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General nonsmooth problems

- We will show that the restricted template captures the familiar composite minimization:

\[
\min_{x \in \mathbb{R}^p} f(x) + g(Ax).
\]

- \(f, g\) are convex, nonsmooth functions; and \(A\) is a linear operator.

**Examples**

- \(g(Ax) = \|Ax - b\|_1\) or \(g(Ax) = \|Ax - b\|_2^2\).

- \(g(Ax) = \delta_{\{b\}}(Ax)\), where \(\delta_{\{b\}}(Ax) = \begin{cases} 0, & \text{if } Ax = b, \\ +\infty, & \text{if } Ax \neq b. \end{cases}\)

**Observations:**

- The indicator example covers constrained problems, such as \(\min_{x \in X} \{f(x) : Ax = b\}\).

- We need a tool, called Fenchel conjugation, to reveal the underlying minimax problem.
Conjugation of functions

◦ Idea: Represent a convex function in max-form:

**Definition**
Let $Q$ be a Euclidean space and $Q^*$ be its dual space. Given a proper, closed and convex function $f : Q \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(y) = \sup_{x \in \text{dom}(f)} \{ y^T x - f(x) \}$$

is called the Fenchel conjugate (or conjugate) of $f$.

**Observations:**
- $y$: slope of the hyperplane
- $-f^*(y)$: intercept of the hyperplane

*Figure:* The conjugate function $f^*(y)$ is the maximum gap between the linear function $x^T y$ (red line) and $f(x)$. 
Conjugation of functions

**Definition**

Given a proper, closed and convex function $f : Q \rightarrow \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(y) = \sup_{x \in \text{dom}(f)} \{ y^T x - f(x) \}$$

is called the Fenchel conjugate (or conjugate) of $f$. 

**Properties**

- $f^*$ is a convex and lower semicontinuous function by construction as the supremum of affine functions of $y$.
- The conjugate of the conjugate of a convex function $f$ is the same function $f$; i.e., $f^{**} = f$ for $f \in F(Q)$.
- The conjugate of the conjugate of a non-convex function $f$ is its lower convex envelope when $Q$ is compact:
  $$f^{**}(x) = \sup\{ g(x) : g \text{ is convex and } g \leq f, \forall x \in Q \}.$$
- For closed convex $f$, $\mu$-strong convexity w.r.t. $\|\cdot\|$ is equivalent to $\frac{1}{\mu}$-smoothness of $f^*$ w.r.t. $\|\cdot\|^*$.

Recall dual norm:

$$\|y\|^* = \sup_{x} \{ \langle x, y \rangle : \|x\| \leq 1 \}.$$
Conjugation of functions

**Definition**

Given a **proper, closed and convex function** \( f : Q \to \mathbb{R} \cup \{+\infty\} \), the function \( f^* : Q^* \to \mathbb{R} \cup \{+\infty\} \) such that

\[
f^*(y) = \sup_{x \in \text{dom}(f)} \left\{ y^T x - f(x) \right\}
\]

is called the **Fenchel conjugate** (or conjugate) of \( f \).

**Properties**

- \( f^* \) is a **convex** and lower semicontinuous function by construction as the supremum of affine functions of \( y \).
- The **conjugate** of the **conjugate** of a convex function \( f \) is the same function \( f \); i.e., \( f^{**} = f \) for \( f \in \mathcal{F}(Q) \).
- The **conjugate** of the **conjugate** of a non-convex function \( f \) is its lower convex envelope when \( Q \) is compact:
  - \( f^{**}(x) = \sup\{g(x) : g \text{ is convex and } g \leq f, \forall x \in Q \} \).
- For closed convex \( f \), \( \mu \)-strong convexity w.r.t. \( \| \cdot \| \) is equivalent to \( \frac{1}{\mu} \) smoothness of \( f^* \) w.r.t. \( \| \cdot \|^* \).
  - Recall dual norm: \( \| y \|_* = \sup_{x} \{ \langle x, y \rangle : \| x \| \leq 1 \} \).
  - See for example Theorem 3 in [12].
Examples

**ℓ₂-norm-squared**

\[ f(x) = \frac{\lambda}{2} \|x\|^2 \Rightarrow f^*(y) = \max_x \langle y, x \rangle - \frac{\lambda}{2} \|x\|^2. \]

- Take the derivative and equate to 0: \(0 = y - \lambda x \iff x = \frac{1}{\lambda} y \iff f^*(y) = \frac{1}{\lambda} \|y\|^2 - \frac{1}{2\lambda} \|y\|^2 = \frac{1}{2\lambda} \|y\|^2.\)

**ℓ₁-norm**

\[ f(x) = \lambda \|x\|_1 \Rightarrow f^*(y) = \max_x \langle y, x \rangle - \lambda \|x\|_1. \]

- By definition of the ℓ₁-norm: \(f^*(y) = \max_x \sum_{i=1}^{n} y_i x_i - \lambda |x_i| = \max_x \sum_{i=1}^{n} y_i \text{sign}(x_i)|x_i| - \lambda |x_i|.\)
- By inspection:
  - If all \(|y_i| \leq \lambda\), then \(\forall i, (y_i \text{sign}(x_i) - \lambda)|x_i| \leq 0\). Taking \(x = 0\) gives the maximum value: \(f^*(y) = 0\).
  - If for at least one \(i\), \(|y_i| > \lambda\), \((y_i \text{sign}(x_i) - \lambda)|x_i| \to +\infty\) as \(|x_i| \to +\infty\).
- \(f^*(y) = \delta_{y: \|\cdot\|_\infty \leq \lambda} = \begin{cases} 0, & \text{if } \|y\|_\infty \leq \lambda \\ +\infty, & \text{if } \|y\|_\infty > \lambda \end{cases}\)

**Remark:**
- See advanced material at the end for non-convex examples, such as \(f(x) = \|x\|_0\).
General nonsmooth problems

\[
\min_{x \in \mathbb{R}^p} f(x) + g(Ax)
\]

- By Fenchel-conjugation, we have \( g(Ax) = \max_y \langle Ax, y \rangle - g^*(y) \), where \( g^* \) is the conjugate of \( g \).
- Min-max formulation:

\[
\min_{x \in \mathbb{R}^p} f(x) + g(Ax) = \min_{x \in \mathbb{R}^p} \max_y \{ \Phi(x, y) := f(x) + \langle Ax, y \rangle - g^*(y) \} 
\]

An example with linear constraints

- If \( g(Ax) = \delta_{\{b\}}(Ax) \) = \[\begin{cases} 0, & \text{if } Ax = b, \\ +\infty, & \text{if } Ax \neq b, \end{cases}\]

Then \( g^*(y) = \max_x \langle y, x \rangle - \delta_{\{b\}}(x) = \max_{x : x = b} \langle y, x \rangle = \langle y, b \rangle \).

- We reach the minimax formulation (or the so-called “Lagrangian”) via conjugation:

\[
\min_{x} \{ f(x) : Ax = b \} = \min_{x} f(x) + g(Ax) = \min_{x} \max_{y} f(x) + \langle Ax - b, y \rangle.
\]
A special case in minimax optimization

Bilinear min-max template

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x) + \langle Ax, y \rangle - h(y),
\]
where \( \mathcal{X} \subseteq \mathbb{R}^p \) and \( \mathcal{Y} \subseteq \mathbb{R}^n \).

- \( f: \mathcal{X} \rightarrow \mathbb{R} \) is convex.
- \( h: \mathcal{Y} \rightarrow \mathbb{R} \) is convex.
Example: Sparse recovery

An example from sparseland $\mathbf{b} = \mathbf{A}\mathbf{x}^q + \mathbf{w}$: constrained formulation

The basis pursuit denoising (BPDN) formulation is given by

$$\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathbb{R}^p} \{ \| \mathbf{x} \|_1 : \| \mathbf{A}\mathbf{x} - \mathbf{b} \|_2 \leq \| \mathbf{w} \|_2, \|\mathbf{x}\|_\infty \leq 1 \}.$$ (BPDN)

A primal problem prototype

$$f^* := \min_{\mathbf{x} \in \mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K} \ \mathbf{x} \in \mathcal{X} \right\},$$

The above template captures BPDN formulation with
- $f(\mathbf{x}) = \|\mathbf{x}\|_1$.
- $\mathcal{K} = \{\|\mathbf{u}\| \in \mathbb{R}^n : \|\mathbf{u}\| \leq \|\mathbf{w}\|_2\}$.
- $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\|_\infty \leq 1\}$. 
An alternative formulation

A primal problem prototype

\[ f^* := \min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax - b \in \mathcal{K}, \ x \in \mathcal{X} \right\}, \tag{1} \]

- \( f \) is a proper, closed and convex function
- \( \mathcal{X} \) and \( \mathcal{K} \) are nonempty, closed convex sets
- \( A \in \mathbb{R}^{n \times p} \) and \( b \in \mathbb{R}^n \) are known
- An optimal solution \( x^* \) to (1) satisfies \( f(x^*) = f^* \), \( Ax^* - b \in \mathcal{K} \) and \( x^* \in \mathcal{X} \)

A simplified template without loss of generality

\[ f^* := \min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax = b \right\}, \tag{2} \]

- \( f \) is a proper, closed and convex function
- \( A \in \mathbb{R}^{n \times p} \) and \( b \in \mathbb{R}^n \) are known
- An optimal solution \( x^* \) to (2) satisfies \( f(x^*) = f^* \), \( Ax^* = b \)
Reformulation between templates

A primal problem template

\[
\min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax - b \in \mathcal{K}, x \in \mathcal{X} \right\}.
\]

First step: Let \( r_1 = Ax - b \in \mathbb{R}^n \) and \( r_2 = x \in \mathbb{R}^p \).

\[
\min_{x, r_1, r_2} \left\{ f(x) : r_1 \in \mathcal{K}, r_2 \in \mathcal{X}, Ax - b = r_1, x = r_2 \right\}.
\]

- Define \( z = \begin{bmatrix} x \\ r_1 \\ r_2 \end{bmatrix} \in \mathbb{R}^{2p+n} \), \( \tilde{A} = \begin{bmatrix} A & -I_{n \times n} & 0_{n \times p} \\ I_{p \times p} & 0_{p \times n} & -I_{p \times p} \end{bmatrix} \), \( \tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} \), \( \tilde{f}(z) = f(x) + \delta_{\mathcal{K}}(r_1) + \delta_{\mathcal{X}}(r_2) \),

where \( \delta_{\mathcal{X}}(x) = 0 \), if \( x \in \mathcal{X} \), and \( \delta_{\mathcal{X}}(x) = +\infty \), o/w.

The simplified template

\[
\min_{z \in \mathbb{R}^{2p+n}} \left\{ \tilde{f}(z) : \tilde{A}z = \tilde{b} \right\}.
\]
From constrained formulation back to minimax

A general template

\[ \min_{\mathbf{x} \in \mathbb{R}^p} \{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \}. \]

Other examples:

- **Standard convex optimization** formulations: *linear programming*, *convex quadratic programming*, *second order cone programming*, *semidefinite programming* and *geometric programming*.
- **Reformulations** of existing unconstrained problems via **convex splitting**: *composite convex minimization*, *consensus optimization*, ... 

Formulating as min-max

\[
\max_{\mathbf{y} \in \mathbb{R}^n} \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle = \begin{cases} 
0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\
+\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}.
\end{cases}
\]

\[
\min_{\mathbf{x} \in \mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \right\} = \min_{\mathbf{x} \in \mathbb{R}^p} \max_{\mathbf{y} \in \mathbb{R}^n} \left\{ \Phi(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle \right\}
\]
Dual problem

\[
\min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax = b \right\} = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \left\{ \Phi(x, y) := f(x) + \langle y, Ax - b \rangle \right\}
\]

- We define the dual problem

\[
\max_{y \in \mathbb{R}^n} d(y) := \max_{y \in \mathbb{R}^n} \left\{ \min_{x \in \mathbb{R}^p} f(x) + \langle y, Ax - b \rangle \right\}.
\]

Concavity of dual problem

Even if \( f(x) \) is not convex, \( d(y) \) is concave:

- For each \( x \), \( d(y) \) is linear; i.e., it is both convex and concave.
- Pointwise minimum of concave functions is still concave.

Remark:
- If we can exchange \( \min \) and \( \max \), we obtain a concave maximization problem.
Example: Nonsmoothness of the dual function

- Consider a constrained convex problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad \left\{ f(x) := x_1^2 + 2x_2 \right\}, \\
\text{s.t.} & \quad 2x_3 - x_1 - x_2 = 1, \\
& \quad x \in \mathcal{X} := [-2, 2] \times [-2, 2] \times [0, 2].
\end{align*}
\]

- The dual function is concave and nonsmooth as written and then illustrated below.

\[
d(\lambda) := \min_{x \in \mathcal{X}} \left\{ x_1^2 + 2x_2 + \lambda(2x_3 - x_1 - x_2 - 1) \right\}
\]
**Exchanging min and max: A dangerous proposal**

- **Weak duality:**

\[
\max_{y \in \mathbb{R}^n} d(y) = \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{R}^p} \Phi(x, y) \leq \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \Phi(x, y) = \min_{x \in \mathbb{R}^p} \{ f(x) : Ax = b \} = \begin{cases} f^*, \text{ if } Ax = b \\ +\infty, \text{ if } Ax \neq b \end{cases}
\]
A proof of weak duality

\[ f^* := \min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax = b \right\} = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \left\{ \Phi(x, y) := f(x) + \langle y, Ax - b \rangle \right\} \]

○ Since \( Ax^* = b \), it holds for any \( y \)

\[
\Phi(x^*, y) = f^* = f(x^*) + \langle y, A x^* - b \rangle
\geq \min_{x \in \mathbb{R}^p} \left\{ f(x) + \langle y, A x - b \rangle \right\}
= \min_{x \in \mathbb{R}^p} \Phi(x, y).
\]

○ Take maximum of both sides in \( y \) and note that \( f^* \) is independent of \( y \):

\[
f^* = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \Phi(x, y) \geq \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{R}^p} \Phi(x, y) =: \max_{y \in \mathbb{R}^n} d(y) = d^*.
\]
Strong duality and saddle points

Strong duality

\[ f^* = f(x^*) = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \Phi(x, y) = \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{R}^p} \Phi(x, y) =: \max_{y \in \mathbb{R}^n} d(y) = d^*. \]

Under strong duality and assuming existence of \( x^* \), \( \Phi(x, y) \) has a saddle point. We have primal and dual optimal values coincide, i.e., \( f^* = d^* \).
Strong duality and saddle points

Strong duality

\[ f^* = f(x^*) = \min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \Phi(x, y) = \max_{y \in \mathbb{R}^n} \min_{x \in \mathbb{R}^p} \Phi(x, y) =: \max_{y \in \mathbb{R}^n} d(y) = d^*. \]

Under strong duality and assuming existence of \( x^* \), \( \Phi(x, y) \) has a saddle point. We have primal and dual optimal values coincide, i.e., \( f^* = d^* \).

Recall saddle point / LNE

A point \( (x^*, y^*) \in \mathbb{R}^p \times \mathbb{R}^n \) is called a saddle point of \( \Phi \) if

\[ \Phi(x^*, y) \leq \Phi(x^*, y^*) \leq \Phi(x, y^*), \ \forall x \in \mathbb{R}^p, \ y \in \mathbb{R}^n. \]
Toy example: Strong duality

Primal problem

- Consider the following primal minimization problem: \( \min_x P(x) := f(x) + g(x) := \frac{1}{2} \|x\|^2 + \|x\|_1 \)
- Using conjugation and strong duality

\[
P(x^*) = \min P(x) = \min_x \max_y f(x) + \langle x, y \rangle - g^*(y), \quad \text{by conjugation}
\]

\[
= \max_y -g^*(y) + \min_x f(x) + \langle x, y \rangle, \quad \text{by changing min-max}
\]

\[
= \max_y -g^*(y) - \max_x \langle x, -y \rangle - f(x), \quad \text{by } \min f = - \max -f
\]

\[
= \max_y -g^*(y) - f^*(-y), \quad \text{by conjugation.}
\]

Dual problem

- Dual problem: \( d^* = \max_y d(y) = -g^*(y) - f^*(-y) \)
- Recall \( f^*(-y) = \frac{1}{2} \|y\|^2 \) and \( g^*(y) = \delta_y : \|y\|_\infty \leq 1(y) \).
Toy example: Strong duality

Primal problem: \( \min_{x} P(x) = \frac{1}{2} \|x\|^2 + \|x\|_1 \)

Dual problem: \( \max_{y} -\frac{1}{2} \|y\|^2 - \delta_{y: \|y\|_\infty \leq 1}(y) \)

\[ d(y) = \begin{cases} -\frac{1}{2} \|y\|^2, & \text{if } \|y\|_\infty \leq 1 \\ -\infty, & \text{if } \|y\|_\infty > 1 \end{cases} \]
Back to convex-concave: Necessary and sufficient condition for strong duality

○ Existence of a saddle point is not automatic even in convex-concave setting!

○ Recall the minimax template:

\[
\min_{x \in \mathbb{R}^p} \max_{y \in \mathbb{R}^n} \{ \Phi(x, y) := f(x) + \langle y, Ax - b \rangle \}
\]

Theorem (Necessary and sufficient optimality condition)

Under the Slater’s condition: \( \text{relint}(\text{dom } f) \cap \{ x : Ax = b \} \neq \emptyset \), strong duality holds, where the primal and dual problems are given by

\[
f^* := \begin{cases} 
\min_{x \in \mathbb{R}^p} & f(x) \\
\text{s.t.} & Ax = b,
\end{cases} \quad \text{and} \quad d^* := \max_{y \in \mathbb{R}^n} d(y).
\]

Remarks:

○ By definition of \( f^* \) and \( d^* \), we always have \( d^* \leq f^* \) (weak duality).

○ If a primal solution exists and the Slater’s condition holds, we have \( d^* = f^* \) (strong duality).
Slater’s qualification condition

- Denote $\text{relint}(\text{dom } f)$ the relative interior of the domain.

- The Slater condition requires

$$\text{relint}(\text{dom } f) \cap \{x : Ax = b\} \neq \emptyset.$$  \hspace{1cm} (3)

Special cases

- If $\text{dom } f = \mathbb{R}^p$, then (3) $\Leftrightarrow \exists \bar{x} : A\bar{x} = b$.

- If $\text{dom } f = \mathbb{R}^p$ and instead of $Ax = b$, we have the feasible set $\{x : h(x) \leq 0\}$, where $h$ is $\mathbb{R}^p \to \mathbb{R}^q$ is convex, then

$$\text{(3)} \Leftrightarrow \exists \bar{x} : h(\bar{x}) < 0.$$
Example: Slater’s condition

Example

Let us consider solving $\min_{x \in D_\alpha} f(x)$ and so the feasible set is $D_\alpha := \mathcal{X} \cap A_\alpha$, where

$$\mathcal{X} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}, \quad A_\alpha := \{x \in \mathbb{R}^2 : x_1 + x_2 = \alpha\},$$

where $\alpha \in \mathbb{R}$. 

Two cases where Slater’s condition holds and does not hold

$x_1 x_2 0 1 1 1 2 1 2 x_1 + x_2 = 1 2 x_2 1 + x_2 \leq 1 \relative \text{interior of } D_1 2 / 2 \sqrt{2} - \text{does not satisfy Slater’s condition}$

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EPFL
Example: Slater’s condition

Example

Let us consider solving \( \min_{x \in D_\alpha} f(x) \) and so the feasible set is \( D_\alpha := \mathcal{X} \cap A_\alpha \), where

\[
\mathcal{X} := \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1 \}, \quad A_\alpha := \{ x \in \mathbb{R}^2 : x_1 + x_2 = \alpha \},
\]

where \( \alpha \in \mathbb{R} \).

Two cases where Slater’s condition holds and does not hold

\( D_{1/2} \) satisfies Slater’s condition – \( D_{\sqrt{2}} \) does not satisfy Slater’s condition
Performance of optimization algorithms

\[ f^* := \min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax = b, \right\}, \]  

(Affine-Constrained)

**Exact vs. approximate solutions**

- Computing an **exact solution** \( x^* \) to (Affine-Constrained) is **impracticable**
- Algorithms seek \( x^*_\epsilon \) that **approximates** \( x^* \) up to \( \epsilon \) in some sense

**A performance metric: Time-to-reach \( \epsilon \)**

\[ \text{time-to-reach } \epsilon = \text{number of iterations to reach } \epsilon \times \text{per iteration time} \]

**A key issue: Number of iterations to reach \( \epsilon \)**

The notion of \( \epsilon \)-accuracy is elusive in constrained optimization!
Numerical $\epsilon$-accuracy

- **Unconstrained case:** All iterates are feasible (no advantage from infeasibility)!
  \[ f(x^*_\epsilon) - f^* \leq \epsilon \]
  \[ f^* = \min_{x \in \mathbb{R}^p} f(x) \]

- **Constrained case:** We need to also measure the infeasibility of the iterates!
  \[ f^* - f(x^*_\epsilon) \leq \epsilon \]
  \[ f^* = \min_{x \in \mathbb{R}^p} \left\{ f(x) : Ax = b \right\} \] (4)

**Our definition of $\epsilon$-accurate solutions [16]**

Given a numerical tolerance $\epsilon \geq 0$, a point $x^*_\epsilon \in \mathbb{R}^p$ is called an $\epsilon$-solution of (4) if

\[
\begin{cases}
  f(x^*_\epsilon) - f^* \leq \epsilon \text{ (objective residual)}, \\
  \|Ax^*_\epsilon - b\| \leq \epsilon \text{ (feasibility gap)},
\end{cases}
\]

- When $x^*$ is unique, we can also obtain $\|x^*_\epsilon - x^*\| \leq \epsilon$ (iterate residual).
**Numerical $\epsilon$-accuracy**

**Constrained problems**

Given a numerical tolerance $\epsilon \geq 0$, a point $x_\epsilon^* \in \mathbb{R}^p$ is called an $\epsilon$-solution of (4) if

$$
\begin{align*}
  f(x_\epsilon^*) - f^* &\leq \epsilon \text{ (objective residual)}, \\
  \|Ax_\epsilon^* - b\| &\leq \epsilon \text{ (feasibility gap)},
\end{align*}
$$

- When $x^*$ is unique, we can also obtain $\|x_\epsilon^* - x^*\| \leq \epsilon$ (iterate residual).

**General minimax problems**

Since duality gap is 0 at the solution, we measure the primal-dual gap

$$
\text{Gap}(\bar{x}, \bar{y}) = \max_{y \in Y} \Phi(\bar{x}, y) - \min_{x \in X} \Phi(x, \bar{y}) \leq \epsilon.
$$

**(5)**

**Remarks:**

- $\epsilon$ can be different for the objective, feasibility gap, or the iterate residual.
- It is easy to show $\text{Gap}(x, y) \geq 0$ and $\text{Gap}(\bar{x}, \bar{y}) = 0$ iff $(\bar{x}, \bar{y})$ is a saddle point.
Primal-dual gap function for nonsmooth minimization

\[
\min_{x \in X} f(x) + g(Ax) = \min_{x \in X} \max_{y \in Y} f(x) + \langle Ax, y \rangle - g^*(y) = \max_{y \in Y} \min_{x \in X} f(x) + \langle Ax, y \rangle - g^*(y)
\]

- Primal problem: \( \min_{x \in X} P(x) \) where
  \[
P(x) = \max_{y \in Y} \Phi(x, y).
  \]
- Dual problem: \( \max_{y \in Y} d(y) \) where
  \[
d(y) = \min_{x \in X} \Phi(x, y).
  \]
- The primal-dual gap, i.e., \( \text{Gap}(\bar{x}, \bar{y}) \), is literally (primal value at \( \bar{x} \)) − (dual value at \( \bar{y} \)):
  \[
  \text{Gap}(\bar{x}, \bar{y}) = P(\bar{x}) - d(\bar{y}) = \max_{y \in Y} \Phi(\bar{x}, y) - \min_{x \in X} \Phi(x, \bar{y}).
  \]
Toy example for nonnegativity of gap

- $P(x) = \frac{1}{2} \|x\|^2 + \|x\|_1$
- $d(y) = -\frac{1}{2} \|y\|^2 - \delta_{y: \|y\|_\infty \leq 1}(y)$

Recall the indicator function

$$\delta_{y: \|y\|_\infty \leq 1}(y) = \begin{cases} 0, & \text{if } \|y\|_\infty \leq 1 \\ +\infty, & \text{if } \|y\|_\infty > 1 \end{cases}$$
Primal-dual gap function in the general case

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} \Phi(x, y)
\]

- Saddle point \((x^*, y^*)\) is such that \(\forall x \in \mathbb{R}^p, \forall y \in \mathbb{R}^n:\)
  \[\Phi(x^*, y) \leq \Phi(x^*, y^*) \leq \Phi(x, y^*)\]  

- Nonnegativity of Gap:
  \[
  \text{Gap}(\bar{x}, \bar{y}) = \max_{y \in \mathcal{Y}} \Phi(\bar{x}, y) - \min_{x \in \mathcal{X}} \Phi(x, \bar{y})
  \geq \Phi(\bar{x}, y^*) - \min_{x \in \mathcal{X}} \Phi(x, \bar{y}), \quad \text{by the definition of maximization}
  \geq \Phi(x^*, y^*) - \min_{x \in \mathcal{X}} \Phi(x, \bar{y}), \quad \text{by the inequality (**)}
  \geq \Phi(x^*, \bar{y}) - \min_{x \in \mathcal{X}} \Phi(x, \bar{y}), \quad \text{by the inequality (*)}
  \geq 0, \quad \text{by the definition of minimization.}
  \]

- If \((\bar{x}, \bar{y}) = (x^*, y^*)\), then all the inequalities will be equalities and \(\text{Gap}(\bar{x}, \bar{y}) = 0\).
**Optimality conditions for minimax**

**Saddle point**

We say \((x^*, y^*)\) is a primal-dual solution corresponding to primal and dual problems

\[
\begin{align*}
\min_{x \in \mathbb{R}^p} & \quad f(x) \\
\text{s.t.} & \quad Ax = b,
\end{align*}
\]

and

\[
\begin{align*}
\max_{y \in \mathbb{R}^n} & \quad d(y) = \max_{y \in \mathbb{R}^n} \min_x \Phi(x, y).
\end{align*}
\]

if it is a saddle point of \(\Phi(x, y) = f(x) + \langle y, Ax - b \rangle\):

\[
\Phi(x^*, y) \leq \Phi(x^*, y^*) \leq \Phi(x, y^*), \quad \forall x \in \mathbb{R}^p, \ y \in \mathbb{R}^n.
\]

**Karush-Khun-Tucker (KKT) conditions**

Under our assumptions, an equivalent characterization of \((x^*, y^*)\) is via the KKT conditions of the problem

\[
\min_{x \in \mathbb{R}^p} f(x) : Ax = b,
\]

which reads

\[
\begin{align*}
0 & \in \partial_x \Phi(x^*, y^*) = A^T y^* + \partial f(x^*), \\
0 & = \nabla_y \Phi(x^*, \lambda^*) = Ax^* - b.
\end{align*}
\]
A naive proposal: Gradient descent-ascent (GDA)

Towards algorithms for minimax optimization

\[ \min_{x \in X} \max_{y \in Y} \Phi(x, y). \]

We assume that

- \( \Phi(\cdot, y) \) is convex,
- \( \Phi(x, \cdot) \) is concave,
- \( \Phi \) is smooth in the following sense:

\[
\| \begin{bmatrix} \nabla_x \Phi(x_1, y_1) \\ -\nabla_y \Phi(x_1, y_1) \end{bmatrix} - \begin{bmatrix} \nabla_x \Phi(x_2, y_2) \\ -\nabla_y \Phi(x_2, y_2) \end{bmatrix} \| \leq L \| \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix} \|. \quad (6)
\]

- Let us try to use gradient descent for \( x \), gradient ascent for \( y \) to obtain a solution

<table>
<thead>
<tr>
<th>GDA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Choose ( x^0, y^0 ) and ( \tau ).</td>
</tr>
<tr>
<td>2. For ( k = 0, 1, \cdots ), perform:</td>
</tr>
<tr>
<td>( x^{k+1} := x^k - \tau \nabla_x \Phi(x^k, y^k) ).</td>
</tr>
<tr>
<td>( y^{k+1} := y^k + \tau \nabla_y \Phi(x^k, y^k) ).</td>
</tr>
</tbody>
</table>
GDA on a simple problem

Min-max problem

\[
\min_{x \in X} \max_{y \in Y} \Phi(x, y).
\]

SimGDA
1. Choose \(x^0, y^0\) and \(\tau\).
2. For \(k = 0, 1, \ldots\), perform:
   \[
   x^{k+1} := x^k - \tau \nabla_x \Phi(x^k, y^k).
   \]
   \[
   y^{k+1} := y^k + \tau \nabla_y \Phi(x^k, y^k).
   \]

AltGDA
1. Choose \(x^0, y^0\) and \(\tau\).
2. For \(k = 0, 1, \ldots\), perform:
   \[
   x^{k+1} := x^k - \tau \nabla_x \Phi(x^k, y^k).
   \]
   \[
   y^{k+1} := y^k + \tau \nabla_y \Phi(x^{k+1}, y^k).
   \]

Example [7]
Let \(\Phi(x, y) = xy\), \(X = Y = \mathbb{R}\), then,

- for the iterates of SimGDA: \(x_{k+1}^2 + y_{k+1}^2 = (1 + \eta^2)(x_k^2 + y_k^2)\),
- for the iterates of AltGDA: \(x_{k+1}^2 + y_{k+1}^2 = C(x_0^2 + y_0^2)\).

\(\circ\) SimGDA diverges and AltGDA does not converge!
Practical performance

\[ \min_{x \in \mathbb{R}} \max_{y \in \mathbb{R}} xy \]

- Simultaneous GDA
- Alternating GDA
Between convex-concave and nonconvex-nonconcave

### Nonconvex-concave problems

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \Phi(x, y)
\]

- $\Phi(x, y)$ is nonconvex in $x$, concave in $y$, smooth in $x$ and $y$.

### Recall

Define $f(x) = \max_{y \in \mathcal{Y}} \Phi(x, y)$.

- Gradient descent applied to nonconvex $f$ requires $\mathcal{O}(\epsilon^{-2})$ iterations to give an $\epsilon$-stationary point.

- (Sub)gradient of $f$ can be computed using Danskin’s theorem:

  \[
  \nabla_x \Phi(\cdot, y^*(\cdot)) \in \partial f(\cdot), \text{ where } y^*(\cdot) \in \arg \max_{y \in \mathcal{Y}} \Phi(\cdot, y),
  \]

  which is tractable since $\Phi$ is concave in $y$ [13].

### Remark:

- “Conceptually” much easier than nonconvex-nonconcave case.
# Epilogue

<table>
<thead>
<tr>
<th>Gradient complexity</th>
<th>Optimality measure</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>convex-concave</td>
<td>$\mathcal{O}\left(\epsilon^{-1}\right)^1$</td>
<td>$\epsilon$ optimality w.r.t. duality gap</td>
</tr>
<tr>
<td>nonconvex-concave</td>
<td>$\tilde{\mathcal{O}}\left(\epsilon^{-2.5}\right)^3$</td>
<td>$\epsilon$-stationarity w.r.t. gradient mapping norm</td>
</tr>
<tr>
<td>nonconvex-nonconcave</td>
<td>HARD</td>
<td>HARD</td>
</tr>
</tbody>
</table>

1 Rates are not directly comparable as duality gap and gradient mapping norm are not necessarily of the same order!


3 The rate is $\tilde{\mathcal{O}}\left(\epsilon^{-2}\right)$ for strongly concave problems.


A new hope

\[ \min_{x \in \mathbb{R}} \max_{y \in \mathbb{R}} xy \]

- Next lecture: Some algorithms that actually converge!

- Convergence of the sequence:

  There exists \( z^* = (x^*, y^*) \), such that \( z_k \to z^* \).

- Convergence rate:

  \[
  \text{Gap} \left( \frac{1}{K} \sum_{k=1}^{K} x^k, \frac{1}{K} \sum_{k=1}^{K} y^k \right) \leq O \left( \frac{1}{K} \right).
  \]
Wrap up!

- Try to finish Homework #2...
A **convex** proto-problem for **structured** sparsity

A combinatorial approach for estimating $\mathbf{x}^\ddagger$ from $\mathbf{b} = A\mathbf{x}^\ddagger + \mathbf{w}$

We may consider the sparsest estimator or its surrogate with a valid sparsity pattern:

$$\hat{x} \in \arg \min_{x \in \mathbb{R}^p} \{ \| x \|_s : \| \mathbf{b} - A\mathbf{x} \|_2 \leq \kappa, \| x \|_\infty \leq 1 \}$$  \hspace{1cm} (P_s)

with some $\kappa \geq 0$. If $\kappa = \| \mathbf{w} \|_2$, then the structured sparse $\mathbf{x}^\ddagger$ is a feasible solution.

**Sparsity and structure together** [5]

Given some weights $\mathbf{d} \in \mathbb{R}^d$, $\mathbf{e} \in \mathbb{R}^p$ and an integer input $c \in \mathbb{Z}^l$, we define

$$\| \mathbf{x} \|_s := \min_{\omega} \{ d^T \omega + e^T s : M \begin{bmatrix} \omega \\ s \end{bmatrix} \leq c, \mathbf{1}_{\text{supp}(\mathbf{x})} = s, \omega \in \{0, 1\}^d \}$$

for all feasible $\mathbf{x}$, $\infty$ otherwise. The parameter $\omega$ is useful for **latent** modeling.
A **convex** proto-problem for **structured** sparsity

A combinatorial approach for estimating $x^\sharp$ from $b = Ax^\sharp + w$

We may consider the sparsest estimator or its surrogate with a valid sparsity pattern:

$$\hat{x} \in \arg\min_{x \in \mathbb{R}^p} \{ \| x \|_s : \| b - Ax \|_2 \leq \kappa, \| x \|_\infty \leq 1 \}$$

($P_s$)

with some $\kappa \geq 0$. If $\kappa = \| w \|_2$, then the structured sparse $x^\sharp$ is a feasible solution.

### Sparsity and structure together [5]

Given some weights $d \in \mathbb{R}^d$, $e \in \mathbb{R}^p$ and an integer input $c \in \mathbb{Z}^l$, we define

$$\| x \|_s := \min_\omega \{ d^T \omega + e^T s : M \begin{bmatrix} \omega \\ s \end{bmatrix} \leq c, 1_{\text{supp}(x)} = s, \omega \in \{0, 1\}^d \}$$

for all feasible $x$, $\infty$ otherwise. The parameter $\omega$ is useful for latent modeling.

A convex candidate solution for $b = Ax^\sharp + w$

We use the convex estimator based on the tightest convex relaxation of $\| x \|_s$:

$$\hat{x} \in \arg\min_{x \in \text{dom}(\| \cdot \|_s)} \{ \| x \|_{s^*} : \| b - Ax \|_2 \leq \kappa \}$$

with some $\kappa \geq 0$, $\text{dom}(\| \cdot \|_s) := \{ x : \| x \|_s < \infty \}$. 
Tractability & tightness of biconjugation

Proposition (Hardness of conjugation)

Let \( F(s) : 2^\mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) be a set function defined on the support \( s = \text{supp}(x) \). Conjugate of \( F \) over the unit infinity ball \( \|x\|_\infty \leq 1 \) is given by

\[
g^*(y) = \sup_{s \in \{0,1\}^p} |y|^T s - F(s).
\]

Observations:

- \( F(s) \) is general set function
  - Computation: NP-Hard
- \( F(s) = \|x\|_s \)
  - Computation: Integer Linear Program (ILP) in general. However, if
    - \( M \) is Totally Unimodular TU
    - \((M, c)\) is Total Dual Integral TDI
  then tight convex relaxations with a linear program (LP, which is “usually” tractable)

  Otherwise, relax to LP anyway!

- \( F(s) \) is submodular
  - Computation: Polynomial-time
Tree sparsity [11, 4, 3, 17]

Structure: We seek the sparsest signal with a rooted connected subtree support.

Linear description: A valid support satisfy \( s_{\text{parent}} \geq s_{\text{child}} \) over tree \( T \)

\[
T 1_{\text{supp}(x)} := Ts \geq 0
\]

where \( T \) is the directed edge-node incidence matrix, which is \( TU \).
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where $T$ is the directed edge-node incidence matrix, which is $TU$.

Biconjugate: $\|x\|_{s}^{**} = \min_{s \in [0,1]^p} \{1^T s : Ts \geq 0, |x| \leq s\}$

for $x \in [-1,1]^p, \infty$ otherwise.
Tree sparsity $[11, 4, 3, 17]$

$\mathcal{G}_H = \{\{1, 2, 3\}, \{2\}, \{3\}\}$

valid selection of nodes

**Structure:** We seek the sparsest signal with a rooted connected subtree support.

**Linear description:** A valid support satisfy $s_{\text{parent}} \geq s_{\text{child}}$ over tree $T$

$$T_{1_{\text{supp}(x)}} := Ts \geq 0$$

where $T$ is the directed edge-node incidence matrix, which is $TU$.

**Biconjugate:**

$$\|x\|_*^* = \min_{s \in [0, 1]^p} \left\{ 1^T s : Ts \geq 0, |x| \leq s \right\} = \sum_{G \in \mathcal{G}_H} \|x_G\|_{\infty}$$

for $x \in [-1, 1]^p$, $\infty$ otherwise.

The set $G \in \mathcal{G}_H$ are defined as each node and all its descendants.
Group knapsack sparsity [19, 8, 6]

**Structure:** We seek the sparsest signal with group allocation constraints.

**Linear description:** A valid support obeys budget constraints over $G$

$$\mathcal{B}^T s \leq c_u$$

where $\mathcal{B}$ is the biadjacency matrix of $G$, i.e., $\mathcal{B}_{ij} = 1$ iff $i$-th coefficient is in $G_j$.

When $\mathcal{B}$ is an interval matrix or $G$ has a loopless group intersection graph, it is $TU$.

**Remark:** We can also budget a lowerbound $c_\ell \leq \mathcal{B}^T s \leq c_u$. 
Group knapsack sparsity [19, 8, 6]

\[ B = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots \\ 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \]

\((p - \Delta + 1) \times p\)

Structure: We seek the sparsest signal with group allocation constraints.

Linear description: A valid support obeys budget constraints over \( G \)

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where \( B \) is the biadjacency matrix of \( G \), i.e., \( B_{ij} = 1 \) iff \( i \)-th coefficient is in \( G_j \).

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Remark: We can also budget a lowerbound \( c_\ell \leq B^T s \leq c_u \).

Biconjugate: \( \|x\|_s^{**} = \begin{cases} \|x\|_1 & \text{if } x \in [-1, 1]^p, B^T |x| \leq c_u, \\ \infty & \text{otherwise} \end{cases} \)

For the neuronal spike example, we have \( c_u = 1 \).
Group knapsack sparsity [19, 8, 6]

\[
\|x\|_s^{**} \leq 1 \quad \text{middle} \quad \|x\|_s^{**} \leq 1.5 \quad \text{right} \quad \|x\|_s^{**} \leq 2 \quad \text{for } S = \{\{1, 2\}, \{2, 3\}\}
\]

**Structure:** We seek the sparsest signal with group allocation constraints.

**Linear description:** A valid support obeys budget constraints over $S$

\[
\mathcal{B}^T s \leq c_u
\]

where $\mathcal{B}$ is the biadjacency matrix of $S$, i.e., $\mathcal{B}_{ij} = 1$ iff $i$-th coefficient is in $G_j$.

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\[
\|x\|_s^{**} = \begin{cases} 
\|x\|_1 & \text{if } x \in [-1, 1]^p, \mathcal{B}^T |x| \leq c_u, \\
\infty & \text{otherwise}
\end{cases}
\]

For the neuronal spike example, we have $c_u = 1$. 
Group knapsack sparsity example: A stylized spike train

- Basis pursuit (BP): $\|x\|_1$
- TU-relax (TU):

$$
\|x\|_s^{**} = \begin{cases} 
\|x\|_1 & \text{if } x \in [-1,1]^p, B^T |x| \leq c_u, \\
\infty & \text{otherwise}
\end{cases}
$$

Figure: Recovery for $n = 0.18p$. 

Relative errors:

- $\|x^B - x^{BP}\|_2 \|x^B\|_2 = 0.200$
- $\|x^B - x^{TU}\|_2 \|x^B\|_2 = 0.067$
Group knapsack sparsity: A simple variation

Structure: We seek the signal with the minimal overall group allocation.

Objective: \( \mathbf{1}^{T} \mathbf{s} \rightarrow \| \mathbf{x} \|_{\omega} = \begin{cases} \min_{\omega \in \mathbb{Z}^+} \omega & \text{if } \mathbf{x} \in [-1, 1]^p, \mathbf{B}^{T} \mathbf{s} \leq \omega \mathbf{1}, \\ \infty & \text{otherwise} \end{cases} \)

Linear description: A valid support obeys budget constraints over \( \mathcal{G} \)

\[ \mathbf{B}^{T} \mathbf{s} \leq \omega \mathbf{1} \]

where \( \mathbf{B} \) is the biadjacency matrix of \( \mathcal{G} \), i.e., \( \mathbf{B}_{ij} = 1 \) iff \( i \)-th coefficient is in \( \mathcal{G}_j \).

When \( \mathbf{B} \) is an interval matrix or \( \mathcal{G} \) has a loopless group intersection graph, it is TU.

Biconjugate: \( \| \mathbf{x} \|_{\omega}^{**} = \begin{cases} \max_{\mathcal{G} \in \mathcal{G}} \| \mathbf{x}^{\mathcal{G}} \|_1 & \text{if } \mathbf{x} \in [-1, 1]^p, \\ \infty & \text{otherwise} \end{cases} \)

Remark: The regularizer is known as exclusive Lasso [19, 15].
Group cover sparsity: **Minimal group cover** [2, 14, 9]

Structure: We seek the signal covered by a minimal number of groups.

Objective: $1^T s \rightarrow d^T \omega$

Linear description: At least one group containing a sparse coefficient is selected

$$\mathcal{B} \omega \geq s$$

where $\mathcal{B}$ is the biadjacency matrix of $\mathcal{G}$, i.e., $\mathcal{B}_{ij} = 1$ iff $i$-th coefficient is in $\mathcal{G}_j$.

When $\mathcal{B}$ is an interval matrix, or $\mathcal{G}$ has a loopless group intersection graph it is **TU**.
Group cover sparsity: **Minimal group cover [2, 14, 9]**

Figure: $\mathcal{G} = \{\{1, 2\}, \{2, 3\}\}$, unit group weights $d = 1$.

**Structure:** *We seek the signal covered by a minimal number of groups.*

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**Biconjugate:** $\|x\|_{\omega}^{**} = \min_{\omega \in [0,1]^M} \{d^T \omega : \mathcal{B} \omega \geq |x|\}$ for $x \in [-1, 1]^p$, $\infty$ otherwise
Group cover sparsity: **Minimal group cover** [2, 14, 9]

Figure: \( \mathcal{G} = \{\{1, 2\}, \{2, 3\}\} \), unit group weights \( d = 1 \).

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\[
\overset{\star}{=} \min_{v_i \in \mathbb{R}^p} \left\{ \sum_{i=1}^M d_i \|v_i\|_\infty : x = \sum_{i=1}^M v_i, \forall \text{supp}(v_i) \subseteq G_i \right\},
\]
Group cover sparsity: **Minimal group cover** [2, 14, 9]

**Figure:** $\mathcal{G} = \{\{1, 2\}, \{2, 3\}\}$, unit group weights $d = 1$.

**Structure:** *We seek the signal covered by a minimal number of groups.*

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When $\mathcal{B}$ is an interval matrix, or $\mathcal{G}$ has a *loopless* group intersection graph it is **TU**.

**Biconjugate:** $\|x\|^{**} = \min_{\omega \in [0,1]^M} \{d^T \omega : \mathcal{B} \omega \geq |x|\}$ for $x \in [-1, 1]^p$, $\infty$ otherwise

$$\overset{*}{=} \min_{v_i \in \mathbb{R}^p} \{\sum_{i=1}^M d_i \|v_i\|_{\infty} : x = \sum_{i=1}^M v_i, \forall \text{supp}(v_i) \subseteq \mathcal{G}_i\},$$

**Remark:** Weights $d$ can depend on the sparsity within each groups (not TU) [5].
**Budgeted group cover sparsity**

**Structure:** We seek the sparsest signal covered by $G$ groups.

**Objective:** $d^T \omega \to 1^T s$

**Linear description:** At least one of the $G$ selected groups cover each sparse coefficient.

$$B \omega \geq s, 1^T \omega \leq G$$

where $B$ is the biadjacency matrix of $G$, i.e., $B_{ij} = 1$ iff $i$-th coefficient is in $G_j$.

When $\begin{bmatrix} B \\ 1 \end{bmatrix}$ is an interval matrix, it is TU.
**Budgeted group cover sparsity**

**Structure:** We seek the sparsest signal covered by $G$ groups.

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When $\begin{bmatrix} \mathcal{B} \\ 1 \end{bmatrix}$ is an interval matrix, it is TU.

**Biconjugate:**

$$\|x\|^{**} = \min_{\omega \in [0,1]^M} \{ \|x\|_1 : \mathcal{B} \omega \geq \|x\|, 1^T \omega \leq G \}$$

for $x \in [-1,1]^P$, $\infty$ otherwise.
Budgeted group cover example: Interval overlapping groups

- Basis pursuit (BP): $\|x\|_1$
- Sparse group Lasso (SGL$q$):
  $$\left(1 - \alpha\right) \sum_{G \in \mathcal{G}} \sqrt{|G|} \|x^G\|_q + \alpha \|x^G\|_1$$
- TU-relax (TU):
  $$\|x\|^{**}_{\omega} = \min_{\omega \in [0,1]^M} \{\|x\|_1 : \exists \omega \geq |x|, 1^T \omega \leq G\}$$
  for $x \in [-1,1]^p$, $\infty$ otherwise.

**Figure:** Recovery for $n = 0.25p$, $s = 15$, $p = 200$, $G = 5$ out of $M = 29$ groups.
Group intersection sparsity \([10, 18, 1]\)

\[ G_2 = \{1, 2, 3, 4, 5\} \]

\[ G_3 = \{5, 6, 7, 8\} \]

\[ G_4 = \{2, 5, 7\} \]

\[ G_5 = \{6, 8\} \]

\[ \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 \mathbf{x}_4 \mathbf{x}_5 \mathbf{x}_6 \mathbf{x}_7 \mathbf{x}_8 \]

1 supp(\( \mathbf{x} \))
support
indicator vector

\[ \mathbf{1}^{T} s \rightarrow d^{T} \mathbf{\omega} \]

**Objective:**

Linear description: All groups containing a sparse coefficient are selected

\[ H_k s \leq \mathbf{\omega}, \forall k \in \Psi \]

where \( H_k(i, j) = \begin{cases} 1 & \text{if } j = k, j \in G_i \\ 0 & \text{otherwise} \end{cases} \), which is TU.

**Structure:** We seek the signal intersecting with minimal number of groups.
Group intersection sparsity [10, 18, 1]

\( \mathcal{G} = \{\{1, 2\}, \{2, 3\}\} \), unit group weights \( d = 1 \)

(left) intersection (right) cover.

Structure: We seek the signal intersecting with minimal number of groups.

Objective: \( 1^T s \rightarrow d^T \omega \)

Linear description: All groups containing a sparse coefficient are selected

\[
H_k s \leq \omega, \forall k \in \Psi
\]

where \( H_k(i, j) = \begin{cases} 1 & \text{if } j = k, j \in G_i \\ 0 & \text{otherwise} \end{cases} \), which is TU.

Biconjugate: \( \| x \|^{**} = \min_{\omega \in [0, 1]^M} \left\{ d^T \omega : H_k |x| \leq \omega, \forall k \in \Psi \right\} \)

for \( x \in [-1, 1]^p \), \( \infty \) otherwise.
Group intersection sparsity \([10, 18, 1]\)

\[ G = \{\{1, 2\}, \{2, 3\}\} \], unit group weights \(d = 1\)

(left) intersection (right) cover.

**Structure:** *We seek the signal intersecting with minimal number of groups.*

**Objective:** \(1^T s \rightarrow d^T \omega\) (*submodular*)

**Linear description:** All groups containing a sparse coefficient are selected

\[
H_k s \leq \omega, \forall k \in \Psi
\]

where \(H_k(i, j) = \begin{cases} 1 & \text{if } j = k, j \in G_i \\ 0 & \text{otherwise} \end{cases}\), which is TU.

**Biconjugate:** \(\|x\|_{\omega^*} = \min_{\omega \in [0, 1]^\Psi} \left\{ d^T \omega : H_k |x| \leq \omega, \forall k \in \Psi \right\} = \sum_{G \in \delta} \|x_G\|_\infty\)

for \(x \in [-1, 1]^p\), \(\infty\) otherwise.
Group intersection sparsity \([10, 18, 1]\)

\[
\mathcal{G} = \{\{1, 2, 3\}, \{2\}, \{3\}\}, \text{ unit group weights } d = 1.
\]

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**Biconjugate:** \(\|x\|^{**} = \min_{\omega \in [0,1]^M} \{d^T \omega : H_k |x| \leq \omega, \forall k \in \Psi\} = \sum_{G \in \mathcal{G}} \|x_G\|_{\infty}\)

for \(x \in [-1, 1]^p, \infty\) otherwise.

**Remark:** For hierarchical \(\mathcal{G}_H\), group intersection and tree sparsity models coincide.
Beyond linear costs: Graph dispersiveness

Figure: (left) $\|x\|_{s}^{\ast} = 0$ (right) $\|x\|_{s}^{\ast} \leq 1$ for $\mathcal{E} = \{\{1, 2\}, \{2, 3\}\}$ (chain graph)

Structure: We seek a signal dispersive over a given graph $\mathcal{G}(\Psi, \mathcal{E})$

Objective: $1^T s \rightarrow \sum_{(i,j) \in \mathcal{E}} s_i s_j$ (non-linear, supermodular function)

Linearization:

$\|x\|_s = \min_{z \in \{0, 1\}^{|\mathcal{E}|}} \left\{ \sum_{(i,j) \in \mathcal{E}} z_{ij} : z_{ij} \geq s_i + s_j - 1 \right\}$

When edge-node incidence matrix of $\mathcal{G}(\Psi, \mathcal{E})$ is TU (e.g., bipartite graphs), it is TU.
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Biconjugate: $\|x\|^{**}_s = \sum_{(i,j) \in E} (|x_i| + |x_j| - 1)_+ \text{ for } x \in [-1, 1]^p, \infty \text{ otherwise.}$
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