Mathematics of Data: From Theory to Computation

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Lecture 11: Primal-dual optimization I

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Outline

- Today
  1. Min-max optimization (continued)
- Next week
  1. Algorithms for solving min-max optimization

A minimax optimization template

Minimax formulation

Consider the following problem that captures adversarial training, GANs, and robust reinforcement learning:

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{1}$$

where Φ is differentiable and nonconvex in $\mathbf{x}$ and nonconcave in $\mathbf{y}$.

◦ Key questions:

1. Where do the algorithms converge?
2. When do the algorithm converge?

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Recall: A buffet of negative results [5]

"Even when the objective is a Lipschitz and smooth differentiable function, deciding whether a min-max point exists, in fact even deciding whether an approximate min-max point exists, is NP-hard. More importantly, an approximate local min-max point of large enough approximation is guaranteed to exist, but finding one such point is PPAD-complete. The same is true of computing an approximate fixed point of the (Projected) Gradient Descent/Ascent update dynamics."

The difficulty of the nonconvex-nonconcave setting

Minimax formulation

Consider the following problem that captures adversarial training, GANs, and robust reinforcement learning:

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{2}$$

where Φ is differentiable and nonconvex in $\mathbf{x}$ and nonconcave in $\mathbf{y}$.

From minimax to minimization

Assume $\Phi(\mathbf{x},\mathbf{y}) = f(\mathbf{x})$ for all $\mathbf{y}$. The minimax optimization problem then seeks to find $\mathbf{x}^\star$ such that

$$f(\mathbf{x}^\star) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^p,$$

where $\mathbf{x}^\star$ is a global minimum of the nonconvex function f.

- Finding $\mathbf{x}^\star$ is NP-Hard even when f is smooth! (see the complexity supplementary material)
- Finding solutions to a nonconvex-nonconvex min-max problem is harder in general.

Question 1 with a twist: Where do want the algorithms to converge?

Definition (Saddle points & Local Nash equilibria)

The point $(\mathbf{x}^\star, \mathbf{y}^\star)$ is called a saddle-point or a local Nash equilibrium (LNE) if it holds that

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \le \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \le \Phi(\mathbf{x}, \mathbf{y}^\star) \qquad \text{(Saddle Point / LNE)}$$

for all $\mathbf{x}$ and $\mathbf{y}$ within some neighborhood of $\mathbf{x}^\star$ and $\mathbf{y}^\star$, i.e., $\|\mathbf{x} - \mathbf{x}^\star\| \le \delta$ and $\|\mathbf{y} - \mathbf{y}^\star\| \le \delta$ for some $\delta > 0$.

Necessary conditions

Through a Taylor expansion around $\mathbf{x}^\star$ and $\mathbf{y}^\star$ one can show that a LNE implies,

$$\nabla_{\mathbf{x}}\Phi(\mathbf{x}, \mathbf{y}), -\nabla_{\mathbf{y}}\Phi(\mathbf{x}, \mathbf{y}) = 0$$
$$\nabla_{\mathbf{xx}}\Phi(\mathbf{x}, \mathbf{y}), -\nabla_{\mathbf{yy}}\Phi(\mathbf{x}, \mathbf{y}) \succeq 0$$

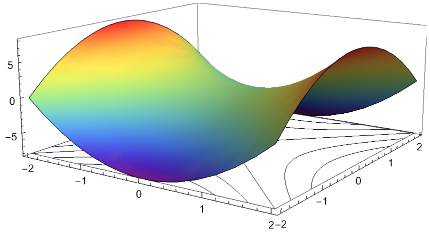


Figure: $\Phi(x, y) = x^2 - y^2$

Question 2 with a twist: When do generalized Robbins-Monro schemes converge?

- Given $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$, define $V(\mathbf{z}) = [-\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), \nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]$ with $\mathbf{z} = [\mathbf{x},\mathbf{y}]^\top$.
- Given $V(\mathbf{z})$, define stochastic estimates of $V(\mathbf{z},\zeta) = V(\mathbf{z}) + U(\mathbf{z},\zeta)$, where
  - $U(\mathbf{z},\zeta)$ is a bias term
  - We often have unbiasedness: $\boldsymbol{E}U(\mathbf{z},\zeta) = 0$
  - The bias term can have bounded moments
  - We often have bounded variance: $P(\|U(\mathbf{z},\zeta)\| \geq t) \leq 2\exp -\frac{t^2}{2\sigma^2}$ for $\sigma > 0$.
- An abstract template for generalized Robbins-Monro schemes, dubbed as $\mathcal{A}$:

$$\mathbf{z}^{k+1} = \mathbf{z}^k + \alpha_k V(\mathbf{z}^k, \zeta^k)$$

The dessert section in the buffet of negative results: [12]

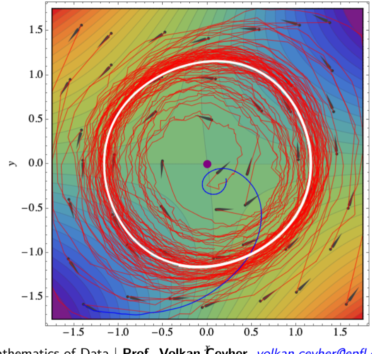
1. Bounded trajectories of $\mathcal{A}$ always converge to an internally chain-transitive (ICT) set.
2. Trajectories of $\mathcal{A}$ may converge with arbitrarily high probability to spurious attractors that contain no critical point of Φ.

Minimax is more difficult than just optimization [11]

- Internally chain-transitive (ICT) sets characterize the convergence of dynamical systems [4].
  - For optimization, {attracting ICT} $\equiv$ {solutions}
  - For minimax, {attracting ICT} $\equiv$ {solutions} $\cup$ {spurious sets}

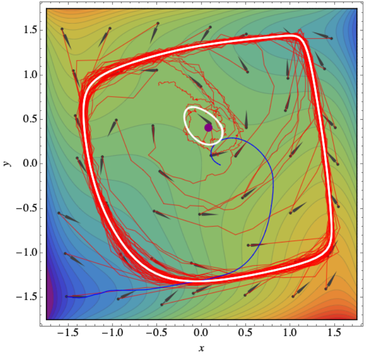
- "Almost" bilinear $\neq$ bilinear:

$$\Phi(x,y) = xy + \epsilon\phi(x), \phi(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$$



- The "forsaken" solutions:

$$\Phi(y,x) = y(x-0.5) + \phi(y) - \phi(x), \phi(u) = \frac{1}{4}u^2 - \frac{1}{2}u^4 + \frac{1}{6}u^6$$



A restricted minimax optimization template

A restricted minimax formulation

Consider the following problem

$$\min_{\mathbf{x}\in\mathcal{X}}\max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{3}$$

where Φ is convex in $\mathbf{x}$ and concave in $\mathbf{y}$.

∘ Key questions:

1. What problems does this template capture?
2. Where do the algorithms converge?
3. When do the algorithm converge?

General nonsmooth problems

◦ We will show that the restricted template captures the familiar composite minimization:

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}).$$

- f, g are convex, nonsmooth functions; and $\mathbf{A}$ is a linear operator.

Examples

- $g(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1$ or $g(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$.
- $g(\mathbf{A}\mathbf{x}) = \delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x})$, where $\delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}. \end{cases}$

Observations: ◦ The indicator example covers constrained problems, such as $\min_{\mathbf{x}\in\mathcal{X}}\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

◦ We need a tool, called Fenchel conjugation, to reveal the underlying minimax problem.

Conjugation of functions

◦ Idea: Represent a convex function in $\max$-form:

Definition

Let $\mathcal{Q}$ be a Euclidean space and $\mathcal{Q}^*$ be its dual space. Given a proper, closed and convex function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : \mathcal{Q}^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \mathrm{dom}(f)} \left\{ \mathbf{y}^T \mathbf{x} - f(\mathbf{x}) \right\}$$

is called the Fenchel conjugate (or conjugate) of f.

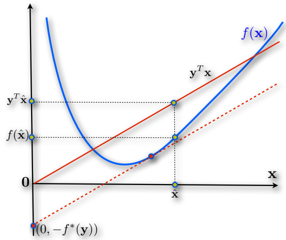


Figure: The conjugate function $f^*(\mathbf{y})$ is the maximum gap between the linear function $\mathbf{x}^T\mathbf{y}$ (red line) and $f(\mathbf{x})$.

Observations: ◦ $\mathbf{y}$: slope of the hyperplane

◦ $-f^*(\mathbf{y})$: intercept of the hyperplane

Conjugation of functions

Definition

Given a proper, closed and convex function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \to \mathbb{R} \cup \{+\infty\}$ such that

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is called the Fenchel conjugate (or conjugate) of f.

Properties

- ◦ f^* is a convex and lower semicontinuous function by construction as the supremum of affine functions of $\mathbf{y}$.
- ◦ The conjugate of the conjugate of a convex function f is the same function f; i.e., $f^{**} = f$ for $f \in \mathcal{F}(\mathcal{Q})$.
- ◦ The conjugate of the conjugate of a non-convex function f is its lower convex envelope when $\mathcal{Q}$ is compact:
  - $f^{**}(\mathbf{x}) = \sup\{g(\mathbf{x}) : g$ is convex and $g \leq f$, $\forall \mathbf{x} \in \mathcal{Q}\ \}$.
- ◦ For closed convex f, μ-strong convexity w.r.t. $\|\cdot\|$ is equivalent to $\frac{1}{\mu}$ smoothness of f^* w.r.t. $\|\cdot\|_*$.
  - Recall dual norm: $\|\mathbf{y}\|_* = \sup_{\mathbf{x}}\{\langle \mathbf{x}, \mathbf{y}\rangle : \|\mathbf{x}\| \leq 1\}$.
  - See for example Theorem 3 in [16].

Examples

ℓ_2-norm-squared

$f(\mathbf{x}) = \frac{\lambda}{2}\|\mathbf{x}\|^2 \Rightarrow f^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x}\rangle - \frac{\lambda}{2}\|\mathbf{x}\|^2$.

∘ Take the derivative and equate to 0: $0 = \mathbf{y} - \lambda\mathbf{x} \iff \mathbf{x} = \frac{1}{\lambda}\mathbf{y} \iff f^*(\mathbf{y}) = \frac{1}{\lambda}\|\mathbf{y}\|^2 - \frac{1}{2\lambda}\|\mathbf{y}\|^2 = \frac{1}{2\lambda}\|\mathbf{y}\|^2$.

ℓ_1-norm

$f(\mathbf{x}) = \lambda\|\mathbf{x}\|_1 \Rightarrow f^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x}\rangle - \lambda\|\mathbf{x}\|_1$.

∘ By definition of the ℓ_1-norm: $f^*(\mathbf{y}) = \max_{\mathbf{x}} \sum_{i=1}^n y_i x_i - \lambda|x_i| = \max_{\mathbf{x}} \sum_{i=1}^n y_i \mathrm{sign}(x_i)|x_i| - \lambda|x_i|$.

∘ By inspection:

- If all $|y_i| \leq \lambda$, then $\forall i, (y_i \mathrm{sign}(x_i) - \lambda)|x_i| \leq 0$. Taking $\mathbf{x} = 0$ gives the maximum value: $f^*(\mathbf{y}) = 0$.
- If for at least one $i, |y_i| > \lambda, (y_i \mathrm{sign}(x_i) - \lambda)|x_i| \to +\infty$ as $|x_i| \to +\infty$.

∘ $f^*(\mathbf{y}) = \delta_{\mathbf{y}:\|\cdot\|_\infty \leq \lambda}(\mathbf{y}) = \begin{cases} 0, & \text{if } \|\mathbf{y}\|_\infty \leq \lambda \\ +\infty, & \text{if } \|\mathbf{y}\|_\infty > \lambda \end{cases}$

Remark: ∘ See advanced material at the end for non-convex examples, such as $f(\mathbf{x}) = \|\mathbf{x}\|_0$.

General nonsmooth problems

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x})}$$

- By Fenchel-conjugation, we have $g(\mathbf{A}\mathbf{x}) = \max_{\mathbf{y}} \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})$, where g^* is the conjugate of g.
- Min-max formulation:

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}} \{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})\}}$$

An example with linear constraints

- If $g(\mathbf{A}\mathbf{x}) = \delta_{\{\mathbf{b}\}}(\mathbf{A}\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}, \end{cases}$

$$\Rightarrow g^*(\mathbf{y}) = \max_{\mathbf{x}} \langle \mathbf{y}, \mathbf{x}\rangle - \delta_{\{\mathbf{b}\}}(\mathbf{x}) = \max_{\mathbf{x}:\mathbf{x}=\mathbf{b}} \langle \mathbf{y}, \mathbf{x}\rangle = \langle \mathbf{y}, \mathbf{b}\rangle.$$

- We reach the minimax formulation (or the so-called “Lagrangian”) via conjugation:

$$\min_{\mathbf{x}} \{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\} = \min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}} \max_{\mathbf{y}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x} - \mathbf{b}, \mathbf{y}\rangle.$$

A special case in minimax optimization

Bilinear min-max template

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - h(\mathbf{y}),$$

where $\mathcal{X} \subseteq R^p$ and $\mathcal{Y} \subseteq \mathbb{R}^n$.

- $f : \mathcal{X} \to \mathbb{R}$ is convex.
- $h : \mathcal{Y} \to \mathbb{R}$ is convex.

Example: Sparse recovery

An example from sparseland $\mathbf{b} = \mathbf{A}\mathbf{x}^{\natural} + \mathbf{w}$: constrained formulation

The basis pursuit denoising (BPDN) formulation is given by

$$\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ \|\mathbf{x}\|_1 : \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \le \|\mathbf{w}\|_2, \|\mathbf{x}\|_\infty \le 1 \right\}. \tag{BPDN}$$

A **primal problem** prototype

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K}\ \mathbf{x} \in \mathcal{X} \right\},$$

The above template captures BPDN formulation with

- $f(\mathbf{x}) = \|\mathbf{x}\|_1$.
- $\mathcal{K} = \{\|\mathbf{u}\| \in \mathbb{R}^n : \|\mathbf{u}\| \le \|\mathbf{w}\|_2\}$.
- $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\|_\infty \le 1\}$.

An alternative formulation

A **primal problem** prototype

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K},\ \mathbf{x} \in \mathcal{X} \right\}, \tag{4}$$

- f is a proper, closed and convex function
- $\mathcal{X}$ and $\mathcal{K}$ are nonempty, closed convex sets
- $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{b} \in \mathbb{R}^n$ are known
- An optimal solution $\mathbf{x}^\star$ to (4) satisfies $f(\mathbf{x}^\star) = f^\star$, $\mathbf{A}\mathbf{x}^\star - \mathbf{b} \in \mathcal{K}$ and $\mathbf{x}^\star \in \mathcal{X}$

A simplified template without loss of generality

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \right\}, \tag{5}$$

- f is a proper, closed and convex function
- $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{b} \in \mathbb{R}^n$ are known
- An optimal solution $\mathbf{x}^\star$ to (5) satisfies $f(\mathbf{x}^\star) = f^\star$, $\mathbf{A}\mathbf{x}^\star = \mathbf{b}$

Reformulation between templates

A primal problem template

$$\min_{\mathbf{x}\in\mathbb{R}^p} \Big\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathcal{K}, \mathbf{x} \in \mathcal{X} \Big\}.$$

First step: Let $\mathbf{r}_1 = \mathbf{A}\mathbf{x} - \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{r}_2 = \mathbf{x} \in \mathbb{R}^p$.

$$\min_{\mathbf{x},\mathbf{r}_1,\mathbf{r}_2} \Big\{ f(\mathbf{x}) : \mathbf{r}_1 \in \mathcal{K}, \mathbf{r}_2 \in \mathcal{X}, \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{r}_1, \mathbf{x} = \mathbf{r}_2 \Big\}.$$

- Define $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \in \mathbb{R}^{2p+n}$, $\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times p} \\ \mathbf{I}_{p\times p} & \mathbf{0}_{p\times n} & -\mathbf{I}_{p\times p} \end{bmatrix}$, $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, $\bar{f}(\mathbf{z}) = f(\mathbf{x}) + \delta_{\mathcal{K}}(\mathbf{r}_1) + \delta_{\mathcal{X}}(\mathbf{r}_2)$,
  where $\delta_{\mathcal{X}}(\mathbf{x}) = 0$, if $\mathbf{x} \in \mathcal{X}$, and $\delta_{\mathcal{X}}(\mathbf{x}) = +\infty$, o/w.

The simplified template

$$\min_{\mathbf{z}\in\mathbb{R}^{2p+n}} \left\{ \bar{f}(\mathbf{z}) : \bar{\mathbf{A}}\mathbf{z} = \bar{\mathbf{b}} \right\}.$$

From constrained formulation back to minimax

A general template

$$\min_{\mathbf{x}\in\mathbb{R}^p} \{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

Other examples:

- Standard convex optimization formulations: *linear programming, convex quadratic programming, second order cone programming, semidefinite programming and geometric programming*.
- Reformulations of existing unconstrained problems via **convex splitting**: *composite convex minimization, consensus optimization, . . .*

Formulating as min-max

$$\max_{\mathbf{y}\in\mathbb{R}^n} \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle = \begin{cases} 0, & \text{if } \mathbf{A}\mathbf{x} = \mathbf{b}, \\ +\infty, & \text{if } \mathbf{A}\mathbf{x} \neq \mathbf{b}. \end{cases}$$

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) \colon \mathbf{A}\mathbf{x} = \mathbf{b} \right\} = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \left\{ \Phi(\mathbf{x}, \mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle \right\}}$$

Dual problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) \colon \mathbf{A}\mathbf{x} = \mathbf{b} \right\} = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \left\{ \Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle \right\}}$$

∘We define the dual problem

$$\max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) := \max_{\mathbf{y}\in\mathbb{R}^n} \{ \underbrace{\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle}_{d(\mathbf{y})} \}.$$

Concavity of dual problem

Even if $f(\mathbf{x})$ is not convex, $d(\mathbf{y})$ is concave:

- For each $\mathbf{x}$, $d(\mathbf{y})$ is linear; i.e., it is both convex and concave.
- Pointwise minimum of concave functions is still concave.

Remark: ∘ If we can exchange $\min$ and $\max$, we obtain a **concave** maximization problem.

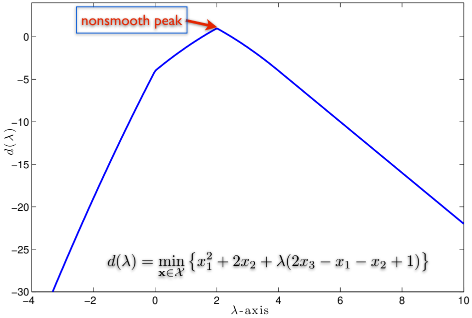
Example: Nonsmoothness of the dual function

◦ Consider a constrained convex problem:

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^3} \quad & \left\{ f(\mathbf{x}) := x_1^2 + 2x_2 \right\}, \\
\text{s.t.} \quad & 2x_3 - x_1 - x_2 = 1, \\
& \mathbf{x} \in \mathcal{X} := [-2,2]\times[-2,2]\times[0,2].
\end{aligned}
$$

◦ The **dual function** is concave and nonsmooth as written and then illustrated below.

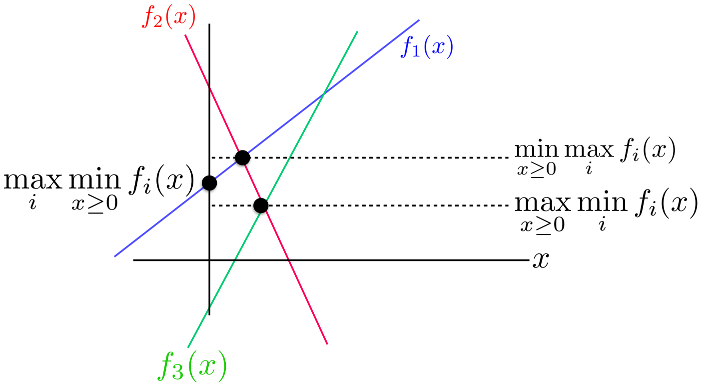
$$
d(\lambda) := \min_{\mathbf{x}\in\mathcal{X}} \left\{ x_1^2 + 2x_2 + \lambda(2x_3 - x_1 - x_2 - 1) \right\}
$$



Exchanging $\min$ and $\max$: A dangerous proposal

◦ Weak duality:

$$\underbrace{\max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y})}_{\text{Dual problem}} =: \boxed{\max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}\in\mathbb{R}^p} \Phi(\mathbf{x},\mathbf{y}) \leq \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \Phi(\mathbf{x},\mathbf{y})} = \underbrace{\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \right\}}_{\text{Primal problem}} = \begin{cases} f^\star, \text{ if } \mathbf{A}\mathbf{x} = \mathbf{b} \\ +\infty, \text{ if } \mathbf{A}\mathbf{x} \neq \mathbf{b} \end{cases}$$



A proof of weak duality

$$\boxed{f^\star := \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\right\} = \min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n}\left\{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\}}$$

∘ Since $\mathbf{A}\mathbf{x}^\star = \mathbf{b}$, it holds for any $\mathbf{y}$

$$\begin{aligned}\Phi(\mathbf{x}^\star,\mathbf{y}) = f^\star &= f(\mathbf{x}^\star) + \langle \mathbf{y}, \mathbf{A}\mathbf{x}^\star - \mathbf{b}\rangle \\ &\geq \min_{\mathbf{x}\in\mathbb{R}^p}\left\{f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b}\rangle\right\} \\ &= \min_{\mathbf{x}\in\mathbb{R}^p}\Phi(\mathbf{x},\mathbf{y}).\end{aligned}$$

∘ Take maximum of both sides in $\mathbf{y}$ and note that $f^\star$ is independent of $\mathbf{y}$:

$$f^\star = \min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n}\Phi(\mathbf{x},\mathbf{y}) \geq \max_{\mathbf{y}\in\mathbb{R}^n}\min_{\mathbf{x}\in\mathbb{R}^p}\Phi(\mathbf{x},\mathbf{y}) =: \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = d^\star.$$

Strong duality and saddle points

Strong duality

$$f^\star = f(\mathbf{x}^\star) = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \Phi(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}\in\mathbb{R}^p} \Phi(\mathbf{x},\mathbf{y}) =: \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = d^\star.$$

Under strong duality and assuming existence of $\mathbf{x}^\star$, $\Phi(\mathbf{x},\mathbf{y})$ has a saddle point. We have primal and dual optimal values coincide, i.e., $f^\star = d^\star$.

Strong duality and saddle points

Strong duality

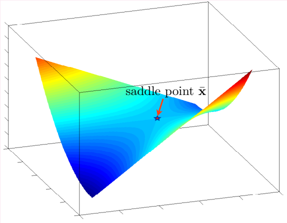
$$f^\star = f(\mathbf{x}^\star) = \min_{\mathbf{x}\in\mathbb{R}^p} \max_{\mathbf{y}\in\mathbb{R}^n} \Phi(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}\in\mathbb{R}^p} \Phi(\mathbf{x},\mathbf{y}) =: \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = d^\star.$$

Under strong duality and assuming existence of $\mathbf{x}^\star$, $\Phi(\mathbf{x},\mathbf{y})$ has a saddle point. We have primal and dual optimal values coincide, i.e., $f^\star = d^\star$.

Recall saddle point / LNE

A point $(\mathbf{x}^\star,\mathbf{y}^\star) \in \mathbb{R}^p \times \mathbb{R}^n$ is called a saddle point of Φ if

$$\Phi(\mathbf{x}^\star,\mathbf{y}) \le \Phi(\mathbf{x}^\star,\mathbf{y}^\star) \le \Phi(\mathbf{x},\mathbf{y}^\star),\ \forall \mathbf{x} \in \mathbb{R}^p,\ \mathbf{y} \in \mathbb{R}^n.$$



Toy example: Strong duality

Primal problem

- Consider the following primal minimization problem: $\min_{\mathbf{x}} P(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x}) := \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1$
- Using conjugation and strong duality

$$
\begin{aligned}
P(\mathbf{x}^\star) = \min_{\mathbf{x}} P(\mathbf{x}) &= \min_{\mathbf{x}} \max_{\mathbf{y}} f(\mathbf{x}) + \langle \mathbf{x}, \mathbf{y} \rangle - g^*(\mathbf{y}), && \text{by conjugation} \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) + \min_{\mathbf{x}} f(\mathbf{x}) + \langle \mathbf{x}, \mathbf{y} \rangle, && \text{by changing min-max} \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) - \max_{\mathbf{x}} \langle \mathbf{x}, -\mathbf{y} \rangle - f(\mathbf{x}), && \text{by } \min f = -\max -f \\
&= \max_{\mathbf{y}} -g^*(\mathbf{y}) - f^*(-\mathbf{y}), && \text{by conjugation.}
\end{aligned}
$$

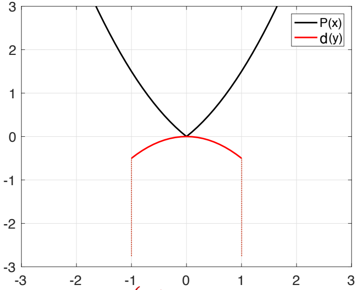
Dual problem

- Dual problem: $d^\star = \max_{\mathbf{y}} d(\mathbf{y}) = -g^*(\mathbf{y}) - f^*(-\mathbf{y})$
- Recall $f^*(-\mathbf{y}) = \frac{1}{2}\|\mathbf{y}\|^2$ and $g^*(\mathbf{y}) = \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \le 1}(\mathbf{y})$.

Toy example: Strong duality

$$\text{Primal problem:} \quad \min_{\mathbf{x}} P(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1$$

$$\text{Dual problem:} \quad \max_{\mathbf{y}} -\frac{1}{2}\|\mathbf{y}\|^2 - \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \leq 1}(\mathbf{y})$$



$$d(\mathbf{y}) = \begin{cases} -\frac{1}{2}\|\mathbf{y}\|^2, & \text{if } \|\mathbf{y}\|_\infty \leq 1 \\ -\infty, & \text{if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$

Back to convex-concave: Necessary and sufficient condition for strong duality

- Existence of a saddle point is not automatic even in convex-concave setting!
- Recall the minimax template:

$$\min_{\mathbf{x}\in\mathbb{R}^p}\max_{\mathbf{y}\in\mathbb{R}^n}\left\{\Phi(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x}-\mathbf{b}\rangle\right\}$$

Theorem (Necessary and sufficient optimality condition)

Under the Slater's condition: $\mathrm{relint}(\mathrm{dom}\, f) \cap \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset$, *strong duality holds, where the primal and dual problems are given by*

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{cases} \qquad \textit{and} \quad d^\star := \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}).$$

Remarks:

- By definition of $f^\star$ and $d^\star$, we always have $\boxed{d^\star \leq f^\star}$ (**weak duality**).
- If a primal solution exists and the Slater's condition holds, we have $\boxed{d^\star = f^\star}$ (**strong duality**).

Slater's qualification condition

- Denote $\mathrm{relint}(\mathrm{dom}\, f)$ the relative interior of the domain.
- The Slater condition requires

$$\boxed{\mathrm{relint}(\mathrm{dom}\, f) \cap \{\mathbf{x} \ : \ \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset.} \tag{6}$$

Special cases

- If $\mathrm{dom}\, f = \mathbb{R}^p$, then (6) $\Leftrightarrow$ $\boxed{\exists \bar{\mathbf{x}} \ : \ \mathbf{A}\bar{\mathbf{x}} = \mathbf{b}}$.
- If $\mathrm{dom}\, f = \mathbb{R}^p$ and instead of $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have the feasible set $\{\mathbf{x} : h(\mathbf{x}) \leq 0\}$, where h is $\mathbb{R}^p \to R^q$ is convex, then

$$(6) \Leftrightarrow \boxed{\exists \bar{\mathbf{x}} \ : \ h(\bar{\mathbf{x}}) < 0.}$$

Example: Slater's condition

Example

Let us consider solving $\min_{\mathbf{x}\in\mathcal{D}_\alpha} f(\mathbf{x})$ and so the feasible set is $\mathcal{D}_\alpha := \mathcal{X} \cap \mathcal{A}_\alpha$, where

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 \leq 1\}, \ \mathcal{A}_\alpha := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_1 + x_2 = \alpha\},$$

where $\alpha \in \mathbb{R}$.

Example: Slater's condition

Example

Let us consider solving $\min_{\mathbf{x}\in\mathcal{D}_\alpha} f(\mathbf{x})$ and so the feasible set is $\mathcal{D}_\alpha := \mathcal{X}\cap\mathcal{A}_\alpha$, where

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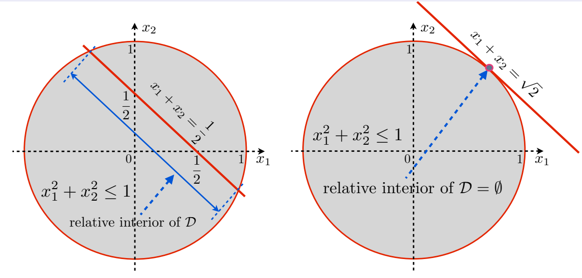
where $\alpha\in\mathbb{R}$.

Two cases where Slater's condition holds and does not hold



$\mathcal{D}_{1/2}$ satisfies Slater's condition – $\mathcal{D}_{\sqrt{2}}$-does not satisfy Slater's condition

Performance of optimization algorithms

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}, \right\}, \qquad \text{(Affine-Constrained)}$$

Exact vs. approximate solutions

- Computing an **exact solution** $\mathbf{x}^\star$ to (Affine-Constrained) is **impracticable**
- Algorithms seek $\mathbf{x}^\star_\epsilon$ that approximates $\mathbf{x}^\star$ up to ϵ in some sense

A performance metric: Time-to-reach ϵ

`time-to-reach` ϵ `= number of iterations to reach` ϵ $\times$ `per iteration time`

A key issue: Number of iterations to reach ϵ

The notion of ϵ-accuracy is elusive in constrained optimization!

Numerical ϵ-accuracy

- **Unconstrained case:** All iterates are feasible (no advantage from infeasibility)!

$$f(\mathbf{x}^\star_\epsilon) - f^\star \leq \epsilon$$

$$\boxed{f^\star = \min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})}$$

- **Constrained case:** We need to also measure the infeasibility of the iterates!

$$f^\star - f(\mathbf{x}^\star_\epsilon) \leq \epsilon \quad !!!$$

$$\boxed{f^\star = \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b} \right\}} \tag{7}$$

Our definition of ϵ-accurate solutions [20]

Given a numerical tolerance $\epsilon \geq 0$, a point $\mathbf{x}^\star_\epsilon \in \mathbb{R}^p$ is called an ϵ-solution of (7) if

$$\begin{cases} f(\mathbf{x}^\star_\epsilon) - f^\star & \leq \epsilon \text{ (objective residual)}, \\ \|\mathbf{A}\mathbf{x}^\star_\epsilon - \mathbf{b}\| & \leq \epsilon \text{ (feasibility gap)}, \end{cases}$$

- When $\mathbf{x}^\star$ is unique, we can also obtain $\|\mathbf{x}^\star_\epsilon - \mathbf{x}^\star\| \leq \epsilon$ (iterate residual).

Numerical ϵ-accuracy

Constrained problems

Given a numerical tolerance $\epsilon \geq 0$, a point $\mathbf{x}^\star_\epsilon \in \mathbb{R}^p$ is called an ϵ-solution of (7) if

$$\begin{cases} f(\mathbf{x}^\star_\epsilon) - f^\star & \leq \epsilon \text{ (objective residual)}, \\ \|\mathbf{A}\mathbf{x}^\star_\epsilon - \mathbf{b}\| & \leq \epsilon \text{ (feasibility gap)}, \end{cases}$$

- When $\mathbf{x}^\star$ is unique, we can also obtain $\|\mathbf{x}^\star_\epsilon - \mathbf{x}^\star\| \leq \epsilon$ (iterate residual).

General minimax problems

Since duality gap is 0 at the solution, we measure the primal-dual gap

$$\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\bar{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}) \leq \epsilon. \tag{8}$$

Remarks:

- ϵ can be different for the objective, feasibility gap, or the iterate residual.
- It is easy to show $\mathrm{Gap}(\mathbf{x}, \mathbf{y}) \geq 0$ and $\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 0$ iff $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a saddle point.

Primal-dual gap function for nonsmooth minimization

$$\boxed{\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) + g(\mathbf{A}\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \underbrace{f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})}_{\Phi(\mathbf{x},\mathbf{y})} = \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - g^*(\mathbf{y})}$$

- Primal problem: $\min_{\mathbf{x}\in\mathcal{X}} P(\mathbf{x})$ where

$$P(\mathbf{x}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y}).$$

- Dual problem: $\max_{\mathbf{y}\in\mathcal{Y}} d(\mathbf{y})$ where

$$d(\mathbf{y}) = \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \mathbf{y}).$$

- The primal-dual gap, i.e., $\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, is literally $(\text{primal value at } \bar{\mathbf{x}}) - (\text{dual value at } \bar{\mathbf{y}})$:

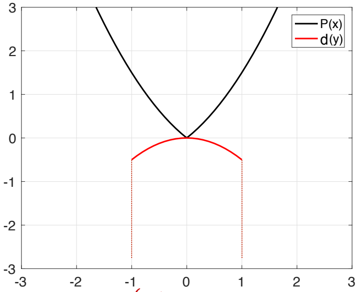
$$\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = P(\bar{\mathbf{x}}) - d(\bar{\mathbf{y}}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\bar{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}).$$

Toy example for nonnegativity of gap

- $P(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2 + \|\mathbf{x}\|_1$
- $d(\mathbf{y}) = -\frac{1}{2}\|\mathbf{y}\|^2 - \delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \leq 1}(\mathbf{y})$

Recall the indicator function

$$\delta_{\mathbf{y}:\|\mathbf{y}\|_\infty \leq 1}(\mathbf{y}) = \begin{cases} 0, \text{ if } \|\mathbf{y}\|_\infty \leq 1 \\ +\infty, \text{ if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$



$$d(\mathbf{y}) = \begin{cases} -\frac{1}{2}\|\mathbf{y}\|^2, & \text{if } \|\mathbf{y}\|_\infty \leq 1 \\ -\infty, & \text{if } \|\mathbf{y}\|_\infty > 1 \end{cases}$$

Primal-dual gap function in the general case

$$\boxed{\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}) = \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x},\mathbf{y})}$$

- Saddle point $(\mathbf{x}^\star, \mathbf{y}^\star)$ is such that $\forall \mathbf{x} \in \mathbb{R}^p$, $\forall \mathbf{y} \in \mathbb{R}^n$:

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \overset{(*)}{\leq} \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \overset{(**)}{\leq} \Phi(\mathbf{x}, \mathbf{y}^\star).$$

- Nonnegativity of Gap:

$$\begin{aligned}
\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) &= \max_{\mathbf{y}\in\mathcal{X}} \Phi(\bar{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}) \\
&\geq \Phi(\bar{\mathbf{x}}, \mathbf{y}^\star) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), \quad && \text{by the definition of maximization} \\
&\geq \Phi(\mathbf{x}^\star, \mathbf{y}^\star) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), \quad && \text{by the inequality } (**) \\
&\geq \Phi(\mathbf{x}^\star, \bar{\mathbf{y}}) - \min_{\mathbf{x}\in\mathcal{X}} \Phi(\mathbf{x}, \bar{\mathbf{y}}), \quad && \text{by the inequality } (*) \\
&\geq 0, && \text{by the definition of minimization.}
\end{aligned}$$

- If $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = (\mathbf{x}^\star, \mathbf{y}^\star)$, then all the inequalities will be equalities and $\mathrm{Gap}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = 0$.

Optimality conditions for minimax

Saddle point

We say $(\mathbf{x}^\star, \mathbf{y}^\star)$ is a primal-dual solution corresponding to primal and dual problems

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{cases} \quad \text{and} \quad d^\star := \max_{\mathbf{y}\in\mathbb{R}^n} d(\mathbf{y}) = \max_{\mathbf{y}\in\mathbb{R}^n} \min_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}).$$

if it is a saddle point of $\Phi(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \langle \mathbf{y}, \mathbf{A}\mathbf{x} - \mathbf{b} \rangle$:

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \leq \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \leq \Phi(\mathbf{x}, \mathbf{y}^\star), \ \forall \mathbf{x} \in \mathbb{R}^p, \ \mathbf{y} \in \mathbb{R}^n.$$

Karush-Khun-Tucker (KKT) conditions

Under our assumptions, an equivalent characterization of $(\mathbf{x}^\star, \mathbf{y}^\star)$ is via the KKT conditions of the problem

$$\min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b},$$

which reads

$$\begin{cases} 0 & \in \partial_{\mathbf{x}} \Phi(\mathbf{x}^\star, \mathbf{y}^\star) & = \mathbf{A}^T \mathbf{y}^\star + \partial f(\mathbf{x}^\star), \\ 0 & = \nabla_{\mathbf{y}} \Phi(\mathbf{x}^\star, \lambda^\star) & = \mathbf{A}\mathbf{x}^\star - \mathbf{b}. \end{cases}$$

A naive proposal: Gradient descent-ascent (GDA)

Towards algorithms for minimax optimization

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}).$$

We assume that

- $\Phi(\cdot,\mathbf{y})$ is convex,
- $\Phi(\mathbf{x},\cdot)$ is concave,
- Φ is smooth in the following sense:

$$\left\| \begin{bmatrix} \nabla_{\mathbf{x}}\Phi(\mathbf{x}_1,\mathbf{y}_1) \\ -\nabla_{\mathbf{y}}\Phi(\mathbf{x}_1,\mathbf{y}_1) \end{bmatrix} - \begin{bmatrix} \nabla_{\mathbf{x}}\Phi(\mathbf{x}_2,\mathbf{y}_2) \\ -\nabla_{\mathbf{y}}\Phi(\mathbf{x}_2,\mathbf{y}_2) \end{bmatrix} \right\| \leq L \left\| \begin{bmatrix} \mathbf{x}_1 - \mathbf{x}_2 \\ \mathbf{y}_1 - \mathbf{y}_2 \end{bmatrix} \right\|. \tag{9}$$

○ Let us try to use gradient descent for $\mathbf{x}$, gradient ascent for $\mathbf{y}$ to obtain a solution

GDA

1. Choose $\mathbf{x}^0, \mathbf{y}^0$ and τ.

2. For $k = 0, 1, \cdots$, perform:

$$\mathbf{x}^{k+1} := \mathbf{x}^k - \tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^k,\mathbf{y}^k).$$
$$\mathbf{y}^{k+1} := \mathbf{y}^k + \tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^k,\mathbf{y}^k).$$

GDA on a simple problem

Min-max problem

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}).$$

SimGDA

1. Choose $\mathbf{x}^0, \mathbf{y}^0$ and τ.
2. For $k = 0, 1, \cdots,$ perform:
   $\mathbf{x}^{k+1} := \mathbf{x}^k - \tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^k, \mathbf{y}^k)$.
   $\mathbf{y}^{k+1} := \mathbf{y}^k + \tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^k, \mathbf{y}^k)$.

AltGDA

1. Choose $\mathbf{x}^0, \mathbf{y}^0$ and τ.
2. For $k = 0, 1, \cdots,$ perform:
   $\mathbf{x}^{k+1} := \mathbf{x}^k - \tau\nabla_{\mathbf{x}}\Phi(\mathbf{x}^k, \mathbf{y}^k)$.
   $\mathbf{y}^{k+1} := \mathbf{y}^k + \tau\nabla_{\mathbf{y}}\Phi(\mathbf{x}^{k+1}, \mathbf{y}^k)$.

Example [9]

Let $\Phi(x,y) = xy$, $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, then,

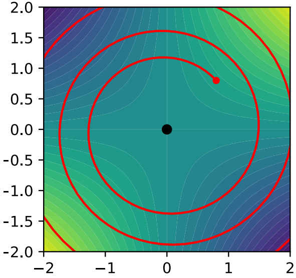
- for the iterates of SimGDA: $x_{k+1}^2 + y_{k+1}^2 = (1+\eta^2)(x_k^2 + y_k^2)$,
- for the iterates of AltGDA: $x_{k+1}^2 + y_{k+1}^2 = C(x_0^2 + y_0^2)$.

◦ SimGDA diverges and AltGDA does not converge!

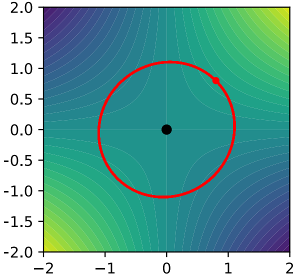
Practical performance

$$\min_{x \in \mathbb{R}} \max_{y \in \mathbb{R}} xy$$

◦ Simultaneous GDA



◦ Alternating GDA



Between convex-concave and nonconvex-nonconcave

Nonconvex-concave problems

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$$

∘ $\Phi(\mathbf{x},\mathbf{y})$ is nonconvex in $\mathbf{x}$, concave in $\mathbf{y}$, smooth in $\mathbf{x}$ and $\mathbf{y}$.

Recall

Define $f(\mathbf{x}) = \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$.

∘ Gradient descent applied to nonconvex f requires $\mathcal{O}(\epsilon^{-2})$ iterations to give an ϵ-stationary point.

∘ (Sub)gradient of f can be computed using Danskin's theorem:

$$\nabla_{\mathbf{x}}\Phi(\cdot, y^\star(\cdot)) \in \partial f(\cdot), \text{ where } y^\star(\cdot) \in \arg\max_{\mathbf{y}\in\mathcal{Y}} \Phi(\cdot,\mathbf{y}),$$

which is tractable since Φ is concave in $\mathbf{y}$ [17].

Remark: ∘ "Conceptually" much easier than nonconvex-nonconcave case.

Epilogue

	Gradient complexity	Optimality measure	Reference
convex-concave	$\mathcal{O}\left(\epsilon^{-1}\right)^1$	ϵ optimality w.r.t. duality gap	Nemirovski, 2004; Chambolle & Pock, 2011; Tran-Dinh & Cevher, 2014.[2]
nonconvex-concave	$\tilde{\mathcal{O}}\left(\epsilon^{-2.5}\right)^3$	ϵ-stationarity w.r.t. gradient mapping norm	Lin, Jin, & Jordan, 2020.[4]
nonconvex-nonconcave	HARD	HARD	Daskalakis, Stratis, & Zampetakis, 2020; Hsieh, Mertikopoulos, & Cevher, 2020.[5]

[1] Rates are not directly comparable as duality gap and gradient mapping norm are not necessarily of the same order!

[2] Arkadi Nemirovski,"Prox-method with rate of convergence $\mathcal{O}1/t)$ for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems." SIAM Journal on Optimization 15.1 (2004): 229-251.

Antonin Chambolle, and Thomas Pock, "A first-order primal-dual algorithm for convex problems with applications to imaging." Journal of mathematical imaging and vision 40.1 (2011): 120-145.

Quoc Tran-Dinh, and Volkan Cevher, "Constrained convex minimization via model-based excessive gap." Advances in Neural Information Processing Systems. 2014.

[3] The rate is $\tilde{\mathcal{O}}\left(\epsilon^{-2}\right)$ for strongly concave problems.

[4] Tianyi Lin, Chi Jin, and Michael Jordan, "Near-optimal algorithms for minimax optimization." arXiv preprint arXiv:2002.02417 (2020).

[5] Constantinos Daskalakis, Stratis Skoulakis, and Manolis Zampetakis, "The complexity of constrained min-max optimization." arXiv preprint arXiv:2009.09623 (2020).

Ya-Ping Hsieh, Panayotis Mertikopoulos, and Volkan Cevher, "The limits of min-max optimization algorithms: convergence to spurious non-critical sets." arXiv preprint arXiv:2006.09065 (2020).

A new hope

$$\min_{x\in\mathbb{R}} \max_{y\in\mathbb{R}} xy$$

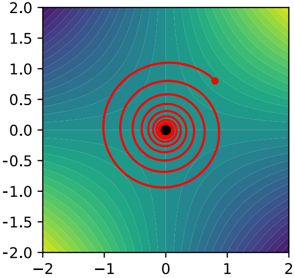
◦ Next lecture: Some algorithms that actually **converge**!

◦ Convergence of the sequence:

There exists $\mathbf{z}^\star = (\mathbf{x}^\star, \mathbf{y}^\star)$, such that $\mathbf{z}_k \to \mathbf{z}^\star$.

◦ Convergence rate:

$$\mathrm{Gap}\left(\frac{1}{K}\sum_{k=1}^{K}\mathbf{x}^k, \frac{1}{K}\sum_{k=1}^{K}\mathbf{y}^k\right) \leq \mathcal{O}\left(\frac{1}{K}\right).$$



Wrap up!

◦ Recitation on Friday is on Homework #2...

A *convex* proto-problem for *structured* sparsity

A combinatorial approach for estimating $\mathbf{x}^\natural$ from $\mathbf{b} = \mathbf{A}\mathbf{x}^\natural + \mathbf{w}$

We may consider the sparsest estimator or its surrogate with a valid sparsity pattern:

$$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathbb{R}^p} \left\{ \|\mathbf{x}\|_{\boldsymbol{s}} : \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \le \kappa, \|\mathbf{x}\|_\infty \le 1 \right\} \qquad (\mathcal{P}_s)$$

with some $\kappa \ge 0$. If $\kappa = \|\mathbf{w}\|_2$, then the structured sparse $\mathbf{x}^\natural$ is a feasible solution.

Sparsity and structure together [7]

Given some weights $\boldsymbol{d} \in \mathbb{R}^d, \boldsymbol{e} \in \mathbb{R}^p$ and an integer input $\boldsymbol{c} \in \mathbb{Z}^l$, we define

$$\|\mathbf{x}\|_{\boldsymbol{s}} := \min_{\boldsymbol{\omega}} \{ \boldsymbol{d}^T\boldsymbol{\omega} + \boldsymbol{e}^T\boldsymbol{s} : \boldsymbol{M}\begin{bmatrix}\boldsymbol{\omega}\\ \boldsymbol{s}\end{bmatrix} \le \boldsymbol{c}, \mathbb{1}_{\mathrm{supp}(\mathbf{x})} = \boldsymbol{s}, \boldsymbol{\omega} \in \{0,1\}^d \}$$

for all feasible $\mathbf{x}$, ∞ otherwise. The parameter $\boldsymbol{\omega}$ is useful for latent modeling.

A *convex* proto-problem for *structured* sparsity

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Given some weights $\boldsymbol{d} \in \mathbb{R}^d, \boldsymbol{e} \in \mathbb{R}^p$ and an integer input $\boldsymbol{c} \in \mathbb{Z}^l$, we define

$$\|\mathbf{x}\|_{\boldsymbol{s}} := \min_{\boldsymbol{\omega}}\{\boldsymbol{d}^T\boldsymbol{\omega} + \boldsymbol{e}^T\boldsymbol{s} : \boldsymbol{M}\begin{bmatrix}\boldsymbol{\omega}\\ \boldsymbol{s}\end{bmatrix} \leq \boldsymbol{c}, \mathbb{1}_{\mathrm{supp}(\mathbf{x})} = \boldsymbol{s}, \boldsymbol{\omega} \in \{0,1\}^d\}$$

for all feasible $\mathbf{x}$, ∞ otherwise. The parameter $\boldsymbol{\omega}$ is useful for latent modeling.

A convex candidate solution for $\mathbf{b} = \mathbf{A}\mathbf{x}^\natural + \mathbf{w}$

We use the convex estimator based on the tightest convex relaxation of $\|\mathbf{x}\|_{\boldsymbol{s}}$:
$\hat{\mathbf{x}} \in \arg\min_{\mathbf{x}\in\mathrm{dom}(\|\cdot\|_{\boldsymbol{s}})} \left\{ \|\mathbf{x}\|_{\boldsymbol{s}}^{**} : \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2 \leq \kappa \right\}$ with some $\kappa \geq 0$, $\mathrm{dom}(\|\cdot\|_{\boldsymbol{s}}) := \{\mathbf{x} : \|\mathbf{x}\|_{\boldsymbol{s}} < \infty\}$.

Tractability & tightness of biconjugation

Proposition (Hardness of conjugation)

Let $F(s) : 2^{\mathfrak{P}} \to \mathbb{R} \cup \{+\infty\}$ be a set function defined on the support $s = supp(\mathbf{x})$. Conjugate of F over the unit infinity ball $\|\mathbf{x}\|_\infty \leq 1$ is given by

$$g^*(\mathbf{y}) = \sup_{s \in \{0,1\}^p} |\mathbf{y}|^T s - F(s).$$

Observations:

- $F(s)$ is general set function

  Computation: NP-Hard

- $F(s) = \|\mathbf{x}\|_s$

  Computation: Integer Linear Program (ILP) in general. However, if
  - $\boldsymbol{M}$ is Totally Unimodular TU
  - $(\boldsymbol{M}, \boldsymbol{c})$ is Total Dual Integral TDI

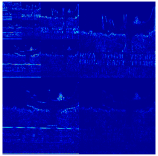
  then tight convex relaxations with a linear program (LP, which is "usually" tractable)

  Otherwise, relax to LP anyway!

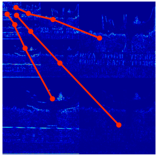
- $F(s)$ is submodular

  Computation: Polynomial-time

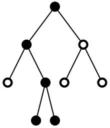
Tree sparsity [15, 6, 3, 21]



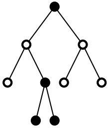
Wavelet coefficients



Wavelet tree



Valid selection of nodes



Invalid selection of nodes

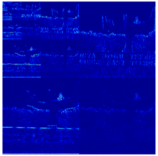
Structure: *We seek the sparsest signal with a rooted connected subtree support.*

Linear description: A valid support satisfy $s_{\text{parent}} \geq s_{\text{child}}$ over tree $\mathcal{T}$

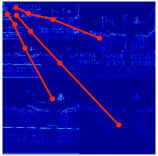
$$\boxed{\boldsymbol{T}\mathbb{1}_{\mathrm{supp}(\mathbf{x})} := \boldsymbol{T}\boldsymbol{s} \geq 0}$$

where $\boldsymbol{T}$ is the directed edge-node incidence matrix, which is TU.

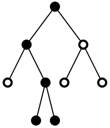
Tree sparsity [15, 6, 3, 21]



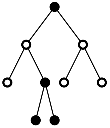
Wavelet coefficients



Wavelet tree



Valid selection of nodes



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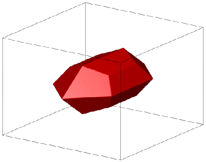
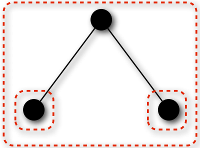
where $\boldsymbol{T}$ is the directed edge-node incidence matrix, which is TU.

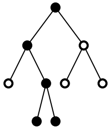
Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \min_{\boldsymbol{s}\in[0,1]^p}\{\mathbb{1}^T\boldsymbol{s} : \boldsymbol{T}\boldsymbol{s} \geq 0, |\mathbf{x}| \leq \boldsymbol{s}\}$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Tree sparsity [15, 6, 3, 21]





$\mathfrak{G}_H = \{\{1,2,3\},\{2\},\{3\}\}$



valid selection of nodes

Structure: *We seek the sparsest signal with a rooted connected subtree support.*

Linear description: A valid support satisfy $s_{\text{parent}} \geq s_{\text{child}}$ over tree $\mathcal{T}$

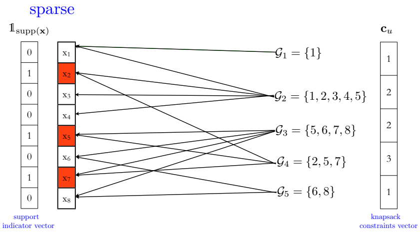
$$\boxed{\boldsymbol{T}\mathbb{1}_{\text{supp}(\mathbf{x})} := \boldsymbol{T}\boldsymbol{s} \geq 0}$$

where $\boldsymbol{T}$ is the directed edge-node incidence matrix, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \min_{\boldsymbol{s}\in[0,1]^p}\{\mathbb{1}^T\boldsymbol{s} : \boldsymbol{T}\boldsymbol{s} \geq 0, |\mathbf{x}| \leq \boldsymbol{s}\} \stackrel{\star}{=} \sum_{\mathcal{G}\in\mathfrak{G}_H} \|x_{\mathcal{G}}\|_\infty$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

The set $\mathcal{G} \in \mathfrak{G}_H$ are defined as each node and all its descendants.

Group knapsack sparsity [23, 10, 8]



Structure: *We seek the sparsest signal with group allocation constraints.*

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

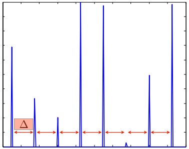
$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Remark: We can also budget a lowerbound $\boldsymbol{c}_\ell \leq \mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u$.

Group knapsack sparsity [23, 10, 8]



$$\mathfrak{B}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & & & & \\ 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_{(p-\Delta+1)\times p}$$

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Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$

For the neuronal spike example, we have $\boldsymbol{c}_u = \mathbb{1}$.

Group knapsack sparsity [23, 10, 8]

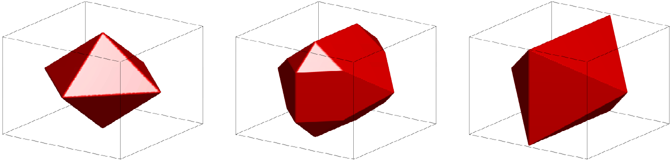

Figure: *

(left) $\|\mathbf{x}\|_s^{**} \leq 1$ (middle) $\|\mathbf{x}\|_s^{**} \leq 1.5$ (right) $\|\mathbf{x}\|_s^{**} \leq 2$ for $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$

Structure: *We seek the sparsest signal with group allocation constraints.*

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

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Remark: We can also budget a lowerbound $\boldsymbol{c}_\ell \leq \mathfrak{B}^T \boldsymbol{s} \leq \boldsymbol{c}_u$.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$

For the neuronal spike example, we have $\boldsymbol{c}_u = \mathbb{1}$.

Group knapsack sparsity example: A stylized spike train

- Basis pursuit (BP): $\|\mathbf{x}\|_1$
- TU-relax (TU):

$$\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \begin{cases} \|\mathbf{x}\|_1 & \text{if } \mathbf{x} \in [-1,1]^p, \mathfrak{B}^T|\mathbf{x}| \leq \boldsymbol{c}_u, \\ \infty & \text{otherwise} \end{cases}$$

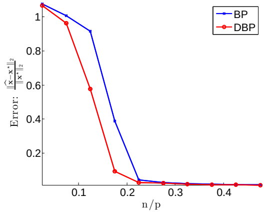
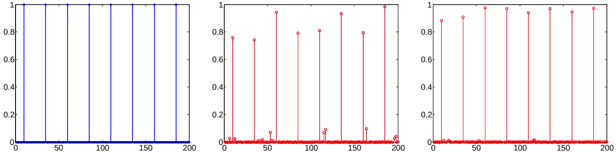


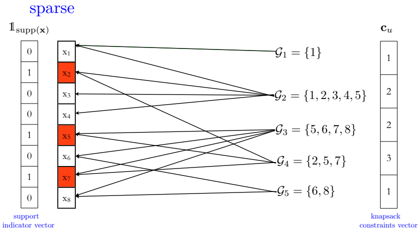
Figure: *Recovery for $n = 0.18p$.*



$\mathbf{x}^\natural$ | $\mathbf{x}^{\mathrm{BP}}$ solution | $\mathbf{x}^{\mathrm{TU}}$ solution

relative errors: $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{BP}}\|_2}{\|\mathbf{x}^\natural\|_2} = .200$ $\quad \frac{\|\mathbf{x}^\natural - \mathbf{x}^{\mathrm{TU}}\|_2}{\|\mathbf{x}^\natural\|_2} = .067$

Group knapsack sparsity: A simple variation



Structure: *We seek the signal with the minimal overall group allocation.*

$$\text{Objective: } \mathbb{1}^T \boldsymbol{s} \to \|\mathbf{x}\|_{\boldsymbol{\omega}} = \begin{cases} \min_{\omega \in \mathbb{Z}_{++}} \omega & \text{if } \mathbf{x} \in [-1, 1]^p, \mathfrak{B}^T \boldsymbol{s} \leq \omega \mathbb{1}, \\ \infty & \text{otherwise} \end{cases}$$

Linear description: A valid support obeys budget constraints over $\mathfrak{G}$

$$\boxed{\mathfrak{B}^T \boldsymbol{s} \leq \omega \mathbb{1}}$$

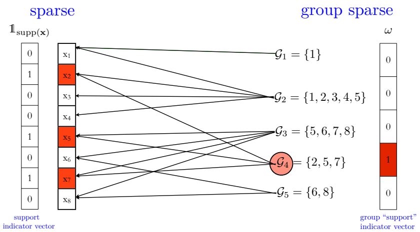
where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix or $\mathfrak{G}$ has a *loopless* group intersection graph, it is TU.

Biconjugate: $\|\mathbf{x}\|_s^{**} = \begin{cases} \max_{\mathcal{G} \in \mathfrak{G}} \|\mathbf{x}^{\mathcal{G}}\|_1 & \text{if } \mathbf{x} \in [-1, 1]^p, \\ \infty & \text{otherwise} \end{cases}$

Remark: The regularizer is known as *exclusive Lasso* [23, 19].

Group cover sparsity: Minimal group cover [2, 18, 13]



Structure: *We seek the signal covered by a minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.
When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Group cover sparsity: Minimal group cover [2, 18, 13]

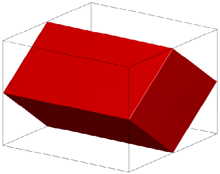


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal covered by a minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T\boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

Group cover sparsity: Minimal group cover [2, 18, 13]

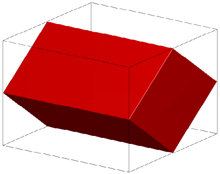


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

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Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M}\{\boldsymbol{d}^T\boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

$$\overset{\star}{=} \min_{\mathbf{v}_i\in\mathbb{R}^p}\{\textstyle\sum_{i=1}^M d_i\|\mathbf{v}_i\|_\infty : \mathbf{x} = \sum_{i=1}^M \mathbf{v}_i, \forall \mathrm{supp}(\mathbf{v}_i) \subseteq \mathcal{G}_i\},$$

Group cover sparsity: Minimal group cover [2, 18, 13]

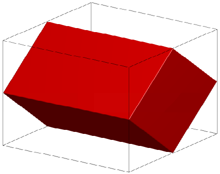


Figure: $\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal covered by a minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \rightarrow \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: At least one group containing a sparse coefficient is selected

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

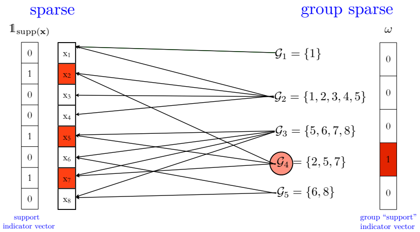
When $\mathfrak{B}$ is an interval matrix, or $\mathfrak{G}$ has a *loopless* group intersection graph it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega} \in [0,1]^M} \{\boldsymbol{d}^T \boldsymbol{\omega} : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|\}$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise

$$\stackrel{\star}{=} \min_{\mathbf{v}_i \in \mathbb{R}^p} \{\textstyle\sum_{i=1}^M d_i \|\mathbf{v}_i\|_\infty : \mathbf{x} = \sum_{i=1}^M \mathbf{v}_i, \forall \mathrm{supp}(\mathbf{v}_i) \subseteq \mathcal{G}_i\},$$

Remark: Weights $\boldsymbol{d}$ can depend on the sparsity within each groups (not TU) [7].

Budgeted group cover sparsity



Structure: *We seek the sparsest signal covered by G groups.*

Objective: $\boldsymbol{d}^T\boldsymbol{\omega} \to \mathbb{1}^T\boldsymbol{s}$

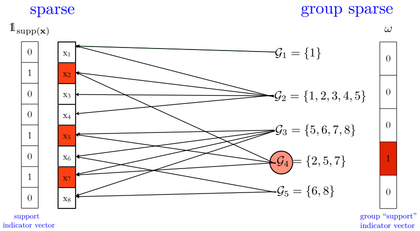
Linear description: At least one of the G selected groups cover each sparse coefficient.

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}, \mathbb{1}^T\boldsymbol{\omega} \leq G}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\begin{bmatrix} \mathfrak{B} \\ \mathbb{1} \end{bmatrix}$ is an interval matrix, it is TU.

Budgeted group cover sparsity



Structure: *We seek the sparsest signal covered by G groups.*

Objective: $\boldsymbol{d}^T\boldsymbol{\omega} \to \mathbb{1}^T\boldsymbol{s}$

Linear description: At least one of the G selected groups cover each sparse coefficient.

$$\boxed{\mathfrak{B}\boldsymbol{\omega} \geq \boldsymbol{s}, \mathbb{1}^T\boldsymbol{\omega} \leq G}$$

where $\mathfrak{B}$ is the biadjacency matrix of $\mathfrak{G}$, i.e., $\mathfrak{B}_{ij} = 1$ iff i-th coefficient is in $\mathcal{G}_j$.

When $\begin{bmatrix}\mathfrak{B} \\ \mathbb{1}\end{bmatrix}$ is an interval matrix, it is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M}\{\|\mathbf{x}\|_1 : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|, \mathbb{1}^T\boldsymbol{\omega} \leq G\}$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Budgeted group cover example: Interval overlapping groups

- Basis pursuit (BP): $\|\mathbf{x}\|_1$
- Sparse group Lasso (SGL$_q$):

$$(1-\alpha)\sum_{\mathcal{G}\in\mathfrak{G}} \sqrt{|\mathcal{G}|}\|\mathbf{x}^{\mathcal{G}}\|_q + \alpha\|\mathbf{x}^{\mathcal{G}}\|_1$$

- TU-relax (TU):

$$\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M}\{\|\mathbf{x}\|_1 : \mathfrak{B}\boldsymbol{\omega} \geq |\mathbf{x}|, \mathbb{1}^T\boldsymbol{\omega} \leq G\}$$

for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise.

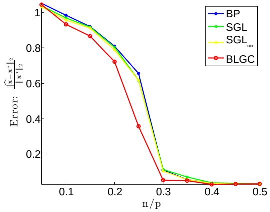
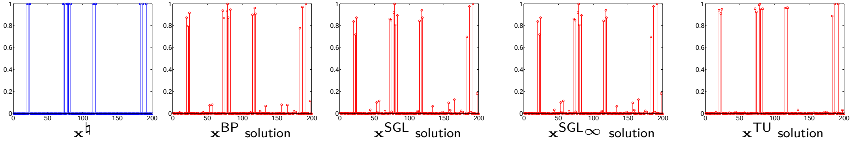
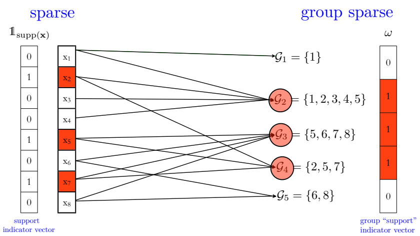


Figure: *Recovery for $n = 0.25p, s = 15, p = 200, G = 5$ out of $M = 29$ groups.*



relative errors: $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{BP}}\|_2}{\|\mathbf{x}^\natural\|_2} = .128$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{SGL}}\|_2}{\|\mathbf{x}^\natural\|_2} = .181$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{SGL}\infty}\|_2}{\|\mathbf{x}^\natural\|_2} = .085$ $\frac{\|\mathbf{x}^\natural - \mathbf{x}^{\text{TU}}\|_2}{\|\mathbf{x}^\natural\|_2} = .058$

Group intersection sparsity [14, 22, 1]



Structure: *We seek the signal intersecting with minimal number of groups.*

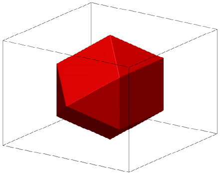
Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Group intersection sparsity [14, 22, 1]



$\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$
(left) intersection (right) cover.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$

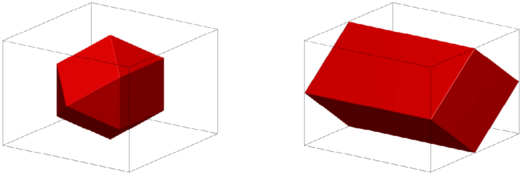
Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M} \{\boldsymbol{d}^T\boldsymbol{\omega} : \boldsymbol{H}_k|\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\}$
for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise.

Group intersection sparsity [14, 22, 1]



$\mathfrak{G} = \{\{1,2\},\{2,3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$
(left) intersection (right) cover.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \boldsymbol{d}^T \boldsymbol{\omega}$ (*submodular*)

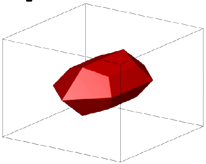
Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M} \{\boldsymbol{d}^T\boldsymbol{\omega} : \boldsymbol{H}_k|\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\} \stackrel{\star}{=} \sum_{\mathcal{G}\in\mathfrak{G}} \|x_{\mathcal{G}}\|_\infty$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Group intersection sparsity [14, 22, 1]



$\mathfrak{G} = \{\{1,2,3\},\{2\},\{3\}\}$, unit group weights $\boldsymbol{d} = \mathbb{1}$.

Structure: *We seek the signal intersecting with minimal number of groups.*

Objective: $\mathbb{1}^T \boldsymbol{s} \rightarrow \boldsymbol{d}^T \boldsymbol{\omega}$ (*submodular*)

Linear description: All groups containing a sparse coefficient are selected

$$\boxed{\boldsymbol{H}_k \boldsymbol{s} \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}}$$

where $\boldsymbol{H}_k(i,j) = \begin{cases} 1 & \text{if } j = k, j \in \mathcal{G}_i \\ 0 & \text{otherwise} \end{cases}$, which is TU.

Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{\omega}}^{**} = \min_{\boldsymbol{\omega}\in[0,1]^M} \{\boldsymbol{d}^T\boldsymbol{\omega} : \boldsymbol{H}_k|\mathbf{x}| \leq \boldsymbol{\omega}, \forall k \in \mathfrak{P}\} \overset{\star}{=} \sum_{\mathcal{G}\in\mathfrak{G}} \|x_{\mathcal{G}}\|_\infty$
for $\mathbf{x} \in [-1,1]^p$, ∞ otherwise.

Remark: For hierarchical $\mathfrak{G}_H$, group intersection and tree sparsity models coincide.

Beyond linear costs: Graph dispersiveness

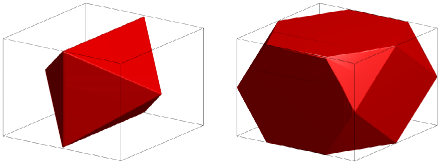


Figure: (left) $\|\mathbf{x}\|_s^{**} = 0$ (right) $\|\mathbf{x}\|_s^{**} \leq 1$ for $\mathcal{E} = \{\{1,2\},\{2,3\}\}$ (chain graph)

Structure: *We seek a signal dispersive over a given graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$*

Objective: $\mathbb{1}^T \boldsymbol{s} \rightarrow \sum_{(i,j)\in\mathcal{E}} s_i s_j$ (non-linear, supermodular function)

Linearization:

$$\boxed{\|\mathbf{x}\|_s = \min_{\mathbf{z}\in\{0,1\}^{|\mathcal{E}|}} \{\textstyle\sum_{(i,j)\in\mathcal{E}} z_{ij} : z_{ij} \geq s_i + s_j - 1\}}$$

When edge-node incidence matrix of $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ is TU (e.g., bipartite graphs), it is TU.

Beyond linear costs: Graph dispersiveness

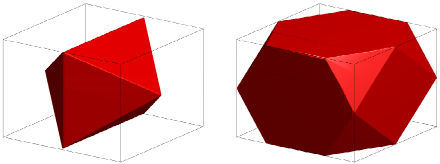


Figure: (left) $\|\mathbf{x}\|_s^{**} = 0$ (right) $\|\mathbf{x}\|_s^{**} \leq 1$ for $\mathcal{E} = \{\{1,2\},\{2,3\}\}$ (chain graph)

Structure: *We seek a signal dispersive over a given graph $\mathcal{G}(\mathfrak{P}, \mathcal{E})$*

Objective: $\mathbb{1}^T \boldsymbol{s} \to \sum_{(i,j)\in\mathcal{E}} s_i s_j$ (non-linear, supermodular function)

Linearization:

$$\boxed{\|\mathbf{x}\|_s = \min_{\mathbf{z}\in\{0,1\}^{|\mathcal{E}|}} \{\textstyle\sum_{(i,j)\in\mathcal{E}} z_{ij} : z_{ij} \geq s_i + s_j - 1\}}$$

When edge-node incidence matrix of $\mathcal{G}(\mathfrak{P}, \mathcal{E})$ is TU (e.g., bipartite graphs), it is TU.
Biconjugate: $\|\mathbf{x}\|_{\boldsymbol{s}}^{**} = \sum_{(i,j)\in\mathcal{E}} (|x_i| + |x_j| - 1)_+$ for $\mathbf{x} \in [-1,1]^p, \infty$ otherwise.

References I

[1] Francis Bach.
Structured sparsity-inducing norms through submodular functions.
Adv. Neur. Inf. Proc. Sys. (NIPS), pages 118–126, 2010.

[2] L. Baldassarre, N. Bhan, V. Cevher, and A. Kyrillidis.
Group-sparse model selection: Hardness and relaxations.
arXiv preprint arXiv:1303.3207, 2013.

[3] R.G. Baraniuk, V. Cevher, M.F. Duarte, and C. Hegde.
Model-based compressive sensing.
IEEE Trans. Inf. Theory, 56(4):1982–2001, April 2010.

[4] Michel Benaïm and Morris W. Hirsch.
Asymptotic pseudotrajectories and chain recurrent flows, with applications.
Journal of Dynamics and Differential Equations, 8(1):141–176, 1996.

[5] Constantinos Daskalakis, Stratis Skoulakis, and Manolis Zampetakis.
The complexity of constrained min-max optimization.
arXiv preprint arXiv:2009.09623, 2020.

References II

[6] Marco F. Duarte, Dharmpal Davenport, Mark A. adn Takhar, Jason N. Laska, Ting Sun, Kevin F. Kelly, and Richard G. Baraniuk.
Single-pixel imaging via compressive sampling.
IEEE Sig. Proc. Mag., 25(2):83–91, March 2008.

[7] Marwa El Halabi and Volkan Cevher.
A totally unimodular view of structured sparsity.
preprint, 2014.
arXiv:1411.1990v1 [cs.LG].

[8] W Gerstner and W. Kistler.
Spiking neuron models: Single neurons, populations, plasticity.
Cambridge university press, 2002.

[9] Gauthier Gidel, Hugo Berard, Gaëtan Vignoud, Pascal Vincent, and Simon Lacoste-Julien.
A variational inequality perspective on generative adversarial networks.
In *International Conference on Learning Representations*, 2018.

[10] C. Hegde, M. Duarte, and V. Cevher.
Compressive sensing recovery of spike trains using a structured sparsity model.
In *Sig. Proc. with Adapative Sparse Struct. Rep. (SPARS)*, 2009.

References III

[11] Ya-Ping Hsieh.
Convergence without convexity: Sampling, optimization, and games.
Technical report, EPFL, 2020.

[12] Ya-Ping Hsieh, Panayotis Mertikopoulos, and Volkan Cevher.
The limits of min-max optimization algorithms: convergence to spurious non-critical sets.
arXiv preprint arXiv:2006.09065, 2020.

[13] J. Huang, T. Zhang, and D. Metaxas.
Learning with structured sparsity.
J. Mach. Learn. Res., 12:3371–3412, 2011.

[14] R. Jenatton, A. Gramfort, V. Michel, G. Obozinski, F. Bach, and B. Thirion.
Multi-scale mining of fmri data with hierarchical structured sparsity.
In *Pattern Recognition in NeuroImaging (PRNI)*, 2011.

[15] R. Jenatton, J. Mairal, G. Obozinski, and F. Bach.
Proximal methods for hierarchical sparse coding.
J. Mach. Learn. Res., 12:2297–2334, 2011.

References IV

[16] Sham M Kakade, Shai Shalev-Shwartz, and Ambuj Tewari.
Regularization techniques for learning with matrices.
The Journal of Machine Learning Research, 13(1):1865–1890, 2012.

[17] Tianyi Lin, Chi Jin, Michael Jordan, et al.
Near-optimal algorithms for minimax optimization.
arXiv preprint arXiv:2002.02417, 2020.

[18] G. Obozinski, L. Jacob, and J.P. Vert.
Group lasso with overlaps: The latent group lasso approach.
arXiv preprint arXiv:1110.0413, 2011.

[19] G. Obozinski, B. Taskar, and M.I. Jordan.
Joint covariate selection and joint subspace selection for multiple classification problems.
Statistics and Computing, 20(2):231–252, 2010.

[20] Q. Tran-Dinh and V. Cevher.
Constrained convex minimization via model-based excessive gap.
In *Proc. the Neural Information Processing Systems Foundation conference (NIPS2014)*, pages 1–9, Montreal, Canada, December 2014.

References V

[21] Peng Zhao, Guilherme Rocha, and Bin Yu.
Grouped and hierarchical model selection through composite absolute penalties.
Department of Statistics, UC Berkeley, Tech. Rep, 703, 2006.

[22] Peng Zhao and Bin Yu.
On model selection consistency of Lasso.
J. Mach. Learn. Res., 7:2541–2563, 2006.

[23] H. Zhou, M.E. Sehl, J.S. Sinsheimer, and K. Lange.
Association screening of common and rare genetic variants by penalized regression.
Bioinformatics, 26(19):2375, 2010.