \tag{OPE}
\]

where \( c(s, a) = \pi(a|s)\mu(s) \) is the joint state-action distribution.

**Remarks:**
- Recall the operator \( \mathcal{L}^\pi \):
  \[
  (\mathcal{L}^\pi Q)(s, a) = Q(s, a) - \gamma \sum_{s', a'} P(s'|s, a)\pi(a'|s')Q(s', a').
  \]
- The problem (OPE) is convex and smooth in \( Q \) because \( g \) is convex.
- The problem (OPE) is unconstrained and \( g \) acts like a loss function.
- A biased objective estimate can be obtained by sampling from \( c \) and \( \lambda^{\tilde{\pi}} \).
- The name \textit{offline} comes from not needing samples from \( \lambda^\pi \).
From policy evaluation to policy optimization

- Maximizing (OPE) objective over $\pi$ gives us a policy optimization objective.
- The resulting formulation is dubbed as AlgaeDICE [23].

### AlgaeDICE

$$\pi^* \in \arg\max_{\pi} \min_Q (1 - \gamma) \langle c, Q \rangle + \mathbb{E}_{\lambda \tilde{\pi}} g (r - \mathcal{L}_\pi Q)$$

### Remarks:
- We only need to sample from the initial distribution $\mu$, the policy $\pi$, and the offline policy $\tilde{\pi}$.
- We only interact with the environment via $\tilde{\pi}$. 
An alternative offline policy evaluation from the Lagrangian perspective [34]

○ The approach in [34] PRO-RL exploits the Lagrangian of (LP) formulation.
○ It has the same underpinnings of REPS adapted for the offline RL.

**PRO-RL [34]**

Let \( h \) be a strongly convex function. The PRO-RL approach uses the following formulation:

\[
\max_{\lambda \in \Delta} \min_{V} \langle \lambda, r + \gamma PV - V \rangle + (1 - \gamma) \langle \mu, V \rangle - \frac{1}{\eta} \mathbb{E}_{(s,a) \sim \lambda} \left( h\left( \frac{\lambda(s,a)}{\hat{\pi}(s,a)} \right) \right)
\]

**Remarks:**

○ The inner product with \( \lambda \) are equivalent to expectations with samples drawn from \( \lambda \):

\[
\langle \lambda, r + \gamma PV - V \rangle = \mathbb{E}_{(s,a) \sim \lambda} \left[ r(s,a) + \gamma PV(s,a) - V(s) \right].
\]

○ [34] proposes to optimize an empirical objective obtained from samples.

○ AlgaeDICE is a \( Q \)-based offline RL approach, whereas PRO-RL is value-based.
Guarantees for PRO-RL

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Main assumptions</th>
<th>Samples for $\epsilon$-optimal policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRO-RL</td>
<td>$\frac{\lambda^*(s,a)}{\lambda^\pi(s,a)} \leq B &lt; \infty$, $h(\cdot)$ is $M_h$-strongly convex</td>
<td>$\mathcal{O}\left(\frac{B</td>
</tr>
</tbody>
</table>

Remarks:
- The assumption $\frac{\lambda^*(s,a)}{\lambda^\pi(s,a)} < \infty$ has the interpretation that the occupancy measure $\lambda^\pi$ has support larger than the support of the optimal occupancy measure $\lambda^*$.
- The sample complexity guarantees worsen as $B$ increases.
- That means that the more “different” $\lambda^\pi$ and $\lambda^*$ are, the more samples are required.
References

Linear programming for large-scale Markov decision problems.

Mirror descent and nonlinear projected subgradient methods for convex optimization.

*N*euo-Dynami*c Programming*.
Athena Scientific, 1996.

A convex analytic approach to Markov decision processes.

A reflected forward-backward splitting method for monotone inclusions involving lipschitzian operators.
References II

Scalable bilinear \( \pi \) learning using state and action features.
31, 33

The linear programming approach to approximate dynamic programming.
19, 21, 22

On constraint sampling in the linear programming approach to approximate dynamic programming.
19, 23, 24

Approximate dynamic programming via a smoothed linear program.
19

On the convergence theory of gradient-based model-agnostic meta-learning algorithms.
63, 65
References III

    *Discrete-Time Markov Control Processes: Basic Optimality Criteria.*
    Springer-Verlag New York, 1996.
    19

    *Further Topics on Discrete-Time Markov Control Processes.*
    19

    Efficiently solving MDPs with stochastic mirror descent.
    31

[14] Sham M. Kakade, Shai Shalev-Shwartz, and Ambuj Tewari.
    Regularization techniques for learning with matrices.
    77, 78

    An extragradient method for finding saddle-points and for other problems.
    62
A linearly relaxed approximate linear program for Markov decision processes.
19

[17] Yura Malitsky and Matthew K Tam.
A forward-backward splitting method for monotone inclusions without cocoercivity.
64

Linear programming and sequential decisions.
19

Human-level control through deep reinforcement learning.
4
References V

A unified analysis of extra-gradient and optimistic gradient methods for saddle point problems: Proximal point approach.

Reinforcement learning via Fenchel-Rockafellar duality.

[22] Ofir Nachum, Yinlam Chow, Bo Dai, and Lihong Li.

Algaedice: Policy gradient from arbitrary experience.

Near optimal policy optimization via REPS.
References VI

Relative entropy policy search.
37, 42

Feature selection using regularization in approximate linear programs for markov decision processes.
19

[27] Marek Petrik and Shlomo Zilberstein.
Constraint relaxation in approximate linear programs.
In International Conference on Machine Learning (ICML), 2009.
19

[28] Leonid Denisovich Popov.
A modification of the arrow-hurwicz method for search of saddle points.
64

15
References VII

Optimization, learning, and games with predictable sequences.
64

[31] R Tyrrell Rockafellar.
Conjugate convex functions in optimal control and the calculus of variations.
67

Monotone operators and the proximal point algorithm.
34, 61

[33] Paul J Schweitzer and Abraham Seidmann.
Generalized polynomial approximations in markovian decision processes.
19, 20

[34] W. Zhan, B. Huang, A. Huang, N. Jiang, and J. D. Lee.
Offline reinforcement learning with realizability and single-policy concentrability, 2022.
49
Supplementary

LP and optimization
Supplementary Material: Bellman Equation for State-action Visitation Distribution

Recall the definition

$$\lambda^\pi(s, a) := \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a | \pi, s_0 \sim \mu).$$

Bellman Equation for $\lambda^\pi$

$$\lambda^\pi(s, a) = \mu(s) \pi(a | s) + \gamma \sum_{s', a'} \pi(a | s) P(s | s', a') \lambda^\pi(s', a').$$
Supplementary Material: Bellman Equation for State-action Visitation Distribution

Proof.

$$\lambda^{\pi}(s, a)$$

$$= P(s_0 = s, a_0 = a) + \sum_{t=1}^{\infty} \gamma^t P(s_t = s, a_t = a | \pi, s_0 \sim \mu)$$

$$= \mu(s)\pi(a|s) + \sum_{t=1}^{\infty} \gamma^t \sum_{s', a'} P(s_t = s, a_t = a | s_{t-1} = s', a_{t-1} = a', \pi, s_0 \sim \mu) P(s_{t-1} = s', a_{t-1} = a' | \pi, s_0 \sim \mu)$$

$$= \mu(s)\pi(a|s) + \gamma \sum_{t=1}^{\infty} P(s_t = s, a_t = a | s_{t-1} = s', a_{t-1} = a') P(s_{t-1} = s', a_{t-1} = a' | \pi, s_0 \sim \mu)$$

$$= \mu(s)\pi(a|s) + \gamma \sum_{t=1}^{\infty} \pi(a|s) P(s_t | s', a') \sum_{t=1}^{\infty} \gamma^{t-1} P(s_{t-1} = s', a_{t-1} = a' | \pi, s_0 \sim \mu)$$

$$= \mu(s)\pi(a|s) + \gamma \sum_{s', a'} \pi(a|s) P(s_t | s', a') \lambda^{\pi}(s', a')$$

where the third equality is due to Markov property. \qed
PPM guarantees for minimax optimization

**Theorem (Convergence of PPM [32])**

Suppose \((x^k, y^k)\) be the iterates generated by PPM (i.e., (5)), then for the averaged iterates, it holds that

\[
\left\| \Phi \left( \frac{1}{K} \sum_{k=1}^{K} x^k, \frac{1}{K} \sum_{k=1}^{K} y^k \right) - \Phi(x^*, y^*) \right\| \leq \frac{\|x^0 - x^*\|^2 + \|y^0 - y^*\|^2}{\tau K}.
\]

**Theorem (Linear convergence [32])**

Suppose \((x^k, y^k)\) be the iterates generated by (5), \(\Phi(\cdot, \cdot)\) is \(\mu_x\)—strongly convex in \(x\) and \(\mu_y\)—strongly concave in \(y\). Let \(\mu = \max\{\mu_x, \mu_y\}\). Then, for any \(\tau > 0\), \((x^k, y^k)\) satisfies the following

\[
r^{k+1} \leq \frac{1}{1 + \mu \tau} r^k,
\]

where \(r^k = \|x^k - x^*\|^2 + \|y^k - y^*\|^2\).

**Remark:**
- Still need an implementable and convergent algorithm beyond the stylized bilinear case.
- Note what happens when \(\tau \to \infty\).
Extra-gradient algorithm (EG) [15]

<table>
<thead>
<tr>
<th>EG method for saddle point problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Choose $x^0, y^0$ and $\tau$.</td>
</tr>
<tr>
<td>2. For $k = 0, 1, \cdots$, perform:</td>
</tr>
<tr>
<td>$\tilde{x}^k := x^k - \tau \nabla_x \Phi(x^k, y^k)$,</td>
</tr>
<tr>
<td>$\tilde{y}^k := y^k + \tau \nabla_y \Phi(x^k, y^k)$.</td>
</tr>
<tr>
<td>$x^{k+1} := x^k - \tau \nabla_x \Phi(\tilde{x}^k, \tilde{y}^k)$.</td>
</tr>
<tr>
<td>$y^{k+1} := y^k + \tau \nabla_y \Phi(\tilde{x}^k, \tilde{y}^k)$.</td>
</tr>
</tbody>
</table>

◦ Idea: Predict the gradient at the next point

$$z^{k+1} = z^k - \tau V(z^k)$$

(EG)

Remark:

◦ 1-extra-gradient computation per iteration
Extra-gradient algorithm: Convergence

**Theorem (General case [10])**

Let $0 < \tau \leq \frac{1}{L}$. It holds that

- Iterates $(x^k, y^k)$ remains bounded in a convex compact set.
- Primal-dual gap reduces: $\text{Gap} \left( \frac{1}{K} \sum_{k=1}^{K} x^k, \frac{1}{K} \sum_{k=1}^{K} y^k \right) \leq O \left( \frac{1}{K} \right)$.

**Theorem (Linear convergence [20])**

Suppose $(x^k, y^k)$ be the iterates generated by Extra-gradient algorithm, $\Phi(\cdot, \cdot)$ is $\mu_x$—strongly convex in $x$ and $\mu_y$—strongly concave in $y$. Let $\mu = \max\{\mu_x, \mu_y\}$. Then, for $\tau = \frac{1}{4L}$, $(x^k, y^k)$ satisfies,

$$r^{k+1} \leq \left(1 - \frac{1}{c\kappa}\right)^k r^0,$$

where $r^k = \|x^k - x^*\|^2 + \|y^k - y^*\|^2$, $\kappa = \frac{L}{\mu}$ is the condition number of the problem, and $c$ is a constant which is independent of the problem parameters.
Optimistic gradient descent ascent algorithm (OGDA) [30]

OGDA for saddle point problems

1. Choose $x^0, y^0, x^1, y^1$ and $\tau$.
2. For $k = 1, \cdots$, perform:
   \[
   x^{k+1} := x^k - 2\tau \nabla_x \Phi(x^k, y^k) + \tau \nabla_x \Phi(x^{k-1}, y^{k-1}).
   \]
   \[
   y^{k+1} := y^k + 2\tau \nabla_y \Phi(x^k, y^k) - \tau \nabla_y \Phi(x^{k-1}, y^{k-1}).
   \]

- Main difference from the GDA: Add a “momentum” or “reflection” term to the updates

\[
z^{k+1} = z^k - \tau \left[ \nabla(z^k) + \frac{1}{2} \left( \nabla(z^k) - \nabla(z^{k-1}) \right) \right].
\]

- Known as Popov’s method [28], it is also a special case of the Forward-Reflected-Backward method [17].

- It has ties to the Reflected-Forward-Backward Splitting (RFBS) method [5]:

\[
z^{k+1} = z^k - \tau \nabla(2z^k - z^{k-1}).
\]

Remark: Advanced material at the end: OGDA is an approximation of PPM for bilinear problems.
OGDA: Convergence

**Theorem (General case [10])**

Let \( 0 < \tau \leq \frac{1}{2L} \), \( x^1 = x^0, y^1 = y^0 \). It holds that

- Iterates \((x^k, y^k)\) remains bounded in a convex compact set.
- Primal-dual gap reduces: \( \text{Gap}\left( \frac{1}{K} \sum_{k=1}^K x^k, \frac{1}{K} \sum_{k=1}^K y^k \right) \leq O\left( \frac{1}{K} \right) \).

**Theorem (Linear convergence [20])**

Suppose \((x^k, y^k)\) be the iterates generated by OGDA, \( \Phi(\cdot, \cdot) \) is \( \mu_x \)—strongly convex in \( x \) and \( \mu_y \)—strongly concave in \( y \). Let \( \mu = \max\{\mu_x, \mu_y\} \). Then, for \( \tau = \frac{1}{4L} \), \((x^k, y^k)\) satisfies,

\[
r^{k+1} \leq \left(1 - \frac{1}{c\kappa}\right)^k r^0,
\]

where \( r^k = \|x^k - x^*\|^2 + \|y^k - y^*\|^2 \), \( \kappa = \frac{L}{\mu} \) is the condition number of the problem, and \( c \) is a constant which is independent of the problem parameters.
**Bregman divergences**

Table: Bregman functions $\psi(x)$ & corresponding Bregman divergences/distances $d_\psi(x,y)^a$.

<table>
<thead>
<tr>
<th>Name (or Loss)</th>
<th>Domain $^b$</th>
<th>$\psi(x)$</th>
<th>$d_\psi(x,y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squared loss</td>
<td>$\mathbb{R}$</td>
<td>$x^2$</td>
<td>$(x-y)^2$</td>
</tr>
<tr>
<td>Itakura-Saito divergence</td>
<td>$\mathbb{R}^{++}$</td>
<td>$-\log x$</td>
<td>$\frac{x}{y} - \log \left( \frac{x}{y} \right) - 1$</td>
</tr>
<tr>
<td>Squared Euclidean distance</td>
<td>$\mathbb{R}^p$</td>
<td>$|x|^2$</td>
<td>$|x-y|^2$</td>
</tr>
<tr>
<td>Squared Mahalanobis distance</td>
<td>$\mathbb{R}^p$</td>
<td>$(x, A x)$</td>
<td>$((x-y), A (x-y))$ $^c$</td>
</tr>
<tr>
<td>Entropy distance</td>
<td>$p$-simplex $^d$</td>
<td>$\sum_i x_i \log x_i$</td>
<td>$\sum_i x_i \log \left( \frac{x_i}{y_i} \right)$</td>
</tr>
<tr>
<td>Generalized I-divergence</td>
<td>$\mathbb{R}^p^{++}$</td>
<td>$\sum_i x_i \log x_i$</td>
<td>$\sum_i \left( \log \left( \frac{x_i}{y_i} \right) - (x_i - y_i) \right)$</td>
</tr>
<tr>
<td>von Neumann divergence</td>
<td>$\mathbb{S}^p^{p \times p}$</td>
<td>$\log X - X$</td>
<td>$\text{tr} \left( X (\log X - \log Y) - X + Y \right)^e$</td>
</tr>
<tr>
<td>logdet divergence</td>
<td>$\mathbb{S}^p^{p \times p}$</td>
<td>$-\log \text{det} X$</td>
<td>$\text{tr} \left( X Y^{-1} \right) - \log \text{det} \left( X Y^{-1} \right) - p$</td>
</tr>
</tbody>
</table>

---

$^a$ $x, y \in \mathbb{R}$, $x, y \in \mathbb{R}^p$ and $X, Y \in \mathbb{R}^{p \times p}$.

$^b$ $\mathbb{R}_+$ and $\mathbb{R}^{++}$ denote non-negative and positive real numbers respectively.

$^c$ $A \in \mathbb{S}^p_+ \times p^p$, the set of symmetric positive semidefinite matrix.

$^d$ $p$-simplex $: = \{x \in \mathbb{R}^p : \sum_{i=1}^p x_i = 1, x_i \geq 0, i = 1, \ldots, p\}$

$^e$ $\text{tr}(A)$ is the trace of $A$. 

---

Theory and Methods for Reinforcement Learning | Prof. Niao He & Prof. Volkan Cevher, niao.he@ethz.ch & volkan.cevher@epfl.ch
What happens if we use a Bregman distance $d_\psi$ in gradient descent?

Let $\psi : \mathbb{R}^p \to \mathbb{R}$ be a $\mu$-strongly convex and continuously differentiable function and let the associated Bregman distance be $d_\psi(x, y) = \psi(x) - \psi(y) - \langle x - y, \nabla \psi(y) \rangle$.

Assume that the inverse mapping $\psi^*$ of $\psi$ is easily computable (i.e., its convex conjugate).

- **Majorize**: Find $\alpha_k$ such that
  \[
  f(x) \leq f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + \frac{1}{\alpha_k} d_\psi(x, x^k) := Q^k_\psi(x, x^k)
  \]

- **Minimize**
  \[
  x^{k+1} = \arg\min_x Q^k_\psi(x, x^k) \Rightarrow \nabla f(x^k) + \frac{1}{\alpha_k} \left( \nabla \psi(x^{k+1}) - \nabla \psi(x^k) \right) = 0
  \]
  \[
  \nabla \psi(x^{k+1}) = \nabla \psi(x^k) - \alpha_k \nabla f(x^k)
  \]
  \[
  x^{k+1} = \nabla \psi^*(\nabla \psi(x^k) - \alpha_k \nabla f(x^k)) \quad (\nabla \psi(\cdot))^{-1} = \nabla \psi^*(\cdot)[31].
  \]

- Mirror descent is a **generalization** of gradient descent for functions that are Lipschitz-gradient in norms other than the Euclidean.
- MD allows to deal with some **constraints** via a proper choice of $\psi$. 

*Mirror descent [2]*
What to keep in mind about mirror descent?

- Approximates the optimum by lower bounding the function via hyperplanes at $x_t$.

- The smaller the gradients, the better the approximation!
Mirror descent example

How can we minimize a convex function over the unit simplex?

\[ \min_{x \in \Delta} f(x), \]

where

- \( \Delta := \{ x \in \mathbb{R}^p : \sum_{j=1}^{p} x_j = 1, x \geq 0 \} \) is the unit simplex;
- \( f \) is convex \( L_f \)-Lipschitz continuous with respect to some norm \( \| \cdot \| \). (not necessarily \( L \)-Lipschitz gradient)

Entropy function

- Define the entropy function

\[ \psi_e(x) = \sum_{j=1}^{p} x_j \ln x_j \quad \text{if} \ x \in \Delta, \quad +\infty \quad \text{otherwise}. \]

- \( \psi_e \) is 1-strongly convex over \( \text{int} \Delta \) with respect to \( \| \cdot \|_1 \).
- \( \psi_e^*(z) = \ln \sum_{j=1}^{p} e^{z_j} \) and \( \| \nabla \psi_e(x) \| \to \infty \) as \( x \to \hat{x} \in \Delta \).
- Let \( x^0 = p^{-1} 1 \), then \( d_\psi(x, x^0) \leq \ln p \) for all \( x \in \Delta \).
## Entropic descent algorithm (EDA)

Let $x^0 = p^{-1}1$ and generate the following sequence

\[
x_j^{k+1} = \frac{x_j^k e^{-t_k f_j'(x^k)}}{\sum_{j=1}^p x_j^k e^{-t_k f_j'(x^k)}}, \quad t_k = \frac{\sqrt{2\ln p}}{L_f} \frac{1}{\sqrt{k}},
\]

where $f'(x) = (f_1(x)', \ldots, f_p(x)')^T \in \partial f(x)$, which is the subdifferential of $f$ at $x$.

- This is an example of **non-smooth** and **constrained** optimization;
- The updates are multiplicative.
Convergence of mirror descent

Problem

\[
\min_{x \in X} f(x) \quad (6)
\]

where

- \(X\) is a closed convex subset of \(\mathbb{R}^p\);
- \(f\) is convex \(L_f\)-Lipschitz continuous with respect to some norm \(\| \cdot \|\).

Theorem ([2])

Let \(\{x^k\}\) be the sequence generated by mirror descent with \(x^0 \in \text{int} X\).

If the step-sizes are chosen as

\[
\alpha_k = \frac{\sqrt{2 \mu d_\psi(x^*, x^0)}}{L_f} \frac{1}{\sqrt{k}}
\]

the following convergence rate holds

\[
\min_{0 \leq s \leq k} f(x^s) - f^* \leq L_f \sqrt{\frac{2 d_\psi(x^*, x^0)}{\mu}} \frac{1}{\sqrt{k}}
\]

- This convergence rate is optimal for solving (6) with a first-order method.
Supplementary material

Offline policy evaluation
A primal LP for policy evaluation.

- Recall that $Q^\pi(s,a)$ is a fixed point for the expectation Bellman operator $T^\pi$.

\[
Q^\pi(s,a) = (T^\pi Q^\pi)(s,a) = r(s,a) + \gamma \sum_{s',a'} P(s'|s,a)\pi(a'|s')Q^\pi(s',a')
\]

**Derivation:**
- It follows that $Q^\pi$ belongs to the set given by

\[
\left\{ Q \in \mathbb{R}^{|S||A|} : Q^\pi(s,a) \geq r(s,a) + \gamma \sum_{s',a'} P(s'|s,a)\pi(a'|s')Q^\pi(s',a') \right\}
\]

- Therefore, we can write the following program for $Q^\pi$:

\[
Q^\pi = \arg\min_Q \langle c, Q \rangle \\
\text{s.t. } Q(s,a) \geq r(s,a) + \gamma \sum_{s',a'} P(s'|s,a)\pi(a'|s')Q(s',a') \forall s,a \in S \times A
\]

- The variable $c$ is a vector of dimension $|S||A|$ defined as $c(s,a) = (1 - \gamma)\pi(a|s)\mu(s)$. 
The corresponding dual LP.

With standard techniques we can derive the following dual formulation over the occupancy measure.

\[
\lambda^\pi = \operatorname{argmax}_{\lambda \geq 0} \langle r, \lambda \rangle \\
\text{s.t. } \lambda(s, a) = \gamma \sum_{s', a'} P(s'|s', a') \pi(a'|s) \lambda(s', a') + c(s, a) \quad \forall s, a \in S \times A
\]

Remark:
- The only feasible point is \( \lambda^\pi \) [21].
- We can change the objective without affecting the maximizer.
- However, we change the objective value.
- Several recent works proposed to add an \( f \)-divergence to the objective. [21, 23, 22]
A modified Dual LP

**Dual LP with $f$-divergences**

$$
\lambda^{\pi} = \arg \max_{\lambda \geq 0} \langle r, \lambda \rangle - \frac{1}{\eta} D_f (\lambda, \lambda^{\tilde{\pi}}) \\
\text{s.t. } \lambda(s, a) = \gamma \sum_{s', a'} P(s'|s, a') \pi(a|s) \lambda(s', a') + c(s, a) \quad \forall s, a \in S \times A
$$

**Remarks:**
- Notice that the constraints are different from the one used in the LP formulation for REPS.
- We use more general $f$-divergences $D_f$ instead than KL divergence.
- The center point is $\lambda^{\tilde{\pi}}$ as opposed to $\lambda_{k-1}$. 
Conjugation of functions

○ Idea: Represent a convex function in max-form:

**Definition**

Let $Q$ be a Euclidean space and $Q^*$ be its dual space. Given a proper, closed and convex function $f : Q \rightarrow \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(y) = \sup_{x \in \text{dom}(f)} \{y^T x - f(x)\}$$

is called the Fenchel conjugate (or conjugate) of $f$.

**Observations:**

○ $y$: slope of the hyperplane

○ $-f^*(y)$: intercept of the hyperplane

---

Figure: The conjugate function $f^*(y)$ is the maximum gap between the linear function $x^T y$ (red line) and $f(x)$. 
Conjugation of functions

Definition

Given a proper, closed and convex function \( f : Q \to \mathbb{R} \cup \{ +\infty \} \), the function \( f^* : Q^* \to \mathbb{R} \cup \{ +\infty \} \) such that

\[
f^*(y) = \sup_{x \in \text{dom}(f)} \{ y^T x - f(x) \}
\]

is called the Fenchel conjugate (or conjugate) of \( f \).
Conjugation of functions

**Definition**
Given a proper, closed and convex function \( f : Q \rightarrow \mathbb{R} \cup \{+\infty\} \), the function \( f^* : Q^* \rightarrow \mathbb{R} \cup \{+\infty\} \) such that

\[
f^*(y) = \sup_{x \in \text{dom}(f)} \left\{ y^T x - f(x) \right\}
\]

is called the Fenchel conjugate (or conjugate) of \( f \).

**Properties**
- \( f^* \) is a convex and lower semicontinuous function by construction as the supremum of affine functions of \( y \).
- The conjugate of the conjugate of a convex function \( f \) is the same function \( f \); i.e., \( f^{**} = f \) for \( f \in \mathcal{F}(Q) \).
- The conjugate of the conjugate of a non-convex function \( f \) is its lower convex envelope when \( Q \) is compact:
  - \( f^{**}(x) = \sup\{g(x) : g \text{ is convex and } g \leq f, \forall x \in Q \} \).
- For closed convex \( f \), \( \mu \)-strong convexity w.r.t. \( \| \cdot \| \) is equivalent to \( \frac{1}{\mu} \) smoothness of \( f^* \) w.r.t. \( \| \cdot \|^* \).
  - Recall dual norm: \( \| y \|^* = \sup_x \{ \langle x, y \rangle : \| x \| \leq 1 \} \).
  - See for example Theorem 3 in [14].
Fenchel duality of $f$-divergence

- Using Fenchel conjugation, we can rewrite an $f$-divergence as follows:

$$D_f(\lambda, \lambda^{\tilde{\pi}}) = \sum_{s,a} \lambda^{\tilde{\pi}}(s, a)f\left(\frac{\lambda(s, a)}{\lambda^{\tilde{\pi}}(s, a)}\right) = \max_u \sum_{s,a} \lambda(s, a)u(s, a) - \lambda^{\tilde{\pi}}(s, a)f^*(u(s, a))$$

where we used the dual function $u : S \times A \rightarrow \mathbb{R}$.

**Remark:**

- When seeing $D_f(\lambda, \lambda^{\tilde{\pi}})$ as a function of $\lambda$, we have that its Fenchel conjugate is given by the following expression:

$$(D_f(\cdot, \lambda^{\tilde{\pi}}))^* = \langle \lambda^{\tilde{\pi}}, f^*(\cdot) \rangle$$
Some additional operators towards the Lagrangian

- For compactness we will consider the Bellman evaluation operator $\mathcal{L}_\pi : \mathbb{R}^{S \times A} \rightarrow \mathbb{R}^{S \times A}$
- The action on $Q(s, a)$ is
  \[(\mathcal{L}^\pi Q)(s, a) = Q(s, a) - \gamma \sum_{s', a'} P(s' | s, a) \pi(a' | s') Q(s', a')\]
- The adjoint operator $\mathcal{L}^*_\pi : \mathbb{R}^{S \times A} \rightarrow \mathbb{R}^{S \times A}$
- The action on $\lambda(s, a)$ is
  \[(\mathcal{L}^*_\pi \lambda)(s, a) = \lambda(s, a) - \gamma \sum_{s', a'} P(s | s', a') \pi(a | s) \lambda(s', a')\]
The Lagrangian

**Derivation:**

○ Thanks to the Bellman evaluation operator we have that

\[
\lambda^\pi = \arg\max_{\lambda \geq 0} \min_Q \langle r, \lambda \rangle - \frac{1}{\eta} D_f(\lambda, \lambda^\pi) - \langle Q, L^*_\pi \lambda \rangle + \langle Q, c \rangle
\]

○ Rearranging the terms:

\[
\lambda^\pi = \arg\max_{\lambda \geq 0} \min_Q \langle r - L_\pi Q, \lambda \rangle - \frac{1}{\eta} D_f(\lambda, \lambda^\pi) + \langle Q, c \rangle
\]

○ Exchanging max and min by strong duality:

\[
Q^\pi = \arg\min_Q \max_{\lambda \geq 0} \langle r - L_\pi Q, \lambda \rangle - \frac{1}{\eta} D_f(\lambda, \lambda^\pi) + \langle Q, c \rangle
\]

○ Recognizing the Fenchel dual:

\[
Q^\pi = \arg\min_Q \langle \lambda^\pi, f^*(\eta (r - L_\pi Q)) \rangle + \langle Q, c \rangle
\]

○ We derived the formulation used in AlgaeDICE for policy evaluation.
References I

Linear programming for large-scale Markov decision problems.
In International Conference on Machine Learning (ICML), 2014.

Mirror descent and nonlinear projected subgradient methods for convex optimization.

Neuro-Dynamic Programming.
Athena Scientific, 1996.

A convex analytic approach to Markov decision processes.

A reflected forward-backward splitting method for monotone inclusions involving lipschitzian operators.
References II

Scalable bilinear $\pi$ learning using state and action features.
31, 33

The linear programming approach to approximate dynamic programming.
19, 21, 22

On constraint sampling in the linear programming approach to approximate dynamic programming.
19, 23, 24

Approximate dynamic programming via a smoothed linear program.
19

On the convergence theory of gradient-based model-agnostic meta-learning algorithms.
63, 65
References III

*Discrete-Time Markov Control Processes: Basic Optimality Criteria.*  
Springer-Verlag New York, 1996.

*Further Topics on Discrete-Time Markov Control Processes.*  

Efficiently solving MDPs with stochastic mirror descent.  

[14] Sham M. Kakade, Shai Shalev-Shwartz, and Ambuj Tewari.  
Regularization techniques for learning with matrices.  

An extragradient method for finding saddle-points and for other problems.  
References IV

A linearly relaxed approximate linear program for Markov decision processes.

[17] Yura Malitsky and Matthew K Tam.
A forward-backward splitting method for monotone inclusions without cocoercivity.

Linear programming and sequential decisions.

Human-level control through deep reinforcement learning.
Reference V

A unified analysis of extra-gradient and optimistic gradient methods for saddle point problems: Proximal point approach.

Reinforcement learning via Fenchel-Rockafellar duality.

[22] Ofir Nachum, Yinlam Chow, Bo Dai, and Lihong Li.

Algaedice: Policy gradient from arbitrary experience.

Near optimal policy optimization via REPS.
References VI

Relative entropy policy search.
37, 42

Feature selection using regularization in approximate linear programs for markov decision processes.
19

[27] Marek Petrik and Shlomo Zilberstein.
Constraint relaxation in approximate linear programs.
19

[28] Leonid Denisovich Popov.
A modification of the arrow-hurwicz method for search of saddle points.
64

15
References VII

Optimization, learning, and games with predictable sequences.

[31] R Tyrrell Rockafellar.
Conjugate convex functions in optimal control and the calculus of variations.

Monotone operators and the proximal point algorithm.

[33] Paul J Schweitzer and Abraham Seidmann.
Generalized polynomial approximations in markovian decision processes.

[34] W. Zhan, B. Huang, A. Huang, N. Jiang, and J. D. Lee.
Offline reinforcement learning with realizability and single-policy concentrability, 2022.