Optimization Challenges in Adversarial Machine Learning

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Today: Adversarial machine learning

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$$

- A seemingly simple optimization formulation
- Critical in machine learning with many applications
  - Adversarial examples and training
  - Generative adversarial networks
  - Robust reinforcement learning

Warm up

$$\Phi^\star = \min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}) \quad (\text{argmin}, \text{argmax} \rightarrow \mathbf{x}^\star, \mathbf{y}^\star)$$

Warm up

$$\Phi^\star = \min_{\mathbf{x}\in\mathcal{X}} \underbrace{\max_{\mathbf{y}:\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})}_{f(\mathbf{x})} \quad (\text{argmin}, \text{argmax} \to \mathbf{x}^\star, \mathbf{y}^\star)$$

$$f^\star = \min_{\mathbf{x}:\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) \quad (\text{argmin} \to \mathbf{x}^\star)$$

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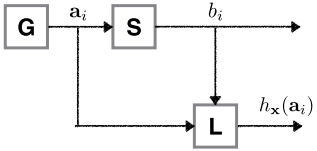
- In the sequel,
  - the set $\mathcal{X}$ is convex and has a tractable projection operator $\pi_{\mathcal{X}}$
  - all convergence characterizations are with feasible iterates $\mathbf{x}^k \in \mathcal{X}$
  - gradient mapping means $G_\alpha(\mathbf{x}^k) = \frac{1}{\alpha}[\mathbf{x}^k - \pi_{\mathcal{X}}(\mathbf{x}^k - \alpha\nabla f(\mathbf{x}^k))]$, where α is the step-size
  - L-smooth means $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x}-\mathbf{y}\|, \forall \mathbf{x},\mathbf{y} \in \mathcal{X}$
  - ∂ may refer to the generalized subdifferential, and $\delta_{\mathcal{X}}$ refers to the indicator function for the set $\mathcal{X}$

A deep learning optimization problem in supervised learning



Definition (Optimization formulation)

The deep-learning training problem is given by

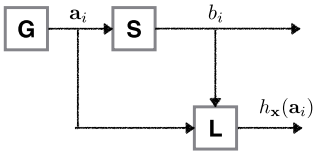
$$\mathbf{x}^\star_{\mathsf{DL}} \in \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ f(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} L(h_{\mathbf{x}}(\mathbf{a}_i), b_i) \right\},$$

where $\mathcal{X}$ denotes the constraints on the parameters.

Some frequently used loss functions

- $L(h_{\mathbf{x}}(\mathbf{a}), b) = \log(1 + \exp(-b h_{\mathbf{x}}(\mathbf{a})))$ *logistic loss*
- $L(h_{\mathbf{x}}(\mathbf{a}), b) = (b - h_{\mathbf{x}}(\mathbf{a}))^2$ *squared error*
- $L(h_{\mathbf{x}}(\mathbf{a}), b) = \max(0, 1 - b h_{\mathbf{x}}(\mathbf{a}))$ *hinge loss*

A deep learning optimization problem in supervised learning



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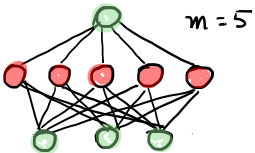
- A single hidden layer neural network with params $\mathbf{x} := [\mathbf{X}_1, \mathbf{X}_2, \mu_1, \mu_2]$



$$h_{\mathbf{x}}(\mathbf{a}) := \begin{bmatrix} \mathbf{X}_2 \end{bmatrix} \sigma \left(\begin{bmatrix} \mathbf{X}_1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} + \begin{bmatrix} \mu_1 \end{bmatrix} \right) + \begin{bmatrix} \mu_2 \end{bmatrix}$$

activation (over σ); weight (over $\mathbf{X}_1$); input (over $\mathbf{a}$); bias (over μ_1); bias (over μ_2)

hidden layer = learned features

An approximation theoretic motivation: Why neural networks?

Theorem (Universal approximation [7])

Let $\sigma(\cdot)$ be a nonconstant, bounded, and increasing continuous function. Let $I_d = [0,1]^d$. The space of continuous functions on I_d is denoted by $\mathcal{C}(I_d)$.

Given $\epsilon > 0$ and $g \in \mathcal{C}(I_d)$ there exists a 1-hidden-layer network h with m neurons such that h is an ϵ-approximation of g, i.e.,

$$\sup_{\mathbf{a} \in I_d} |g(\mathbf{a}) - h(\mathbf{a})| \leq \epsilon$$

Caveat

The number of neurons m needed to approximate some function g can be arbitrarily large!

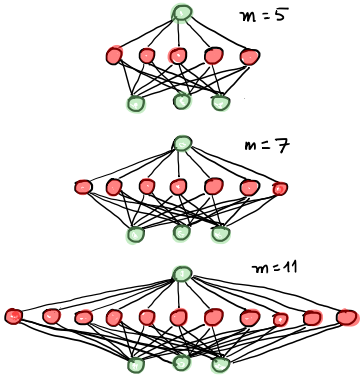
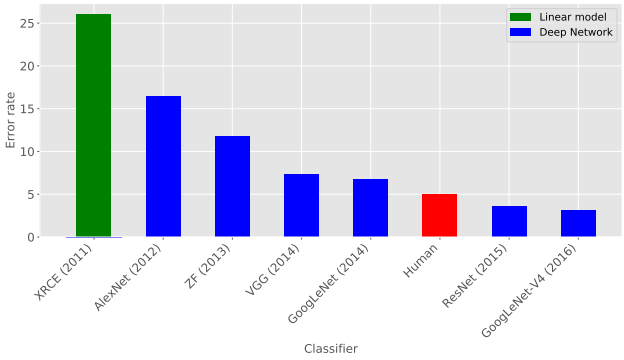
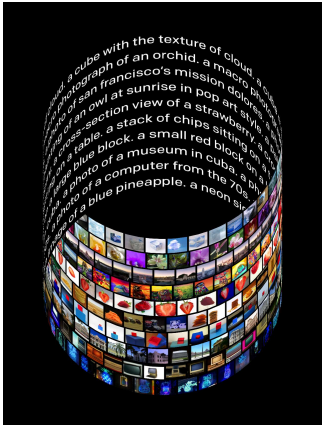


Figure: Neural networks of increasing width

A more realistic motivation: Why neural networks?

- Beyond human performance in applications





Size of neural networks grows exponentially!

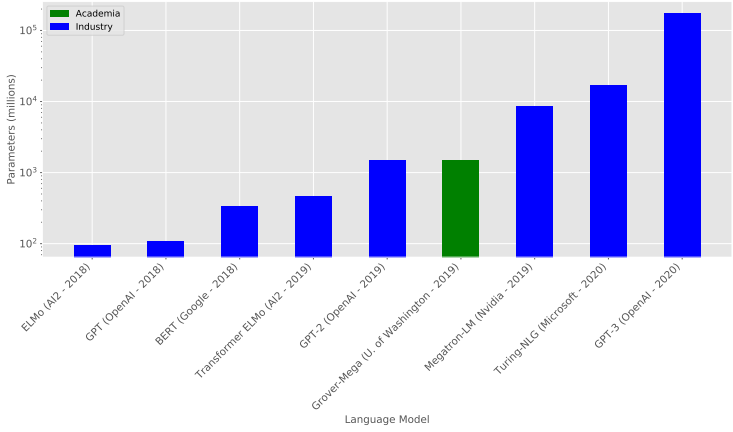


Figure: Number of parameters in language models based on deep learning.

Towards training with neural networks

- What do we have at hand?
  1. The optimization objective $f(\mathbf{x})$ from multi-layer, multi-class, convolutions, transformers, etc.
  2. First-order gradient via backpropagation $\nabla f(\mathbf{x})$

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- Barriers to training of neural networks:
  1. Curse-of-dimensionality
  2. Non-convexity
  3. Ill-conditioning

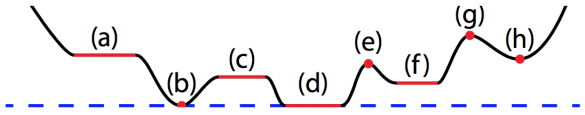

Figure: A non-convex function. (a) and (c) are plateaus, (b) and (d) are global minima, (f) and (h) are local minima, (e) and (g) are local maxima. [19]

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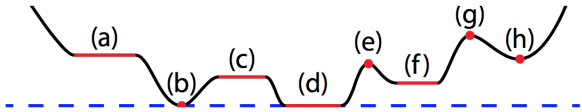
- Barriers to training of neural networks:
  1. Curse-of-dimensionality → first-order methods
  2. Non-convexity → stochasticity + momentum
  3. Ill-conditioning → adaptive gradient methods



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Basic first-order methods: GD and SGD

- Consider the finite sum (e.g., ERM) setting

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) := \frac{1}{n}\sum_{j=1}^{n} f_j(\mathbf{x}) \right\}.$$

Algorithms in the finite sum setting

Gradient Descent	**Stochastic Gradient Descent**
$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k)$	$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k G(\mathbf{x}^k, \theta_k)$
$\nabla f(\mathbf{x}^k) = \frac{1}{n}\sum_{j=1}^{n} \nabla f_j(\mathbf{x}^k)$	$G(\mathbf{x}^k, \theta_k) = \nabla f_j(\mathbf{x}^k)$, $j \sim \text{Uniform}(\{1, \cdots, n\})$
α_k can be constant	$\mathbb{E}[G(\mathbf{x}^k, \theta_k)] = \nabla f(\mathbf{x}^k)$

- We will mostly focus on SGD in the sequel due to its scalability and generalization performance.

Stochastic Gradient Descent (SGD) and some key variants

Vanilla (Minibatch) SGD
Input: Stochastic gradient oracle $\mathbf{g}$, initial point $\mathbf{x}^0$, step size α_k
1. For $k = 0, 1, \ldots$:   obtain the (minibatch) stochastic gradient $\mathbf{g}^k$ update $\mathbf{x}^{k+1} \leftarrow \mathbf{x}^k - \alpha_k \mathbf{g}^k$

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Perturbed Stochastic Gradient Descent [12]
Input: Stochastic gradient oracle $\mathbf{g}$, initial point $\mathbf{x}^0$, step size α_k
1. For $k = 0, 1, \ldots$:   sample noise ξ uniformly from unit sphere   update $\mathbf{x}^{k+1} \leftarrow \mathbf{x}^k - \alpha_k(\mathbf{g}^k + \xi)$

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Stochastic Gradient Langevin Dynamics [53]

Input: Stochastic gradient oracle g, initial point $\mathbf{x}^0$, step size α_k

1. For $k = 0, 1, \ldots$

sample noise ξ standard Gaussian

update $\mathbf{x}^{k+1} \leftarrow \mathbf{x}^k - \alpha_k \mathbf{g}^k + \sqrt{2\alpha_k}\xi$

Basic questions:

1. Does SGD converge with probability 1?
2. Does SGD avoid non-minimum points with probability 1?
3. How fast does SGD converge to local minimizers?

Critical points

Recall (Classification of critical points)

Let $f : \mathbb{R}^p \to \mathbb{R}$ be twice differentiable and let $\bar{\mathbf{x}}$ be a critical point, i.e., $\nabla f(\bar{\mathbf{x}}) = 0$. Let $\{\lambda_i\}_{i=1}^d$ be the eigenvalues of the hessian $\nabla^2 f(\bar{\mathbf{x}})$, then

- $\lambda_i > 0$ for all $i \Rightarrow \bar{\mathbf{x}}$ is a local minimum
- $\lambda_i < 0$ for all $i \Rightarrow \bar{\mathbf{x}}$ is a local maximum
- $\lambda_i > 0$, $\lambda_j < 0$ for some i, j and $\lambda_i \neq 0$ for all $i \Rightarrow \bar{\mathbf{x}}$ is a saddle point
- Other cases $\Rightarrow$ inconclusive

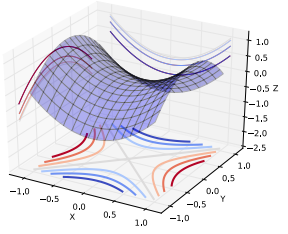


Figure: Minmax saddle ($\lambda_i \neq 0$ for all i)

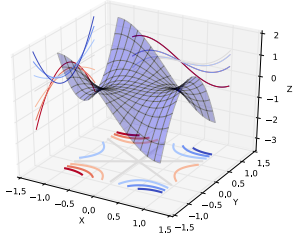


Figure: Monkey saddle ($\lambda_i = 0$ for some i)

Q1: Does SGD converge?

- SGD converges to the critical points of f as $k \to \infty$.

1. GD converges from any intialization with constant step-size and full gradients
2. With probability 1, (P)SGD does not converge with constant step-size α [2, 45]
3. With probability 1, SGD converges with vanishing step-size if $\mathbf{x}^k$ is bounded with probability 1 [34, 2]

Boundedness is not required (Theorem 1 of [37])

Assume Lipschitzness, sublevel regularity, $\mathbb{E}\|\mathbf{g}\|^q \leq \sigma^q$ and $\sum_k \alpha_k^{1+q/2} < \infty$ ($q \geq 2$). Then, $\mathbf{x}^k$ converges with probability 1.

Q2: Does SGD avoid saddle points?

- SGD avoids strict saddles ($\lambda_{\min}(\nabla^2 f(\bar{\mathbf{x}})) < 0$)

1. GD avoids strict saddles from almost all initializations [30]
2. With probability $1-\zeta$, PSGD with constant α escapes strict saddles after $\Omega\left(\log(1/\zeta)/\alpha^2\right)$ iterations [13]
   - However, SGD does not converge with constant α
   - We cannot take $\zeta = 0$

SGD avoids traps almost surely (Theorem 3 of [37])

Assume bounded uniformly exciting noise and $\alpha_k = \mathcal{O}\left(\frac{1}{k^\kappa}\right)$ for $\kappa \in (0, 1]$. Then, SGD avoids strict saddles from any initial condition with probability 1.

Q3: How fast does SGD converge to local minimizers?

- SGD remains close to Hurwicz minimizers (i.e., $\mathbf{x}^* : \lambda_{\min}(\nabla^2 f(\mathbf{x}^*)) > 0$)

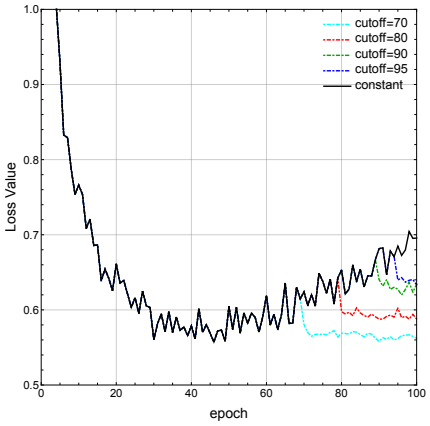
1. SGD with constant α can obtain objective value ϵ-close to a Hurwicz minimizer in $\mathcal{O}(1/\epsilon^2)$-iterations [13, 14]
   - However, SGD does not converge with constant α
   - Need averaging which is problematic in non-convex optimization

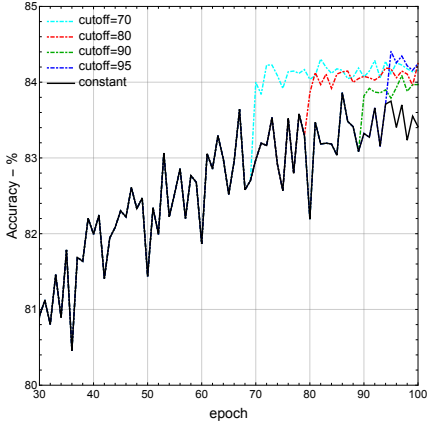
Using a vanishing step-size helps! (Theorem 4 of [37])

Using $\alpha_k = \mathcal{O}\left(\frac{1}{k}\right)$, SGD enjoys a $\mathcal{O}\left(\frac{1}{k}\right)$ convergence rate in objective value.

Using $1/k$ step-size decrease helps in practice

- ResNet training at different cool-down cut-offs





Worst-case iteration complexities of classical projected first-order methods[1][2]

$f(\mathbf{x})$	gradient oracle	L-smooth	Stationarity measure	GD/SGD	Accelerated GD/SGD
Convex	stochastic	yes	$f(\mathbf{x}^k) - f^\star =$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$
Convex	deterministic	yes	$f(\mathbf{x}^k) - f^\star =$	$\mathcal{O}\left(\frac{1}{k}\right)$	$\mathcal{O}\left(\frac{1}{k^2}\right)$
Convex	stochastic	no	$f(\mathbf{x}^k) - f^\star =$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$
Nonconvex	stochastic	yes	$\|G_\alpha(\mathbf{x}^k)\|^2 =$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)^3$	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)^3$
Nonconvex	deterministic	yes	$\|G_\alpha(\mathbf{x}^k)\|^2 =$	$\mathcal{O}\left(\frac{1}{k}\right)^4$	$\mathcal{O}\left(\frac{1}{k}\right)^4$
Nonconvex	stochastic	no	$\text{dist}(0, \partial(f(\mathbf{x}^k) + \delta_{\mathcal{X}}(\mathbf{x}^k)))^2 =$	?[3][5][6]	?[3][5][6]

- Basic structures, such as smoothness or strong convexity, help, but there are more structures that can be used:
  - $\max$-form, metric subregularity, Polyak-Lojasiewicz, Kurdyka-Lojasiewicz, weak convexity,[3] growth cond...

[1] Y. Nesterov, "Introductory lectures on convex optimization: A basic course," Springer Science, 2013.

[2] Y. Carmon, J.C. Duchi, O. Hinder, and A. Sidford, "Lower bounds for finding stationary points I–II." Mathematical Programming, 2019.

[3] D. Davis and D. Drusvyatskiy, "Stochastic model-based minimization of weakly convex functions," SIOPT, 2019.

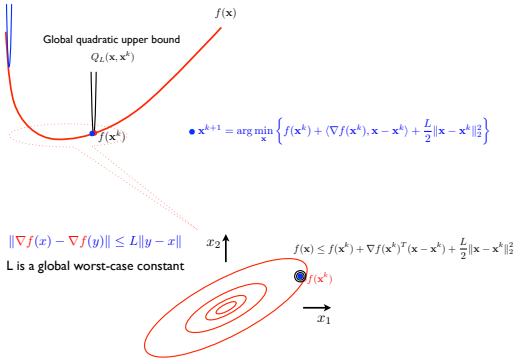
[4] S. Ghadimi and G. Lan, "Accelerated gradient methods for nonconvex nonlinear and stochastic programming," MathProg, 2016.

[5] J. Zhang, et al., "On complexity of finding stationary points of nonsmooth nonconvex functions," arXiv:2002.04130, 2020.

[6] O. Shamir, "Can We Find Near-Approximately-Stationary Points of Nonsmooth Nonconvex Functions?" arXiv:2002.11962, 2020.

Worst-case is often too pessimistic

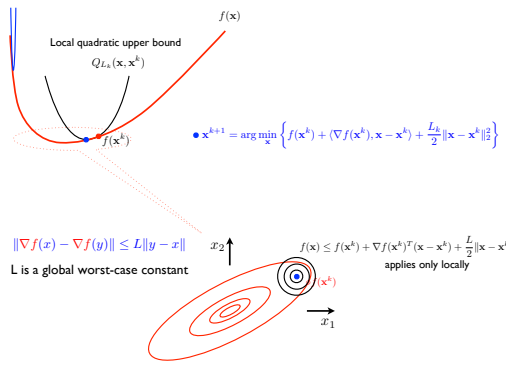
- GD: $x^{k+1} = x^k - \frac{1}{L}\nabla f(x^k)$



- Rates are not everything!
  - overall computational effort is what matters
  - constants &implementations are key
- Knowledge of smoothness, the value of L,...
  - challenging
- Must "somehow" adapt to a "different" function
  - online and without knowing L
  - can reduce overall computational effort!

A classical approach: Line-search

- Long history: Backtracking, Armijo, steepest descent...



- Universal accelerated gradient method[1]

$$f(x^k) - f^\star = \mathcal{O}\left(\frac{L_\nu \|x^0 - x^\star\|^{1+\nu}}{k^{\frac{1+3\nu}{2}}} \right)$$

  - adapts to Hölder smoothness ($\nu \in [0, 1]$)

$$\boxed{\|\nabla f(x) - \nabla f(y)\|_2 \leq L_\nu \|x - y\|_2^\nu}$$

  - has extensions to primal-dual optimization[2]
  - sets accuracy a priori & monotonic step-sizes

- Not as universal as we wish it to be
  - different procedures for stochastic gradients[3]

[1] Y. Nesterov, "Universal Gradient Methods for Convex Optimization Problems," Mathematical Programming, 2015.
[2] A. Yurtsever, Q. Tran-Dinh, and V. Cevher, "A Universal Primal-Dual Convex Optimization Framework," NeurIPS, 2015.
[3] S. Vaswani et al., "Painless Stochastic Gradient: Interpolation, Line-Search, and Convergence Rates," NeurIPS, 2019.

Stochastic adaptive first-order methods

Adaptive methods

Stochastic adaptive methods converge without knowing the smoothness constant.

They do so by making use of the information from stochastic gradients and their norms.

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- Additional resources:
  - My OPT-ML talk on universality and adaptation at NeurIPS: https://t.co/sw9qfiXYfW?amp=1
  - Eusipco tutorial with Ahmet Alacaoglu, Ali Kavis, and Kfir Levy: https://tinyurl.com/y2nu6m5c

Variable metric stochastic gradient descent algorithm

Variable metric stochastic gradient descent algorithm

1. Choose $\mathbf{x}^0 \in \mathbb{R}^p$ as a starting point and $\mathbf{H}_0 \succ 0$.
2. For $k = 0, 1, \cdots$, perform:

$$\begin{cases} \mathbf{d}^k & := -\mathbf{H}_k^{-1}\mathbf{g}^k, \\ \mathbf{x}^{k+1} & := \mathbf{x}^k + \alpha_k \mathbf{d}^k, \end{cases}$$

where $\alpha_k \in (0, 1]$ is a given step size.

3. Update $\mathbf{H}_{k+1} \succ 0$ if necessary.

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Common choices of the variable metric $\mathbf{H}_k$

- $\mathbf{H}_k := \lambda_k \mathbf{I}$ $\implies$ stochastic gradient descent method.
- $\mathbf{H}_k := \mathbf{D}_k$ (a positive diagonal matrix) $\implies$ stochastic adaptive gradient methods.

Adaptive gradient methods

Intuition

Adaptive gradient methods adapt locally by setting $\mathbf{H}_k$ as a function of past stochastic gradient information.

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Adaptive gradient methods adapt locally by setting $\mathbf{H}_k$ as a function of past stochastic gradient information.

- Roughly speaking, $\mathbf{H}_k = \text{function}(\mathbf{g}^1, \mathbf{g}^2, \cdots, \mathbf{g}^k)$
- Some well-known examples:

AdaGrad [9]

$$\mathbf{H}_k = \sqrt{\sum_{t=1}^{k} \mathbf{g}^k \mathbf{g}^{k\top}}$$

RmsProp [50]

$$\mathbf{H}_k = \sqrt{\beta \mathbf{H}_{k-1} + (1-\beta)\text{diag}(\mathbf{g}^k)^2}$$

ADAM [25]

$$\hat{\mathbf{H}}_k = \beta \hat{\mathbf{H}}_{k-1} + (1-\beta)\text{diag}(\mathbf{g}^k)^2$$
$$\mathbf{H}_k = \sqrt{\hat{\mathbf{H}}_k/(1-\beta^k)}$$

AdaGrad - Adaptive gradient method with $\mathbf{H}_k = \lambda_k \mathbf{I}$

- If $\mathbf{H}_k = \lambda_k \mathbf{I}$, it becomes stochastic gradient descent method with adaptive step-size $\frac{\alpha_k}{\lambda_k}$.

How does the step-size adapt?

If the stochastic gradient $\|\mathbf{g}^k\|$ is large/small $\rightarrow$ AdaGrad adjusts step-size α_k/λ_k smaller/larger

Adaptive gradient descent (AdaGrad with $\mathbf{H}_k = \lambda_k \mathbf{I}$) [31]

1. Set $Q^0 = 0$.
2. For $k = 0, 1, \ldots$, iterate

$$\begin{cases} Q^k & = Q^{k-1} + \|\mathbf{g}^k\|^2 \\ \mathbf{H}_k & = \sqrt{Q^k} I \\ \mathbf{x}^{k+1} & = \mathbf{x}^k - \alpha_k \mathbf{H}_k^{-1} \mathbf{g}^k \end{cases}$$

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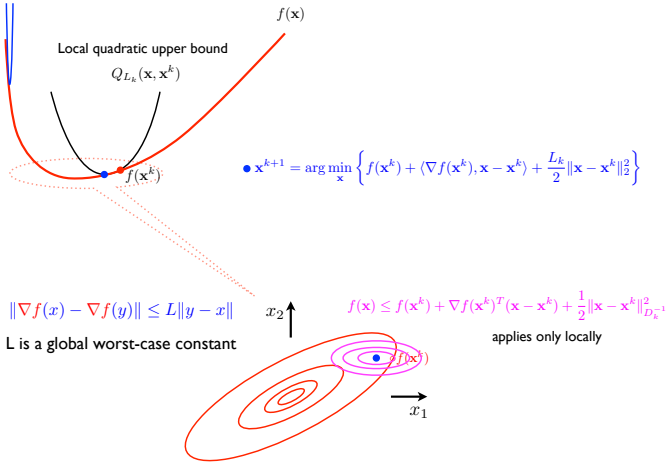
Adaptation through first-order information

- When $H_k = \lambda_k I$, AdaGrad estimates local geometry through stochastic gradient norms.
- Akin to estimating a local quadratic upper bound (majorization / minimization) using gradient history.

AdaGrad - Adaptive gradient method with $\mathbf{H}_k = \mathbf{D}_k$

Adaptation strategy with a positive diagonal matrix $\mathbf{D}_k$

Adaptive step-size + coordinate-wise extension = adaptive step-size for each coordinate



AdaGrad - Adaptive gradient method with $\mathbf{H}_k = \mathbf{D}_k$

- Suppose $\mathbf{H}_k$ is diagonal,

$$\mathbf{H}_k := \begin{bmatrix} \lambda_{k,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{k,d} \end{bmatrix},$$

- For each coordinate i, we have different step-size $\frac{\alpha_k}{\lambda_{k,i}}$ is the step-size.

Adaptive gradient descent(AdaGrad with $\mathbf{H}_k = \mathbf{D}_k$)

1. Set $\mathbf{Q}^0 = 0$.

2. For $k = 0, 1, \ldots,$ iterate

$$\begin{cases} \mathbf{Q}^k & = \mathbf{Q}^{k-1} + \operatorname{diag}(\mathbf{g}^k)^2 \\ \mathbf{H}_k & = \sqrt{\mathbf{Q}^k} \\ \mathbf{x}^{k+1} & = \mathbf{x}^k - \alpha_k \mathbf{H}_k^{-1} \mathbf{g}^k \end{cases}$$

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Adaptation across each coordinate

- When $\mathbf{H}_k = \mathbf{D}_k$, we adapt across each coordinate individually.
- Essentially, we have a finer treatment of the function we want to optimize.

RMSProp - Adaptive gradient method with $\mathbf{H}_k = \mathbf{D}_k$

What could be improved over AdaGrad?

1. Stochastic gradients have equal weights in step size.
2. Consider a *steep* function, flat around minimum $\rightarrow$ slow convergence at flat region.

RMSProp - Adaptive gradient method with $\mathbf{H}_k = \mathbf{D}_k$

What could be improved over AdaGrad?

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AdaGrad with $\mathbf{H}_k = \mathbf{D}_k$

1. Set $\mathbf{Q}_0 = 0$.
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RMSProp

1. Set $\mathbf{Q}_0 = 0$.
2. For $k = 0, 1, \ldots$, iterate

$$\begin{cases} \mathbf{Q}^k & = \beta \mathbf{Q}^{k-1} + (1-\beta)\mathrm{diag}(\mathbf{g}^k)^2 \\ \mathbf{H}_k & = \sqrt{\mathbf{Q}^k} \\ \mathbf{x}^{k+1} & = \mathbf{x}^k - \alpha_k \mathbf{H}_k^{-1} \mathbf{g}^k \end{cases}$$

RMSProp - Adaptive gradient method with $\mathbf{H}_k = \mathbf{D}_k$

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Remarks:

- RMSProp uses weighted averaging with constant β
- Recent gradients have greater importance

Example: AdaGrad vs. RMSProp

Setting:

- $f(x) = x^4$ (one-dimensional function)
- $x^0 = 10$, $x^\star = 0$

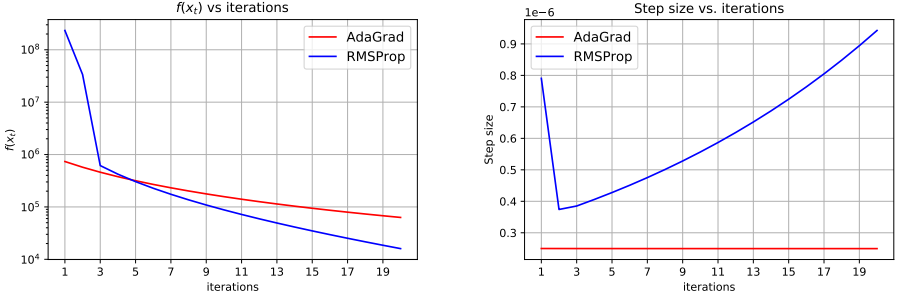

Figure: RMSProp vs. AdaGrad

ADAM - Adaptive moment estimation

Over-simplified idea of ADAM

RMSProp + 2nd order moment estimation = ADAM

ADAM - Adaptive moment estimation

Over-simplified idea of ADAM

RMSProp + 2nd order moment estimation = ADAM

ADAM

Input. Step size α, exponential decay rates $\beta_1, \beta_2 \in [0, 1)$

1. Set $\mathbf{m}_0, \mathbf{v}_0 = 0$

2. For $k = 0, 1, \ldots$, iterate

$$\begin{cases} \mathbf{g}_k & = \nabla f(\mathbf{x}^{k-1}) \\ \mathbf{m}_k & = \beta_1 \mathbf{m}_{k-1} + (1-\beta_1)\mathbf{g}_k \leftarrow \text{1st order estimate} \\ \mathbf{v}_k & = \beta_2 \mathbf{v}_{k-1} + (1-\beta_2)\mathbf{g}_k^{\cdot 2} \leftarrow \text{2nd order estimate} \\ \hat{\mathbf{m}}_k & = \mathbf{m}_k/(1-\beta_1^k) \leftarrow \text{Bias correction} \\ \hat{\mathbf{v}}_k & = \mathbf{v}_k/(1-\beta_2^k) \leftarrow \text{Bias correction} \\ \mathbf{H}_k & = \sqrt{\hat{\mathbf{v}}_k} + \epsilon \\ \mathbf{x}^{k+1} & = \mathbf{x}^k - \alpha \hat{\mathbf{m}}_k ./ \mathbf{H}_k \end{cases}$$

Output : $\mathbf{x}^k$

(Every vector operation is an element-wise operation)

Non-convergence of ADAM and a new method: AmsGrad

- It has been shown that ADAM may not converge for *some* objective functions [56].
- An ADAM alternative is proposed that is proved to be convergent [43].

AmsGrad

Input. Step size $\{\gamma_k\}_{k\in\mathbb{N}}$, exponential decay rates $\{\beta_{1,k}\}_{k\in\mathbb{N}}$, $\beta_2 \in [0,1)$

1. Set $\mathbf{m}_0 = 0, \mathbf{v}_0 = 0$ and $\hat{\mathbf{v}}_0 > 0$

2. For $k = 1, 2, \ldots$, iterate

$$\begin{cases} \mathbf{g}_k & = G(\mathbf{x}^k, \theta) \\ \mathbf{m}_k & = \beta_{1,k}\mathbf{m}_{k-1} + (1-\beta_{1,k})\mathbf{g}_k \leftarrow \text{1st order estimate} \\ \mathbf{v}_k & = \beta_2\mathbf{v}_{k-1} + (1-\beta_2)\mathbf{g}_k^{\cdot 2} \leftarrow \text{2nd order estimate} \\ \hat{\mathbf{v}}_k & = \max\{\hat{\mathbf{v}}_{k-1}, \mathbf{v}_k\} \text{ and } \hat{\mathbf{V}}_k = \mathsf{diag}(\hat{\mathbf{v}}_k) \\ \mathbf{H}_k & = \sqrt{\hat{\mathbf{v}}_k} \\ \mathbf{x}^{k+1} & = \Pi_{\mathcal{X}}^{\sqrt{\hat{\mathbf{V}}_k}}(\mathbf{x}^k - \gamma_k \hat{\mathbf{m}}_k ./ \mathbf{H}_k) \end{cases}$$

Output : $\mathbf{x}^k$

where $\pi_{\mathcal{K}}^{\mathbf{A}}(\mathbf{y}) = \arg\min_{\mathbf{x}\in\mathcal{K}} \langle (\mathbf{x}-\mathbf{y}), \mathbf{A}(\mathbf{x}-\mathbf{y})\rangle$ (weighted Euclidean projection onto $\mathcal{K}$).
(Every vector operation is an element-wise operation)

Example: Least squares with synthetic data

Setting:

- $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$
- $\mathbf{A} \in \mathbb{R}^{n \times p}$, $\mathbf{A} \sim N(\mu, \sigma^2 I)$
- $n = 1000$, $p = 1000$

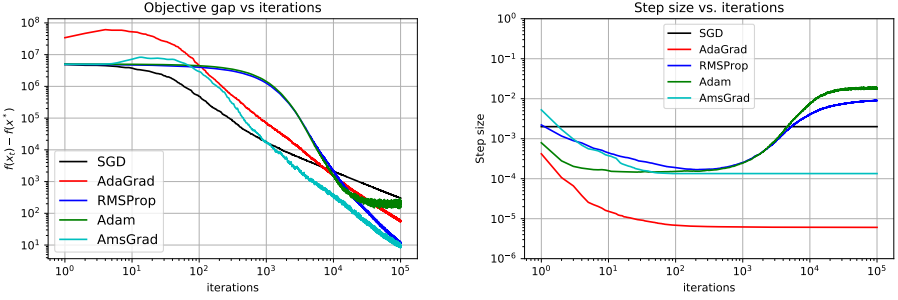

Figure: Comparison of convergence rate and stepsize evolution. Mini-batch stochastic gradients with a batch size of 20

A comparison of algorithms

	GD/SGD	Accelerated GD/SGD	AdaGrad	AcceleGrad/UniXgrad	Adam/AMSGrad
Convex, stochastic	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[1]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[1]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[2]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[3,4]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[5]
Convex, deterministic, L-smooth	$\mathcal{O}\left(\frac{1}{k}\right)$[1]	$\mathcal{O}\left(\frac{1}{k2}\right)$[1]	$\mathcal{O}\left(\frac{1}{k}\right)$[3]	$\mathcal{O}\left(\frac{1}{k2}\right)$[3,4]	$\mathcal{O}\left(\frac{1}{k}\right)$[6]
Nonconvex, stochastic, L-smooth	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[1]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[1]	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[7]	?	$\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$[8]
Nonconvex, deterministic, L-smooth	$\mathcal{O}\left(\frac{1}{k}\right)$[1]	$\mathcal{O}\left(\frac{1}{k}\right)$[1]	$\mathcal{O}\left(\frac{1}{k}\right)$[7]	?	$\mathcal{O}\left(\frac{1}{k}\right)$[6]

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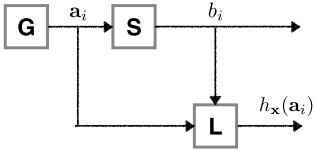
From empirical risk minimization...



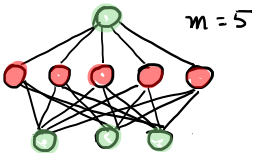
Definition (Optimization formulation)

The deep-learning training problem is given by

$$\mathbf{x}^\star_{\mathsf{DL}} \in \arg\min_{\mathbf{x}\in\mathcal{X}} \{\, f(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} L(h_{\mathbf{x}}(\mathbf{a}_i), b_i) \,\},$$

where $\mathcal{X}$ denotes the constraints on the parameters.

- A single hidden layer neural network with params $\mathbf{x} := [\mathbf{X}_1, \mathbf{X}_2, \mu_1, \mu_2]$



$$h_{\mathbf{x}}(\mathbf{a}) := \begin{bmatrix} \mathbf{X}_2 \end{bmatrix} \sigma \left(\begin{bmatrix} \mathbf{X}_1 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} + \begin{bmatrix} \mu_1 \end{bmatrix} \right) + \begin{bmatrix} \mu_2 \end{bmatrix}$$

activation (over σ); weight (over $\mathbf{X}_1$); input (over $\mathbf{a}$); bias (over μ_1); bias (over μ_2)

hidden layer = learned features

...Into adversarial examples

Definition (Adversarial examples [49])

Let $h_{\mathbf{x}^\star} : \mathbb{R}^p \to \mathbb{R}$ be a model trained through empirical risk minimization, with optimal parameters $\mathbf{x}^\star$. Let $(\mathbf{a}, b)$ be a sample with $b \in \{-1, 1\}$ and $\mathbf{a} \in \mathbb{R}^p$. An **adversarial example** is a perturbation $\boldsymbol{\eta} \in \mathbb{R}^n$ designed to lead the trained model $h_{\mathbf{x}^\star}$ to misclassify a given input $\mathbf{a}$. Given an $\epsilon > 0$, it is constructed by solving

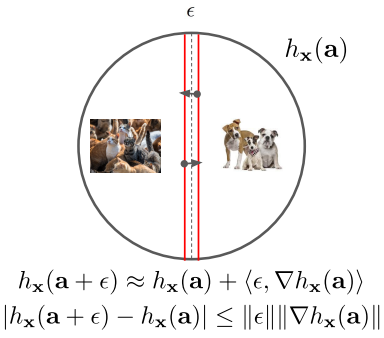
$$\boldsymbol{\eta} \in \underset{\boldsymbol{\eta}:\|\boldsymbol{\eta}\| \leq \epsilon}{\arg\max}\, L(h_{\mathbf{x}^\star}(\mathbf{a} + \boldsymbol{\eta}), \mathbf{b})$$

Example norms frequently used in adversarial attacks

- The most commonly used norm is the ℓ_∞-norm [17, 36].
- The use of ℓ_1-norm leads to sparse attacks.

Figure: (Left) An ℓ_∞-attack: The alteration is hard to perceive. (Right) An ℓ_1-attack: The alteration in this case is obvious.

Challenge: Robustness to adversarial examples



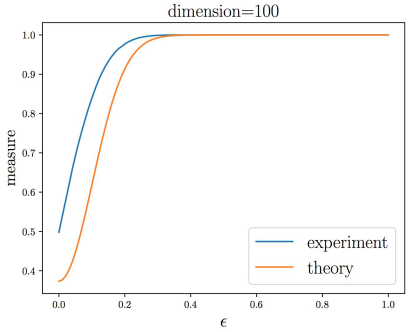


Figure: Understanding the robustness of a classifier in high-dimensional spaces. Shafahi et al. 2019.

A robustness example: Linear prediction

Linear model

Consider a linear model $h_{\mathbf{x}^\star}(\mathbf{a}) = \langle \mathbf{x}^\star, \mathbf{a} \rangle$ with weights $\mathbf{x}^\star \in \mathbb{R}^p$, for some input $\mathbf{a}$.

An adversarial perturbation

We aim at finding the perturbation $\boldsymbol{\eta} \in \mathbb{R}^n$ subject to $\|\boldsymbol{\eta}\|_\infty \leq \epsilon$ that produces the largest change on $h_{\mathbf{x}^\star}(\mathbf{a})$:

$$\begin{aligned}
\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\eta}) &= \max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} \langle \mathbf{x}^\star, \mathbf{a}+\boldsymbol{\eta} \rangle \\
&= \langle \mathbf{x}^\star, \mathbf{a} \rangle + \max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} \langle \mathbf{x}^\star, \boldsymbol{\eta} \rangle && \triangleright \text{ As } \mathbf{a} \text{ does not influence the optimization.} \\
&= \langle \mathbf{x}^\star, \mathbf{a} \rangle + \max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq 1} \langle \mathbf{x}^\star, \epsilon\boldsymbol{\eta} \rangle && \triangleright \text{ By the change of variables } \boldsymbol{\eta} := \boldsymbol{\eta}/\epsilon \\
&= \langle \mathbf{x}^\star, \mathbf{a} \rangle + \epsilon \|\mathbf{x}^\star\|_1 && \triangleright \text{ Definition of the dual norm } \|\mathbf{x}\|_1 := \max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq 1} \langle \mathbf{x}, \boldsymbol{\eta} \rangle
\end{aligned}$$

Taking $\boldsymbol{\eta}^\star = \operatorname{sign}(\mathbf{x}^\star)$ achieves this maximum: $\langle \mathbf{x}, \epsilon \operatorname{sign}(\mathbf{x}^\star) \rangle = \epsilon \sum_{i=1}^n \operatorname{sign}(x_i^\star) x_i^\star = \epsilon \sum_{i=1}^n |x_i^\star| = \epsilon \|\mathbf{x}^\star\|_1$.

Remarks:

- For the linear model, we have $\nabla_{\mathbf{a}} h_{\mathbf{x}^\star}(\mathbf{a}) = \mathbf{x}^\star$.
- *The gradient sign* of $h_{\mathbf{x}^\star}$ with respect to the input $\mathbf{a}$ achieves the worst perturbation.
- Sparse models are robust in linear prediction.

Adversarial examples in neural networks

- Target problem:

$$\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\eta}), \mathbf{b})$$

- Historically, researchers first tried to find approximate solutions that empirically perform well [17, 36].

Fast Gradient Sign Method (FGSM) [17]

Let $h_{\mathbf{x}^\star} : \mathbb{R}^p \to \mathbb{R}$ be a model trained through empirical risk minimization on the loss L, with optimal parameters $\mathbf{x}^\star$. Let $(\mathbf{a}, b)$ be a sample with $b \in \{-1, 1\}$ and $\mathbf{a} \in \mathbb{R}^p$. The *Fast Gradient Sign Method* computes the adversarial example

$$\boldsymbol{\eta} = \epsilon \operatorname{sign}\left(\nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), b)\right) = \epsilon \operatorname{sign}\left(\nabla_{\mathbf{a}} h_{\mathbf{x}^\star}(\mathbf{a}) \nabla_h L(h_{\mathbf{x}^\star}(\mathbf{a}), b)\right)$$

Remarks:

- The FGSM obtains adversarial examples by using *sign of the gradient of the loss*.
- Such an approach can be viewed as a linearization of the objective L around the data $\mathbf{a}$.
- For single output $h_{\mathbf{x}}(\mathbf{a})$, $\nabla_h L(h_{\mathbf{x}^\star}(\mathbf{a}), b)$ is a scalar,
  - $\operatorname{sign}\left(\nabla_{\mathbf{a}} h_{\mathbf{x}^\star}(\mathbf{a})\right)$ pattern is important

Results of FGSM on MNIST

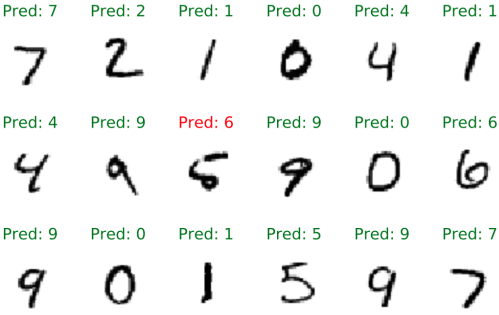


Figure: MNIST images with the predicted digit.

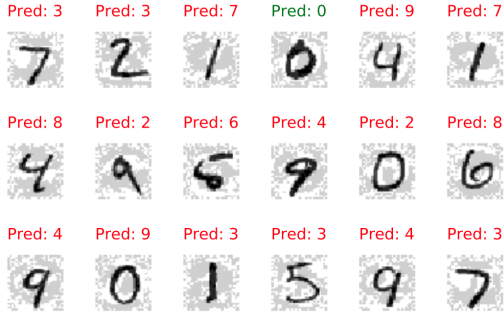


Figure: MNIST images perturbed by a FGSM attack.

Taken from `https://adversarial-ml-tutorial.org/adversarial_examples/`

Adversarial examples and proximal gradient ascent

- Target problem:

$$\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\eta}), \mathbf{b})$$

- We can do better than FGSM via proximal gradient methods for composite minimization:

$$\max_{\boldsymbol{\eta}\in\mathbb{R}^p} \underbrace{L(h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\eta}), \mathbf{b})}_{f(\boldsymbol{\eta})} + \underbrace{\delta_{\mathcal{N}}(\boldsymbol{\eta})}_{g(\boldsymbol{\eta})},$$

where $\delta_{\mathcal{N}}(\boldsymbol{\eta})$ is the indicator function of the ball $\mathcal{N} := \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \epsilon\}$.

Recall: Proximal operator of indicator functions

For the indicator functions of simple sets, e.g., $g(\boldsymbol{\eta}) := \delta_{\mathcal{N}}(\boldsymbol{\eta})$, the prox-operator is the projection operator

$$\text{prox}_{\alpha g}(\boldsymbol{\eta}) := \pi_{\mathcal{N}}(\boldsymbol{\eta}),$$

where $\pi_{\mathcal{N}}(\boldsymbol{\eta})$ denotes the projection of $\boldsymbol{\eta}$ onto $\mathcal{N}$. When $\mathcal{N} = \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \lambda\}$, $\pi_{\mathcal{N}}(\boldsymbol{\eta}) = \text{clip}(\boldsymbol{\eta}, [-\lambda, \lambda])$.

Adversarial examples and proximal gradient ascent (cont'd)

- Target non-convex problem:

$$\max_{\boldsymbol{\eta}\in\mathbb{R}^p} \underbrace{L(h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\eta}),\mathbf{b})}_{f(\boldsymbol{\eta})} + \underbrace{\delta_{\mathcal{N}}(\boldsymbol{\eta})}_{g(\boldsymbol{\eta})},$$

where $\delta_{\mathcal{N}}(\boldsymbol{\eta})$ is the indicator function of the ball $\mathcal{N} := \{\mathbf{y} : \|\mathbf{y}\|_\infty \leq \epsilon\}$.

Proximal gradient ascent (PGA)

1. Choose $\boldsymbol{\eta}^0 \in \operatorname{dom} f(\boldsymbol{\eta}) + g(\boldsymbol{\eta})$ as initialization.
2. For $k = 0, 1, \cdots$, generate a sequence $\{\boldsymbol{\eta}^k\}_{k\geq 0}$ as:

$$\boldsymbol{\eta}^{k+1} := \operatorname{prox}_{\alpha_k g}\left(\boldsymbol{\eta}^k + \alpha_k \nabla f(\boldsymbol{\eta}^k)\right).$$

Remarks:

- PGA results in more powerful adversarial "attacks" than FGSM [26].
- The PGA is incorrectly referred to as projected gradient descent in this literature.
- Practitioners prefer to use several steps of FGSM instead of PGA [28, 29, 36]:

$$\boldsymbol{\eta}^{k+1} = \pi_{\mathcal{N}}\left(\boldsymbol{\eta}^k + \alpha_k \ \textbf{sign}\left(\nabla f(\boldsymbol{\eta}^k)\right)\right).$$

A proposed link between FGSM and PGD

- Recall
  - A single step of PGA reads $\boldsymbol{\eta}_{\text{PGA}}^{k+1} := \pi_{\mathcal{N}}\left(\boldsymbol{\eta}^k + \alpha \nabla f(\boldsymbol{\eta})\right)$
  - The FGSM attack is defined as $\boldsymbol{\eta}_{\text{FGSM}} := \epsilon \ \text{sign}\left(\nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})\right)$
  - When $\mathcal{N} = \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \lambda\}$, $\pi_{\mathcal{N}}(\boldsymbol{\eta}) = \text{clip}(\boldsymbol{\eta}, [-\lambda, \lambda])$

FGSM as one step of PGA

Let $\boldsymbol{\eta}^0 = \mathbf{0}$ and $\alpha > 0$ such that $(\alpha \left|\nabla f(\mathbf{0})\right|)_i > \epsilon$ for $i = 1, \ldots, n$. Then, one step of PGA yields

$$
\begin{aligned}
\boldsymbol{\eta}_{\text{PGD}}^1 &= \pi_{\mathcal{N}}\left(\boldsymbol{\eta}^0 + \alpha \nabla_{\boldsymbol{\eta}} \nabla f(\boldsymbol{\eta}^0)\right) \\
&= \text{clip}\left(\alpha \nabla f(\mathbf{0}), [-\epsilon, \epsilon]\right) && \triangleright\ \boldsymbol{\eta}^0 = \mathbf{0} \\
&= \epsilon \ \text{sign}\left(\nabla f(\mathbf{0})\right) && \triangleright\ \text{All values are outside of the interval } [-\epsilon, \epsilon] \\
&= \epsilon \ \text{sign}\left(\nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})\right) = \boldsymbol{\eta}_{\text{FGSM}} && \triangleright\ \nabla f(\mathbf{0}) = \nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})
\end{aligned}
$$

A proposed link between FGSM and PGD

- Recall
  - A single step of PGA reads $\boldsymbol{\eta}_{\text{PGA}}^{k+1} := \pi_{\mathcal{N}}\left(\boldsymbol{\eta}^k + \alpha \nabla f(\boldsymbol{\eta})\right)$
  - The FGSM attack is defined as $\boldsymbol{\eta}_{\text{FGSM}} := \epsilon \ \text{sign}\left(\nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})\right)$
  - When $\mathcal{N} = \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \lambda\}$, $\pi_{\mathcal{N}}(\boldsymbol{\eta}) = \text{clip}(\boldsymbol{\eta}, [-\lambda, \lambda])$

FGSM as one step of PGA

Let $\boldsymbol{\eta}^0 = \mathbf{0}$ and $\alpha > 0$ such that $(\alpha \left|\nabla f(\mathbf{0})\right|)_i > \epsilon$ for $i = 1, \ldots, n$. Then, one step of PGA yields

$$
\begin{aligned}
\boldsymbol{\eta}_{\text{PGD}}^1 &= \pi_{\mathcal{N}}\left(\boldsymbol{\eta}^0 + \alpha \nabla_{\boldsymbol{\eta}} \nabla f(\boldsymbol{\eta}^0)\right) \\
&= \text{clip}\left(\alpha \nabla f(\mathbf{0}), [-\epsilon, \epsilon]\right) && \triangleright\ \boldsymbol{\eta}^0 = \mathbf{0} \\
&= \epsilon \ \text{sign}\left(\nabla f(\mathbf{0})\right) && \triangleright\ \text{All values are outside of the interval } [-\epsilon, \epsilon] \\
&= \epsilon \ \text{sign}\left(\nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})\right) = \boldsymbol{\eta}_{\text{FGSM}} && \triangleright\ \nabla f(\mathbf{0}) = \nabla_{\mathbf{a}} L(h_{\mathbf{x}^\star}(\mathbf{a}), \mathbf{b})
\end{aligned}
$$

Multiple steps of FGSM: A connection to majorization-minimization

Minimization-majorization for concave functions

Let f be a concave function which is smooth in the ℓ_∞-norm with constant L_∞. Our target non-convex problem is given by

$$\max_{\boldsymbol{\eta}} f(\boldsymbol{\eta}) + \delta_{\mathcal{N}}(\boldsymbol{\eta})$$

where $\delta_{\mathcal{N}}(\boldsymbol{\eta})$ is the indicator function of the ball $\mathcal{N} := \{\boldsymbol{\eta} : \|\boldsymbol{\eta}\|_\infty \leq \epsilon\}$. Smoothness in ℓ_∞-norm implies

$$f(\boldsymbol{\eta}) + \delta_{\mathcal{N}}(\boldsymbol{\eta}) \geq \underbrace{f(\boldsymbol{\zeta}) + \langle \nabla_{\boldsymbol{\eta}} f(\boldsymbol{\zeta}), \boldsymbol{\eta} - \boldsymbol{\zeta} \rangle - \frac{L_\infty}{2} \|\boldsymbol{\eta} - \boldsymbol{\zeta}\|_\infty^2 + \delta_{\mathcal{X}}(\boldsymbol{\eta})}_{\boldsymbol{\eta}^\star \leftarrow \arg\max_{\boldsymbol{\eta}}} .$$

Maximizing the RHS with respect to $\boldsymbol{\eta}$ leads to the following (non trivial) solution [11]:

$$\boldsymbol{\eta}^\star = \text{clip}\left(\boldsymbol{\zeta} - t^\star \text{sign}(\nabla f(\boldsymbol{\zeta})), [-\epsilon, \epsilon]\right)$$

where $t^\star = \arg\max_{t:\|\boldsymbol{\eta}-\boldsymbol{\zeta}\|_\infty \leq t} \max_{\boldsymbol{\zeta}:\|\boldsymbol{\zeta}\|_\infty \leq \epsilon} \langle \nabla f(\boldsymbol{\zeta}), \boldsymbol{\eta} - \boldsymbol{\zeta} \rangle$ can be found by linear search.

Remarks:

- Setting $\boldsymbol{\zeta} = \boldsymbol{\eta}^k$ and $\boldsymbol{\eta}^\star = \boldsymbol{\eta}^{k+1}$ with a fixed step size $\alpha = t^\star$, we obtain the update in [28, 29, 36]
$$\boldsymbol{\eta}^{k+1} = \text{clip}\left(\boldsymbol{\eta}^k - t^\star \text{sign}(\nabla f(\boldsymbol{\eta}^k)), [-\epsilon, \epsilon]\right).$$
- This proof holds for **concave** and smooth functions, and need further quantification for our setting.

Towards adversarial training

Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with the data $\mathbf{a}_i \in \mathbb{R}^p$ and the labels $\mathbf{b}_i$. The problem of adversarial training is the following adversarial optimization problem

$$\min_{\mathbf{x}} \frac{1}{n} \sum_{i=1}^{n} \left[\max_{\boldsymbol{\eta}: \|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i) \right] \approx \min_{\mathbf{x}} \mathbb{E}_{(\mathbf{a},\mathbf{b})\sim\mathbb{P}} \left[\max_{\boldsymbol{\eta}: \|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i) \right].$$

Note the similarity with the template $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$.

Solving the outer problem

Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with $\mathbf{a}_i \in \mathbb{R}^p$ and $\mathbf{b}_i$ be the corresponding labels. The adversarial training optimization problem is given by

$$\min_{\mathbf{x}} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} \underbrace{\left[\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i) \right]}_{=: f_i(\mathbf{x})} \right\}.$$

Note that L is not continuously differentiable due to ReLU, max-pooling, etc.

Solving the outer problem

Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with $\mathbf{a}_i \in \mathbb{R}^p$ and $\mathbf{b}_i$ be the corresponding labels. The adversarial training optimization problem is given by

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Note that L is not continuously differentiable due to ReLU, max-pooling, etc.

Question

How can we compute the gradient

$$\nabla_{\mathbf{x}} f_i(\mathbf{x}) := \nabla_{\mathbf{x}} \left(\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i) \right)?$$

- **Challenge:** It involves differentiating with respect to a maximization.
- **A solution:** We can use Danskin's theorem under some conditions.

Danskin's theorem

Danskin's theorem (Bertsekas variant)

Let $\Phi(\mathbf{x}, \mathbf{y}) : \mathbb{R}^p \times \mathcal{Y} \to \mathbb{R}$, where $\mathcal{Y} \subset \mathbb{R}^m$ is a compact set and define $f(\mathbf{x}) := \max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$. Let $\Phi(\mathbf{x}, \mathbf{y})$ is an extended real-valued closed proper convex function for each $\mathbf{y}$ in the compact set $\mathcal{Y}$; the interior of the domain of f is nonempty; $\Phi(\mathbf{x}, \mathbf{y})$ is jointly continuous on the relative interior of the domain of f and $\mathcal{Y}$.

Define $\mathcal{Y}^\star := \arg\max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$ as the set of maximizers and $\mathbf{y}^\star \in \mathcal{Y}^\star$ as an element of this set. We have

1. $f(\mathbf{x})$ is a convex function.
2. If $\mathbf{y}^\star = \arg\max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$ is unique, then the function $f(\mathbf{x}) = \max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$ is differentiable at $\mathbf{x}$:

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \nabla_{\mathbf{x}} \left(\max_{\mathbf{y} \in \mathcal{Y}} \phi(\mathbf{x}, \mathbf{y}) \right) = \nabla_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}^\star).$$

3. If $\mathbf{y}^\star = \arg\max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$ is not unique, then the subdifferential $\partial_{\mathbf{x}} f(\mathbf{x})$ of f is given by

$$\partial_{\mathbf{x}} f(\mathbf{x}) = \operatorname{conv} \left\{ \partial_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}^\star) : \mathbf{y}^\star \in \mathcal{Y}^\star \right\}.$$

Remarks:

- The adversarial problem is not convex in $\mathbf{x}$ in general.
- With proper initialization, overparameterization works argue that it is effectively convex.
- (Sub)Gradients of f_i are calculated as $\partial f_i(\mathbf{x}) = \nabla_{\mathbf{x}} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}^\star(\mathbf{x})), \mathbf{b}_i)$.

A corollary to Danskin's theorem

Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with $\mathbf{a}_i \in \mathbb{R}^p$ and $\mathbf{b}_i$ be the corresponding labels. The adversarial training optimization problem is given by

$$\min_{\mathbf{x}} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} \underbrace{\left[\max_{\boldsymbol{\eta}: \|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i) \right]}_{=: f_i(\mathbf{x})} \right\}.$$

L is not continuously differentiable due to ReLU, max-pooling, etc.

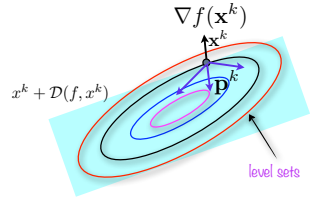


Figure: Descent directions in 2D should be an element of the cone of descent directions $\mathcal{D}(f, \cdot)$.

Descent directions [36]

Define $\mathcal{Y}^\star := \arg\max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$ as the set of maximizers, $\mathbf{y}^\star \in \mathcal{Y}^\star$, and $f(\mathbf{x}) := \max_{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$. As long as $\nabla_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}^\star)$ is non-zero, it is a descent direction (and not a subgradient!) for $f(\mathbf{x})$.

Remarks:

- $\nabla_{\mathbf{x}} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}^\star(\mathbf{x})), \mathbf{b}_i)$ is a descent direction for $f_i(\mathbf{x})$.
- We cannot find global maximizers $\mathcal{Y}^\star$.
- Only when $\mathbf{y}^\star$ is a singleton, $\nabla_{\mathbf{x}} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\eta}^\star(\mathbf{x})), \mathbf{b}_i)$ is a (sub)gradient [4].

A practical implementation of adversarial training: Stochastic subgradient descent

Stochastic Adversarial Training [36]

Input: learning rate α_k, iterations T, batch size K.

1. initialize neural network parameters $\mathbf{x}^0$

2. **For** $k = 0, 1, ..., T$:

i. initialize (sub)gradient vector $\mathbf{g}^k := 0$

ii. select a mini-batch of data $B \subset \{1, \ldots, n\}$ with $|B| = K$

iii. **For** $i \in B$:

a. Find an attack $\boldsymbol{\eta}^\star$ by (approximately) solving

$$\boldsymbol{\eta}^\star \in \arg\max_{\boldsymbol{\eta}:\|\boldsymbol{\eta}\|_\infty \leq \epsilon} L(h_{\mathbf{x}^k}(\mathbf{a}_i + \boldsymbol{\eta}), \mathbf{b}_i)$$

b. Store optimal (sub)gradient

$$\mathbf{g}^k := \mathbf{g}^k + \nabla_{\mathbf{x}} L(h_{\mathbf{x}^k}(\mathbf{a}_i + \boldsymbol{\eta}^\star), \mathbf{b}_i)$$

iv. Update parameters

$$\mathbf{x}^{k+1} := \mathbf{x}^k - \frac{\alpha_k}{K}\mathbf{g}^k$$

Remarks:

- Expensive but worth it!
- Inner problem **iii.a** cannot be solved to optimality (non-convex).
- Practitioners use FGSM or PGA or PGA-ℓ_∞ to approximate the true $\boldsymbol{\eta}^\star$.
- (Sub)Gradient computation in step **iii.b** is motivated by Danskin's theorem.

Application: Adversarial training for better interpretability

- Retinopathy classification problem: Given a retinal image (left), predict whether there is a disease.
- **Zeiss:** How can we interpret the prediction of a model $h_{\mathbf{x}}(\mathbf{a})$?
- **Solution:** Look at $\nabla_{\mathbf{x}} h_{\mathbf{x}}(\mathbf{a})$, called the saliency map [48]. Adversarial training helps!

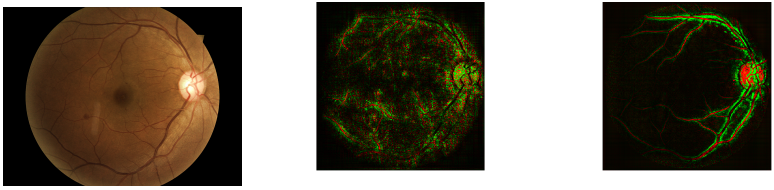
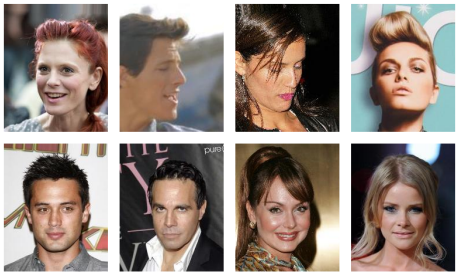


Table: **Left:** Ground truth image, **Middle:** Saliency map, **Right:** Saliency map with adversarial training.

Adversarial machine learning: Introduction to Generative Adversarial Networks (GANs)

- Recall the parametric density estimation setting

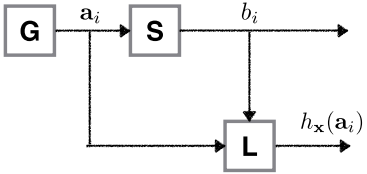


(source: http://mmlab.ie.cuhk.edu.hk/projects/CelebA.html)

$\mathbf{a}_i = [$ ...images...$]$

$b_i = [$ ...probability... $]$

- Goal: Games, denoising, image recovery...



- Generator $\mathbb{P}_{\mathbf{a}}$
  - Nature
- Supervisor $\mathbb{P}_{B|\mathbf{a}}$
  - Frequency data
- Learning Machine $h_{\mathbf{x}}(\mathbf{a}_i)$
  - Data scientist: Mathematics of Data

A notion of distance between distributions

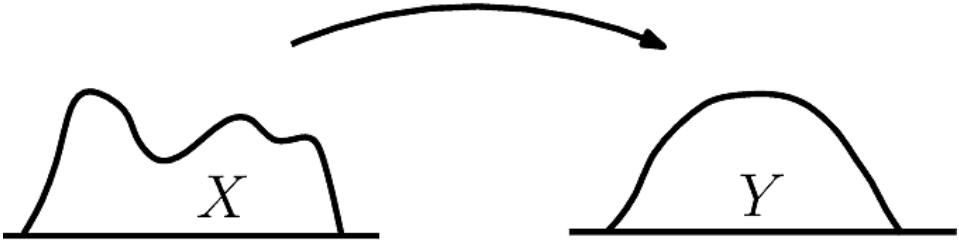


Figure: The Earth Mover's distance

Minimum cost transportation problem (Monge's problem)

Find a *transport map* $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(X) \sim Y$, minimizing the cost

$$\text{cost}(T) := \boldsymbol{E}_X \left\| X - T(X) \right\| . \tag{1}$$

The Wasserstein distance

Definition

Let μ and ν be two probability measures on $\mathbb{R}^d$. Their set of couplings is defined as

$$\Gamma(\mu,\nu) := \{\pi \text{ probability measure on } \mathbb{R}^d \times \mathbb{R}^d \text{ with marginals } \mu,\nu\} \tag{2}$$

Definition (Primal form of the q-Wasserstein distance)

$$W_q(\mu,\nu) := \left(\inf_{\pi\in\Gamma(\mu,\nu)} \boldsymbol{E}_{(\mathbf{a},\mathbf{a}')\sim\pi} \left\|\mathbf{a}-\mathbf{a}'\right\|^q \right)^{1/q}, \tag{3}$$

where $q = 1, 2$.

Problem

Given a true probability measure μ on $\mathbb{R}^d$ we are interested in solving the following optimization problem,

$$\min_{\nu\in\Delta(\mathbb{R}^d)} W_q(\mu,\nu), \tag{4}$$

where $\Delta(\mathbb{R}^d)$ is the set of all probability measures on $\mathbb{R}^d$ and q is often selected as 1.

A way to model complex distributions: The push-forward measure

- Traditionally, we use analytical distributions: Restricts what we could model in real applications.
- Now, we use more expressive probability measures via *push-forward measures* with neural networks

Definition

- Let $\omega \sim \mathsf{p}_\Omega$ be a random variable.
- $h_{\mathbf{x}}(\cdot) : \mathbb{R}^p \to \mathbb{R}^m$ a function parameterized by parameters $\mathbf{x}$.

The pushforward measure of p_Ω under $h_{\mathbf{x}}$, denoted by $h_{\mathbf{x}}\#\mathsf{p}_\Omega$ is the distribution of $h_{\mathbf{x}}(\omega)$.

Example: Chi-square distribution

Let $\omega \sim \mathsf{p}_\Omega := \mathcal{N}(0,1)$ be the normal distribution. Let $h_x : \mathbb{R} \to \mathbb{R}$, $h_x(\omega) = w^x$. Let us fix $x = 2$. Then, $h\#\mathsf{p}_\Omega$ is the chi-square distribution with one degree of freedom.

Explanation: Change of variables.

Assume that $h : \mathbb{R}^n \to \mathbb{R}^n$ is monotonic. Given the random variable $\omega \sim \mathsf{p}_\Omega$ with probability density function $\mathsf{p}_\Omega(\omega)$, the density $\mathsf{p}_Y(\mathbf{y})$ of $\mathbf{y} = h_{\mathbf{x}}(\omega)$ reads

$$\mathsf{p}_Y(\mathbf{y}) = \mathsf{p}_\Omega(h_{\mathbf{x}}^{-1}(\mathbf{y}))\det\left(\mathbf{J}_{\mathbf{y}} h_{\mathbf{x}}^{-1}(\mathbf{y})\right)$$

where det denotes the determinant operation.

Towards an optimization problem

Problem (Ideal parametric density estimator)

Given a true distribution $\mu^\natural$, we can solve the following optimization problem,

$$\min_{\mathbf{x}} W_1(\mu^\natural, h_{\mathbf{x}} \# p_\Omega), \tag{5}$$

where the measurable function $h_{\mathbf{x}}$ is parameterized by $\mathbf{x}$ and $\omega \sim p_\Omega$ is "simple."

- Issues:
  - We only have access to empirical samples $\hat{\mu}_n$ of $\mu^\natural$.
  - W_1 is non-smooth.
  - $h_{\mathbf{x}}$ is non-convex.

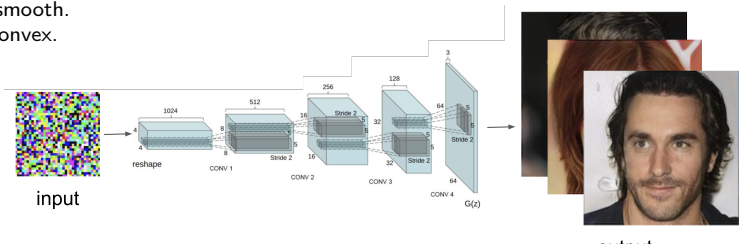
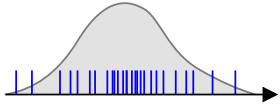


Figure: Schematic of a generative model, $h_{\mathbf{x}} \# \omega$ [16, 24].

Learning without concentration

- We can minimize $W_1\left(\hat{\mu}_n, h_{\mathbf{x}}\#\mathsf{p}_\Omega\right)$ with respect to $\mathbf{x}$.
- Figure: Empirical distribution (blue), $\hat{\mu}_n = \sum_{i=1}^n \delta_i$



A plug-in empirical estimator

Using the triangle inequality for Wasserstein distances we can upper bound in the follow way,

$$W_1(\mu^\natural, h_{\mathbf{x}}\#\mathsf{p}_\Omega) \leq W_1(\mu^\natural, \hat{\mu}_n) + W_1(\hat{\mu}_n, h_{\mathbf{x}}\#\mathsf{p}_\Omega), \tag{6}$$

where $\hat{\mu}_n$ is the empirical estimator of $\mu^\natural$ obtained from n independent samples from $\mu^\natural$.

Theorem (Slow convergence of empirical measures in 1-Wasserstein [52, 10])

Let $\mu^\natural$ be a measure defined on $\mathbb{R}^p$ and let $\hat{\mu}_n$ be its empirical measure. Then the $\hat{\mu}_n$ converges, in the worst case, at the following rate,

$$W_1(\mu^\natural, \hat{\mu}_n) \gtrsim n^{-1/p}. \tag{7}$$

Remarks:

- Using an empirical estimator in high-dimensions is terrible in the worst case.
- However, it does not directly say that $W_1\left(\mu^\natural, h_{\mathbf{x}}\#\mathsf{p}_\Omega\right)$ will be large.
- So we can still proceed and hope our parameterization interpolates harmlessly.

Duality of 1-Wasserstein

- How do we get a sub-gradient of $W_1\left(\hat{\mu}_n, h_{\mathbf{x}} \# \mathsf{p}_\Omega\right)$ with respect to $\mathbf{x}$?

Theorem (Kantorovich-Rubinstein duality)

$$W_1(\mu, \nu) = \sup_{\mathtt{d}} \{\langle \mathtt{d}, \mu \rangle - \langle \mathtt{d}, \nu \rangle : \mathtt{d} \textit{ is 1-Lipschitz}\} \tag{8}$$

Remark: - $\mathtt{d}$ is the "dual" variable. In the literature, it is commonly referred to as the "discriminator."

Inner product is an expectation

$$\langle \mathtt{d}, \mu \rangle = \int \mathtt{d} \mathrm{d}\mu = \int \mathtt{d}(\mathbf{a}) \mathrm{d}\mu(\mathbf{a}) = \boldsymbol{E}_{\mathbf{a} \sim \mu} \left[\mathtt{d}(\mathbf{a})\right]. \tag{9}$$

Kantorovich-Rubinstein duality applied to our objective

$$W_1\left(\hat{\mu}_n, h_{\mathbf{x}} \# \omega\right) = \sup \left\{ \boldsymbol{E}_{\mathbf{a} \sim \hat{\mu}_n} [\mathtt{d}(\mathbf{a})] - \boldsymbol{E}_{\mathbf{a} \sim h_{\mathbf{x}} \# \omega} [\mathtt{d}(\mathbf{a})] : \mathtt{d} \text{ is 1-Lipschitz} \right\} \tag{10}$$

Wasserstein GANs formulation

- Ingredients:
  - fixed *noise* distribution p_Ω (e.g., normal)
  - target distribution $\hat{\mu}_n$ (natural images)
  - $\mathcal{X}$ parameter class inducing a class of functions (generators)
  - $\mathcal{Y}$ parameter class inducing a class of functions (dual variables)

Wasserstein GANs formulation [1]

Define a parameterized function $\mathsf{d}_{\mathbf{y}}(\mathbf{a})$, where $\mathbf{y} \in \mathcal{Y}$ such that $\mathsf{d}_{\mathbf{y}}(\mathbf{a})$ is 1-Lipschitz. In this case, the Wasserstein GAN optimization problem is given by

$$\min_{\mathbf{x}\in\mathcal{X}} \left(\max_{\mathbf{y}\in\mathcal{Y}} \boldsymbol{E}_{\mathbf{a}\sim\hat{\mu}_n} \left[\mathsf{d}_{\mathbf{y}}(\mathbf{a})\right] - \boldsymbol{E}_{\boldsymbol{\omega}\sim\mathsf{p}_\Omega} \left[\mathsf{d}_{\mathbf{y}}(h_{\mathbf{x}}(\boldsymbol{\omega}))\right] \right) \tag{11}$$

Obtaining a stochastic sub-gradient with respect to $\mathbf{x}$

- Recall Danskin's theorem
- For fixed $\mathbf{x}$, we obtain an optimal solution $\mathbf{y}^\star$ for the inner problem, e.g., with gradient ascent.
- Then, we can use the (sub)gradient for $\mathbf{x}$ at $(\mathbf{x}, \mathbf{y}^\star)$ in the outer problem.

General diagram of GANs

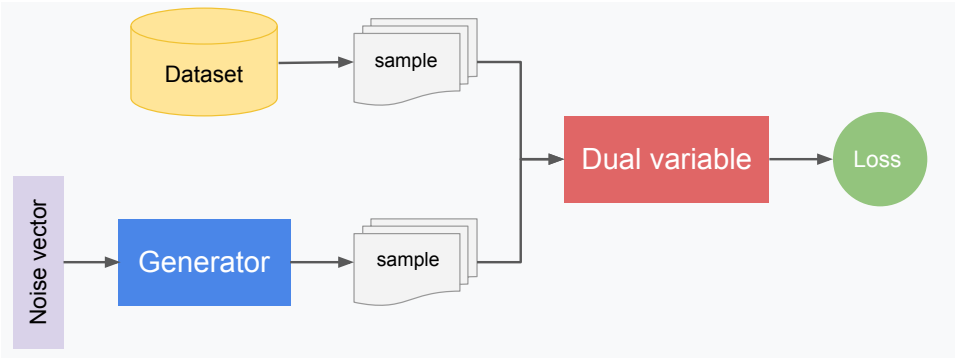


Figure: Generator/dual variable/dataset relation in GANs

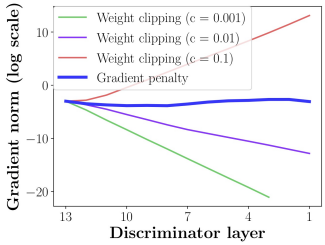
The theory-practice gap: Enforcing 1-Lipschitz of the discriminator

Weight clipping [1]

The "dual" or the "discriminator" $\mathrm{d}_{\mathbf{y}}$ weights $\mathbf{y}$ are constrained by an ℓ_∞-ball with radius $c > 0$, denoted as $\mathcal{B}$, at every iteration with

$$\pi_{\mathcal{B}}(\mathbf{y}) = \mathrm{clip}(\mathbf{y}, [-c, c]). \tag{12}$$

This trick is used to pseudo-enforce the constraint.



Remark: ○ *"Weight clipping is a clearly terrible way to enforce a Lipschitz constraint"* – original authors.

Gradient penalty [18]

Recall that 1-Lipschitz is equivalent to $\|\nabla_{\mathbf{a}} \mathrm{d}_{\mathbf{y}}(\mathbf{a})\|_* \leq 1$. This can be enforced directly through

$$\boldsymbol{E}_{\mathbf{a}\sim\hat{\mu}_n}\left[\mathrm{d}_{\mathbf{y}}(\mathbf{a})\right] - \boldsymbol{E}_{\boldsymbol{\omega}\sim\Omega}\left[\mathrm{d}_{\mathbf{y}}(h_{\mathbf{x}}(\boldsymbol{\omega}))\right] + \lambda \boldsymbol{E}_{\mathbf{a}\sim\nu}\left[\left(\|\nabla_{\mathbf{a}} \mathrm{d}_{\mathbf{y}}(\mathbf{a})\|_* - 1\right)^2\right]. \tag{13}$$

Remarks: ○ In practice the distribution ν mimicks uniform (linearly interpolated) sampling as follows:

$$\mathbf{a} \sim \mathrm{Uniform}(\mathbf{a}_i, h_{\mathbf{x}}(\boldsymbol{\omega}_i)).$$

○ Spectral normalization: Divide each weight matrix by their spectral norm [38].

Practical implementation of GANs

Stochastic training of Wasserstein GANs

Input: primal and "dual" learning rates γ_t and α_m, primal iterations T, "dual" network $\mathsf{d}_{\mathbf{y}}$, generator network $h_{\mathbf{x}}$, noise distribution p_Ω, real distribution $\hat{\mu}_n$, primal and dual batch sizes B, K, "dual" iterations M.

1. initialize $\mathbf{x}^0$
2. **For** $t = 0, 1, ..., T-1$:
   - **For** $m = 0, 1, ..., M-1$:
     - initialize $\mathbf{y}^0$,
     - draw noise sample $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_K \sim \mathsf{p}_\Omega$
     - draw real samples $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_K \sim \hat{\mu}_n$
     - "dual" pseudo-loss $L(\mathbf{y}) := K^{-1} \sum_{i=1}^K \mathsf{d}_{\mathbf{y}}(\boldsymbol{r}_i) - \mathsf{d}_{\mathbf{y}}(h_{\mathbf{x}^t}(\boldsymbol{\omega}_i))$
     - $\sharp$update "dual" parameters $\mathbf{y}^{m+1} = \mathbf{y}^m + \gamma_m \nabla_{\mathbf{y}} L(\mathbf{y}^m)$
     - $\sharp$enforce 1-Lipschitz constraint on $\mathsf{d}_{\mathbf{y}^{m+1}}$
   - **end-For**
   - draw noise sample $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_B \sim \mathsf{p}_\Omega$
   - generator pseudo-loss $L(\mathbf{x}) := -B^{-1} \sum_{i=1}^B \mathsf{d}_{\mathbf{y}^M}(h_{\mathbf{x}}(\boldsymbol{\omega}_i))$
   - update generator parameters $\mathbf{x}^{t+1} = \mathbf{x}^t - \alpha_t \nabla_{\mathbf{x}} L(\mathbf{x}^t)$
3. **end-For**

$\sharp$: Ideally, should be performed jointly.

Some historical background for a Turing award

Vanilla GAN [16]

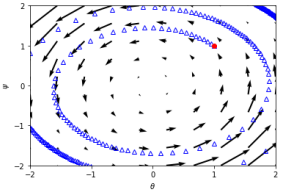
$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \boldsymbol{E}_{\mathbf{a}\sim\hat{\mu}_n} \left[\log \mathtt{d}_{\mathbf{y}}(\mathbf{a})\right] + \boldsymbol{E}_{\boldsymbol{\omega}\sim\mathtt{p}_\Omega} \left[\log\left(1-\mathtt{d}_{\mathbf{y}}(h_{\mathbf{x}}(\boldsymbol{\omega}))\right)\right] \tag{14}$$

- Binary cross-entropy modeling.
- $\mathtt{d}_{\mathbf{y}}(\mathbf{a}) : \mathcal{Y} \to [0,1]$ represents the probability that $\mathbf{a}$ came from the real data distribution $\mu^\natural$.

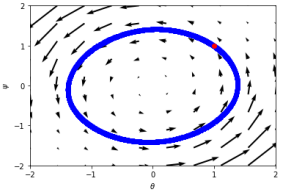
Observation: ○ Minimizes Jensen-Shannon divergence:

$$\mathrm{JSD}(\hat{\mu}_n \| h_{\mathbf{x}} \# \mathtt{p}_\Omega) = \frac{1}{2} D(\hat{\mu}_n \| h_{\mathbf{x}} \# \mathtt{p}_\Omega) + \frac{1}{2} D(h_{\mathbf{x}} \# \mathtt{p}_\Omega \| \hat{\mu}_n).$$

Difficulties of GAN training



(a) SimGD



(b) AltGD

Figure: Mode collapse (left). Simultaneous vs alternating generator/discriminator updates (right).

- Heuristics galore!
- Difficult to enforce 1-Lipschitz constraint
- Overall a difficult minimax problem: Scalability, mode collapse, periodic cycling...
- Privacy concerns due to memorization

Abstract minmax formulation

Minimax formulation

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{15}$$

where

- Φ is differentiable and nonconvex in $\mathbf{x}$ and nonconcave in $\mathbf{y}$,
- The domain is unconstrained, specifically $\mathcal{X}=\mathbb{R}^m$ and $\mathcal{Y}=\mathbb{R}^n$.

○ Key questions:

1. Where do the algorithms converge?
2. When do the algorithm converge?

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1. Where do the algorithms converge?
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A buffet of negative results [8]

"Even when the objective is a Lipschitz and smooth differentiable function, deciding whether a min-max point exists, in fact even deciding whether an approximate min-max point exists, is NP-hard. More importantly, an approximate local min-max point of large enough approximation is guaranteed to exist, but finding one such point is PPAD-complete. The same is true of computing an approximate fixed point of the (Projected) Gradient Descent/Ascent update dynamics."

Solution concept

- Like for nonconvex problems in minimization we try to find a *local* solution.

Definition (Local Nash Equilibrium)

A pure strategy $(\mathbf{x}^\star, \mathbf{y}^\star)$ is called a Local Nash Equilibrium (LNE) if,

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \leq \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \leq \Phi(\mathbf{x}, \mathbf{y}^\star) \tag{LNE}$$

for all $\mathbf{x}$ and $\mathbf{y}$ within some neighborhood of $\mathbf{x}^\star$ and $\mathbf{y}^\star$, i.e., $\|\mathbf{x} - \mathbf{x}^\star\| \leq \delta$ and $\|\mathbf{y} - \mathbf{y}^\star\| \leq \delta$ for some $\delta > 0$.

Necessary conditions

Through a Taylor expansion around $\mathbf{x}^\star$ and $\mathbf{y}^\star$ one can show that a LNE implies,

$$\nabla_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}), -\nabla_{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}) = 0$$
$$\nabla_{\mathbf{xx}} \Phi(\mathbf{x}, \mathbf{y}), -\nabla_{\mathbf{yy}} \Phi(\mathbf{x}, \mathbf{y}) \succeq 0$$

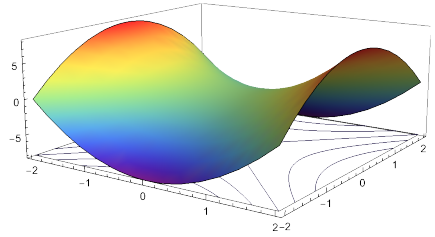

Figure: $\Phi(x, y) = x^2 - y^2$

Recall SGD results

$$\min_{\mathbf{x}:\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

- For a non-convex, smooth f, we have that

1. SGD converges to the critical points of f as $N \to \infty$.
2. SGD avoids strict saddles/traps ($\lambda_{\min}(\nabla^2 f(\mathbf{x}^*)) < 0$) almost surely.
3. SGD remains close to Hurwicz minimizers (i.e., $\mathbf{x}^* : \lambda_{\min}(\nabla^2 f(\mathbf{x}^*)) > 0$ almost surely.

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- Nail in the coffin:
  - not even sure if we obtain stochastic descent directions by approximately solving inner problems in GANs.
  - GANs are fundamentally different from adversarial training!
- Need more direct approaches with the stochastic gradient estimates.

Generalized Robbins-Monro schemes

- Given $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$, define $V(\mathbf{z}) = [-\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), \nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]$ with $\mathbf{z} = [\mathbf{x},\mathbf{y}]$.
- Given $V(\mathbf{z})$, define stochastic estimates of $V(\mathbf{z},\zeta) = V(\mathbf{z}) + U(\mathbf{z},\zeta)$, where
  - $U(\mathbf{z},\zeta)$ is a bias term
  - We often have unbiasedness: $\boldsymbol{E}U(\mathbf{z},\zeta) = 0$
  - The bias term can have bounded moments
  - We often have bounded variance: $P(\|U(\mathbf{z},\zeta)\| \geq t) \leq 2\exp -\frac{t^2}{2\sigma^2}$ for $\sigma > 0$.
- An abstract template for generalized Robbins-Monro schemes, dubbed as $\mathcal{A}$:

$$\mathbf{z}^{k+1} = \mathbf{z}^k + \alpha_k V(\mathbf{z}^k, \zeta^k)$$

The dessert section in the buffet of negative results: [21]

1. Bounded trajectories of $\mathcal{A}$ always converge to an internally chain-transitive (ICT) set.
2. Trajectories of $\mathcal{A}$ may converge with arbitrarily high probability to spurious attractors that contain no critical point of Φ.

Basic algorithms for minimax

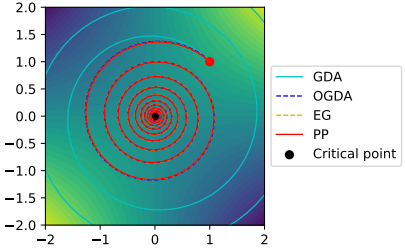

Figure: Trajectory of different algorithms for a simple bilinear game $\min_x \max_y xy$.

- (In)Famous algorithms
  - Gradient Descent Ascent (GDA)
  - Proximal point method (PPM)
  - Extra-gradient (EG)
  - Optimistic Gradient Descent Ascent (OGDA)
  - Reflected-Forward-Backward-Splitting (RFBS)

- EG and OGDA are approximations of the PPM
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha \mathbf{V}(\mathbf{z}^k)$.
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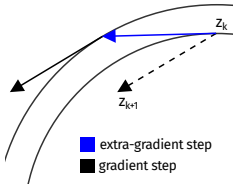
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha \mathbf{V}(\mathbf{z}^k - \alpha \mathbf{V}(\mathbf{z}^{k-1}))$
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha [2\mathbf{V}(\mathbf{z}^k) - \mathbf{V}(\mathbf{z}^{k-1})]$
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha \mathbf{V}(2\mathbf{z}^k - \mathbf{z}^{k-1})$

Extra-gradient algorithm (EG)[1]

EG method for saddle point problems

1. Choose $\mathbf{x}^0, \mathbf{y}^0$ and α.
2. For $k = 0, 1, \cdots$, perform:

$$\tilde{\mathbf{x}}^k := \mathbf{x}^k - \alpha \nabla_{\mathbf{x}} \Phi(\mathbf{x}^k, \mathbf{y}^k),$$
$$\tilde{\mathbf{y}}^k := \mathbf{y}^k + \alpha \nabla_{\mathbf{y}} \Phi(\mathbf{x}^k, \mathbf{y}^k).$$
$$\mathbf{x}^{k+1} := \mathbf{x}^k - \alpha \nabla_{\mathbf{x}} \Phi(\tilde{\mathbf{x}}^k, \tilde{\mathbf{y}}^k).$$
$$\mathbf{y}^{k+1} := \mathbf{y}^k + \alpha \nabla_{\mathbf{y}} \Phi(\tilde{\mathbf{x}}^k, \tilde{\mathbf{y}}^k).$$



- Idea: Predict the gradient at the next point

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha \mathbf{V}\Big(\underbrace{\mathbf{z}^k - \alpha \mathbf{V}(\mathbf{z}^k)}_{\text{prediction of } \mathbf{z}^{k+1}} \Big) \qquad \text{(EG)}$$

Remark: - 1-extra-gradient computation per iteration

[1] G.M. Korpelevic, "An extragradient method for finding saddle points and for other problems," Ekonomika i Matematischeskie Metody, vol. 12, no. 4, pp. 747-756, 1976.

ExtraAdam

ExtraAdam for GANs [15]

Input. Step size γ, exponential decay rates $\eta_1, \eta_2 \in [0, 1)$

1. Set $\mathbf{m}_0, \mathbf{v}_0 = 0$
2. For $k = 0, 1, \ldots,$ iterate

$$
\begin{cases}
\mathbf{g}_k & = V(\mathbf{z}^k, \zeta^k) \\
\mathbf{m}_{k-1/2} & = \eta_1 \mathbf{m}_{k-1} + (1-\eta_1)\mathbf{g}_k \leftarrow \text{1st order estimate} \\
\mathbf{v}_{k-1/2} & = \eta_2 \mathbf{v}_{k-1} + (1-\eta_2)\mathbf{g}_k^2 \leftarrow \text{2nd order estimate} \\
\hat{\mathbf{m}}_{k-1/2} & = \mathbf{m}_{k-1/2}/(1-\eta_1^k) \leftarrow \text{Bias correction} \\
\hat{\mathbf{v}}_{k-1/2} & = \mathbf{v}_{k-1/2}/(1-\eta_2^k) \leftarrow \text{Bias correction} \\
\mathbf{z}^{k+1/2} & = \mathbf{z}^k - \gamma \hat{\mathbf{m}}_{k-1/2}/(\sqrt{\hat{\mathbf{v}}_{k-1/2}} + \epsilon) \leftarrow \text{Extrapolation step} \\
\mathbf{g}_{k+1/2} & = V(\mathbf{z}^{k+1/2}, \zeta^{k+1/2}) \\
\mathbf{m}_k & = \eta_1 \mathbf{m}_{k-1/2} + (1-\eta_1)\mathbf{g}_{k+1/2} \leftarrow \text{1st order estimate} \\
\mathbf{v}_k & = \eta_2 \mathbf{v}_{k-1/2} + (1-\eta_2)\mathbf{g}_{k+1/2}^2 \leftarrow \text{2nd order estimate} \\
\hat{\mathbf{m}}_k & = \mathbf{m}_k/(1-\eta_1^k) \leftarrow \text{Bias correction} \\
\hat{\mathbf{v}}_k & = \mathbf{v}_k/(1-\eta_2^k) \leftarrow \text{Bias correction} \\
\mathbf{z}^{k+1} & = \mathbf{z}^k - \gamma \hat{\mathbf{m}}_k/(\sqrt{\hat{\mathbf{v}}_k} + \epsilon) \leftarrow \text{Update step}
\end{cases}
$$

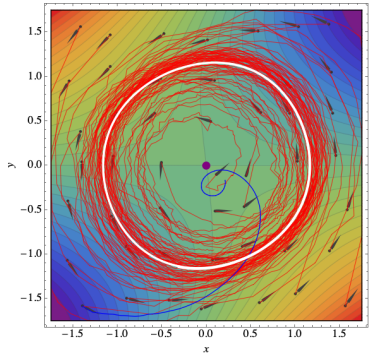
Output : $\mathbf{z}^k$

Minimax is more difficult than just optimization [21]

- Internally chain-transitive (ICT) sets characterize the convergence of dynamical systems [3].
  - For optimization, {attracting ICT} $\equiv$ {solutions}
  - For minimax, {attracting ICT} $\equiv$ {solutions} $\cup$ {spurious sets}

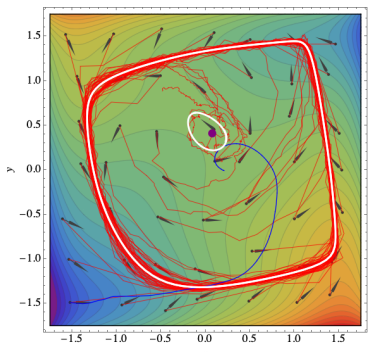
- "Almost" bilinear $\neq$ bilinear:

$$\Phi(x,y) = xy + \epsilon\phi(x), \phi(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$$



- The "forsaken" solutions:

$$\Phi(y,x) = y(x-0.5)+\phi(y)-\phi(x), \phi(u) = \frac{1}{4}u^2-\frac{1}{2}u^4+\frac{1}{6}u^6$$



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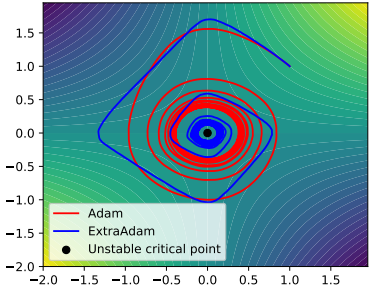

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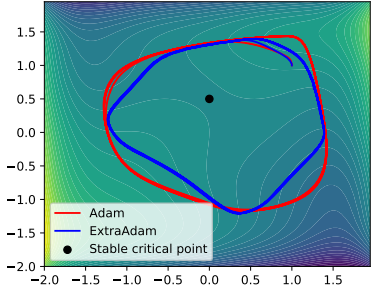


Comparison of convergence rates for smooth convex-concave minimax

Method	Assumption on $\Phi(\cdot,\cdot)$	Convergence rate	Reference	Note
PP	convex-concave	$\mathcal{O}\left(\epsilon^{-1}\right)$	[44]	
PP	strongly convex- strongly concave	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[44]	Implicit algorithm
PP	Bilinear	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[44]	
EG	convex-concave	$\mathcal{O}\left(\epsilon^{-1}\right)$	[39]	
EG	strongly convex- strongly concave	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[40, 39]	1 extra-gradient evaluation per iteration
EG	Bilinear	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[40, 39]	
OGDA	convex-concave	$\mathcal{O}\left(\epsilon^{-1}\right)$	[39]	
OGDA	strongly convex- strongly concave	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[40, 39]	no obvious downside
OGDA	Bilinear	$\mathcal{O}\left(\kappa \log(\epsilon^{-1})\right)$	[40, 39]	

An abridged collection of algorithms for convex-concave minimax optimization

A restricted minimax formulation

Let us consider

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{16}$$

where $\Phi(\mathbf{x},\mathbf{y})$ is convex in $\mathbf{x}$ and concave in $\mathbf{y}$.

1. $\mathcal{X}\subset\mathbb{R}^p$ and $\mathcal{Y}\subset\mathbb{R}^n$; and $\Phi(\mathbf{x},\mathbf{y})$ is smooth, or bilinear, or strongly convex/strongly concave
   - Algorithms: Proximal-Point [44], Extra-gradient [27, 40, 39], OGDA [40, 39]
2. $\mathcal{X}\subset\mathbb{R}^p$ and $\mathcal{Y}\subset\mathbb{R}^n$ with tractable "mirror maps"; and $\Phi(\mathbf{x},\mathbf{y})$ is smooth and continuously differentiable
   - Algorithm: Mirror-Prox [41]
3. $\mathcal{X}=\mathbb{R}^p$ and $\mathcal{Y}=\mathbb{R}^n$; and $\Phi(\mathbf{x},\mathbf{y})=h(\mathbf{x})+f(\mathbf{x})+\langle\mathbf{A}\mathbf{x},\mathbf{y}\rangle-g^*(\mathbf{y})$
   - Algorithms: Chambolle-Pock [5], Condat-Vu [6, 51], PD3O [55]

Between convex-concave and nonconvex-nonconcave

Nonconvex-concave problems

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$$

- $\Phi(\mathbf{x},\mathbf{y})$ is nonconvex in $\mathbf{x}$, concave in $\mathbf{y}$, smooth in $\mathbf{x}$ and $\mathbf{y}$.

Recall

Define $f(\mathbf{x}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$.

- Gradient descent applied to nonconvex f requires $\mathcal{O}(\epsilon^{-2})$ iterations to give an ϵ-stationary point.
- (Sub)gradient of f can be computed using Danskin's theorem:

$$\nabla_{\mathbf{x}}\Phi(\cdot, y^\star(\cdot)) \in \partial f(\cdot), \text{ where } y^\star(\cdot) \in \arg\max_{\mathbf{y}\in\mathcal{Y}} \Phi(\cdot,\mathbf{y}),$$

which is tractable since Φ is concave in $\mathbf{y}$ [33].

Remark: ○ "Conceptually" much easier than nonconvex-nonconcave case.

A summary of results for nonconvex-concave setting

○ A summary of gradient complexities to reach $\epsilon-$first order stationary point in terms of **gradient mapping.**

Method	Assumption on $\Phi(\cdot,\cdot)$	Convergence rate	Reference
GDA	noconvex-concave	$\tilde{\mathcal{O}}\left(\epsilon^{-6}\right)$	[33]
GDA	nonconvex- strongly concave	$\mathcal{O}\left(\epsilon^{-2}\right)$	[33]
GDmax	nonconvex-concave	$\tilde{\mathcal{O}}\left(\epsilon^{-6}\right)$	[22]
GDmax	nonconvex- strongly concave	$\mathcal{O}\left(\epsilon^{-2}\right)$	[22]
HiBSA, AGP	nonconvex-concave	$\tilde{\mathcal{O}}\left(\epsilon^{-4}\right)$	[35], [54]
HiBSA, AGP	nonconvex- strongly concave	$\mathcal{O}\left(\epsilon^{-2}\right)$	[35], [54]

The elephant in the room: Nonsmooth, nonconvex optimization

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})$$

- Finding a stationary point of nonsmooth nonconvex minimization problems are hard [57]
  - A traditional $\epsilon-$stationarity can not be obtained in finite time
- Even the relax notions are hard [46]
- Really puzzling how deep learning approaches with ReLu etc. work...

Detour: Weak convexity (WeCo) & approximate stationarity[1]

- Smooth: Gradient mapping norm
  - $\|G_\alpha(\mathbf{x}^k)\|^2 = \frac{1}{\alpha^2}\|x^k - \pi_{\mathcal{X}}(\mathbf{x}^k - \alpha\nabla f(\mathbf{x}^k))\|^2$
  - possible to compute
- Non-smooth: Generalized subdifferential distance
  - $\text{dist}(0, \partial(f(\mathbf{x}^k) + \delta_{\mathcal{X}}(\mathbf{x}^k)))^2$
  - hard in general (even approximately)[2][3]

- f is ρ-weakly convex if $f(\mathbf{x}) + \frac{\rho}{2}\|\mathbf{x}\|^2$ is convex.

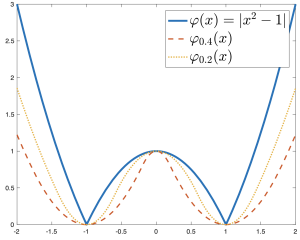


Figure: ME with $f(x) = |x^2 - 1|$, $\mathcal{X} = \mathbb{R}$, and $\hat{v}_t = \mathbb{I}$.[1]

- Moreau envelope (ME):

$$\varphi_{1/\rho}(\mathbf{x}) = \min_{y\in\mathcal{X}} \left\{ f(\mathbf{y}) + \frac{\rho}{2}\|\mathbf{y} - \mathbf{x}\|^2 \right\}$$

$$\hat{\mathbf{x}} \leftarrow \arg\min$$

$$\nabla\varphi_{1/\rho}(x) = \rho(\mathbf{x} - \hat{\mathbf{x}})$$

- Small $\|\nabla\phi_{1/\rho}(\mathbf{x})\|$ implies near-stationarity:[1]

$$\text{dist}(0, \partial(f(\mathbf{x}^k) + \delta_{\mathcal{X}}(\mathbf{x}^k)))^2 \leq \|\nabla\phi_{1/\rho}(\mathbf{x}^k)\|^2$$

  - also implies small $\|G_\alpha(\mathbf{x}^k)\|^2$ if f is smooth

[1] D. Davis and D. Drusvyatskiy, "Stochastic model-based minimization of weakly convex functions," SIOPT, 2019.

[3] J. Zhang, et al., "On complexity of finding stationary points of nonsmooth nonconvex functions," arXiv:2002.04130, 2020.

[3] O. Shamir, "Can We Find Near-Approximately-Stationary Points of Nonsmooth Nonconvex Functions?" arXiv:2002.11962, 2020.

Lifting minimax optimization: From pure to mixed Nash equilibrium

- Rethinking minimax problem as pure strategy game formulation

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$$

- A corresponding **mixed** strategy formulation

$$\min_{p\in\mathcal{M}(\mathcal{X})} \max_{q\in\mathcal{M}(\mathcal{Y})} \mathbb{E}_{\mathbf{x}\sim p}\mathbb{E}_{\mathbf{y}\sim q}\left[\Phi(\mathbf{x},\mathbf{y})\right]$$

  - $\mathcal{M}(\mathcal{Z}) \coloneqq \{\text{all randomized strategies on } \mathcal{Z}\}$
  - In the sequel, a way to solve this infinite dimensional problem: Mirror descent + Langevin dynamics [20]

From pure to mixed Nash equilibrium

- Key ingredients
  - $\langle p, h \rangle \coloneqq \int h \, \mathrm{d}p$ for a measure p and function h (Riesz representation)
  - the linear operator G and its adjoint $G^\dagger$:

$$\begin{aligned} (Gq)(\mathbf{x}) &\coloneqq \mathbb{E}_{\mathbf{y}\sim q}\left[\Phi(\mathbf{x}, \mathbf{y})\right] \\ (G^\dagger p)(\mathbf{y}) &\coloneqq \mathbb{E}_{\mathbf{x}\sim p}\left[\Phi(\mathbf{x}, \mathbf{y})\right], \end{aligned}$$

- Mixed NE formulation $\simeq$ finite two-player games

$$\min_{p\in\mathcal{M}(\mathcal{X})} \max_{q\in\mathcal{M}(\mathcal{Y})} \mathbb{E}_{\mathbf{x}\sim p}\mathbb{E}_{\mathbf{y}\sim q}\left[\Phi(\mathbf{x}, \mathbf{y})\right]$$

$$\Updownarrow$$

$$\min_{p\in\mathcal{M}(\mathcal{X})} \max_{q\in\mathcal{M}(\mathcal{Y})} \langle p, Gq \rangle$$

  - If $\mathcal{X}$ and $\mathcal{Y}$ are finite $\Rightarrow$ mirror descent
  - Caveat: **infinite dimension!!!**

Mirror descent in infinite dimension

Theorem (Informal, Main result in [20])

For a learning rate η, a probability measure p, and an arbitrary function f, define

$$p_+ = \mathsf{MD}_\eta(p, f) \quad \equiv \quad \mathrm{d}p_+ = \frac{e^{-\eta f}\,\mathrm{d}p}{\int e^{-\eta f}\,\mathrm{d}p}.$$

Then (inf-dimensional) entropic mirror descent solves the mixed Nash equilibrium.

- **Issue:** Cannot update probability measures in practice...
  - But maybe we can sample by Langevin dynamics!

Langevin dynamics for inf-dimensional mirror descent

- Pick p_1, q_1 (say Gaussians), need to draw samples from:

$$\mathrm{d}p_2(\mathbf{x}) \propto e^{-\eta f(\mathbf{x})}\mathrm{d}p_1, \quad f(\mathbf{x}) = \mathbb{E}_{\mathbf{y}\sim q_1}[\Phi(\mathbf{x},\mathbf{y})]$$

  - LD solves the problem with (stochastic) gradients of f:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma\tilde{\nabla} f(\mathbf{x}_k) + \sqrt{2\gamma}\mathcal{N}(0, I)$$

  - Query $\tilde{\nabla} f$ by drawing n samples $\{\mathbf{y}_i\}_{i=1}^n$ from q_1 and estimate

$$\tilde{\nabla} f(\mathbf{x}) = \nabla\left(\hat{\mathbb{E}}_{\mathbf{y}\sim q_1}[\Phi(\mathbf{x},\mathbf{y})]\right) \equiv \frac{1}{n}\sum_{i=1}^n \nabla\Phi(\mathbf{x},\mathbf{y}_i)$$

- Therefore,

$$\text{samples from } (p_1, q_1) \overset{\text{LD}}{\Rightarrow} \text{samples from } (p_2, q_2) \overset{\text{LD}}{\Rightarrow} \text{samples from } (p_3, q_3) \overset{\text{LD}}{\Rightarrow} \cdots$$

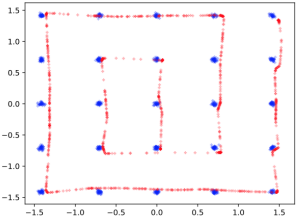
Langevin dynamics for inf-dimensional mirror descent

- With additional simplifications, we can write

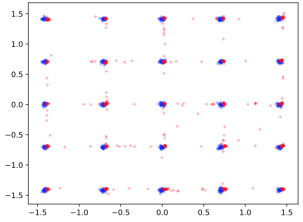
$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma_k \tilde{\nabla}_{\mathbf{x}} \Phi(\mathbf{x}_k, \mathbf{y}_k) + \sqrt{2\gamma_k}\mathcal{N}(0, I)$$
$$\mathbf{y}_{k+1} = \mathbf{y}_k + \gamma_k \tilde{\nabla}_{\mathbf{y}} \Phi(\mathbf{x}_k, \mathbf{y}_k) + \sqrt{2\gamma_k}\mathcal{N}(0, I) \qquad \text{(SGLD)}$$

- Per-iteration cost $\simeq$ SGDA

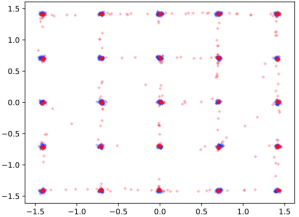
Application to 25 Gaussians: Algorithms matter [20]



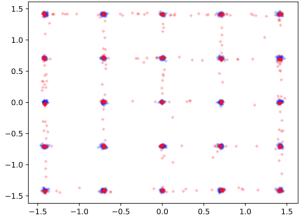
(a) SGD



(b) Adam

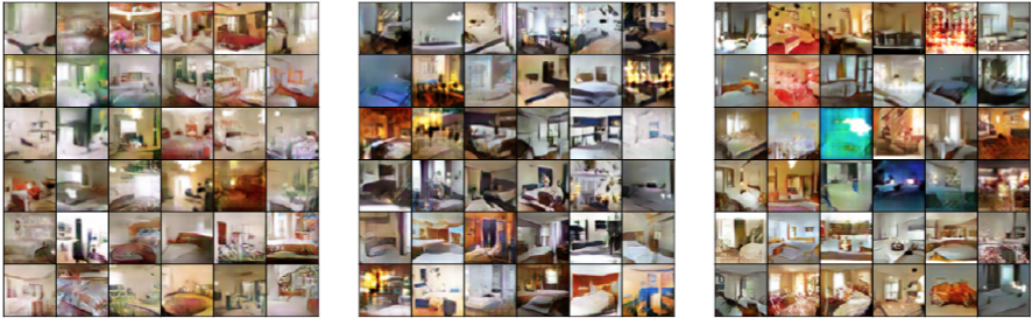


(c) Mirror-GAN



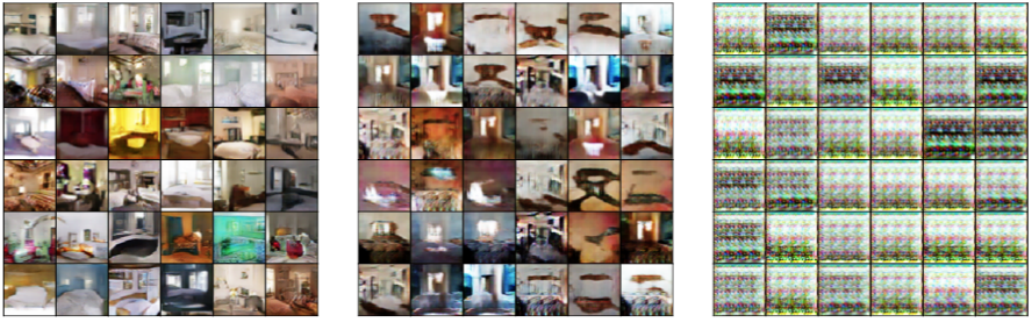
(d) Mirror-Prox-GAN

Real LSUN Dataset: RMSProp, $4 \times 10^4, 8 \times 10^4, \times 10^5$ iterations [20]



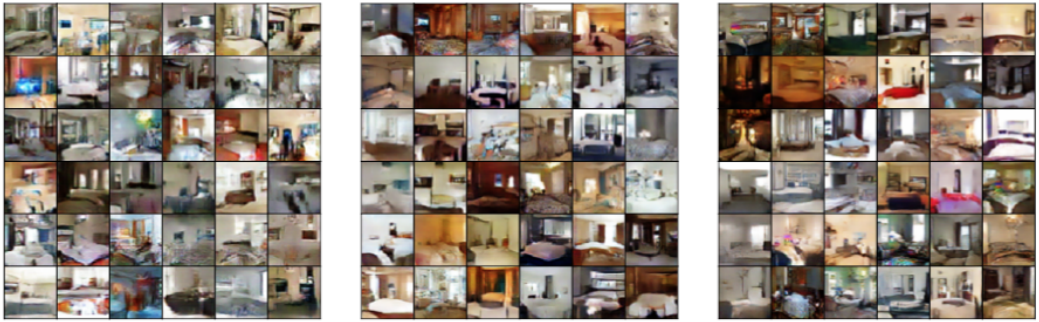
(a) RMSProp

Real LSUN Dataset: Adam, $4 \times 10^4, 8 \times 10^4, \times 10^5$ iterations [20]



(b) Adam

Real LSUN Dataset: Mirror-GAN, $4 \times 10^4, 8 \times 10^4, \times 10^5$ iterations [20]



(c) Mirror-GAN, **Algorithm 3**

Real LSUN Dataset: Extra-Adam, $4 \times 10^4, 8 \times 10^4, \times 10^5$ iterations [20]

(d) Simultaneous Extra-Adam

(e) Alternated Extra-Adam

Application: Noisy action robust reinforcement learning[1]

- Train RL agent in the presence of an adversary
- Adversary budget: $\alpha \in [0, 0.5]$

Noisy action robust MDP game

for $t = 1, 2, \ldots$ do:

both players observe state $S_t \in \mathcal{S}$

both players choose actions $A_t = \mu(S_t) \in \mathcal{A}$, and $A'_t = \nu(S_t) \in \mathcal{A}$

execute the noisy action $\bar{A}_t = (1 - \alpha)A_t + \alpha A'_t$

agent gets reward $R_{t+1} = R(S_t, \bar{A}_t)$, adversary gets $-R_{t+1}$

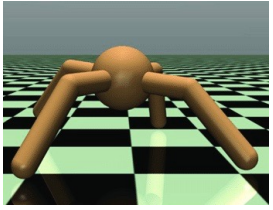
both players enter new state S_{t+1}

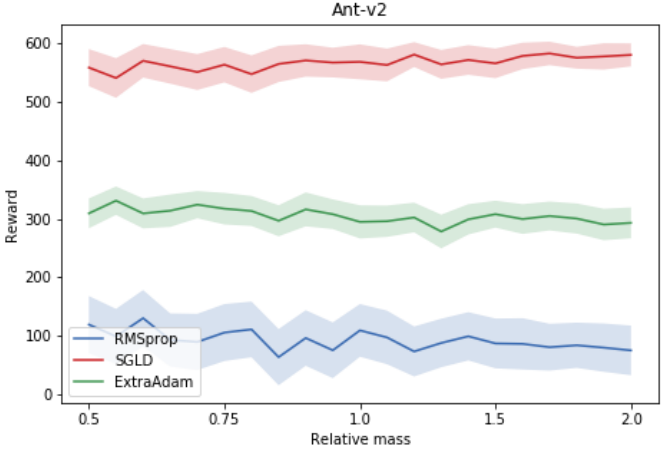
- Hope: Train in one environment, generalize to others

[1] K. Parameswaran, Y-T. Huang, Y-P. Hsieh, P. Rolland, C. Shi, and V. Cevher, "Robust Reinforcement Learning via Adversarial training with Langevin Dynamics" In NeurIPS, 2020.

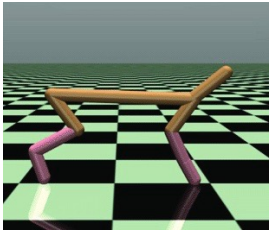
Experimental evaluation via MuJoCo

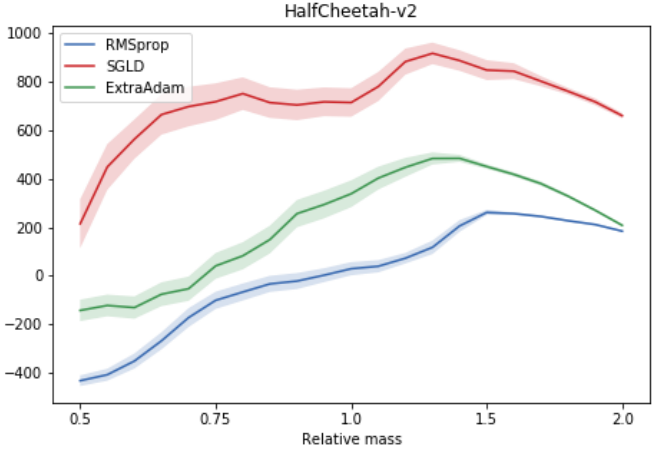
- Standard MuJoCo datasets





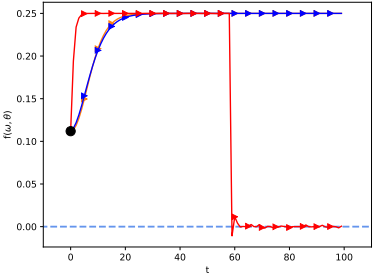
Experimental evaluation via MuJoCo

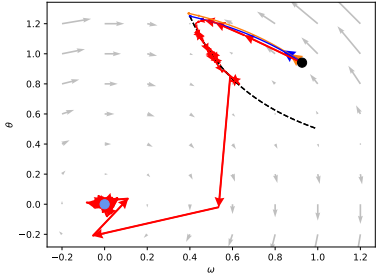


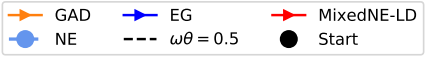


Simple Nash Equilibrium Test

$$\max_{\omega\in[-2,2]} \min_{\theta\in[-2,2]} -\omega^2\theta^2 + \omega\theta$$







Thank you for your attention!

- Minimax is more difficult than just optimization!
- Universal adaptation is an open research topic
- Interpretability, robustness, and adversarial training are all entangled...

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