Mathematics of Data: From Theory to Computation

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Outline

- Today
  1. Convex constrained optimization and motivating examples
  2. Optimality condition
  3. Conjugate functions
  4. Monotone inclusion and monotone mixed variational inequality formulations
  5. Chambolle-Pock's primal-dual method
  6. Primal-dual hybrid gradient method
  7. Splitting methods
  8. Model-based excessive gap primal-dual method
- Next week
  1. Disciplined convex programming

Motivation

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- **Primal-dual convex optimization methods** are powerful for solving saddle point problems as well as constrained convex optimization problems.
- This lecture aims at presenting some emerging primal-dual methods which have been recently used to solve many practical problems in signal/image processing, machine learning and statistics.
- This lecture is a continuation of Lecture 7.

Motivating example: image denoising via anisotropicTV-norm

We consider an image denoising problem with anisotropic total variation norm:

- Given a noisy image $\mathbf{b} \in \mathbb{R}^{m \times n}$. The goal is to recover a clean image from $\mathbf{b}$ using anisotropic total variation norm.

Original image

Noisy image

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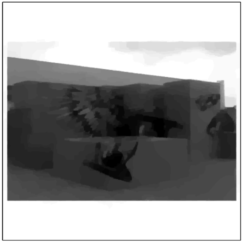
Anisotropic TV denoising ($\rho = 20$)

Anisotropic TV denoising ($\rho = 40$)



Motivating example: image denoising via anisotropicTV-norm

We consider an image denoising problem with anisotropic total variation norm:

- Given a noisy image $\mathbf{b} \in \mathbb{R}^{m \times n}$. The goal is to recover a clean image from $\mathbf{b}$ using anisotropic total variation norm.
- **This problem** can be formulated as a convex optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^{m \times n}} \frac{1}{2}\|\mathbf{x} - \mathbf{b}\|_F^2 + \rho\|\mathbf{D}\mathbf{x}\|_1, \tag{1}$$

  where $\rho > 0$ is a regularization parameter and $\mathbf{D}$ is a given matrix representing the total variation of $\mathbf{x}$.

There are different ways to reformulate problem (1), for example:

- Since $\|\mathbf{z}\|_1 = \max_{\|\mathbf{u}\|_\infty \leq 1} \mathbf{u}^T\mathbf{z}$, we can reformulate (1) as a saddle point problem:

$$\min_{\mathbf{x} \in \mathbb{R}^{m \times n}} \max_{\|\mathbf{u}\|_\infty \leq 1} \left\{ \frac{1}{2}\|\mathbf{x} - \mathbf{b}\|_F^2 + \rho\mathbf{u}^T\mathbf{D}\mathbf{x} \right\},$$

- We can also reformulate (1) as a constrained convex minimization problem:

$$\begin{array}{ll} \min\limits_{\mathbf{x},\mathbf{r}} & \left\{ \frac{1}{2}\|\mathbf{x} - \mathbf{b}\|_F^2 + \rho\|\mathbf{r}\|_1 \right\}, \\ \text{s.t.} & \mathbf{D}\mathbf{x} - \mathbf{r} = 0. \end{array}$$

In this lecture, we present several emerging methods to solve both the saddle point formulation and the constrained formulation of (1).

Mathematical form of constrained convex optimization

Constrained convex optimization setting

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{cases} \tag{2}$$

- $f \in \mathcal{F}(\mathbb{R}^p)$ is a proper, closed and convex function (see Lecture 2).
- $\mathbf{A} \in \mathbb{R}^{n\times p}$ is full-row rank $(n < p)$, $\mathbf{b} \in \mathbb{R}^n$.

We can incorporating constraints $\mathbf{x} \in \mathcal{X}$ for a given closed and convex set $\mathcal{X}$ via its indicator function $\iota_{\mathcal{X}}$, i.e.:

$$\boxed{f(\mathbf{x}) \leftarrow f(\mathbf{x}) + \iota_{\mathcal{X}}(\mathbf{x})} \quad \text{where} \quad \iota_{\mathcal{X}}(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{X} \\ +\infty & \text{otherwise.} \end{cases}$$

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Common structures

As in Lecture 7, methods presented in this lecture also rely on the two common structures:

- **Decomposability** of f.
- **Tractable proximity** of f.

Structures of constrained convex optimization problems

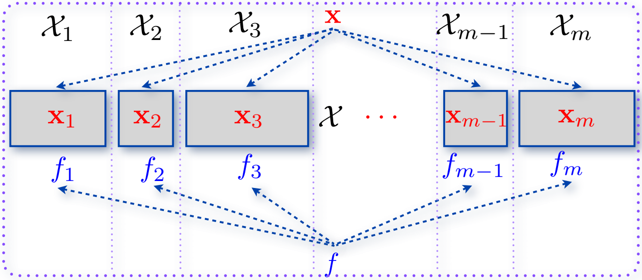
Decomposable structure

The function f can be decomposed as

$$f(\mathbf{x}) := \sum_{i=1}^{m} f_i(\mathbf{x}_i) \tag{3}$$

where $m \geq 1$ is the number of components, $\mathbf{x}_i$ is a sub-vector (component) of $\mathbf{x}$, $f_i : \mathbb{R}^{p_i} \to \mathbb{R} \cup \{+\infty\}$ is convex and $\sum_{i=1}^{m} p_i = p$.

Special case: $m = 2$, which already covers many important instances (see Lecture 7).



Structures of constrained convex optimization problems

Tractable proximity

Each **component** f_i has a **"tractablly proximal operator"** $(i = 1, \ldots, m)$, i.e.:

$$\boxed{\mathrm{prox}_{f_i}(\mathbf{x}_i) := \underset{\mathbf{z}_i \in \mathbb{R}^{p_i}}{\mathrm{argmin}} \left\{ f_i(\mathbf{z}_i) + (1/2)\|\mathbf{z}_i - \mathbf{x}_i\|_2^2 \right\}} \quad (4)$$

can be solved "efficiently":

- (4) has a closed form solution (with low computational cost)
- (4) can be solved in polynomial time.

Example (Tractable proximity functions)

- One-variable functions
  - Smooth functions, e.g., $f(x) := x - 2\log(1 + x)$
  - Nonsmooth functions, e.g., $f(x) := |x|$
- Separable functions, e.g., $f(\mathbf{x}) := \sum_{i=1}^{p} \|\mathbf{x}_i\|_2$, where $\mathbf{x} := (\mathbf{x}_1^T, \cdots, \mathbf{x}_p^T)^T$.
- The indicator function ι of boxes, cones ($\mathbb{R}_+^p$, $\mathbb{S}_+^p$ and Lorentz cone) and simplex.
- More examples can be found in Lectures 4 and 5.

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Solutions and solution set

Optimal solutions and optimal solution set

We define the feasible set of (2): (cf. $f^\star := \min_{\mathbf{x}\in\mathbb{R}^p}\{f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$) as:

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^p \ : \ \mathbf{A}\mathbf{x} = \mathbf{b}\}.$$

A feasible point $\mathbf{x}^\star \in \mathcal{D}$ is called a globally optimal solution (or solution) of (2) if

$$f(\mathbf{x}^\star) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{D}.$$

All solutions of (2) forms the solution set $\mathcal{X}^\star$ of (2).

- The solution set $\mathcal{X}^\star$ is closed and convex.
- Numerical solution methods often try to find an approximation $\mathbf{x}^\star_\epsilon$ of one solution $\mathbf{x}^\star \in \mathcal{X}^\star$ in the following sense:

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Approximate solution

Given a tolerance $\epsilon \geq 0$, a point $\mathbf{x}^\star_\epsilon \in \mathbb{R}^p$ is called an ϵ-solution of (2) if

$$\begin{cases} |f(\mathbf{x}^\star_\epsilon) - f^\star| \leq \epsilon & \text{(objective residual)}, \\ \|\mathbf{A}\mathbf{x}^\star_\epsilon - \mathbf{b}\| \leq \epsilon & \text{(feasibility gap)}. \end{cases}$$

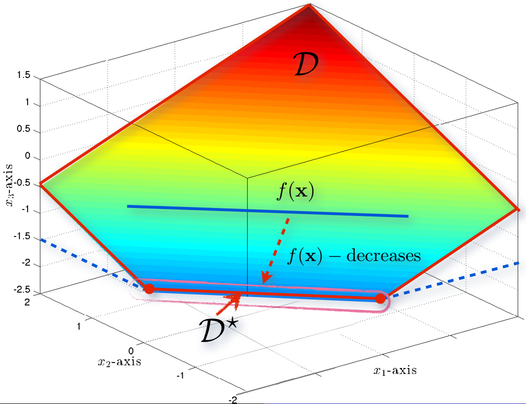
Remark: we can use different tolerances for the objective residual and feasibility gap.

Example: Feasible set and solution set

Consider a constrained convex problem:

$$\min_{\mathbf{x}\in\mathbb{R}^3} \quad \{f(\mathbf{x}) := x_3\},$$
$$\text{s.t.} \quad 2x_3 - x_1 - x_2 = -1,$$
$$\mathbf{x} \in \mathcal{X} := [-2,2]\times[-2,2]\times[-1.5,1.5].$$

The **feasible set** $\mathcal{D} := \{\mathbf{x}\in\mathbb{R}^3 : 2x_3 - x_1 - x_2 = -1, \mathbf{x}\in\mathcal{X}\}$ and the **solution set** $\mathcal{D}^\star$ of this problem are plotted in the figure below. $\mathcal{D}^\star$ is in fact a segment (many solutions), which is the lowest edge of the polytope $\mathcal{D}$.



Optimality condition

Lagrange function

$$\mathcal{L}(\mathbf{x}, \lambda) := f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}).$$

Here, $\lambda \in \mathbb{R}^n$ is the vector of Lagrange multipliers (or dual variables).

Optimality condition

The **optimality condition** of (2) can be written as

$$\begin{cases} 0 & \in \mathbf{A}^T\lambda^\star + \partial f(\mathbf{x}^\star), \\ 0 & = \mathbf{A}\mathbf{x}^\star - \mathbf{b}. \end{cases} \tag{5}$$

Here the subdifferential of f at $\mathbf{x}^\star$ is defined as (see Lecture 2):

$$\partial f(\mathbf{x}^\star) := \{\mathbf{z} \in \mathbb{R}^p \ : \ f(\mathbf{y}) \geq f(\mathbf{x}^\star) + \mathbf{z}^T(\mathbf{y} - \mathbf{x}^\star), \ \forall \mathbf{y} \in \mathbb{R}^p\}.$$

- The condition (5) is the KKT (Karush-Kuhn-Tucker) condition.
- Any point $(\mathbf{x}^\star, \lambda^\star)$ satisfying (5) is called a KKT point.
- $\mathbf{x}^\star$ is called a stationary point and $\lambda^\star$ is the corresponding multipliers.

Conjugation of functions

- **Duality** is a central concept in optimization, especially in convex optimization.
- We review the notion of Fenchel's conjugate function and its basic properties which will be used to define the dual problem.
- We limit our definition to the class of convex functions $f \in \mathcal{F}(\mathbb{R}^p)$.

Definition

Let $\mathcal{Q}$ be a predefined Euclidean space and Q^* be its dual space. Given a proper, closed and convex function $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : Q^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \text{dom}(f)} \left\{ \mathbf{y}^T \mathbf{x} - f(\mathbf{x}) \right\}$$

is called the Fenchel conjugate (or conjugate) of f.

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is called the Fenchel conjugate (or conjugate) of f.

- f^* is a convex and lower, semicontinuous function by construction (as the supremum of affine functions of $\mathbf{y}$).
- The conjugate of the conjugate of a convex function f is ... the same function f; i.e., $f^{**} = f$ for $f \in \mathcal{F}(\mathcal{Q})$.

Two basic properties of the function and its conjugation

Lemma (Fenchel-Young inequality (Property 1))

Let $f : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ *and* $f^* : Q^* \to \mathbb{R} \cup \{+\infty\}$ *be a function and its conjugation; here* Q^* *be the dual space of* $\mathcal{Q}$*. Then, the following inequality holds true:*

$$f(\mathbf{x}) + f^*(\mathbf{y}) \geq \mathbf{x}^T\mathbf{y}, \quad \forall \mathbf{x} \in Q, \mathbf{y} \in Q^*.$$

Two basic properties of the function and its conjugation

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$$f(\mathbf{x}) + f^*(\mathbf{y}) \geq \mathbf{x}^T\mathbf{y}, \quad \forall \mathbf{x} \in Q, \mathbf{y} \in Q^*.$$

- Since f^* is proper, closed and convex, its subdifferential ∂f^* exists for any $\mathbf{y}$ in the relative interior of its domain.
- For $f \in \mathcal{F}(\mathcal{Q})$, if the subdifferential of f and f^* exists, then we have the following relation:

Lemma (Subgradient property (Property 2))

Let $\mathbf{y} \in \partial f(\mathbf{x})$ for some $\mathbf{x} \in dom(f)$. Then $\mathbf{y} \in dom(f^*)$ and vise versa. Moreover, we have

$$\mathbf{u} \in \partial f(\mathbf{x}) \Leftrightarrow \mathbf{x} \in \partial f^*(\mathbf{u}).$$

Conjugation of functions

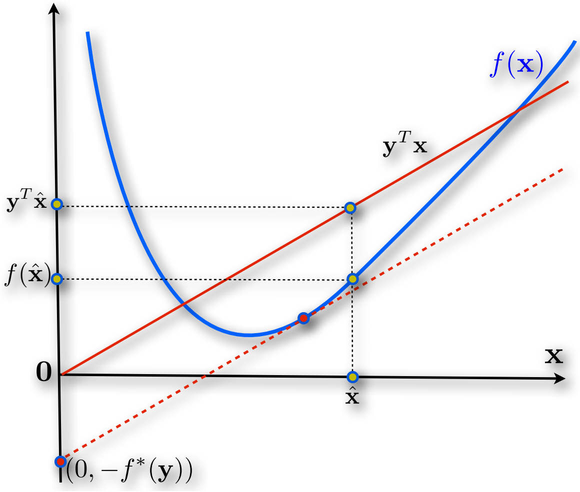

Figure: The conjugate function $f^*(\mathbf{y})$ is the maximum gap between the linear function $\mathbf{x}^T\mathbf{y}$ (red line) and $f(\mathbf{x})$, as shown in dashed line.

Example 1: Convex quadratic function

Example (Strictly convex quadratic function)

- Given a symmetric positive definite matrix $\mathbf{\Phi}$, $(\mathbf{\Phi} \succ 0)$ and a vector $\mathbf{b}$.
- Let $f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T\mathbf{\Phi}\mathbf{x} + \mathbf{b}^T\mathbf{x}$ be a quadratic function for $\mathbf{x} \in \mathbb{R}^p$.
- It is clear that $\text{dom}(f) = \mathbf{R}^p$.
- By definition,

$$f^*(\mathbf{y}) := \sup_{\mathbf{x}\in\mathbb{R}^p} \left\{ \mathbf{y}^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\mathbf{\Phi}\mathbf{x} - \mathbf{b}^T\mathbf{x} \right\}.$$

- Since this is an unconstrained convex problem, the maximum is attained when $\mathbf{\Phi}\mathbf{x}^* + \mathbf{b} = \mathbf{y}$, which leads to $\mathbf{x}^\star = \mathbf{\Phi}^{-1}(\mathbf{y} - \mathbf{b})$.
- Hence, we have

$$f^*(\mathbf{y}) = \frac{1}{2}(\mathbf{y} - \mathbf{b})^T\mathbf{\Phi}^{-1}(\mathbf{y} - \mathbf{b}) = \frac{1}{2}\mathbf{y}^T\mathbf{\Phi}^{-1}\mathbf{y} - (\mathbf{\Phi}^{-1}\mathbf{b})^T\mathbf{y} + \frac{1}{2}\mathbf{b}^T\mathbf{\Phi}^{-1}\mathbf{b},$$

  which is again a convex quadratic function with $\text{dom}(f^*) = \mathbb{R}^p$.

- Since $\nabla f(\mathbf{x}) = \mathbf{\Phi}\mathbf{x} + \mathbf{b} := \mathbf{u}$ and $\nabla f^*(\mathbf{y}) = \mathbf{\Phi}^{-1}(\mathbf{y} - \mathbf{b})$, we can see that $\mathbf{x} = \mathbf{\Phi}^{-1}(\mathbf{u} - \mathbf{b}) = \nabla f^*(\mathbf{u})$.

Example 2: Log-determinant

Example (Log-determinant function)

- Let $f(\mathbf{X}) := -\log\det(\mathbf{X})$, where $\mathsf{dom}(f) \equiv \mathbb{S}^p_{++}$.
- By definition, we have

$$f^*(\mathbf{Y}) = \sup_{\mathbf{X}\in\mathsf{dom}(f)} \left\{\mathsf{tr}(\mathbf{YX}) + \log\det(\mathbf{X})\right\},$$

- One can show that the above is unbounded above unless $\mathbf{Y} \prec 0$.
- To find the maximum of the above problem, we have:

$$\nabla\left(\mathsf{tr}(\mathbf{YX}) + \log\det(\mathbf{X})\right) = 0 \Rightarrow \mathbf{X}^\star = -\mathbf{Y}^{-1},$$

and thus,

$$f^*(\mathbf{Y}) = -\log\det\left(-\mathbf{Y}\right) - p, \quad \text{with} \quad \mathsf{dom}(f^*) = -\mathbb{S}^p_{++}.$$

- Since $\nabla f(\mathbf{X}) = -\mathbf{X}^{-1} := \mathbf{U}$ and $\nabla f^*(\mathbf{Y}) = -\mathbf{Y}^{-1}$, we have $\mathbf{X} = -\mathbf{U}^{-1} = \nabla f^*(\mathbf{U})$.

Conjugation of functions

Example

$f(\mathbf{x})$	dom(f)/dom(f^*)	$f^*(\mathbf{y})$
$f(\alpha\mathbf{x})$ (where $a \neq 0$)	$\mathcal{Q}/\mathcal{Q}^*$	$f^*\left(\frac{\mathbf{y}}{\alpha}\right)$
$f(\mathbf{x}+\beta)$	$\mathcal{Q}/\mathcal{Q}^*$	$f^*(\mathbf{y}) - \langle\beta, \mathbf{y}\rangle$
$\alpha f(\mathbf{x})$ (where $\alpha > 0$)	$\mathcal{Q}/\mathcal{Q}^*$	$\alpha f^*\left(\frac{\mathbf{y}}{\alpha}\right)$
$\frac{\\|\mathbf{x}\\|^r}{r}$ (where $r > 1$)	$\mathbb{R}^p/\mathbb{R}^p$	$\frac{\\|\mathbf{y}\\|^q}{q}$ (where $\frac{1}{r}+\frac{1}{q}=1$)
$-\log(x)$	$\mathbb{R}_{++}/\mathbb{R}_{--}$	$-(1+\log(y))$
e^x	$\mathbb{R}/\mathbb{R}_+$	$\begin{cases} y\log(y) - y & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$

Table: Legendre transforms (conjugations) for many common functions as well as a few useful properties.

Dual problem

- In Lecture 7 we have used the Lagrange duality theory to present **methods of multipliers**.
- In this lecture, we use the Frenchel duality theory to define the **dual problem** formulation and develop **primal-dual methods** for solving (2).

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Dual formulation

- From the optimality condition (5): $\begin{cases} 0 & \in \mathbf{A}^T\lambda^\star + \partial f(\mathbf{x}^\star) \\ 0 & = \mathbf{A}\mathbf{x}^\star - \mathbf{b} \end{cases}$ we have

$$-\mathbf{A}^T\lambda^\star \in \partial f(\mathbf{x}^\star).$$

- The last expression is equivalent to $\mathbf{x}^\star \in \partial f^*(-\mathbf{A}^T\lambda^\star)$ (see Property 2).
- Since $\mathbf{A}\mathbf{x}^\star - \mathbf{b} = 0$, using $\mathbf{x}^\star \in \partial f^*(-\mathbf{A}^T\lambda^\star)$, we have

$$\boxed{0 \in \mathbf{A}\partial f^*(-\mathbf{A}^T\lambda^\star) - \mathbf{b}.} \tag{6}$$

(6) is exactly the optimality condition of

$$\boxed{\max_{\lambda\in\mathbb{R}^n} \left\{-f^*(-\mathbf{A}^T\lambda) - \mathbf{b}^T\lambda\right\}.} \tag{7}$$

(7) is still a convex problem and equivalent to $\min_\lambda\{f^*(-\mathbf{A}^T\lambda) + \mathbf{b}^T\lambda\}$.

Decomposable structure

Dual problem of the decomposable objective function

If $f(\mathbf{x}) := \sum_{i=1}^{m} f_i(\mathbf{x}_i)$ then

$$\max_{\lambda \in \mathbb{R}^n} \Big\{ -\sum_{i=1}^{m} f_i^*(\mathbf{A}_i^T \lambda) - \mathbf{b}^T \lambda \Big\}.$$

where $\mathbf{A} \equiv [\mathbf{A}_1, \cdots, \mathbf{A}_m]$.

Note: The evaluation of the **dual objective function** and its **gradient** can be computed in **parallel**.

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Dual formulation of empirical risk minimization

An empirical risk minimization problem can be expressed as

$$\min_{\mathbf{z} \in \mathbb{R}^n} \sum_{i=1}^{m} f_i(\mathbf{A}_i^T \mathbf{z} + \mathbf{b}_i).$$

Its Fenchel dual problem therefore can be written as:

$$\min_{\mathbf{x} \in \mathbb{R}^p} \sum_{i=1}^{m} \Big\{ f_i^*(\mathbf{x}_i) - \mathbf{b}_i^T \mathbf{x}_i \Big\} \quad \text{s.t.} \quad \sum_{i=1}^{m} \mathbf{A}_i \mathbf{x}_i = 0.$$

Example: Dual problem of the basis pursuit

Basis pursuit

Consider the following basis pursuit problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^p} \quad & \left\{ f(\mathbf{x}) := \|\mathbf{x}\|_1 = \sum_{i=1}^{p} |x_i| \right\} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned}$$

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Dual problem of basis pursuit

- The Fenchel dual function $f^*(\mathbf{u}) := \sup_{\mathbf{x}\in\mathbb{R}^p} \{\mathbf{u}^T\mathbf{x} - \|\mathbf{x}\|_1\}$ becomes

$$f^*(\mathbf{u}) = \iota_{\{\|\mathbf{u}\|_\infty \le 1\}}(\mathbf{u}) = \begin{cases} 0 & \text{if } \|\mathbf{u}\|_\infty \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

- The dual problem

$$\max_{\lambda} \left\{ -\iota_{\{\|\mathbf{A}^T\lambda\|_\infty \le 1\}}(-\mathbf{A}^T\lambda) - \mathbf{b}^T\lambda \right\}$$

- Equivalent expression:

$$\boxed{\min_{\lambda\in\mathbb{R}^n} \mathbf{b}^T\lambda \;\; \text{s.t.} \;\; \|\mathbf{A}^T\lambda\|_\infty \le 1.}$$

Set-valued mappings

In the previous slide, we have seen the subdifferential $\partial f(\mathbf{x})$ of a convex function f at a given point $\mathbf{x}$:

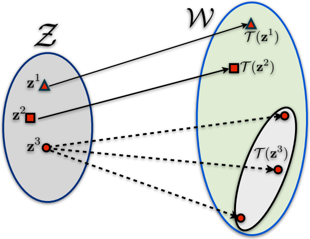
$$\partial f(\mathbf{x}) := \text{set of all subgradients of } f \text{ at } \mathbf{x}$$

Example: Subdifferential of $|\mathbf{x}|$ in $\mathbb{R}$:

$$\mathcal{T}(\mathbf{x}) = \partial|\mathbf{x}| = \begin{cases} +1 & \text{if } x > 0 \quad \text{(single value)}, \\ -1 & \text{if } x < 0 \quad \text{(single value)}, \\ [-1, 1] & \text{if } x = 0 \quad \text{(multiple values)}. \end{cases}$$

Set-valued mappings:

- For any convex set $\mathcal{W}$, we denote by $2^{\mathcal{W}}$ the set of all subsets of $\mathcal{W}$.
- $\mathcal{T} : \mathcal{Z} \rightrightarrows 2^{\mathcal{W}}$ is a set-valued mapping if for $\mathbf{z} \in \mathcal{Z}$, $\mathcal{T}(\mathbf{z})$ is a subset in $\mathcal{W}$.



Roughly speaking, a mapping that produces more than one output values for at least one input is called a set-valued mapping.

Monotone operators

For a **set-valued mapping** $\mathcal{T} : \mathcal{Z} \rightrightarrows 2^{\mathcal{Z}}$, we define

- The domain of $\mathcal{T}$ as $\mathrm{dom}(\mathcal{T}) := \{\mathbf{z} \in \mathcal{Z} \ : \ \mathcal{T}(\mathbf{z}) \neq \emptyset\}$.
- The graph of $\mathcal{T}$ is $\mathrm{graph}(\mathcal{T}) := \{(\mathbf{z}, \mathbf{v}) \ : \ \mathbf{v} \in \mathcal{T}(\mathbf{z}), \ \mathbf{z} \in \mathsf{dom}(\mathcal{T})\}$.

Monotone operators

For a **set-valued mapping** $\mathcal{T} : \mathcal{Z} \rightrightarrows 2^{\mathcal{Z}}$, we define

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Monotonicity can be considered as an equivalent property of convexity acting on the differential ∇f or subdifferential ∂f of the function instead of the function value.

For smooth and convex function, monotonicity of ∇f means that

$$(\nabla f(\mathbf{x}) - \nabla f(\hat{\mathbf{x}}))^T(\mathbf{x} - \hat{\mathbf{x}}) \geq 0, \quad \forall \mathbf{x}, \hat{\mathbf{x}} \in \mathrm{dom}(f).$$

This inequality is the sum of $f(\mathbf{x}) \geq f(\hat{\mathbf{x}}) + \nabla f(\hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}})$ and $f(\hat{\mathbf{x}}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\hat{\mathbf{x}} - \mathbf{x})$.

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Monotonicity

Given a convex set $\mathcal{Z}$ and a set-valued mapping $\mathcal{T} : \mathcal{Z} \rightrightarrows 2^{\mathcal{Z}}$.

- $\mathcal{T}$ is called μ-strongly monotone on $\mathcal{Z}$ if for any $\mathbf{z}$ and $\hat{\mathbf{z}}$ in $\mathcal{Z}$:

$$\boxed{(\mathbf{u} - \hat{\mathbf{u}})^T(\mathbf{z} - \hat{\mathbf{z}}) \geq \mu\|\mathbf{z} - \hat{\mathbf{z}}\|^2, \ \forall \mathbf{u} \in \mathcal{T}(\mathbf{z}), \hat{\mathbf{u}} \in \mathcal{T}(\hat{\mathbf{z}}).}$$

- If $\mu = 0$, then we say that $\mathcal{T}$ is monotone.
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If $\mathcal{T}$ is single-valued, then the condition reduces to

$$(\mathbf{z} - \hat{\mathbf{z}})^T(\mathcal{T}(\mathbf{z}) - \mathcal{T}(\hat{\mathbf{z}})) \geq \mu\|\mathbf{z} - \hat{\mathbf{z}}\|^2, \quad \forall \ \mathbf{z}, \hat{\mathbf{z}} \in \mathcal{Z}.$$

Monotone function vs nonmonotone functions

$(T(\hat{\mathbf{x}}) - T(\mathbf{x}))^T(\hat{\mathbf{x}} - \mathbf{x}) \geq 0$ $\quad$ $(T(\hat{\mathbf{x}}) - T(\mathbf{x}))^T(\hat{\mathbf{x}} - \mathbf{x}) \leq 0$ $\quad$ $(T(\hat{\mathbf{x}}) - T(\mathbf{x}))^T(\hat{\mathbf{x}} - \mathbf{x}) \leq 0$

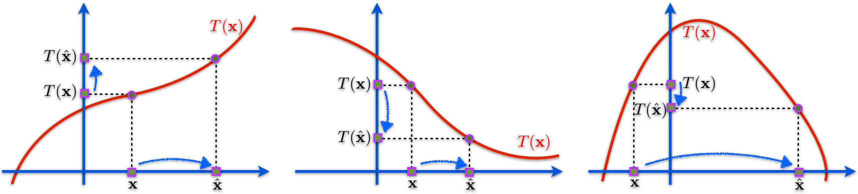


Figure: Monotone function - Nonmonotone function - Nonmonotone function

increasing function - decreasing function -

Maximal monotone operators

Definition (Maximal monotonicity)

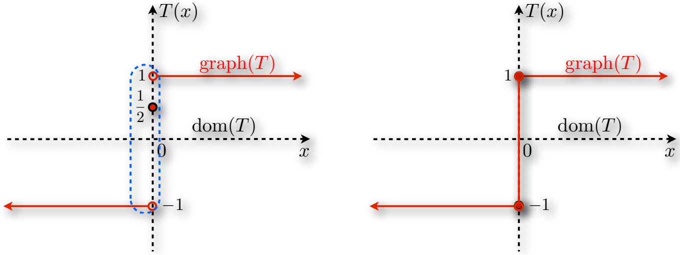
A set-valued mapping $\mathcal{T}$ is called maximal monotone if:

- $\mathcal{T}$ is monotone
- There exists no other monotone mapping $\tilde{\mathcal{T}}$ such that $\mathrm{graph}(\mathcal{T}) \subset \mathrm{graph}(\tilde{\mathcal{T}})$.

Example (Visualization of a maximal monotone operator)

Consider the mapping $\mathcal{T}$ from $\mathbb{R}$ to $2^{\mathbb{R}}$ as follows:

- Nonmaximal monotone (left figure): $T(x) = 1$ if $x > 0$, $T(x) = -1$ if $x < 0$ and $T(x) = 1/2$ if $x = 0 \Rightarrow T$ is not maximal monotone.
- Maximal monotone (right figure): $T(x) = 1$ if $x > 0$, $T(x) = -1$ if $x < 0$ and $T(x) = [-1, 1]$ if $x = 0 \Rightarrow T$ is maximal monotone.



Example of maximal monotone operators

- Affine mapping $\mathcal{T}(\mathbf{z}) := \mathbf{H}\mathbf{z} + \mathbf{h}$ is maximal monotone if $\mathbf{H}$ is positive semidefinite.
  - We have
$$(\mathcal{T}(\mathbf{z}) - \mathcal{T}(\hat{\mathbf{z}}))^T(\mathbf{z} - \hat{\mathbf{z}}) = (\mathbf{z} - \hat{\mathbf{z}})^T\mathbf{H}(\mathbf{z} - \hat{\mathbf{z}}) \geq \sigma_{\min}(\mathbf{H})\|\mathbf{z} - \hat{\mathbf{z}}\|^2,$$
    where $\mu = \sigma_{\min}(\mathbf{H})$ the smallest singular value of $\mathbf{H}$
  - $\mathcal{T}$ is strongly monotone if $\mathbf{H}$ is positive definite.
- Any nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ is monotone.

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- Any nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ is monotone.
- The subdifferential ∂f of a convex function f is maximal monotone.
  - By definition, any $\mathbf{u} \in \partial f(\mathbf{x})$, one has $f(\hat{\mathbf{x}}) - f(\mathbf{x}) \geq \mathbf{u}^T(\hat{\mathbf{x}} - \mathbf{x})$ for any $\hat{\mathbf{x}}$.
  - Similarly, $\hat{\mathbf{u}} \in \partial f(\hat{\mathbf{x}})$, then $f(\mathbf{x}) - f(\hat{\mathbf{x}}) \geq \hat{\mathbf{u}}^T(\mathbf{x} - \hat{\mathbf{x}})$.
  - Summing up these inequalities, we obtain $(\mathbf{u} - \hat{\mathbf{u}})^T(\mathbf{x} - \hat{\mathbf{x}}) \geq 0$.
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- The normal cone $\mathcal{N}_{\mathcal{X}}$ of a nonempty, closed and convex set $\mathcal{X}$ is also a monotone mapping
  - Since it is the subdifferential of the indicator function $\iota_{\mathcal{X}}$, which is proper, closed and convex.

Monotonicity of the normal cone

Given a nonempty, closed and convex set $\mathcal{X}$. The **normal cone** of $\mathcal{X}$ at $\mathbf{x}$ is defined as

$$\mathcal{N}_{\mathcal{X}}(\mathbf{x}) := \begin{cases} \{\mathbf{u} \ : \ \mathbf{u}^T(\mathbf{x}-\mathbf{y}) \geq 0, \ \forall \mathbf{y} \in \mathcal{X}\} & \text{if } \mathbf{x} \in \mathcal{X}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{N}_{\mathcal{X}}(\cdot)$ is a set-valued mapping and is monotone.

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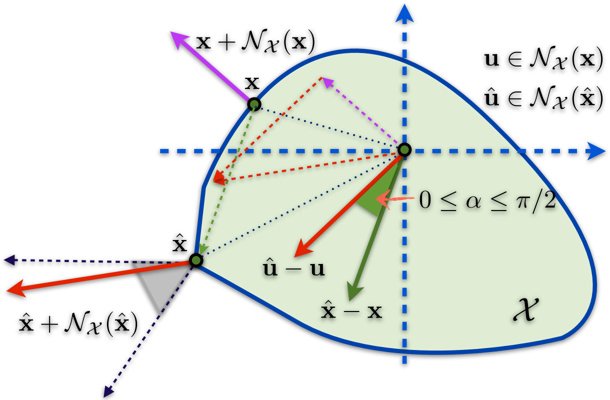

Resolvent and relation to prox-operator

Resolvent of a maximal monotone mapping

Given a maximal monotone mapping $\mathcal{T}$.

- The **resolvent** $\mathcal{J}_{\mathcal{T}}$ of $\mathcal{T}$ at $\mathbf{w}$ is defined as a solution of the *inclusion* w.r.t. $\mathbf{z}$:

$$\mathbf{w} \in \mathbf{z} + \mathcal{T}(\mathbf{z}).$$

- Conventionally, we can write $\boxed{\mathcal{J}_{\mathcal{T}}(\mathbf{w}) := (\mathbb{I} + \mathcal{T})^{-1}(\mathbf{w})}$.

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Lemma (Well-definedness [12])

If $\mathcal{T}$ is maximal monotone then $\mathcal{J}_{\mathcal{T}}(\mathbf{w})$ is well-defined and single-valued.

Remark: If $\mathcal{T}$ is not maximal monotone, then $\mathcal{J}_{\mathcal{T}}(\mathbf{w})$ may not be well-defined.

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Relation to prox operator

Let $\mathcal{T} := \partial f$ the subdifferential of a proper, closed and convex function $f \in \mathcal{F}(\mathbb{R}^p)$. Then $\mathcal{T}$ is maximal monotone and

$$\boxed{\mathcal{J}_{\partial f}(\cdot) \equiv \mathrm{prox}_f(\cdot).}$$

Example: Resolvent of the normal cone

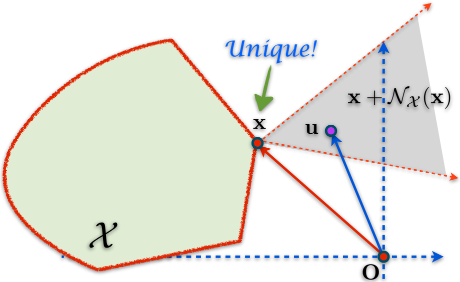
Consider the inclusion $\mathbf{u} \in \mathbf{x} + \mathcal{N}_{\mathcal{X}}(\mathbf{x})$. We can write $\mathbf{u} - \mathbf{x} \in \mathcal{N}_{\mathcal{X}}(\mathbf{x})$. By definition of $\mathcal{N}_{\mathcal{X}}(\mathbf{x})$ we have $(\mathbf{u}-\mathbf{x})^T\mathbf{x} \geq (\mathbf{u}-\mathbf{x})^T\mathbf{y}$ for all $\mathbf{y} \in \mathcal{X}$. Hence, we can write

$$(\mathbf{x}-\mathbf{u})^T(\mathbf{y}-\mathbf{u}) \geq \|\mathbf{x}-\mathbf{u}\|_2^2, \quad \forall \mathbf{y} \in \mathcal{X}.$$

This inequality shows that $\mathbf{x}$ is the solution of

$$\mathbf{x} = \arg\min_{\mathbf{y}\in\mathcal{X}} \|\mathbf{y}-\mathbf{u}\|_2^2$$

which is indeed the projection of $\mathbf{u}$ onto $\mathcal{X}$, i.e. $\mathcal{S}_{\mathcal{N}_{\mathcal{X}}(\cdot)}(\mathbf{u}) = \mathbf{x} = \pi_{\mathcal{X}}(\mathbf{u})$.



From equations to inclusions

Before presenting methods for solving (2), we review a notion in convex analysis called inclusion. Let us motivate this concept by starting from a **system of equations**.

For a single-valued mapping $J : \mathbb{R}^p \to \mathbb{R}^p$, we consider the **system of equations**:

$$\mathbf{x} = J(\mathbf{x}). \tag{8}$$

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Fixed point iteration scheme

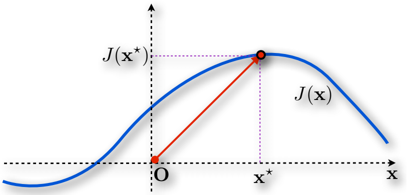
If the mapping J is contractive, i.e., $\exists \kappa \in [0, 1)$ such that

$$\|J(\mathbf{x}) - J(\hat{\mathbf{x}})\| \leq \kappa \|\mathbf{x} - \hat{\mathbf{x}}\|, \quad \forall \mathbf{x}, \hat{\mathbf{x}},$$

then, by the Banach contraction mapping principle, the sequence $\{\mathbf{x}^k\}$ generated by

$$\mathbf{x}^{k+1} := J(\mathbf{x}^k), \quad k \geq 0,$$

starting from $\mathbf{x}^0$ converges to the unique fixed point $\mathbf{x}^\star$ of J, i.e., $\mathbf{x}^\star = J(\mathbf{x}^\star)$.



From equations ...

Now, given a single-valued mapping $T : \mathbb{R}^p \to \mathbb{R}^p$, let us consider a **general system of equations**:

$$T(\mathbf{x}) = 0, \tag{9}$$

A simple way to transform (9) into (8), i.e., $\mathbf{x} = J(\mathbf{x})$, is:

$$\mathbf{x} = \mathbf{x} - \gamma T(\mathbf{x}) := J_T^{\gamma}(\mathbf{x}), \quad \gamma \neq 0.$$

Then, we can generate a fixed-point scheme for solving (9) as

$$\mathbf{x}^{k+1} := J_T^{\gamma}(\mathbf{x}^k). \tag{10}$$

If J_T^{γ} is contractive for given $\gamma \neq 0$, then $\{\mathbf{x}^k\}$ generated by (10) converges to a solution $\mathbf{x}^\star$ of (8).

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Example (Gradient method)

Let us consider the **optimality condition** of the unconstrained smooth convex problem: $f^\star = \min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})$ as

$$\nabla f(\mathbf{x}) = 0.$$

Since ∇f is single-valued, using the same trick as in the previous slide, we can write the fixed-point iterative scheme as

$$\mathbf{x}^{k+1} := \mathbf{x}^k - \gamma_k \nabla f(\mathbf{x}^k) = J_T^{\gamma_k}(\mathbf{x}).$$

If f is μ-strongly convex with $\mu > 0$ and ∇f is L_f-Lipschitz continuous, then J_T^{γ} is contractive for any $\gamma \in (0, 2\mu/L^2)$.

Proof of contractivity of J_T^γ

Proof.

1. If f is smooth and strongly convex with the strong convexity parameter μ. Then
$$(\nabla f(\mathbf{x}) - \nabla f(\hat{\mathbf{x}}))^T(\mathbf{x} - \hat{\mathbf{x}}) \geq \mu\|\mathbf{x} - \hat{\mathbf{x}}\|^2, \ \forall \mathbf{x}, \hat{\mathbf{x}} \in \mathbb{R}^p.$$
2. Since $J_T^\gamma(\mathbf{x}) := \mathbf{x} - \gamma\nabla f(\mathbf{x})$, for any $\mathbf{x}, \hat{\mathbf{x}} \in \mathbb{R}^p$, we have
$$\begin{aligned}
\|J_T^\gamma(\mathbf{x}) - J_T^\gamma(\hat{\mathbf{x}})\|^2 &= \|\mathbf{x} - \hat{\mathbf{x}} - \gamma(\nabla f(\mathbf{x}) - \nabla f(\hat{\mathbf{x}}))\|^2 \\
&= \|\mathbf{x} - \hat{\mathbf{x}}\|^2 - 2\gamma(\mathbf{x} - \hat{\mathbf{x}})^T(\nabla f(\mathbf{x}) - \nabla f(\hat{\mathbf{x}})) \\
&+ \gamma^2\|\nabla f(\mathbf{x}) - \nabla f(\hat{\mathbf{x}})\|^2 \\
&\leq \|\mathbf{x} - \hat{\mathbf{x}}\|^2 - 2\gamma \underbrace{\mu\|\mathbf{x} - \hat{\mathbf{x}}\|^2}_{\text{strong convexity}} + \gamma^2 \underbrace{L^2\|\mathbf{x} - \hat{\mathbf{x}}\|^2}_{\text{Lipschitz gradient}} \\
&\leq (1 - 2\gamma\mu + \gamma^2 L^2)\|\mathbf{x} - \hat{\mathbf{x}}\|^2.
\end{aligned}$$
3. Hence, J_T^γ is contractive if $1 - 2\gamma\mu + \gamma^2 L^2 \in [0, 1)$, which implies $\gamma \in (0, 2\mu/L^2)$.
4. We note that $1 - 2\gamma\mu + \gamma^2 L^2$ is minimized if $\gamma_\star = \mu/L^2$ and hence $1 - 2\gamma\mu + \gamma^2 L^2 = 1 - \mu^2/L^2$.

□

... To inclusions

In the example presented previously, if we no longer assume f to be smooth, then the **optimality condition** turns out to be

$$0 \in \partial f(\mathbf{x}).$$

Since ∂f is a set-valued mapping, this condition is called an **inclusion**.

We can generalize this inclusion to any set-valued mapping $\mathcal{T}$ from $\mathbb{R}^p$ to $2^{\mathbb{R}^p}$ as

$$0 \in \mathcal{T}(\mathbf{x}). \tag{11}$$

In general, solving the inclusion (11) is much more difficulty than solving the equation system $T(\mathbf{x}) = 0$. Methods for solving (11) on the one hand can inherit from methods of solving equations, but on the other hand, require new mathematical tools.

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Additional mathematical tools

Since we are working with the possibly nonsmooth and constrained convex problem (2), where $\mathcal{X}$ is not specified, its optimality condition will be reformulated as an inclusion. We will use additional mathematical tools from variational analysis such as:

- Monotone inclusions
- Monotone mixed variational inequalities
- Gap functions

Mixed variational inequality (MVI) formulation

Primal-dual mapping

We introduce a new primal-dual variable $\mathbf{z} := (\mathbf{x}^T, \lambda^T)^T \in \mathbb{R}^{p+n}$ and two mappings:

$$M(\mathbf{z}) := \begin{bmatrix} \mathbf{A}^T\lambda \\ \mathbf{b} - \mathbf{A}\mathbf{x} \end{bmatrix} \text{ and } \mathcal{T}(\mathbf{z}) := \left\{ \begin{pmatrix} \xi \\ 0^n \end{pmatrix} \in \mathbb{R}^{p+n} \;:\; \xi \in \partial f(\mathbf{x}) \right\}. \quad (12)$$

- Then $M : \mathbb{R}^{p+n} \to \mathbb{R}^{p+n}$ is a single-valued mapping (linear mapping).
- If f is not differentiable, then $\mathcal{T} : \mathbb{R}^{p+n} \rightrightarrows 2^{\mathbb{R}^{p+n}}$ is a set-valued mapping.

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Inclusion and MVI formulation

- The **optimality condition** (5) can be written as an **inclusion**:

$$\boxed{0 \in \mathcal{P}(\mathbf{z}) := M(\mathbf{z}) + \mathcal{T}(\mathbf{z}).}$$

- (5) can also be expressed as a **mixed variational inequality** (MVI):

$$\boxed{f(\mathbf{x}) - f(\mathbf{x}^\star) + M(\mathbf{z}^\star)^T(\mathbf{z} - \mathbf{z}^\star) \geq 0, \quad \forall \mathbf{z} \in \mathbb{R}^{p+n}.} \tag{13}$$

Optimality condition as a monotone inclusion/VI problem

Lemma (Monotonicity of primal-dual mapping)

The mapping M and $\mathcal{T}$ defined in (12)*:*

$$M(\mathbf{z}) := \begin{bmatrix} \mathbf{A}^T\lambda \\ \mathbf{b} - \mathbf{A}\mathbf{x} \end{bmatrix} \quad \textit{and} \quad \mathcal{T}(\mathbf{z}) := \left\{ \begin{pmatrix} \xi \\ 0^n \end{pmatrix} \in \mathbb{R}^{p+n} \ : \ \xi \in \partial f(\mathbf{x}) \right\}.$$

are **maximal monotone**. *Consequently, $\mathcal{P} := M + \mathcal{T}$ is also* ***maximal monotone***.

To show the monotonicity of M, we can write $M(\mathbf{z})$ as

$$M(\mathbf{z}) := \mathbf{H}\mathbf{z} + \mathbf{h} \equiv \begin{bmatrix} 0 & \mathbf{A}^T \\ -\mathbf{A} & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

It is clear that

$$\mathbf{z}^T\mathbf{H}\mathbf{z} = \mathbf{x}^T(\mathbf{A}^T\lambda) - \lambda^T\mathbf{A}\mathbf{x} = 0,$$

which shows that M is monotone.

Optimality condition as a monotone inclusion/VI problem

Lemma (Monotonicity of primal-dual mapping)

The mapping M and $\mathcal{T}$ defined in (12)*:*

$$M(\mathbf{z}) := \begin{bmatrix} \mathbf{A}^T\lambda \\ \mathbf{b} - \mathbf{A}\mathbf{x} \end{bmatrix} \quad \textit{and} \quad \mathcal{T}(\mathbf{z}) := \left\{ \begin{pmatrix} \xi \\ 0^n \end{pmatrix} \in \mathbb{R}^{p+n} \ : \ \xi \in \partial f(\mathbf{x}) \right\}.$$

are **maximal monotone***. Consequently, $\mathcal{P} := M + \mathcal{T}$ is also* ***maximal monotone****.*

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It is clear that

$$\mathbf{z}^T\mathbf{H}\mathbf{z} = \mathbf{x}^T(\mathbf{A}^T\lambda) - \lambda^T\mathbf{A}\mathbf{x} = 0,$$

which shows that M is monotone.

Monotone inclusion and monotone variational inequality

- The **inclusion** $0 \in \mathcal{P}(\mathbf{z}^\star)$ is called a monotone inclusion if $\mathcal{P}$ is maximal monotone.
- The **mixed variational inequality**

$$f(\mathbf{x}) - f(\mathbf{x}^\star) + M(\mathbf{z}^\star)^T(\mathbf{z} - \mathbf{z}^\star) \geq 0, \quad \forall \mathbf{z} \in \mathbb{R}^{p+n}$$

  is called monotone if f is proper, closed and convex and M is maximal monotone.

Gap function for the MVI problem

Gap function

Let us consider a monotone MVIP problem of the form:

$$f(\mathbf{x}) - f(\mathbf{x}^\star) + M(\mathbf{z}^\star)^T(\mathbf{z} - \mathbf{z}^\star) \geq 0, \quad \forall \mathbf{z} \in \mathbb{R}^{p+n}$$

The **gap function** associated with this problem is defined as follows

$$\boxed{G(\mathbf{z}) := \max_{\hat{\mathbf{z}} \in \mathbb{R}^{p+n}} \Big\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \Big\}.} \tag{14}$$

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Properties

- Computing G and its **gradient** require to solve the convex problem in (14).
- G is nonnegative, i.e.: $G(\mathbf{z}) \geq 0$ for all $\mathbf{z} \in \mathbb{R}^{p+n}$.
  - Indeed, we have $G(\mathbf{z}) = \max_{\hat{\mathbf{z}} \in \mathbb{R}^{p+n}} \left\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \right\} \geq f(\mathbf{x}) - f(\mathbf{x}) + M(\mathbf{z})^T(\mathbf{z} - \mathbf{z}) = 0,$ with $\hat{\mathbf{z}} = \mathbf{z}$.
- $\boxed{G(\mathbf{z}^\star) = 0}$ iff $\mathbf{z}^\star$ is a KKT point of (5).

  Exercise: By writing the optimality condition of the maximization problem (14) and rearranging it, we obtain exactly the KKT condition (5).

Example: Gap function

Consider the following constrained convex problem:

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^2} \quad & \{f(\mathbf{x}) := |x_1| + x_2^2\}, \\ \text{s.t.} \quad & x_1 + x_2 = 1.\end{aligned}$$

We have $\mathbf{z} = (x_1, x_2, \lambda)^T \in \mathbb{R}^3$ and $M(\mathbf{z}) := (\lambda, \lambda, 1 - x_1 - x_2)^T$.

The **gap function** associated with the optimality condition of this problem becomes:

$$\begin{aligned}G(\mathbf{z}) &:= \max_{(y_1,y_2,\eta)^T\in\mathbb{R}^3} \Big\{|x_1| + x_2^2 - |y_1| - y_2^2 + \eta(x_1 + x_2 - 1) - \lambda(y_1 + y_2 - 1)\Big\} \\ &= \max_{(y_1,y_2)^T\in\mathbb{R}^2} \Big\{ - |y_1| - y_2^2 - \lambda(y_1 + y_2 - 1)\Big\} + \max_{\eta\in\mathbb{R}} \Big\{|x_1| + x_2^2 + \eta(x_1 + x_2 - 1)\Big\} \\ &= \begin{cases} |x_1| + x_2^2 - d(\lambda) & \text{if } x_1 + x_2 - 1, \\ +\infty & \text{otherwise,}\end{cases}\end{aligned}$$

where $d(\lambda) := \min_{(y_1,y_2)^T\in\mathbb{R}^2} \Big\{|y_1| + y_2^2 + \lambda(y_1 + y_2 - 1)\Big\}$ is the dual function.

Outline

- Today
  1. Convex constrained optimization and motivating examples
  2. Optimality condition
  3. Conjugate functions
  4. Monotone inclusion and monotone mixed variational inequality formulations
  5. Chambolle-Pock's primal-dual method
  6. Primal-dual hybrid gradient method
  7. Splitting methods
  8. Model-based excessive gap primal-dual method
- Next week
  1. Disciplined convex programming

A special class of constrained convex problems (2)

We first consider a constrained reformulation of composite convex minimization problems considered in Lecture 5.

Constrained convex reformulation

We consider the following special case of (2):

$$\begin{cases} \min\limits_{\mathbf{x}:=(\mathbf{u},\mathbf{v})\in\mathbb{R}^{p_1+p_2}} & \left\{F(\mathbf{x}) := f(\mathbf{u}) + g(\mathbf{v})\right\} \\ \text{s.t.} & \mathbf{K}\mathbf{u} - \mathbf{v} = 0. \end{cases} \tag{15}$$

where $\mathbf{K}$ is a **linear operator**, $f \in \mathcal{F}(\mathbb{R}^{p_1})$ and $g \in \mathcal{F}(\mathbb{R}^{p_2})$ are two convex functions.

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where $\mathbf{K}$ is a **linear operator**, $f \in \mathcal{F}(\mathbb{R}^{p_1})$ and $g \in \mathcal{F}(\mathbb{R}^{p_2})$ are two convex functions.

- By setting $\mathbf{A} := [\mathbf{K}, -\mathbb{I}]$ and $\mathbf{b} := 0^n$, we can formulate (15) into (2):

$$F^\star := \begin{cases} \min_{\mathbf{x}\in\mathbb{R}^p} & F(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{cases}$$

- Problem (15) can be written as a composite convex minimization:

$$F^\star := \min_{\mathbf{u}\in\mathbb{R}^{p_1}} \left\{F(\mathbf{u}) := f(\mathbf{u}) + g(\mathbf{K}\mathbf{u})\right\}. \tag{16}$$

Min-max formulation and dual problem

The min-max (saddle point) problem

By using the Fenchel conjugate g^* of g, we can write

$$g(\mathbf{K}\mathbf{u}) = \max_{\mathbf{v}\in\mathbb{R}^n}\{\langle \mathbf{K}\mathbf{u}, \mathbf{v}\rangle - g^*(\mathbf{v})\}.$$

Substituting this function into (16), we obtain:

$$\boxed{\min_{\mathbf{u}\in\mathbb{R}^{p}1}\ \max_{\mathbf{v}\in\mathbb{R}^n}\Big\{\ \langle \mathbf{K}\mathbf{u}, \mathbf{v}\rangle + f(\mathbf{u}) - g^*(\mathbf{v})\Big\}} \tag{17}$$

where g^* is the conjugate of g.

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Substituting this function into (16), we obtain:

$$\boxed{\min_{\mathbf{u}\in\mathbb{R}^{p1}} \max_{\mathbf{v}\in\mathbb{R}^n} \Big\{ \langle \mathbf{K}\mathbf{u}, \mathbf{v}\rangle + f(\mathbf{u}) - g^*(\mathbf{v}) \Big\}} \tag{17}$$

where g^* is the conjugate of g.

Dual problem

By exchanging the min-max in (17) and note that

$$\min_{\mathbf{u}\in\mathbb{R}^{p1}} \{\langle \mathbf{K}\mathbf{u}, \mathbf{v}\rangle + f(\mathbf{u})\} = -\max_{\mathbf{u}\in\mathbb{R}^{p1}} \{\Big\langle -\mathbf{K}^T\mathbf{v}, \mathbf{u}\Big\rangle - f(\mathbf{u})\} = -f^*(-\mathbf{K}^T\mathbf{v})$$

we have

$$\boxed{\max_{\mathbf{v}\in\mathbb{R}^q} \Big\{ -f^*(-\mathbf{K}^T\mathbf{v}) - g^*(\mathbf{v}) \Big\}.} \tag{18}$$

Chambolle-Pock's algorithm: the main idea

Optimality condition

First, we write the optimality condition of (17) as follows:

$$\begin{cases}\mathbf{K}\mathbf{u}^\star & \in \partial g^*(\mathbf{v}^\star)\\ -\mathbf{K}^T\mathbf{v}^\star & \in \partial f(\mathbf{u}^\star).\end{cases} \tag{19}$$

Since problem is convex, condition (19) is necessary and sufficient for $(\mathbf{u}^\star, \mathbf{v}^\star)$ to be primal and dual optimal to (17):

$$\min_{\mathbf{u}\in\mathbb{R}^p}\max_{\mathbf{v}\in\mathbb{R}^q}\Big\{\langle\mathbf{K}\mathbf{u},\mathbf{v}\rangle + f(\mathbf{u}) - g^*(\mathbf{v})\Big\}.$$

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Fixed-point expression

Second, from (19), for any $\sigma > 0$ and $\tau > 0$, we can write

$$\mathbf{v}^\star + \sigma\mathbf{K}\mathbf{u}^\star \in (\mathbb{I} + \sigma\partial g^*)(\mathbf{v}^\star) \text{ and } \mathbf{u}^\star - \tau\mathbf{K}^T\mathbf{v}^\star \in (\mathbb{I} + \tau\partial f)(\mathbf{u}^\star).$$

Using the proximal operator of τf and σg^*, we can write the last expression as

$$\begin{cases} \mathbf{v}^\star & = \mathrm{prox}_{\sigma g^*}\left(\mathbf{v}^\star + \sigma\mathbf{K}\mathbf{u}^\star\right) \\ \mathbf{u}^\star & = \mathrm{prox}_{\tau f}\left(\mathbf{u}^\star - \tau\mathbf{K}^T\mathbf{v}^\star\right). \end{cases} \tag{20}$$

This relation shows that $\mathbf{x}^\star := (\mathbf{u}^\star, \mathbf{v}^\star)$ is a fixed point of the **mapping** $\mathcal{T}$ with:

$$\mathcal{T}(\mathbf{x}) := (\mathrm{prox}_{\sigma g^*}(\mathbf{v} + \sigma\mathbf{K}\mathbf{u}), \mathrm{prox}_{\tau f}(\mathbf{u} - \tau\mathbf{K}^T\mathbf{v})).$$

The Chambolle-Pock algorithm

The **Chambolle-Pock algorithm** is rooted from the classical Arrow-Hurwicz method, which is based on the fixed-point expression (20).

Chambolle-Pock's algorithm (CPA) [2]

1. Choose $\tau > 0$, $\sigma > 0$, $\theta \in [0, 1]$, $\mathbf{u}^0 \in \mathbb{R}^p$ and $\mathbf{v}^0 \in \mathbb{R}^q$. Set $\hat{\mathbf{u}}^0 := \mathbf{u}^0$.
2. For $k = 0, 1, \cdots$, perform:

$$\begin{cases} \mathbf{v}^{k+1} & := \mathrm{prox}_{\sigma g^*}\left(\mathbf{v}^k + \sigma \mathbf{K}\hat{\mathbf{u}}^k\right) \\ \mathbf{u}^{k+1} & := \mathrm{prox}_{\tau f}\left(\mathbf{u}^k - \tau \mathbf{K}^T \mathbf{v}^k\right) \\ \hat{\mathbf{u}}^{k+1} & := \mathbf{u}^{k+1} + \theta(\mathbf{u}^{k+1} - \mathbf{u}^k). \end{cases} \tag{21}$$

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Remarks

- If $\theta = 0$, then $\hat{\mathbf{u}}^k = \mathbf{u}^k$ and (21) collapses to the Arrow-Hurwicz method:

$$\begin{cases} \mathbf{v}^{k+1} & := \mathrm{prox}_{\sigma g^*}\left(\mathbf{v}^k + \sigma \mathbf{K}\mathbf{u}^k\right) \\ \mathbf{u}^{k+1} & := \mathrm{prox}_{\tau f}\left(\mathbf{u}^k - \tau \mathbf{K}^T\mathbf{v}^k\right). \end{cases}$$

- The step sizes σ and τ and the parameter θ can keep constantly or adaptively updates.
- When $\mathbf{K} = \mathbb{I}$, the **Chambolle-Pock algorithm** is equivalent to **ADMM**.

Restricted gap function for optimality certification

Let us define $\mathbf{x} := (\mathbf{u}, \mathbf{v}) \equiv (\mathbf{u}^T, \mathbf{v}^T)^T$, $\phi(\mathbf{x}) := f(\mathbf{u}) + g^\star(\mathbf{v})$ and

$$M(\mathbf{x}) := \begin{bmatrix} -\mathbf{K}^T\mathbf{v} \\ \mathbf{K}\mathbf{u} \end{bmatrix}.$$

Then (19): $\begin{cases} \mathbf{K}\mathbf{u}^\star \in \partial g^*(\mathbf{v}^\star) \\ -\mathbf{K}^T\mathbf{v}^\star \in \partial f(\mathbf{u}^\star) \end{cases}$ can be written as a **monotone MVI problem**:

$$\boxed{\phi(\mathbf{x}) - \phi(\mathbf{x}^\star) + M(\mathbf{x}^\star)^T(\mathbf{x} - \mathbf{x}^\star) \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^{p_1+p_2}} \qquad (22)$$

where $\mathbf{x}^\star := (\mathbf{u}^\star, \mathbf{v}^\star)$.

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where $\mathbf{x}^\star := (\mathbf{u}^\star, \mathbf{v}^\star)$.

Definition (Restricted gap function)

Let $\mathcal{X} \subseteq \mathbb{R}^{p_1+p_2}$ be a nonempty, closed, convex and bounded set. We define a **restricted gap function** of (22) restricted on $\mathcal{X}$ as

$$\boxed{G_{\mathcal{X}}(\mathbf{x}) := \max_{\tilde{\mathbf{x}} \in \mathcal{X} \subseteq \mathbb{R}^{p_1+p_2}} \left\{ \phi(\mathbf{x}) - \phi(\tilde{\mathbf{x}}) + M(\mathbf{x})^T(\mathbf{x} - \tilde{\mathbf{x}}) \right\}} \tag{23}$$

Convergence theorem

Theorem (Convergence)

Assumptions:

- (17): $\min_{\mathbf{u}\in\mathbb{R}^{p1}} \max_{\mathbf{v}\in\mathbb{R}^{p2}} \Big\{ \langle \mathbf{K}\mathbf{u}, \mathbf{v}\rangle + f(\mathbf{u}) - g^*(\mathbf{v}) \Big\}$ *has a saddle point* $\mathbf{z}^\star := (\mathbf{u}^\star, \mathbf{v}^\star)$.
- $\{(\mathbf{u}^k, \mathbf{v}^k)\}$ *be the sequence generated by the* ***Chambolle-Pock algorithm****.*
- *If we choose* $\theta := 1$, $\sigma > 0$ *and* $\tau > 0$ *such that* $\tau\sigma\|\mathbf{K}\|^2 < 1$.

Conclusion:

- *The sequence* $\{\bar{\mathbf{x}}^k\}_{k\geq 0}$ *defined by*

$$\bar{\mathbf{x}}^k = (\bar{\mathbf{u}}^k, \bar{\mathbf{v}}^k) := \frac{1}{(k+1)} \sum_{j=0}^{k} (\mathbf{u}^j, \mathbf{v}^j)$$

satisfies

$$G_{\mathcal{X}}(\bar{\mathbf{x}}^k) \leq \frac{1}{k+1}\Big[\max_{\mathbf{x}:=(\mathbf{u},\mathbf{v})\in\mathcal{X}} \Big\{ (1/(2\tau))\|\mathbf{u}-\mathbf{u}^0\|^2 + (1/(2\sigma))\|\mathbf{v}-\mathbf{v}^0\|^2 \Big\} \Big]. \quad (24)$$

- $\{\bar{\mathbf{z}}^k\}_{k\geq 0}$ *converges to a saddle point* $\mathbf{z}^\star$ *of* (17) *at the* $\mathcal{O}(1/k)$ *rate w.r.t. the restricted gap function* $G_{\mathcal{X}}$ *(in the ergodic sense).*

Convergence theorem: Remarks

$$G_{\mathcal{X}}(\bar{\mathbf{x}}^k) \leq \frac{1}{k+1}\Big[\max_{\mathbf{x}:=(\mathbf{u},\mathbf{v})\in\mathcal{X}}\Big\{(1/(2\tau))\|\mathbf{u}-\mathbf{u}^0\|^2 + (1/(2\sigma))\|\mathbf{v}-\mathbf{v}^0\|^2\Big\}\Big]. \quad (24)$$

- The right-hand side of the estimate (24) depends on the choice of $\mathcal{X}$. Theoretically, we need to choose $\mathcal{X}$ such that $\mathcal{X}^\star \subseteq \mathcal{X}$, where $\mathcal{X}^\star$ is the solution set of (17), which is unknown.
- The estimate (24) does not imply the convergence rate of $\{\bar{\mathbf{u}}^k\}$ and $\{\bar{\mathbf{v}}^k\}$ separately.

Acceleration - Case 1: f or $g^\star$ is strongly convex

If either f or $g^\star$ is strongly convex, the **Chambolle-Pock algorithm** can be accelerated to get better convergence rate.

Assumption A.1.

The function f is strongly convex with a strong convexity parameter $\sigma_f > 0$.

- Under Assumption A.1., the conjugate f^* is smooth and has the Lipschitz gradient.
- The **Chambolle-Pock algorithm** can be accelerated to get $\mathcal{O}(1/k^2)$ convergence rate.

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Chambolle-Pock's algorithm for strongly convex (CPA$_1$)

1. Choose $\tau > 0$, $\sigma > 0$ such that $\sigma_0\tau_0 L^2 \leq 1$, $\mathbf{u}^0 \in \mathbb{R}^p$ and $\mathbf{v}^0 \in \mathbb{R}^q$. Set $\hat{\mathbf{u}}^0 := \mathbf{u}^0$.

2. For $k = 0, 1, \cdots$, perform:

$$\begin{cases} \mathbf{v}^{k+1} & := \mathrm{prox}_{\sigma_k g^*}\left(\mathbf{v}^k + \sigma_k \mathbf{K}\hat{\mathbf{u}}^k\right) \\ \mathbf{u}^{k+1} & := \mathrm{prox}_{\tau_k f}\left(\mathbf{u}^k - \tau_k \mathbf{K}^T\mathbf{v}^k\right) \\ \theta_k & := (1 + 2\sigma_f\tau_k)^{-1/2}, \\ \tau_{k+1} & := \theta_k\tau_k, \\ \sigma_{k+1} & := \theta_k^{-1}\sigma_k, \\ \hat{\mathbf{u}}^{k+1} & := \mathbf{u}^{k+1} + \theta_k(\mathbf{u}^{k+1} - \mathbf{u}^k). \end{cases} \tag{25}$$

Acceleration - Case 2: Both f and g^* are strongly convex

If both f and g^* are strongly convex, we can accelerate the **Chambolle-Pock algorithm** to obtain the linear convergence rate.

Assumption A.2.

The functions f and g^* are strongly convex with a strong convexity parameters $\sigma_f > 0$ and $\sigma_{g^*} > 0$, respectively.

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Chambolle-Pock's algorithm for strongly convex f and g^* (CPA$_2$)

1. Choose $\omega \leq 2\sqrt{\sigma_f \sigma_{g^*}} L$.
2. Set $\tau := \omega/(2\sigma_f)$, $\sigma := \omega/(2\sigma_{g^*})$ and choose $\theta \in [(1+\omega)^{-1}, 1]$.
3. Find $\mathbf{u}^0 \in \mathbb{R}^p$ and $\mathbf{v}^0 \in \mathbb{R}^q$. Set $\hat{\mathbf{u}}^0 := \mathbf{u}^0$.
4. For $k = 0, 1, \cdots$, perform:

$$\begin{cases} \mathbf{v}^{k+1} & := \mathrm{prox}_{\sigma g^*}\left(\mathbf{v}^k + \sigma \mathbf{K} \hat{\mathbf{u}}^k\right) \\ \mathbf{u}^{k+1} & := \mathrm{prox}_{\tau f}\left(\mathbf{u}^k - \tau \mathbf{K}^T \mathbf{v}^k\right) \\ \hat{\mathbf{u}}^{k+1} & := \mathbf{u}^{k+1} + \theta(\mathbf{u}^{k+1} - \mathbf{u}^k). \end{cases} \quad (26)$$

Example: Strong convexity of f/g^*

Example (Strong convexity of f)

We consider the following image ℓ_1-TV denoising problem:

$$\min_{\mathbf{u}\in\mathbb{R}^{p_1}} (1/2)\|\mathbf{u}-\mathbf{b}\|_F^2 + \rho\|\mathbf{D}\mathbf{u}\|_1. \tag{27}$$

Here $\mathbf{b}$ is a noisy image, $\rho > 0$ is a regularization parameter, and $\mathbf{D}$ is a given matrix. We can write this problem into the following minmax form:

$$\min_{\mathbf{u}\in\mathbb{R}^{p_1}} \max_{\|\mathbf{v}\|_\infty\le 1} \left\{\rho\mathbf{v}^T\mathbf{D}\mathbf{u} + (1/2)\|\mathbf{u}-\mathbf{b}\|_F^2\right\}.$$

In this case, we have $f(\mathbf{u}) := (1/2)\|\mathbf{u}-\mathbf{b}\|_F^2$, which is strongly convex with the parameter $\mu_f = 1$, and $g^*(\mathbf{v}) = 0$.

Example (Strong convexity of both f and g^*)

If we apply Nesterov's smoothing technique to (27) with a simple prox-function $(1/2)\|\mathbf{v}\|_F^2$, we obtain the following problem:

$$\min_{\mathbf{u}\in\mathbb{R}^{p_1}} \max_{\|\mathbf{v}\|_\infty\le 1} \left\{\rho\mathbf{v}^T\mathbf{D}\mathbf{u} + (1/2)\|\mathbf{u}-\mathbf{b}\|_F^2 - (\gamma/2)\|\mathbf{v}\|_F^2\right\}.$$

where $\gamma > 0$ is a smoothness parameter. In this case, we can denote $g^*(\mathbf{v}) := (\gamma/2)\|\mathbf{v}\|_F^2$, which is strongly convex with the parameter $\gamma > 0$.

Convergence of CPA_1 and CPA_2

- **Assumptions**:
  - f is strongly convex with a strong convexity parameter $\sigma_f > 0$.
  - Let $\{(\mathbf{u}^k, \mathbf{v}^k)\}_{k\geq 0}$ be the sequence generated by CPA_1.
  - Let $\tau_0 > 0$ and $\sigma_0 := 1/(\tau_0 L^2)$.

Conclusion: Then for any $\epsilon > 0$, the exists K_0 (depending on ϵ and $\sigma_f \tau_0$) such that for any $k \geq K_0$,

$$\|\bar{\mathbf{u}}^k - \mathbf{u}^\star\|^2 \leq \frac{1+\epsilon}{(k+1)^2}\left(\frac{\|\mathbf{u}^0 - \mathbf{u}^\star\|^2}{\sigma_f^2 \tau_0^2} + \frac{L^2}{\sigma_f^2}\|\mathbf{v}^0 - \mathbf{v}^\star\|^2\right),$$

where $\bar{\mathbf{u}}^k := (k+1)^{-1}\sum_{j=0}^k \mathbf{u}^j$. The sequence $\{\bar{\mathbf{u}}^k\}_{k\geq 0}$ converges to $\mathbf{u}^\star$ at the $\mathcal{O}(1/k^2)$ rate.

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- **Assumptions**:
  - f is strongly convex with a strong convexity parameter $\sigma_f > 0$.
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  - Let $\tau_0 > 0$ and $\sigma_0 := 1/(\tau_0 L^2)$.

  Conclusion: Then for any $\epsilon > 0$, the exists K_0 (depending on ϵ and $\sigma_f \tau_0$) such that for any $k \geq K_0$,

$$\|\bar{\mathbf{u}}^k - \mathbf{u}^\star\|^2 \leq \frac{1+\epsilon}{(k+1)^2}\left(\frac{\|\mathbf{u}^0 - \mathbf{u}^\star\|^2}{\sigma_f^2 \tau_0^2} + \frac{L^2}{\sigma_f^2}\|\mathbf{v}^0 - \mathbf{v}^\star\|^2\right),$$

  where $\bar{\mathbf{u}}^k := (k+1)^{-1}\sum_{j=0}^k \mathbf{u}^j$. The sequence $\{\bar{\mathbf{u}}^k\}_{k\geq 0}$ converges to $\mathbf{u}^\star$ at the $\mathcal{O}(1/k^2)$ rate.

- **Assumptions:**
  - f and g^* are strongly convex with a strong convexity parameters $\sigma_f > 0$ and $\sigma_{g^*} > 0$, respectively.
  - Let $\{(\mathbf{u}^k, \mathbf{v}^k)\}_{k\geq 0}$ be the sequence generated by CPA_2.
  - Let

$$c := \frac{1+\theta}{2(1+\sqrt{\sigma_f \sigma_{g^*}}/L)} < 1.$$

  Conclusion: Then $\{(\mathbf{u}^k, \mathbf{v}^k)\}_{k\geq 0}$ converges to $(\mathbf{u}^\star, \mathbf{v}^\star)$ at linear rate $\mathcal{O}(c^k)$.

Example: Image inpainting

Mathematical formulation

Given a damaged image $\mathbf{b} \in \mathbb{R}^{m\times n}$, where the missed pixels b_{ij} are in certain region, i.e., $(i,j) \in \mathcal{M} \subset \mathcal{I} := \{1, \cdots, m\} \times \{1, \cdots, n\}$. The aim is to recover a undamaged image $\mathbf{x}$ by using the total variation operator. This problem can be formulated as:

$$\min_{\mathbf{u}\in\mathbb{R}^{m\times n}} \|\mathbf{K}\mathbf{u}\|_1 + (\rho/2) \sum_{(i,j)\in\mathcal{I}\setminus\mathcal{M}} (u_{ij} - b_{ij})^2 \tag{28}$$

where $\rho > 0$ is a regularization parameter and $\mathbf{K}$ is the total variation linear transform.

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How to apply the **Chambolle-Pock algorithm**?

- This problem is of the form $F^\star := \min_{\mathbf{u} \in \mathbb{R}^{p_1}} \left\{ F(\mathbf{u}) := f(\mathbf{u}) + g(\mathbf{K}\mathbf{u}) \right\}$, where $f(\mathbf{u}) := (\rho/2) \sum_{(i,j) \in \mathcal{I} \setminus \mathcal{M}} (u_{ij} - b_{ij})^2$ and $g(\mathbf{v}) := \|\mathbf{v}\|_1$.
- Both f and g have closed form prox-operators:

$$\begin{aligned} \mathrm{prox}_{\sigma g^*}(\mathbf{v}) &= \mathbf{v}./\max(1, |v|) \\ \mathrm{prox}_{\tau f}(\mathbf{u}) &= \begin{cases} u_{ij} & \text{if} (i,j) \in \mathcal{M} \\ \frac{u_{ij} + \tau\rho b_{ij}}{1 + \tau\rho} & \text{otherwise} \end{cases} \end{aligned}$$

Example: Image inpainting - configuration

We implement Chambolle-Pock's algorithm for solving the inpainting problem (28) using the following configurations:

- **Parameter selection:**
  - $\sigma = 10$, $\tau = 0.01125$ and $\theta = 1$.
  - The initial point $\mathbf{u}^0 := \mathbf{b}$ and $\mathbf{v}^0 := 0$.
  - The tolerance $\epsilon = 10^{-5}$.
- **Stopping criterion:**
$$\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_F \leq \epsilon \|\mathbf{u}^k\|_F.$$
- **Data generating:**
  - We take a real gray image of size 255×255
  - The image is damaged by a mask of 30 lines crossing from the left to the right.
  - The regularization parameter ρ is chosen as $\rho = 1$ and $\rho = 0.75$ for two cases.

Convergence behavior: Left: $\rho = 1$ – Right: $\rho = 0.75$.

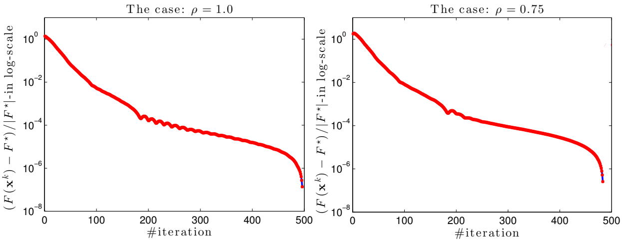


Image inpainting: outputs

Original image

Damaged image

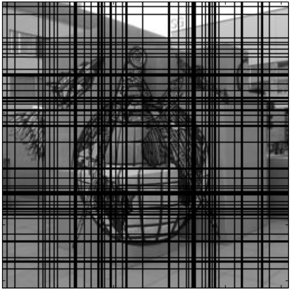
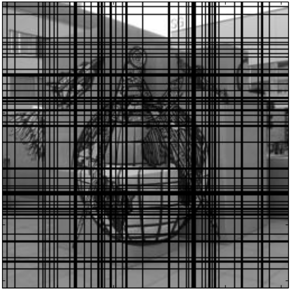


Image inpainting: outputs

Original image

Damaged image



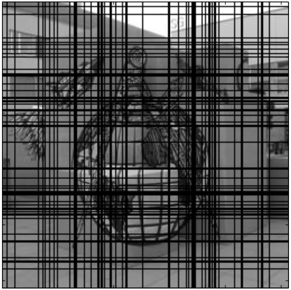
- The objective value: 9.1323×10^3
- Relative error:
  $$\|\mathbf{x}^k - \mathbf{x}^\natural\| / \|\mathbf{x}^\natural\| = 0.076398$$
  where $\mathbf{x}^\natural$ is the original image
- The number of iterations: 497
- The CPU time: 2.864s.

- The objective value: 5.9100×10^3
- Relative error:
  $$\|\mathbf{x}^k - \mathbf{x}^\natural\| / \|\mathbf{x}^\natural\| = 0.078396$$
  where $\mathbf{x}^\natural$ is the original image
- The number of iterations: 484
- The CPU time: 2.707s.

Image inpainting: outputs

Original image

Damaged image



Recovered image ($\rho = 1$)

Recovered image ($\rho = 0.75$)

Outline

- Today
  1. Convex constrained optimization and motivating examples
  2. Optimality condition
  3. Conjugate functions
  4. Monotone inclusion and monotone mixed variational inequality formulations
  5. Chambolle-Pock's primal-dual method
  6. Primal-dual hybrid gradient method
  7. Splitting methods
  8. Model-based excessive gap primal-dual method
- Next week
  1. Disciplined convex programming

Splitting methods

- **Splitting methods** have been widely used to solve monotone inclusions involving the **sum of two** maximal monotone operators.
- They can be used to solve the constrained convex optimization problem (2).

From the first line of (19) we have $\mathbf{v}^\star \in \partial g(\mathbf{K}\mathbf{u}^\star)$. Plug this into the second line of (19) to get

$$\boxed{0 \in \partial f(\mathbf{u}^\star) + \mathbf{K}^T \partial g(\mathbf{K}\mathbf{u}^\star)} \tag{29}$$

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$$\boxed{0 \in \partial f(\mathbf{u}^\star) + \mathbf{K}^T \partial g(\mathbf{K}\mathbf{u}^\star)} \quad (29)$$

Splitting monotone inclusion

Assumptions:

- Define $A(\mathbf{u}) := \mathbf{K}^T \partial g(\mathbf{K}\mathbf{u})$ and $B(\mathbf{u}) := \partial f(\mathbf{u})$.
- Assume that $\mathbf{K}$ is **full rank**.

Conclusion:

- A and B are two maximal monotone operators.
- (29) can be expressed as:

$$\boxed{0 \in A(\mathbf{u}^\star) + B(\mathbf{u}^\star)} \quad (30)$$

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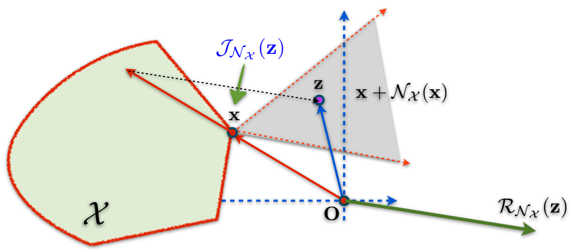
- They can be used to solve the constrained convex optimization problem (2).

Reflection operator of the **resolvent**

Let $\mathcal{J}_{\mathcal{T}}(\cdot) := (\mathbb{I} + \mathcal{T})^{-1}(\cdot)$ be the **resolvent** of a maximal monotone operator $\mathcal{T}$. We define the reflection operator of $\mathcal{J}_{\mathcal{T}}$ as

$$\boxed{\mathcal{R}_{\mathcal{T}}(\mathbf{z}) := 2\mathcal{J}_{\mathcal{T}}(\mathbf{z}) - \mathbf{z}.}$$

Example (The reflection operator of the normal cone $\mathcal{N}_{\mathcal{X}}$)



Deriving a fixed-point formulation

- **A splitting method** generates an iterative sequence $\{\mathbf{u}^k\}$ by using a fixed-point derivation of the inclusion (30): $0 \in A(\mathbf{u}) + B(\mathbf{u})$.
- In addition, it splits the **computations** such that one can exploit the individual computations of A and B separately.

Deriving a fixed-point formulation

Starting from $0 \in A(\mathbf{u}) + B(\mathbf{u})$, we can rewrite

$$2\mathbf{u} \in (\mathbb{I} + A)(\mathbf{u}) + (\mathbb{I} + B)(\mathbf{u}). \quad (31)$$

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$$2\mathbf{u} - \mathbf{z} \in (\mathbb{I} + A)(\mathbf{u}) \quad \Leftrightarrow \quad 2\mathcal{J}_B(\mathbf{z}) - \mathbf{z} \in (\mathbb{I} + A)(\mathbf{u}).$$

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Using the resolvent of B, we can express the last inclusion as

$$\mathbf{u} = \mathcal{J}_A(2\mathcal{J}_B(\mathbf{z}) - \mathbf{z}) = \mathcal{J}_A(\mathcal{R}_B(\mathbf{z})).$$

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Hence, by using the reflection operator of B, we can rewrite this equivalently to

$$\mathbf{z} = 2\mathcal{J}_A(\mathcal{R}_B(\mathbf{z})) - (2\mathbf{u} - \mathbf{z}) = 2\mathcal{J}_A(\mathcal{R}_B(\mathbf{z})) - \mathcal{R}_B(\mathbf{z}).$$

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Using the reflection operator of A we finally get

$$\boxed{\mathbf{z} = \mathcal{R}_A(\mathcal{R}_B(\mathbf{z})).}$$

Fixed-point characterization and splitting computation

Fixed-point formulation

- If $\mathbf{u}^\star$ is a solution of (30) (cf. $0 \in A(\mathbf{u}^\star) + B(\mathbf{u}^\star)$) then

$$\boxed{\mathbf{z}^\star = \mathcal{R}_A\left(\mathcal{R}_B(\mathbf{z}^\star)\right) \quad \text{and} \quad \mathbf{u}^\star = \mathcal{J}_B(\mathbf{z}^\star).} \tag{32}$$

- Alternatively, if $\mathbf{u}^\star$ is a solution of (30) then for any $\beta \neq 0$, we have

$$\boxed{\mathbf{z}^\star = (1-\beta)\mathbf{z}^\star + \beta\mathcal{R}_A\left(\mathcal{R}_B(\mathbf{z}^\star)\right) \quad \text{and} \quad \mathbf{u}^\star = \mathcal{J}_B(\mathbf{z}^\star).} \tag{33}$$

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Splitting computation

Let assume that our iterative scheme is based on the fixed point formulation (33) as:

$$\mathbf{z}^{k+1} := (1-\beta)\mathbf{z}^k + \beta\mathcal{R}_A\left(\mathcal{R}_B(\mathbf{z}^k)\right).$$

We can split this formula by using the definition of $\mathcal{R}$ and $\mathcal{J}$.

- Let $\mathbf{u}^k := \mathcal{J}_B(\mathbf{z}^k)$. Then $\mathcal{R}_B(\mathbf{z}^k) = 2\mathbf{u}^k - \mathbf{z}^k$.

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Splitting: Douglas-Rachford's method

Fixed-point iteration

The **Douglas-Rachford method** bases on the fixed-point formulation (33) to generate an iterative sequence as:

$$\mathbf{z}^{k+1} := (1-\beta_k)\mathbf{z}^k + \beta_k \mathcal{R}_A\left(\mathcal{R}_B(\mathbf{z}^k)\right) \quad \text{and} \quad \mathbf{u}^k = \mathcal{J}_B(\mathbf{z}^k).$$

By **splitting the computation** as in the previous slide, we can summarize this scheme as:

$$\begin{cases} \mathbf{u}^k & := \mathcal{J}_B(\mathbf{z}^k) \\ \mathbf{v}^k & := \mathcal{J}_A(2\mathbf{u}^k - \mathbf{z}^k) \\ \mathbf{z}^{k+1} & := \mathbf{z}^k + \eta_k(\mathbf{v}^k - \mathbf{u}^k) \end{cases}$$

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Douglas-Rachford's algorithm (DRA)
1. Given $\mathbf{z}^0 \in \text{dom}(B)$ as an initial point and $\eta_0 \neq 0$.
2. For $k = 0, 1, \cdots$, perform: $\begin{cases} \mathbf{u}^k & := \mathcal{J}_B(\mathbf{z}^k) \\ \mathbf{v}^k & := \mathcal{J}_A(2\mathbf{u}^k - \mathbf{z}^k) \\ \mathbf{z}^{k+1} & := \mathbf{z}^k + \eta_k(\mathbf{v}^k - \mathbf{u}^k) \end{cases}$
and update η_k if required.

Douglas-Rachford method for convex problem (15)

In order to apply DRA for solving (15), we define $A(\mathbf{u}) := \tau\mathbf{K}^T\partial g(\mathbf{K}\mathbf{u})$ and $B(\mathbf{u}) := \tau\partial f(\mathbf{u})$ for a given scaling factor $\tau > 0$.

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2. For $k = 0, 1, \cdots$, perform:

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Remark: Quadratic loss and diagonalizable operator

- **Assumptions:** $g(\mathbf{K}\mathbf{v}) := (1/2)\|\mathbf{K}\mathbf{v} - \mathbf{b}\|_2^2$ and $\mathbf{K}^T\mathbf{K} = \Sigma$, where Σ is diagonal.
- **Conclusion:** $\boxed{\mathbf{v}^k := (\tau_k^{-1}\mathbb{I} + \Sigma)^{-1}(\mathbf{K}^T\mathbf{b} + \tau_k^{-1}\mathbf{s}^k)}$.
  - Let $\mathbf{s}^k := 2\mathbf{u}^k - \mathbf{z}^k$. We can write the optimality condition of $\mathbf{v}^k$ as

$$(\tau_k^{-1}\mathbb{I} + \mathbf{K}^T\mathbf{K})\mathbf{v}^k = \mathbf{K}^T\mathbf{b} + \tau_k^{-1}\mathbf{s}^k.$$

  - Since $\mathbf{K}^T\mathbf{K} = \Sigma$, we can compute $\mathbf{v}^k$ explicitly as

$$\mathbf{v}^k = (\tau_k^{-1}\mathbb{I} + \Sigma)^{-1}(\mathbf{K}^T\mathbf{b} + \tau_k^{-1}\mathbf{s}^k).$$

Convergence of splitting methods

Theorem (Convergence of Douglas-Rachford's method [4])

Assume that the ***solution set*** $\mathcal{U}^\star$ *of* (30) *is nonempty and the sequence* $\{\eta_k\}$ *is chosen such that*

$$\eta_k \in [0,2] \ \text{ and } \ \sum_{k=0}^{+\infty} \eta_k(2-\eta_k) = \infty. \tag{34}$$

Then, the sequence $\{\mathbf{u}^k\}$ *generated by the* ***Douglas-Rachford algorithm*** *converges to a* ***solution*** $\mathbf{u}^\star$ *in* $\mathcal{U}^\star$.

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Remarks

- We can choose η_k as a constant step, e.g., $\eta_k = 1$ for $k \geq 0$.
- If $\eta_k = 1$ for all $k \geq 0$, then **Douglas-Rachford's algorithm** coincides with **Peaceman-Rachford's method** [10].
- When **Douglas-Rachford's algorithm** is applied to solve (15), it coincides with **ADMM** [3].

Alternative derivation

Assumptions

- The **solution set** $\mathcal{U}^\star$ of $0 \in A(\mathbf{u}) + B(\mathbf{u})$ is nonempty
- For simplicity of discussion, we assume that B is single-valued and $\beta = 1/2$.

By using the definition of $\mathcal{R}_A = 2\mathcal{J}_A - \mathbb{I}$ and $\mathcal{R}_B = 2\mathcal{J}_B - \mathbb{I}$, the **Douglas-Rachford** iterative scheme

$$\begin{cases} \mathbf{u}^k & := \mathcal{J}_B(\mathbf{z}^k), \\ \mathbf{z}^{k+1} & := (1/2)\mathbf{z}^k + (1/2)\mathcal{R}_A(\mathcal{R}_B(\mathbf{z}^k)). \end{cases}$$

can be expressed as

$$\mathbf{u}^{k+1} := \mathcal{J}_B\left(\mathcal{J}_A(2\mathcal{J}_B(\mathcal{J}_B^{-1}(\mathbf{u}^k)) - \mathcal{J}_B^{-1}(\mathbf{u}^k)) + \mathcal{J}_B^{-1}(\mathbf{u}^k) - \mathcal{J}_B(\mathcal{J}_B^{-1}(\mathbf{u}^k))\right). \quad (35)$$

- First, it is obvious that $\mathcal{J}_B(\mathcal{J}_B^{-1}(\mathbf{u}^k)) = \mathbf{u}^k$
- Second, by definition of $\mathcal{J}_B$, we also have $\mathcal{J}_B^{-1}(\mathbf{u}^k) = \mathbf{u}^k + B(\mathbf{u}^k)$.

Substituting these relations into (35), we obtain

$$\boxed{\mathbf{u}^{k+1} := \mathcal{J}_B\left(\mathcal{J}_A(\mathbf{u}^k - B(\mathbf{u}^k)) + B(\mathbf{u}^k)\right)} \quad (36)$$

We can rewrite (36) as

$$\boxed{(\mathbb{I} + B)(\mathbf{u}^{k+1}) \in (\mathbb{I} + B)(\mathbf{u}^k) - e(\mathbf{u}^k)} \quad (37)$$

where $e(\mathbf{u}^k) := \mathbf{u}^k - \mathcal{J}_A(\mathbf{u}^k - B(\mathbf{u}^k))$.

Convergence rate

Facts:

- It is obvious that $\mathbf{u}^\star = \mathcal{J}_A(\mathbf{u}^\star - B(\mathbf{u}^\star))$ for any $\mathbf{u}^\star \in \mathcal{U}^\star$.
- We define $e(\mathbf{u}) := \mathbf{u} - \mathcal{J}_A(\mathbf{u} - B(\mathbf{u}))$ the residual operator at $\mathbf{u}$. Then $e(\mathbf{u}^\star) = 0$.
- Let $\mathcal{T} := \mathbb{I} + A$. The **Douglas-Rachford** scheme (37) can be written as

$$\boxed{\mathcal{T}(\mathbf{u}^{k+1}) \in \mathcal{T}(\mathbf{u}^k) - e(\mathbf{u}^k)} \tag{38}$$

Theorem (Convergence rate [7])

*Assume that B is single-valued and the **solution set** $\mathcal{U}^\star$ of $0 \in A(\mathbf{u}) + B(\mathbf{u})$ is nonempty. Let $\{\mathbf{u}^k\}_{k\geq 0}$ be the sequence generated by the **Douglas-Rachford algorithm** with $\eta_k = 1$ for all $k \geq 0$. Then for any **solution** $\mathbf{u}^\star$ of $0 \in A(\mathbf{u}) + B(\mathbf{u})$, we have*

$$\|e(\mathbf{u}^k)\|^2 \leq \frac{1}{k+1}\|(\mathbf{u}^0 - \mathbf{u}^\star) + (B(\mathbf{u}^0) - B(\mathbf{u}^\star))\|^2. \tag{39}$$

The sequence $\{\|e(\mathbf{u}^k)\|^2\}$ converges to zero at the $\mathcal{O}(1/k)$ rate. Consequently, $\{\mathbf{u}^k\}$ converges to a solution $\mathbf{u}^\star$ of $0 \in A(\mathbf{u}) + B(\mathbf{u})$.

Remarks

- Since we assume that B is single-valued, the right-hand side (39) is bounded.
- In the case B is a set-valued mapping, the **right-hand side** may be no longer bounded. For example, if $B = \mathcal{N}_{\mathcal{Z}}$ the normal cone of a convex set $\mathcal{Z}$.

Example: Image denoising

TV-denoising

Given a noisy image $\mathbf{b} \in \mathbb{R}^{m\times n}$, we want to recover a clean image $\mathbf{x}$ by using the total variation operator. This problem can be formulated as:

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^{m\times n}} \frac{1}{2}\|\mathbf{x}-\mathbf{b}\|^2 + \rho\|\mathbf{x}\|_{\mathrm{TV}},}$$

where $\rho > 0$ is a regularization parameter and

$$\|\mathbf{x}\|_{\mathrm{TV}} := \begin{cases} \sum_{i,j} |x_{i,j+1} - x_{i,j}| + |x_{i,j+1} - x_{i,j}| & \text{anisotropic (ATV)} \\ \sum_{i,j} \sqrt{(x_{i,j+1} - x_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2} & \text{isotropic (ITV)} \end{cases}$$

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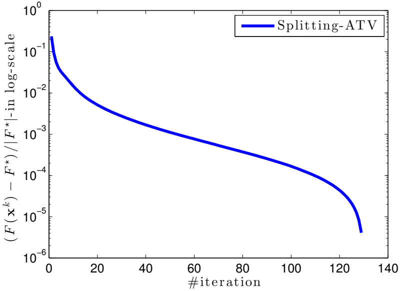
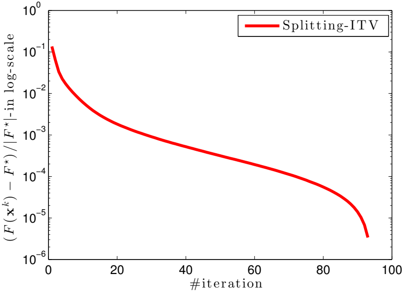
How to apply the **splitting algorithm**?

- By letting $\mathbf{z} = \mathbf{D}\mathbf{x}$, we can convert the TV-denoising problem into (2), where $\mathbf{D}$ is a matrix representing the **total variation**.
- **Splitting algorithm** is now applied to the resulting problem.
- We choose $\tau_k = \eta_k = 1$ in our test.

Example: Image denoising - convergence behavior

- The convergence of the splitting method using ATV and ITV norms on a gray image of size 512×512.
- The regularization parameter $\rho = 20$.





- The objective value: 9.9319×10^7
- Relative error:
$$\|\mathbf{x}^k - \mathbf{x}^\natural\|/\|\mathbf{x}^\natural\| = 0.0727258$$
where $\mathbf{x}^\natural$ is the original image
- The number of iterations: 131
- The CPU time: 30.6s.
- The PSNR $= 23.1983$.

- The objective value: 9.4897×10^7
- Relative error:
$$\|\mathbf{x}^k - \mathbf{x}^\natural\|/\|\mathbf{x}^\natural\| = 0.0676706$$
where $\mathbf{x}^\natural$ is the original image
- The number of iterations: 95
- The CPU time: 26.0s.
- The PSNR $= 23.6192$.

Example: Image denoising

Original

Noisy

Example: Image denoising

Original

Noisy

Anisotropic TV denoising

Isotropic TV denoising

Primal-dual hybrid gradient (PDHG) algorithm

The idea of PDHG

- Originally, PDHG is a combination of a primal and dual proximal-gradient descent step applying to the min-max problem (17) [17]:

$$\min_{\mathbf{u}\in\mathbb{R}^p}\max_{\mathbf{v}\in\mathbb{R}^q}\Big\{\langle\mathbf{K}\mathbf{u},\mathbf{v}\rangle + f(\mathbf{u}) - g^*(\mathbf{v})\Big\}.$$

- First, PDHG performs a primal proximal-gradient step on the minimization problem w.r.t. $\mathbf{u}$ given $\mathbf{v}^k$ to compute $\mathbf{u}^{k+1}$.
- Second, PDHG performs a dual proximal-gradient step on the maximization problem w.r.t. $\mathbf{v}$ given $\mathbf{u}^{k+1}$ to compute $\mathbf{v}^{k+1}$.
- We can also add an intermediate step $\bar{\mathbf{u}}^{k+1}$ before performing the dual step.

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Primal-Dual Hybrid Gradient algorithm (PDHG) for (15)

1. Given $\mathbf{u}^0 \in \mathbb{R}^p$ and $\mathbf{v}^0 \in \mathbb{R}^n$ as an initial point.
2. Choose $\tau_0 > 0$, $\sigma_0 > 0$ and $\eta_0 \neq 0$.
3. For $k = 0, 1, \cdots$, perform:

$$\begin{cases} \mathbf{u}^{k+1} & := \mathrm{prox}_{\tau_k f}(\mathbf{u}^k - \tau_k\mathbf{K}^T\mathbf{v}^k), \\ \bar{\mathbf{u}}^{k+1} & := \mathbf{u}^{k+1} + \eta_k(\mathbf{u}^{k+1} - \mathbf{u}^k), \\ \mathbf{v}^{k+1} & := \mathrm{prox}_{\sigma_k g^*}(\mathbf{v}^k + \sigma_k\mathbf{K}\bar{\mathbf{u}}^{k+1}). \end{cases}$$

and update τ_k, σ_k and η_k if necessary.

Connection to Chambolle-Pock's algorithm and enhancement

Connection to Chambolle-Pock's algorithm

- **PDHG** is very similar to **Chambolle-Pock's algorithm**.
  - **Chambolle-Pock's algorithm** performs the dual step on $\mathbf{v}$ and then the primal step on $\mathbf{u}$.
  - **PDHG** performs the primal step on $\mathbf{u}$ and then the dual step on $\mathbf{v}$.
  - Symmetrically, we can say that both methods are equivalent.
- The convergence theory of **Chambolle-Pock's algorithm** is applicable to **PDHG**.

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Adaptive PDHG

- If we select the parameters τ_k and σ_k based on the convergence theory, PDHG often has poor performance in practice.
- We can enhance the performance of PDHG by adaptively updating τ_k and σ_k [5].
  - Define the primal and dual residuals $(\mathbf{p}^{k+1}, \mathbf{d}^{k+1})$:

$$\begin{cases} \mathbf{p}^{k+1} & := \tau_k^{-1}(\mathbf{u}^k - \mathbf{u}^{k+1}) - \mathbf{K}^T(\mathbf{v}^k - \mathbf{v}^{k+1}), \\ \mathbf{d}^{k+1} & := \sigma_k^{-1}(\mathbf{v}^k - \mathbf{v}^{k+1}) + \mathbf{K}(\mathbf{u}^k - \mathbf{u}^{k+1}). \end{cases}$$

  - Adaptively update τ_k and σ_k by trading-off the primal residual $\|\mathbf{p}^{k+1}\|$ and dual residual $\|\mathbf{d}^{k+1}\|$ at each iteration.

Outline

- Today
  1. Convex constrained optimization and motivating examples
  2. Optimality condition
  3. Conjugate functions
  4. Monotone inclusion and monotone mixed variational inequality formulations
  5. Chambolle-Pock's primal-dual method
  6. Primal-dual hybrid gradient method
  7. Splitting methods
  8. Model-based excessive gap primal-dual method
- Next week
  1. Disciplined convex programming

Primal-dual method using model-based excessive gap technique

Problem restatement

We consider again problem (2) with additional convex constraint as:

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}\in\mathbb{R}^p} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in \mathcal{X} \end{cases} \tag{40}$$

where f, $\mathbf{A}$ and $\mathbf{b}$ are defined as in (2) and $\mathcal{X}$ is a nonempty, closed, convex and bounded set in $\mathbb{R}^p$.

We recall the variational inequality presenting the optimality condition of (40) as

$$f(\mathbf{x}) - f(\mathbf{x}^\star) + M(\mathbf{z}^\star)^T(\mathbf{z} - \mathbf{z}^\star) \geq 0, \quad \forall \mathbf{z} \in \mathcal{X} \times \mathbb{R}^n \tag{41}$$

where $\mathbf{z}^\star := (\mathbf{x}^\star, \lambda^\star) \in \mathcal{X} \times \mathbb{R}^n$ is a primal-dual solution of (40) and

$$M(\mathbf{z}) := \begin{bmatrix} \mathbf{A}^T\lambda \\ \mathbf{b} - \mathbf{A}\mathbf{x} \end{bmatrix}.$$

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Gap function

The **gap function** of (41) is redefined as (different from (14) at $\hat{\mathbf{z}} \in \mathcal{X} \times \mathbb{R}^n$):

$$G(\mathbf{z}) := \max_{\hat{\mathbf{z}}\in\mathcal{X}\times\mathbb{R}^n} \left\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \right\} \tag{42}$$

Smoothed gap function

- Let b be a prox-function of $\mathcal{X}$ with strong convexity parameter $\sigma_b = 1$.
  - i.e., b is a smooth, strongly convex function with strong convexity parameter $\sigma_b = 1$.
- We define ξ the Bregman distance corresponding to b as

$$\xi(\mathbf{x}, \hat{\mathbf{x}}) := b(\mathbf{x}) - b(\hat{\mathbf{x}}) - \nabla b(\hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}}).$$

- For $\mathbf{x}_c \in \mathbb{R}^p$ and two positive parameters γ and β we define

$$\xi_{\gamma\beta}(\mathbf{z}) := \gamma\xi(\mathbf{x}, \mathbf{x}_c) + (\beta/2)\|\lambda\|_2^2 \tag{43}$$

  a smoother for the gap function (42).
- $\xi_{\gamma\beta}$ is strongly convex and satisfies $\xi_{\gamma\beta}(\mathbf{z}) \geq (\gamma/2)\|\mathbf{x} - \mathbf{x}_c\|_2^2 + (\beta/2)\|\lambda\|_2^2$.

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We define a smoothed version for the gap function G given by (42) as follows:

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Properties of $G_{\gamma\beta}$

- Evaluating $G_{\gamma\beta}$ and its gradient requires to solve a strongly convex program (42)
- $G_{\gamma\beta}(\mathbf{z}) \to G(\mathbf{z})$ as γ and β go to zero for all $\mathbf{z} \in \mathbb{R}^{p+n}$.

Comments on the gap function G and its smooth version $G_{\gamma\beta}$

- The gap function G defined by (42):

$$G(\mathbf{z}) := \max_{\hat{\mathbf{z}} \in \mathcal{X} \times \mathbb{R}^n} \left\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \right\}$$

can be written as

$$G(\mathbf{z}) := \max_{\hat{\mathbf{z}} \in \mathbb{R}^p \times \mathbb{R}^n} \left\{ (f(\mathbf{x}) + \iota_{\mathcal{X}}(\mathbf{x})) - (f(\hat{\mathbf{x}}) + \iota_{\mathcal{X}}(\hat{\mathbf{x}})) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \right\}$$

which is exactly the gap function G in the unconstrained form (14), where $\iota_{\mathcal{X}}$ is the indicator function of $\mathcal{X}$.

- If we define

$$\begin{aligned} d_\gamma(\lambda) &:= \min_{\mathbf{x} \in \mathcal{X}} \left\{ f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) + \gamma \xi(\mathbf{x}; \mathbf{x}_c) \right\} \\ f_\beta(\mathbf{x}) &:= f(\mathbf{x}) + \max_{\lambda \in \mathbb{R}^n} \{ \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) - (\beta/2)\|\lambda\|^2 \} \\ &= f(\mathbf{x}) + (1/(2\beta))\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 \end{aligned}$$

then:

  - d_γ is an approximation of the **dual function** d
  - f_β is an approximation of the **objective function** f.

Moreover, we have

$$G_{\gamma\beta}(\mathbf{z}) = f_\beta(\mathbf{x}) - d_\gamma(\lambda).$$

Illustration of smooth gap function $G_{\gamma\beta}$

- The **objective function** $f(\mathbf{x})$ does not depend on β
- The **dual function** $d(\lambda)$ does not depend on γ:

$$d(\bar{\lambda}) := \min_{\mathbf{x}\in\mathcal{X}}\{f(\mathbf{x}) + \bar{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b})\}.$$

- $f(x)$ is decreasing over $\mathbf{x}$ and $d(\lambda)$ is increasing over λ.

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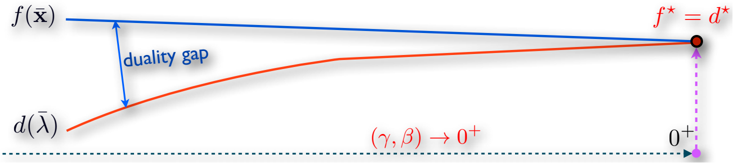

- The duality gap is defined as $G(\bar{\mathbf{z}}) := f(\bar{\mathbf{x}}) - d(\bar{\lambda}) \geq 0$.
- At the optimal solution $\mathbf{z}^\star := (\mathbf{x}^\star, \lambda^\star)$, one has $f(\mathbf{x}^\star) = d(\lambda^\star)$ and $G(\mathbf{z}^\star) = 0$.

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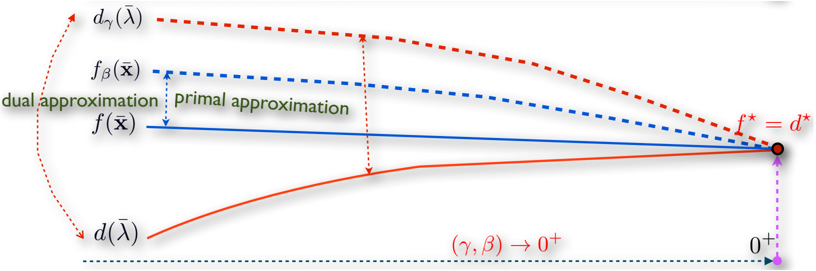

- The augmented function f_β approximates f: $f_\beta(\mathbf{x}) = f(\mathbf{x}) + (1/(2\beta))\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$
- The smoothed dual function d_γ approximates d:

$$d_\gamma(\lambda) := \min_{\mathbf{x}\in\mathcal{X}} \left\{f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) + \gamma\xi(\mathbf{x}; \mathbf{x}_c)\right\}$$

Illustration of smooth gap function $G_{\gamma\beta}$

- The **objective function** $f(\mathbf{x})$ does not depend on β
- The **dual function** $d(\lambda)$ does not depend on γ:

$$d(\bar{\lambda}) := \min_{\mathbf{x}\in\mathcal{X}}\{f(\mathbf{x}) + \bar{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b})\}.$$

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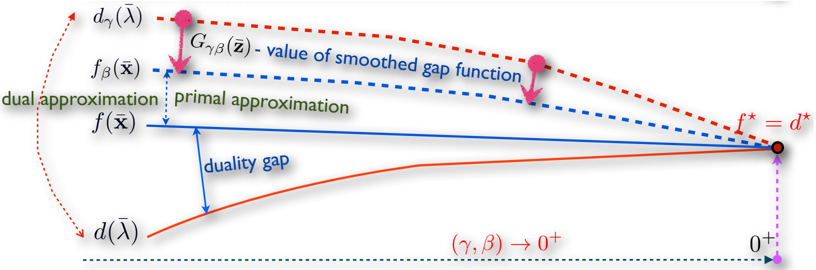

- The smoothed duality gap is defined as $G_{\gamma\beta}(\bar{\mathbf{z}}) := f_\beta(\bar{\mathbf{x}}) - d_\gamma(\bar{\lambda}) \leq 0$.
- At the optimal solution $\mathbf{z}^\star := (\mathbf{x}^\star, \lambda^\star)$, one has $f(\mathbf{x}^\star) = d(\lambda^\star)$ and

$$G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k) \to G(\mathbf{z}^\star) = 0 \text{ as } \gamma_k\beta_k \to 0^+.$$

Model-based excessive gap technique

What is the smoothed gap function used for?

Aim: To generate a primal-dual sequence $\{\bar{\mathbf{z}}^k\}_{k\geq 0}$ with $\bar{\mathbf{z}}^k := (\bar{\mathbf{x}}^k, \bar{\lambda}^k)$ such that

$$G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k) \to 0^+$$

by controlling γ_k and $\beta_k \to 0^+$.

- When γ_k and β_k go to zero, we have $G_{\gamma_k\beta_k}(\cdot) \to G(\cdot)$.
- **Consequence**: $\boxed{G(\mathbf{z}^k) \to 0^+ \quad \Rightarrow \quad \bar{\mathbf{z}}^k \to \mathbf{z}^\star = (\mathbf{x}^\star, \lambda^\star)}$ (primal-dual solution).

Model-based excessive gap condition

A **sequence** $\{\bar{\mathbf{z}}^k\}_{k\geq 0} \subset \mathcal{X} \times \mathbb{R}^n$ is said to satisfy the model-based excessive gap condition if

$$\boxed{G_{\gamma_{k+1}\beta_{k+1}}(\bar{\mathbf{z}}^{k+1}) \leq (1-\tau_k)G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k) - \psi_k} \tag{45}$$

where $\psi_k \geq 0$, $\tau_k \in (0,1)$ and $\gamma_k\beta_{k+1} < \gamma_k\beta_k$ for $k \geq 0$.

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where $\psi_k \geq 0$, $\tau_k \in (0,1)$ and $\gamma_k\beta_{k+1} < \gamma_k\beta_k$ for $k \geq 0$.

Let $\bar{G}_k := G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k)$. By induction, we have

$$\bar{G}_{k+1} \leq \prod_{j=0}^{k}(1-\tau_j)\bar{G}_0 - \left[\psi_0 + \sum_{j=1}^{k-1}\prod_{l=0}^{j-1}(1-\tau_l)\psi_j\right].$$

$\Rightarrow$ The convergence rate of $\{\bar{G}_k\}$ depends on the convergence rate of $\{\tau_k\}$.

Key estimates

- For a bounded set $\mathcal{X}$ and $\hat{\mathbf{x}} \in \mathcal{X}$, the quality $D_{\mathcal{X}}$ defined below is finite

$$D_{\mathcal{X}} := \max_{\mathbf{x} \in \mathcal{X}} \xi(\mathbf{x}, \hat{\mathbf{x}}) < +\infty.$$

- Denote

$$\omega_k := \prod_{j=0}^{k} (1 - \tau_j) \quad \text{and} \quad \Psi_k := \psi_0 + \sum_{j=1}^{k-1} \prod_{l=0}^{j-1} (1 - \tau_l) \psi_j.$$

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Theorem (Bounds on the objective residual and primal feasibility)

Assume that $\{\bar{\mathbf{z}}^k\}_{k\geq 0}$ *is a sequence satisfying* (45). *Then*

$$\begin{cases} -\|\lambda^\star\|\|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| \leq f(\bar{\mathbf{x}}^k) - f^\star \leq C_k, \\ \|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| \leq \beta_k\left[\|\lambda^\star\| + \sqrt{\|\lambda^\star\|^2 + 2\beta_k^{-1}C_k}\right], \end{cases} \tag{46}$$

where $C_k := \omega_{k-1}G_{\gamma_0\beta_0}(\bar{\mathbf{z}}^0) + \gamma_k D_{\mathcal{X}} - \Psi_{k-1}$, *provided that* $\beta_k\|\lambda^\star\| + 2C_k \geq 0$.

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As a consequence, we have

$$\begin{cases} |f(\bar{\mathbf{x}}^k) - f^\star| \leq \max\Big\{\gamma_k D_{\mathcal{X}}, \Big(2\beta_k D_{\Lambda^\star} + \sqrt{2\gamma_k\beta_k D_{\mathcal{X}}}\Big)D_{\Lambda^\star}\Big\}, \\ \|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| \leq 2\beta_k D_{\Lambda^\star} + \sqrt{2\gamma_k\beta_k D_{\mathcal{X}}}, \end{cases} \tag{47}$$

where $D_{\Lambda^\star} := \min\{\|\lambda^\star\| \ : \ \lambda^\star \in \Lambda^\star\}$ the **norm** of the minimum norm solution of the **dual problem**.

Sketch of proof

- From the saddle point inequalities, we have $f^\star = \mathcal{L}(\mathbf{x}^\star, \lambda^\star) \leq \mathcal{L}(\mathbf{x}, \lambda^\star)$. Hence,

$$d(\lambda) \leq f^\star \leq f(\mathbf{x}) + (\mathbf{A}\mathbf{x} - \mathbf{b})^T\lambda^\star \leq f(\mathbf{x}) + \|\mathbf{A}\mathbf{x} - \mathbf{b}\|\|\lambda^\star\|, \ \forall \mathbf{x} \in \mathcal{X}.$$

  We finally get $-\|\lambda^\star\|\|\mathbf{A}\mathbf{x} - \mathbf{b}\| \leq f(\mathbf{x}) - f^\star \leq f(\mathbf{x}) - d(\lambda)$ for all $\mathbf{x} \in \mathcal{X}$.
- Since $\xi(\mathbf{x}, \mathbf{x}_c) \geq 0$ and $d(\lambda) = \min_{\mathbf{x} \in \mathcal{X}} \{f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b})\}$, we have

$$d(\lambda) \leq d_\gamma(\lambda) \leq d(\lambda) + \gamma D_{\mathcal{X}}$$

- Putting things together, we get

$$\begin{aligned} -\|\lambda^\star\|\|\mathbf{A}\mathbf{x} - \mathbf{b}\| &\leq f(\mathbf{x}) - f^\star \leq f(\mathbf{x}) - d(\lambda) \\ &\leq f_\beta(\mathbf{x}) - d_\gamma(\lambda) + \gamma D_{\mathcal{X}} - (1/(2\beta))\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 \\ &= G_{\gamma\beta}(\mathbf{z}) + \gamma D_{\mathcal{X}} - (1/(2\beta))\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2. \end{aligned} \quad (48)$$

- Since $G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k) \leq \omega_{k-1} G_{\gamma_0\beta_0}(\bar{\mathbf{z}}^0) - \Psi_{k-1}$ due to (45), we obtain from (48)

$$-\|\lambda^\star\|\|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| \leq f(\bar{\mathbf{x}}^k) - f^\star \leq \omega_{k-1} G_{\gamma_0\beta_0}(\bar{\mathbf{z}}^0) - \Psi_{k-1} + \gamma_k D_{\mathcal{X}} = C_k. \quad (49)$$

  which is the **first inequality of** (46).
- Let $s := \|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\|$. From (49) and (48) we have $s^2 - 2\beta_k\|\lambda^\star\|s - 2\beta_k C_k \leq 0$. Solving this in equation, we obtain the **second inequality of** (46).

Evaluating the smoothed gap function $G_{\gamma\beta}$

Evaluation of $G_{\gamma\beta}$

In order to evaluate $G_{\gamma\beta}$, we need to solve the maximization problem:

$$G_{\gamma\beta}(\mathbf{z}) := \max_{\hat{\mathbf{z}} \in \mathcal{X} \times \mathbb{R}^n} \left\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) - d_{\gamma\beta}(\hat{\mathbf{z}}) \right\}$$

The solution $\mathbf{z}^\star_{\gamma\beta}(\mathbf{z}) := (\mathbf{x}^\star_\gamma(\lambda), \lambda^\star_\beta(\mathbf{x}))$ of this problem is given as

$$\begin{cases} \mathbf{x}^\star_\gamma(\lambda) & := \arg\min_{\mathbf{x} \in \mathcal{X}} \left\{ f(\mathbf{x}) + (\mathbf{A}^T\lambda)^T\mathbf{x} + \gamma d(\mathbf{x}, \mathbf{x}_c) \right\} \\ \lambda^\star_\beta(\mathbf{x}) & := \beta^{-1}(\mathbf{A}\mathbf{x} - \mathbf{b}). \end{cases} \tag{50}$$

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Given $\bar{\mathbf{z}}^k := (\bar{\mathbf{x}}^k, \bar{\lambda}^k)$ and (γ_k, β_k). The **idea of the algorithms** is to:

- Update $\bar{\mathbf{z}}^{k+1} := (\bar{\mathbf{x}}^{k+1}, \bar{\lambda}^{k+1})$ from $\bar{\mathbf{z}}^k$ and $\mathbf{z}^\star_{\gamma_k\beta_k}(\mathbf{z})$.
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Proximal-gradient step

In our algorithms, we need to compute $\bar{\mathbf{x}}^{k+1}$ using the following mapping:

$$\text{prox}_{\beta f}(\mathbf{x},\lambda) := \arg\min_{\hat{\mathbf{x}}\in\mathcal{X}} \left\{ f(\hat{\mathbf{x}}) + (\mathbf{A}^T\lambda)^T\hat{\mathbf{x}} + (\|\mathbf{A}\|^2/(2\beta))\|\hat{\mathbf{x}}-\mathbf{x}\|_2^2 \right\} \tag{51}$$

How can we generate $\{\bar{\mathbf{z}}^k\}$?

The main idea of generating the **sequence** $\{\bar{\mathbf{z}}^k\}$ such that $\{G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k)\}$ decreases come from the following **observations**:

- Since $G_{\gamma\beta}(\mathbf{z}) = f_\beta(\mathbf{x}) - d_\gamma(\lambda)$, then $G_{\gamma\beta}(\mathbf{z})$ decreases if at least $f_\beta(\mathbf{x})$ decrease or $d_\gamma(\lambda)$ increases.

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  - this function is concave and has Lipschitz gradient ∇d_γ.

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- Every iteration, we can perform one scheme or both in order to decrease $G_{\gamma\beta}(\mathbf{z})$, while simultaneously decrease the **product** $\gamma\beta$.

This observations lead to the following two update schemes in the next slides.

How can we generate $\{\bar{\mathbf{z}}^k\}$?

Expression of the idea

We denote by $\mathbf{s}$ a variable standing for either $\mathbf{x}$ or λ. The **main step** of the algorithmic scheme consists of

- For given $\mathbf{s}$, $\mathbf{s}^\star$ and $\tau \in (0, 1]$ at the current iteration. We perform one interpolation step:

$$\hat{\mathbf{s}} := (1 - \tau)\mathbf{s} + \tau\mathbf{s}^\star.$$

- When $\hat{\mathbf{s}}$ is available, we perform a proximal-gradient or gradient step to compute the next iteration $\mathbf{s}^+$:

$$\mathbf{s}^+ := \mathrm{prox}_\psi(\hat{\mathbf{s}} - (1/L)\nabla\varphi(\hat{\mathbf{s}})).$$

  where
  - $\varphi := d_\gamma$ the smoothed dual function and $\psi = f$ in the primal step
  - $\varphi := (1/(2\beta))\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$ and $\psi = 0$ and in the dual scheme.
  - L is the Lipschitz constant of $\nabla\varphi$.

Remarks

- The above scheme looks very similar to FISTA [1] in the context of Nesterov's accelerating method.

How can we generate $\{\bar{\mathbf{z}}^k\}$?

We propose two schemes to generate the sequence $\{\bar{\mathbf{z}}^k\}$

- The **primal-dual** scheme with two primal steps and one dual step (2P1D);
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Primal-dual schemes

- The (2P1D) scheme generates $\{\bar{z}^k\}$ with $\bar{\mathbf{z}}^k := (\bar{\mathbf{x}}^k, \bar{\lambda}^k)$ as follows:

$$\begin{cases} \hat{\mathbf{x}}^k & := (1-\tau_k)\bar{\mathbf{x}}^k + \tau_k \mathbf{x}^\star_{\gamma_k}(\bar{\lambda}^k) \\ \bar{\mathbf{x}}^{k+1} & := \mathrm{prox}_{\beta_{k+1} f}(\hat{\mathbf{x}}^k, \lambda^\star_{\beta_{k+1}}(\hat{\mathbf{x}}^k)) \\ \bar{\lambda}^{k+1} & := (1-\tau_k)\bar{\lambda}^k + \tau_k \lambda^\star_{\beta_{k+1}}(\hat{\mathbf{x}}^k) \end{cases} \qquad \text{(2P1D)}$$

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- Symmetrically, the (1P2D) scheme generates $\{\bar{z}^k\}$ as follows:

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where $\alpha_k := \gamma_{k+1}\|\mathbf{A}\|^{-2}$.

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  where $\alpha_k := \gamma_{k+1}\|\mathbf{A}\|^{-2}$.
- The parameters β_k and γ_k are updated as ($c_k \in (-1, 1]$ given):

$$\gamma_{k+1} := (1 - c_k\tau_k)\gamma_k \quad \text{and} \quad \beta_{k+1} = (1-\tau_k)\beta_k \qquad (52)$$

Remarks on the computational complexity of both schemes

- (2P1D) requires two primal steps: **one** to compute $\mathbf{x}^\star_{\gamma_k}(\bar{\lambda}^k)$ and **one** to compute $\bar{\mathbf{x}}^{k+1}$.
  - The first step corresponds to solving
  $$\mathbf{x}^\star_{\gamma_k}(\bar{\lambda}^k) := \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ f(\mathbf{x}) + (\mathbf{A}^T\bar{\lambda}^k)^T\mathbf{x} + \gamma_k d(\mathbf{x}, \mathbf{x}_c) \right\}.$$
  - The second step corresponds to solving
  $$\bar{\mathbf{x}}^{k+1} := \arg\min_{\hat{\mathbf{x}}\in\mathcal{X}} \left\{ f(\hat{\mathbf{x}}) + (\mathbf{A}^T\lambda^\star_{\beta_{k+1}}(\hat{\mathbf{x}}^k))^T\hat{\mathbf{x}} + (\|\mathbf{A}\|^2/(2\beta))\|\hat{\mathbf{x}} - \hat{\mathbf{x}}^k\|_2^2 \right\}$$
  - If b is a quadratic prox-function and $\mathcal{X}$ is absent, then solving both problems corresponds to computing the proximal operator of f.
- (1P2D) only requires one primal step to compute $\mathbf{x}^\star_{\gamma_{k+1}}(\hat{\lambda}^k)$.
- (1P2D) requires two dual steps corresponding to two matrix-vector multiplications $\mathbf{A}\bar{\mathbf{x}}^k$ and $\mathbf{A}\mathbf{x}^\star_{\gamma_{k+1}}(\hat{\lambda}^k)$.
- (2P1D) requires only one $\mathbf{A}\hat{\mathbf{x}}^k$.

Updating step-size

The key point in both schemes (1P2D) and (2P1D) is to update the step size τ_k:

- The model-based excessive gap condition (45) shows that $G_{\gamma_k\beta_k}(\bar{\mathbf{z}}^k) \to 0^+$.
- The convergence rate of $\{\beta_k\}$ and $\{\gamma_k\}$ depends on the convergence rate of $\{\tau_k\}$ and hence, the convergence rate of the algorithms.

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Theorem (Key condition)

Let $\{\bar{\mathbf{z}}^k\}$ be the sequence generated by either (1P2D) or (2P1D) and $G_{\gamma_k \beta_k}(\bar{\mathbf{z}}^k) \leq 0$. Then, under the condition:

$$\tau^2 \|\mathbf{A}\|^2 \leq \gamma_{k+1}\beta_{k+1} \tag{53}$$

we have $G_{\gamma_{k+1}\beta_{k+1}}(\bar{\mathbf{z}}^{k+1}) \leq 0$.

Condition (53) and the update rules $\gamma_{k+1} := (1 - c_k\tau_k)\gamma_k$ and $\beta_{k+1} = (1 - \tau_k)\beta_k$ allow us to derive the update rule for τ_k:

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Update the step-size τ_k

- **Initialization:** $\tau_0 := a_0^{-1}$, where $a_0 := \left(1 + c_0 + [4(1 - c_0) + (1 + c_0)^2]^{1/2}\right)/2$.
- **Update:** τ_{k+1} is updated from τ_k as

$$\tau_k = a_k^{-1}, \quad a_{k+1} := \left(1 + c_{k+1} + \sqrt{4a_k^2 + (1 - c_{k+1})^2}\right)/2$$

Primal-dual framework using model-based excessive gap technique

Putting all ingredients together, we can describe the complete algorithm as below:

Primal-dual method using model-based excessive gap technique (PDM)
Initialization
1.1. Given $\gamma_0 > 0$, $c_0 \in (-1, 1]$ and $\bar{L}_d := \|\mathbf{A}\|^2$.
1.2. $a_0 := (1 + c_0 + \sqrt{4(1 - c_0) + (1 + c_0)^2})/2$, $\tau_0 := a_0^{-1}$ and $\beta_0 := \bar{L}_g \gamma^{-1}$.
1.3. Compute a starting point $\bar{\mathbf{z}}^0 := (\bar{\mathbf{x}}^0, \bar{\lambda}^0)$.

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Iterations: **For $k = 0, 1, \cdots, K$, perform:**

2.1. Given $(\bar{\mathbf{x}}^k, \bar{\lambda}^k)$, compute $(\bar{\mathbf{x}}^{k+1}, \bar{\lambda}^{k+1})$ by either (2P1D) or (1P2D).

2.2. Update $\gamma_{k+1} := (1 - c_k \tau_k)\gamma_k$ and $\beta_{k+1} := (1 - \tau_k)\beta_k$

2.3. Update c_{k+1} from c_k if necessary (optional).

2.4. Update $a_{k+1} := \left(1 + c_{k+1} + \sqrt{4a_k^2 + (1 - c_{k+1})^2}\right)/2$ and $\tau_{k+1} = a_{k+1}^{-1}$.

Convergence guarantee

Theorem (Convergence)

Let $\{\bar{\mathbf{z}}^k := (\bar{\mathbf{x}}^k, \bar{\lambda}^k)\}$ *be the sequence generated by* ***PDM*** *after* $k \geq 1$ *iterations.*

a) *If* (2P1D) *is used and* $\gamma_0 := \|\mathbf{A}\|$ *and* $c_k = 1$ *for all* $k \geq 0$*, then*

$$\begin{cases} & \|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| & \leq \frac{\|\mathbf{A}\|(2D_{\Lambda^\star} + \sqrt{D_{\mathcal{X}}})}{k+1}, \\ -D_{\Lambda^\star}\|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| & \leq f(\bar{\mathbf{x}}^k) - f^\star & \leq \frac{2\|\mathbf{A}\|D_{\mathcal{X}}}{k+1}. \end{cases}$$

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b) *If* (1P2D) *is used and* $\gamma_0 := \frac{2\sqrt{2}\|\mathbf{A}\|}{K+1}$ *and* $c_k = 0$ *for all* $0 \leq k \leq K$*, then*

$$\begin{cases} & \|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq \frac{2\sqrt{2}\|\mathbf{A}\|(D_{\Lambda^\star} + \sqrt{D_{\mathcal{X}}})}{K+1}, \\ -D_{\Lambda^\star}\|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq f(\bar{\mathbf{x}}^K) - f^\star & \leq \frac{2\sqrt{2}\|\mathbf{A}\|D_{\mathcal{X}}}{K+1}. \end{cases}$$

The worst-case complexity of ***PDM*** *to reach an* ϵ*-solution* $\mathbf{x}^\star$ *of* (40) *is* $\mathcal{O}\left(\frac{\|\mathbf{A}\|R}{\epsilon}\right)$*, where* $R := \max\{D_{\mathcal{X}}, D_{\Lambda^\star} + \sqrt{D_{\mathcal{X}}}\}$.

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Remarks:

- If (1P2D) is used, then $\gamma_0 := \frac{2\sqrt{2}\|\mathbf{A}\|}{K+1}$, which requires to fix the number of iterations priori

Strongly convex case

PDM can be accelerated from $\mathcal{O}(1/k)$ to $\mathcal{O}(1/k^2)$ if f is strongly convex.

Assumption A.2.

The objective function f is strongly convex with the convexity parameter $\mu_f > 0$.

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Dual function

We define the **dual function** of (40) as

$$d(\lambda) := \min_{\mathbf{x}\in\mathcal{X}} \Big\{ f(\mathbf{x}) + \lambda^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \Big\}. \tag{54}$$

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Properties of d

- d is concave and smooth.
- Gradient of d is given by $\nabla d(\lambda) := \mathbf{A}\mathbf{x}^\star(\lambda) - \mathbf{b}$.
- ∇d is Lipschitz continuous with a Lipschitz constant $\bar{L}_d := \frac{\|\mathbf{A}\|^2}{\mu_f}$.

Algorithm

When specifying **PDM** to solve the strongly convex case, some steps in the algorithm are changed:

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Primal-dual method for strongly convex case (PDM$_\mu$)

Initialization:

1.1. Compute $\bar{L}_d := \mu_f^{-1}\|\mathbf{A}\|^2$, $\tau_0 := (\sqrt{5}-1)/2$ and $\beta_0 := \sqrt{\bar{L}_d}$.

1.2. Compute a starting point $\bar{\mathbf{z}}^0 := (\bar{\mathbf{x}}^0, \bar{\lambda}^0)$ as:

$$\bar{\mathbf{x}}^0 := \mathbf{x}^\star(0^n) \quad \text{and} \quad \bar{\lambda}^0 := \beta_0^{-1}(\mathbf{A}\bar{\mathbf{x}}^0 - \mathbf{b}).$$

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2.1. Given $(\bar{\mathbf{x}}^k, \bar{\lambda}^k)$, compute $(\bar{\mathbf{x}}^{k+1}, \bar{\lambda}^{k+1})$ as $$\begin{cases} \hat{\lambda}^k & := (1-\tau_k)\bar{\lambda}^k + \tau_k \lambda^\star_{\beta_k}(\bar{\mathbf{x}}^k) \\ \bar{\mathbf{x}}^{k+1} & := (1-\tau_k)\bar{\mathbf{x}}^k + \tau_k \mathbf{x}^\star(\hat{\lambda}^k) \\ \bar{\lambda}^{k+1} & := \hat{\lambda}^k + \bar{L}_d^{-1}(\mathbf{A}\mathbf{x}^\star(\hat{\lambda}^k) - \mathbf{b}). \end{cases} \quad (1\text{P}2\text{D}_\mu)$$
2.2. Update $\beta_{k+1} := (1-\tau_k)\beta_k$ and $\tau_{k+1} := \tau_k(\sqrt{\tau_k^2+4} - \tau_k)/2$.

Convergence guarantee

Theorem (Convergence guarantee)

Assumptions:

- *f is strongly convex with a strong convexity parameter $\mu_f > 0$.*
- *$\{\bar{\mathbf{z}}^k\}$ is generated by PDM$_\mu$.*

Conclusions:

- *We have estimates:*

$$\begin{cases} -D_{\Lambda^\star}\|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| & \leq f(\bar{\mathbf{x}}^k) - f^\star \quad \leq 0 \\ & \|\mathbf{A}\bar{\mathbf{x}}^k - \mathbf{b}\| \quad \leq \frac{\|\mathbf{A}\|^2}{(k+2)^2\mu_f} D_{\Lambda^\star} \\ & \|\bar{\mathbf{x}}^k - \mathbf{x}^\star\| \quad \leq \frac{\|\mathbf{A}\|}{(k+2)^2\mu_f} D_{\Lambda^\star} \end{cases}$$

- *The bounds do not depend on $\mathcal{D}_{\mathcal{X}}$ the prox-diameter of $\mathcal{X}$.*
- *$\{\mathbf{x}^k\}$ converges to the unique solution $\mathbf{x}^\star$ of (40) at $\mathcal{O}(1/k^2)$ rate.*

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Remarks:

- We always have $f(\bar{\mathbf{x}}^k) \leq f^\star$ in PDM$_\mu$, which is different from the unconstrained case, i.e. $f(\mathbf{x}^k) \geq f^\star$.
- The convergence rate is **optimal** in the sense of black-box first order methods.

ADMM variant

ADMM was originally developed to solve a special case of (40):

$$f^\star := \min_{\mathbf{x}\in\mathcal{X}}\{f(\mathbf{x}) \ : \ \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

where f and $\mathcal{X}$ is decomposable with $g = 2$.

Problem setting: When f and $\mathcal{X}$ are 2-decomposable

$$f^\star := \begin{cases} \min\limits_{\mathbf{x}:=(\mathbf{x}_1,\mathbf{x}_2)} & \Big\{ f(\mathbf{x}) := f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) \Big\}, \\ \text{s.t.} & \mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\mathbf{x}_2 = b, \\ & \mathbf{x}_1 \in \mathcal{X}_1, \ \mathbf{x}_2 \in \mathcal{X}_2. \end{cases} \quad (55)$$

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Augmented Lagrangian smoother

- When $\tilde{\mathbf{x}}_1^k$ and $\tilde{\mathbf{x}}_2^{k+1}$ are available, we use

$$d_{\gamma\beta}(\mathbf{z}) := \frac{\gamma}{2}\left[\|\mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\tilde{\mathbf{x}}_2^k - \mathbf{b}\|^2 + \|\mathbf{A}_1\tilde{\mathbf{x}}_1^{k+1} + \mathbf{A}_2\mathbf{x}_2 - \mathbf{b}\|^2 \right] + \frac{\beta}{2}\|\lambda\|^2.$$

  to smooth the gap function $G(\mathbf{z}) := \max_{\hat{\mathbf{z}}\in\mathcal{X}\times\mathbb{R}^n}\{ f(\mathbf{x}) - f(\hat{\mathbf{x}}) + M(\mathbf{z})^T(\mathbf{z} - \hat{\mathbf{z}}) \}$.
- Modify the (1P2D) scheme to obtain a new variant of ADMM by alternating the computation of $\mathbf{x}^\star_\gamma(\lambda)$.

ADMM variant

By alternating the step of $\mathbf{x}^\star_\gamma(\lambda)$, the main step of the **new ADMM variant** becomes:

New ADMM scheme (ADMM$_1$)

$$\begin{cases} \hat{\lambda}^k & := (1-\tau_k)\bar{\lambda}^k + \tau_k \lambda^\star_{\beta_k}(\bar{\mathbf{x}}^k) \\ \tilde{\mathbf{x}}_1^{k+1} & := \operatorname*{argmin}_{\mathbf{x}_1 \in \mathcal{X}_1} \left\{ f_1(\mathbf{x}_1) + (\mathbf{A}_1^T\hat{\lambda}^k)^T\mathbf{x}_1 + (\gamma_k/2)\|\mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2\tilde{\mathbf{x}}_2^k - \mathbf{b}\|^2 \right\} \\ \tilde{\mathbf{x}}_2^{k+1} & := \operatorname*{argmin}_{\mathbf{x}_2 \in \mathcal{X}_2} \left\{ f_2(\mathbf{x}_2) + (\mathbf{A}_2^T\hat{\lambda}^k)^T\mathbf{x}_2 + (\gamma_k/2)\|\mathbf{A}_1\tilde{\mathbf{x}}_1^{k+1} + \mathbf{A}_2\bar{\mathbf{x}}_2 - \mathbf{b}\|^2 \right\} \\ \bar{\mathbf{x}}^{k+1} & := (1-\tau_k)\bar{\mathbf{x}}^k + \tau_k\tilde{\mathbf{x}}^{k+1}, \quad \text{where} \quad \tilde{\mathbf{x}}^{k+1} := (\tilde{\mathbf{x}}_1^{k+1}, \tilde{\mathbf{x}}_2^{k+1}) \\ \bar{\lambda}^{k+1} & := \hat{\lambda}^k + (\gamma_k/2)(\mathbf{A}\tilde{\mathbf{x}}^{k+1} - \mathbf{b}). \end{cases}$$

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By alternating the step of $\mathbf{x}^\star_\gamma(\lambda)$, the main step of the **new ADMM variant** becomes:

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$$\begin{cases} \hat{\lambda}^k & := (1-\tau_k)\bar{\lambda}^k + \tau_k \lambda^\star_{\beta_k}(\bar{\mathbf{x}}^k) \\ \tilde{\mathbf{x}}_1^{k+1} & := \underset{\mathbf{x}_1 \in \mathcal{X}_1}{\mathrm{argmin}} \left\{ f_1(\mathbf{x}_1) + (\mathbf{A}_1^T \hat{\lambda}^k)^T \mathbf{x}_1 + (\gamma_k/2)\|\mathbf{A}_1\mathbf{x}_1 + \mathbf{A}_2 \tilde{\mathbf{x}}_2^k - \mathbf{b}\|^2 \right\} \\ \tilde{\mathbf{x}}_2^{k+1} & := \underset{\mathbf{x}_2 \in \mathcal{X}_2}{\mathrm{argmin}} \left\{ f_2(\mathbf{x}_2) + (\mathbf{A}_2^T \hat{\lambda}^k)^T \mathbf{x}_2 + (\gamma_k/2)\|\mathbf{A}_1\tilde{\mathbf{x}}_1^{k+1} + \mathbf{A}_2 \bar{\mathbf{x}}_2 - \mathbf{b}\|^2 \right\} \\ \bar{\mathbf{x}}^{k+1} & := (1-\tau_k)\bar{\mathbf{x}}^k + \tau_k \tilde{\mathbf{x}}^{k+1}, \quad \text{where } \tilde{\mathbf{x}}^{k+1} := (\tilde{\mathbf{x}}_1^{k+1}, \tilde{\mathbf{x}}_2^{k+1}) \\ \bar{\lambda}^{k+1} & := \hat{\lambda}^k + (\gamma_k/2)(\mathbf{A}\tilde{\mathbf{x}}^{k+1} - \mathbf{b}). \end{cases}$$

Convergence of ADMM_1 [14]

Assumptions:

- Let $\{(\mathbf{x}^k, \lambda^k)\}_{k\geq 0}$ be the sequence generated by **PDM** using ADMM_1.
- Let $\gamma_k = \gamma_0 := \frac{2\sqrt{2}\|\mathbf{A}\|}{K+3}$ and $\beta_{k+1} := (1-\tau_k)\beta_k$ for $k = 0, \cdots, K$.

Conclusion:

$$\begin{cases} & \|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq \frac{2\sqrt{2}\|\mathbf{A}\|(D_{\Lambda^\star} + \bar{D}_{\mathcal{X}})}{K+3} \\ -D_{\Lambda^\star}\|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq f(\bar{\mathbf{x}}^K) - f^\star & \leq \frac{2\sqrt{2}\|\mathbf{A}\|}{K+3}(\bar{D}_{\mathcal{X}})^2 \end{cases}$$

where $\bar{D}_{\mathcal{X}} := 2 \max_{\mathbf{x}, \hat{\mathbf{x}} \in \mathcal{X}} \|\mathbf{A}(\mathbf{x} - \hat{\mathbf{x}})\|$.

Preconditioned ADMM variant

When f_1 and f_2 are proximally tractable and $\mathcal{X}_1$ and $\mathcal{X}_2$ are absent

- We can linearize the quadratic terms in lines 2 and 3 of ADMM_1.
- Then, by using the gradient step, we obtain a preconditioned ADMM variant.

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Preconditioned ADMM variant (PADMM$_1$)

$$\begin{cases} \tilde{\mathbf{x}}_1^{k+1} & := \mathrm{prox}_{\gamma_k^{-1}\alpha_{1k}f_1}\left(\mathbf{g}_1^k + \gamma_k^{-1}\mathbf{A}_1^T\lambda^k\right) \\ \tilde{\mathbf{x}}_2^{k+1} & := \mathrm{prox}_{\gamma_k^{-1}\alpha_{2k}f_2}\left(\mathbf{g}_2^k + \gamma_k^{-1}\mathbf{A}_2^T\lambda^k\right) \end{cases}$$

where g_1^k and g_2^k are the gradient step of the quadratic term computed as

$$\begin{cases} \mathbf{g}_1^k & := \tilde{\mathbf{x}}_1^k - \alpha_{1k}\mathbf{A}_1^T(\mathbf{A}_1\tilde{\mathbf{x}}_1^k + \mathbf{A}_2\tilde{\mathbf{x}}_2^k - \mathbf{b}) \\ \mathbf{g}_2^k & := \tilde{\mathbf{x}}_2^k - \alpha_{2k}\mathbf{A}_2^T(\mathbf{A}_1\tilde{\mathbf{x}}_1^{k+1} + \mathbf{A}_2\tilde{\mathbf{x}}_2^k - \mathbf{b}). \end{cases}$$

Here α_{1k} and α_{2k} are given step-sizes.

There are at least two ways of computing the step-sizes:

- **Constant step size**: We can take $\alpha_{1k} := \|\mathbf{A}_1\|^{-1}$ and $\alpha_{2k} := \|\mathbf{A}_2\|^{-1}$.
- **Adaptive step-size**: α_{1k} and α_{2k} are computed from the exact line-search condition of the form:

$$\alpha := \arg\min_{\alpha > 0} \xi(\mathbf{u}^k - \alpha\nabla\xi(\mathbf{u}^k))$$

where $\mathbf{u}$ can be $\mathbf{x}_1$ or $\mathbf{x}_2$, and ξ is the quadratic function of $\mathbf{x}_1$ or $\mathbf{x}_2$ in PADMM$_1$.

Convergence of PAMMM

Convergence of PADMM$_1$ [14]

Assumptions:

- Let $\{(\mathbf{x}^k, \lambda^k)\}_{k\geq 0}$ be the sequence generated by **PDM** using PADMM$_1$.
- Let $\gamma_k = \gamma_0 := \frac{2\sqrt{2}\|\mathbf{A}\|}{K+3}$ and $\beta_{k+1} := (1-\tau_k)\beta_k$ for $k = 0, \cdots, K$.

Conclusion:

$$\begin{cases} & \|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq \frac{2\sqrt{2}\|\mathbf{A}\|(D_{\Lambda^\star} + \bar{D}_{\mathcal{X}})}{K+3} \\ -D_{\Lambda^\star}\|\mathbf{A}\bar{\mathbf{x}}^K - \mathbf{b}\| & \leq f(\bar{\mathbf{x}}^K) - f^\star & \leq \frac{2\sqrt{2}\|\mathbf{A}\|}{K+3}(\bar{D}_{\mathcal{X}})^2 \end{cases}$$

where $\bar{D}_{\mathcal{X}} := 4 \max_{\mathbf{x},\hat{\mathbf{x}}\in\mathcal{X}} \|\mathbf{x} - \hat{\mathbf{x}}\|$.

Enhancements of the PDM algorithm:

- There is a freedom of choosing the center point $\mathbf{x}_c$ for computing $\mathbf{x}^\star_\gamma(\lambda)$.
  - $\mathbf{x}_c$ can be fixed for all the iterations.
  - One can choose $\mathbf{x}_c$ as the previous iteration, i.e., $\mathbf{x}_c := \mathbf{x}^\star_{\gamma_k}(\lambda^{k-1})$.
  - Or choose $\mathbf{x}_c$ adaptively as in the PADMM variant.
- The smoothness parameter γ can be increased as long as the objective values does not increase substantially.
  - When $\mathbf{x}_c$ is adaptively chosen, we can slightly increase γ_k as $\gamma_{k+1} := c\gamma_k$, for e.g., $c_k := 1.05$.

Comparison

We summarize the convergence rate of 5 different methods and the assumptions where the methods use in the following table:

- The average sequence $\{\widehat{\mathbf{x}}^k\}$ is computed as $\widehat{\mathbf{x}}^k := (k+1)^{-1}\sum_{j=0}^{k}\mathbf{x}^j$.
- Convergence guarantee using this sequence is referred to as an ergodic convergence.

Method name	Assumptions	Convergence	References
ADMM	$\leq$ 2-decomposable	$\mathcal{O}(1/k)$ on the joint $(\mathbf{x}^k, \mathbf{y}^k)$ using a gap function	[2, 8, 9]
[Fast] ADMM	$\leq$ 2-decomposable and f_1 or $f_2 \in \mathcal{F}_\mu$	$[\mathcal{O}(1/k^2)]$ $\mathcal{O}(1/k)$ on the dual-objective	[6]
Decomposition methods with 1P2D and 2P1D	p-decomposable	$\lvert f(\mathbf{x}^k) - f^\star\rvert \leq \mathcal{O}(1/k)$ and $\lVert\mathbf{A}\mathbf{x}^k - \mathbf{b}\rVert_2 \leq \mathcal{O}(1/k)$ (*non-ergodic*)	[14]
	p-decomposable and $f_i \in \mathcal{F}_\mu$	$\lvert f(\mathbf{x}^k) - f^\star\rvert \leq \mathcal{O}(1/k^2)$, $\lVert\mathbf{A}\mathbf{x}^k - \mathbf{b}\rVert_2 \leq \mathcal{O}(1/k^2)$, and $\lVert\mathbf{x}^k - \mathbf{x}^\star\rVert_2 \leq \mathcal{O}(1/k)$ (*non-ergodic*)	
New ADMM	$\leq$ 2-decomposable	$\lvert f(\mathbf{x}^k) - f^\star\rvert \leq \mathcal{O}(1/k)$ and $\lVert\mathbf{A}\mathbf{x}^k - \mathbf{b}\rVert_2 \leq \mathcal{O}(1/k)$ (*non-ergodic*)	[14]
New preconditioned ADMM	$\leq$ 2-decomposable	$\lvert f(\mathbf{x}^k) - f^\star\rvert \leq \mathcal{O}(1/k)$ and $\lVert\mathbf{A}\mathbf{x}^k - \mathbf{b}\rVert_2 \leq \mathcal{O}(1/k)$ (*non-ergodic*)	[14]

Example 1: Group sparse recovery

Sparse recovery

- Let $\mathcal{I} := \{1, \ldots, p\}$ be the set of indices. Let $\mathfrak{G} := \{\mathcal{G}_1, \ldots, \mathcal{G}_g\}$ be the set of g groups $\mathcal{G}_i \subseteq \mathcal{I}$ and $\mathcal{I} \subseteq \cup_{i=1}^{g} \mathcal{U}_i$.
- For given group $\mathcal{G}_i$, and a vector $\mathbf{x} \in \mathbb{R}^p$, we use $\mathbf{x}_{\mathcal{G}_i} = \{x_j \ : \ j \in \mathcal{G}_i\}$.
- For fixed group structure $\mathfrak{G}$, $\mathbf{x} \in \mathbb{R}^p$ is called group sparse vector if the number of groups in $\mathcal{G}$ is **small**.
- Given a **linear operator** $\mathbf{A}$ and an **observed/measurement** vector $\mathbf{b} \in \mathbb{R}^n$. We want to recover the **group sparse** input vector $\mathbf{x} \in \mathbb{R}^p$ such that $\mathbf{b} = \mathbf{A}\mathbf{x}$.

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Optimization formulation

$$\begin{array}{ll} \min\limits_{\mathbf{x} \in \mathbb{R}^p} & \sum_{\mathcal{G}_i \in \mathfrak{G}} \|\mathbf{x}_{\mathcal{G}_i}\|_2 \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{array} \tag{56}$$

Here, $f(\mathbf{x}) := \sum_{\mathcal{G}_i \in \mathfrak{G}} \|\mathbf{x}_{\mathcal{G}_i}\|_2$ and $\mathcal{X} := \mathbb{R}^p$. This problem possesses two common structures: decomposability and tractable proximity.

When $g = p$ and $\mathcal{G}_i = \{i\}$, (56) reduces to the well-known linear sparse recovery problem (basis pursuit):

$$\min_{\mathbf{x} \in \mathbb{R}^p} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}. \tag{57}$$

Example 1: Group sparse recovery - Numerical results

Algorithm configuration:

- Assume that (56) is constrained by a boxed constraint $\mathbf{x} \in \mathcal{X} := [\mathbf{l}, \mathbf{u}]$.
- The Bregman distance is chosen as $d(\mathbf{x}, \mathbf{x}_c) := (1/2)\|\mathbf{x} - \mathbf{x}_c\|_2^2$ and $\mathbf{x}_c = 0 \in [\mathbf{l}, \mathbf{u}]$.
- $\beta_0 = \gamma_0 = \|\mathbf{A}\|$ in 2P1D and $\gamma_0 := 2\sqrt{2}\|\mathbf{A}\|/(K+1)$ with $K = 10^4$.

Data generation:

- $p = 1024$, $n = 341$ and $g = 128$.
- $\mathbf{A}$ is a random matrix generated via the standard Gaussian distribution.
- $\mathbf{b} := \mathbf{A}\mathbf{x}^\natural$, where $\mathbf{x}^\natural$ is a 128-group sparse vector.
- The group $\mathfrak{G}$ is also generated randomly.
- $\mathbf{l} := \min(\mathbf{x}^\natural)$ and $\mathbf{u} := \max(\mathbf{x}^\natural)$.

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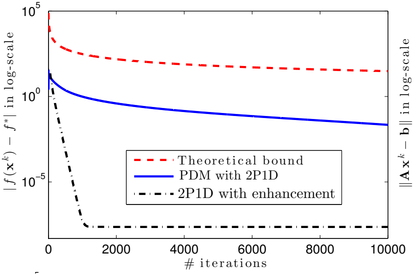
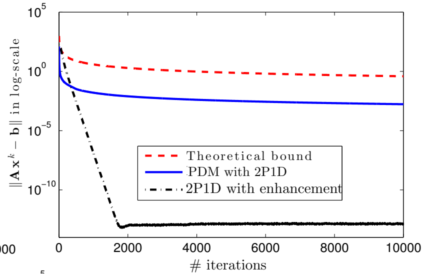
Comparison We compare the following three quantities:

- The theoretical bounds given in the right-hand side of the convergence theorem
- The PDM algorithm with 2P1D or 1P2D - i.e., follow the theory.
- The 2P1D or 1P2D with enhancement - i.e., updating the parameter γ_k by $\gamma_{k+1} := 1.05\gamma_k$ and using the adaptive center point $\mathbf{x}_c$ as in **PADMM**.

Example 1: Theoretical bounds vs actual performance

The performance of two variants of PDM using 2P1D





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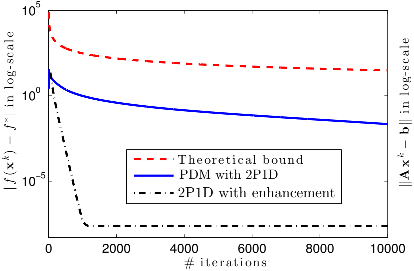
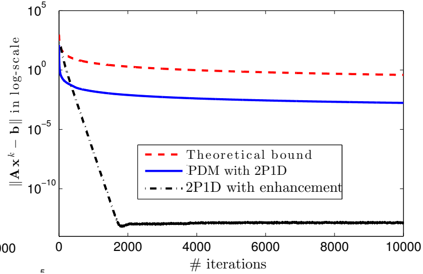
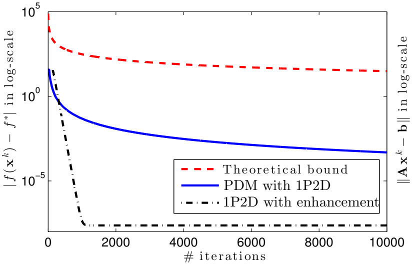
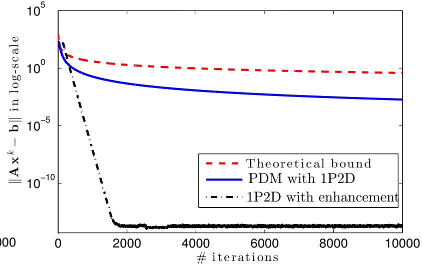




The performance of two variants of PDM using 1P2D





Example 2: Image processing

Problem (Imaging denoising/deblurring)

Our goal is to obtain a clean image $\mathbf{x}$ *given "dirty" observations* $\mathbf{b} \in \mathbb{R}^{n \times 1}$ *via* $\mathbf{b} = \mathcal{A}(\mathbf{x}) + \mathbf{w}$*, where* $\mathcal{A}$ *is a linear operator, which, e.g., captures camera blur as well as image subsampling, and* $\mathbf{w}$ *models Gaussian perturbations.*

Optimization formulation

$$\boxed{\text{Gaussian}: \quad \min_{\mathbf{z} \in \mathcal{Z}} \left\{ (1/2)\|\mathcal{A}(\mathbf{z}) - \mathbf{b}\|_2^2 + \rho\|\mathbf{z}\|_{\text{TV}} \right\}} \tag{58}$$

where $\|\mathbf{z}\|_{\text{TV}} := \sum_{i,j} |z_{i,j+1} - z_{i,j}| + |z_{i+1,j} - z_{i,j}|$, $\rho > 0$ is a regularization parameter and $\mathcal{Z} := [0, 255]^{n \times p}$.

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Reformulation: Since $\|\mathbf{z}\|_{\mathrm{TV}} = \|\mathbf{D}\mathbf{z}\|_1$ for a given matrix $\mathbf{D}$. By letting $\mathbf{r} = \mathbf{D}\mathbf{z}$, we can reformulate (58) as

$$\begin{aligned} \min_{\mathbf{z} \in \mathcal{Z}, \mathbf{r}} \quad & \left\{ (1/2)\|\mathcal{A}(\mathbf{z}) - \mathbf{b}\|_2^2 + \rho\|\mathbf{r}\|_1 \right\} \\ \text{s.t.} \quad & \mathbf{D}\mathbf{z} - \mathbf{r} = 0. \end{aligned}$$

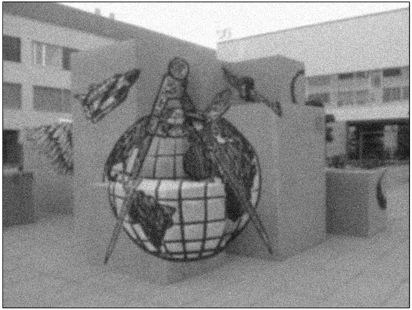
This problem is a constrained convex minimization problem with 2-decomposable objective $f(\mathbf{x}) := (1/2)\|\mathcal{A}(\mathbf{z}) - \mathbf{b}\|_2^2 + \rho\|\mathbf{r}\|_1$ and $\mathbf{x} := (\mathbf{z}, \mathbf{r})$.

Example 2: Image processing - Input data

The original image and Gaussian noise image

original image

input: noise image



Data generation:

- The original image is filtered with a multidimensional filter H of size 9×9 (circulant).
- 0.5% Gaussian noise is added to the output.

Parameter configuration:

- The number of iterations: 200 and the relative tolerance: 10^{-8}.

Example 2: Image processing - Numerical results

The performance of the new ADMM variant of PDM for two values of ρ.

output: 1P2D with $\rho = 100$

output: 1P2D with $\rho = 50$

- $f(\mathbf{x}^\natural) = 138097.919259$ and 77284.828237, where $\mathbf{x}^\natural$ is the original image.
- The **objective values**: $f(\mathbf{x}^k) := 122557.13880$ and $f(\mathbf{x}^k) := 64456.44963$
- **Relative error** between original image to clean image: 0.089152 and 0.089167
- PSNR: 26.011 and 25.994.

Example 3: Binary linear support vector machine

Problem (**Binary classification**)

Given a sample vector $\mathbf{a} \in \mathbb{R}^p$ *and a binary class label vector* $\mathbf{b} \in \{-1, +1\}^n$*.*
The goal is to find a separating hyperplane $\varphi(\mathbf{a}, \mathbf{z}) := \mathbf{a}^T\mathbf{z} + \mu$ *such that*

$$b_i = \begin{cases} +1 & \textit{if } \varphi(\mathbf{a}, \mathbf{z}) \geq 0 \\ -1 & \textit{otherwise} \end{cases}$$

where $\mathbf{z} \in \mathbb{R}^p$ *is a weight vector,* μ *is called a bias.*

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Optimization formulation

$$\min_{\mathbf{z} \in \mathbb{R}^p} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathcal{H}(b_i, \mathbf{a}_i^T \mathbf{z} + \mu) + \rho \|\mathbf{z}\|_1 \right\} \tag{59}$$

where $\mathbf{a}_i$ is the i-th row of the observed data matrix $\mathbf{A}$ in $\mathbb{R}^{n \times p}$, $\rho > 0$ is a regularization parameter, and $\mathcal{H}$ is the Hingle loss function $\mathcal{H}(s, \tau) := \max\{0, 1 - s\tau\}$.

Constrained reformulation: By introducing a slack variable $\mathbf{r} := \mathbf{A}\mathbf{z} + \mu$, we have

$$\begin{aligned} \min_{\mathbf{z} \in \mathbb{R}^p, \mathbf{r}} \quad & \left\{ \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \mathcal{H}(b_i, r_i) + \rho \|\mathbf{z}\|_1 \right\} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{z} + \mu - \mathbf{r} = 0. \end{aligned}$$

Example 3: Binary linear support vector machine

Testing data

- **Test problems**: Two real-world problems `a1a` and `news20` from
  `http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/`
- The data size of `a1a`: $p = 119$ features and $n = 1605$ data points
- The data size of `news20`: $p = 1355191$ features and $n = 19996$ data points
- The parameter ρ changes from $\rho^{-1} = 10^{-3}$ to $\rho^{-1} = 10^{3}$.

Comparison: We compare the new PADMM variant with **LibSVM**.

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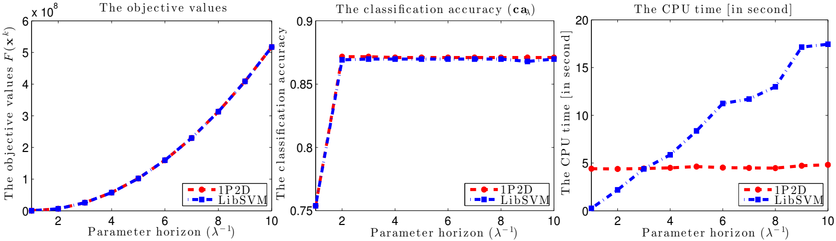
Problem	The parameter values									
λ^{-1}	10^{-3}	111.1	222.2	333.3	444.4	555.6	666.7	777.8	888.9	10^{3}
The accuracy of problem `a1a`										
(1P2D)	0.7539	0.8717	0.8717	0.8710	0.8710	0.8710	0.8710	0.8710	0.8710	0.8710
`LibSVM`	0.7539	0.8692	0.8698	0.8698	0.8698	0.8698	0.8698	0.8698	0.8679	0.8698
The CPU time [in second] of problem `a1a`										
(1P2D)	4.4045	4.3769	4.4246	4.4941	4.6238	4.5175	4.4836	4.4719	4.7179	4.8097
`LibSVM`	0.2549	2.1909	4.3884	5.8583	8.3662	11.2350	11.7036	12.9832	17.1424	17.4362
The accuracy of problem `news20`										
(1P2D)	0.5001	0.9987	0.9987	0.9987	0.9987	0.9987	0.9987	0.9987	0.9987	0.9987
`LibSVM`	0.5001	0.9987	0.9987	0.9987	0.9987	0.9988	0.9988	0.9988	0.9988	0.9988
The CPU time [in second] of problem `news20`										
(1P2D)	762.31	1023.22	994.64	1043.06	984.24	989.70	1064.33	1073.94	984.47	1018.35
`LibSVM`	890.26	1440.28	1449.23	1439.77	1434.27	1518.56	1560.38	1557.48	1535.19	1530.71

- The **accuracy** is computed as $\mathrm{ca}_{\rho} := 1 - n^{-1}\sum_{j=1}^{n}[\mathrm{sign}(\mathbf{A}\mathbf{z}^k + \mu)_i \neq b_i)]$.

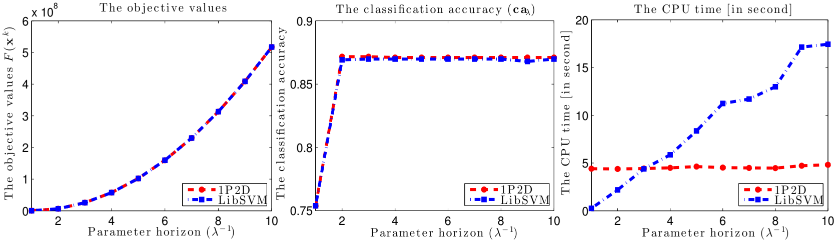
Example 3: Binary linear support vector machine

The results of two algorithms on the real-world problem a1a

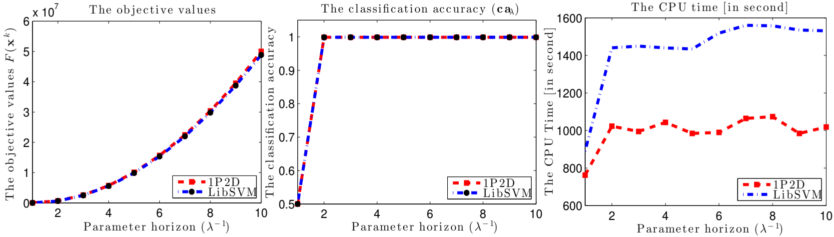


Example 3: Binary linear support vector machine

The results of two algorithms on the real-world problem `a1a`



The results of two algorithms on the real-world problem `news20`



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