Doctoral class on neurophysics
June 15, 2023

Neural population dynamics of finite-size spiking neural networks

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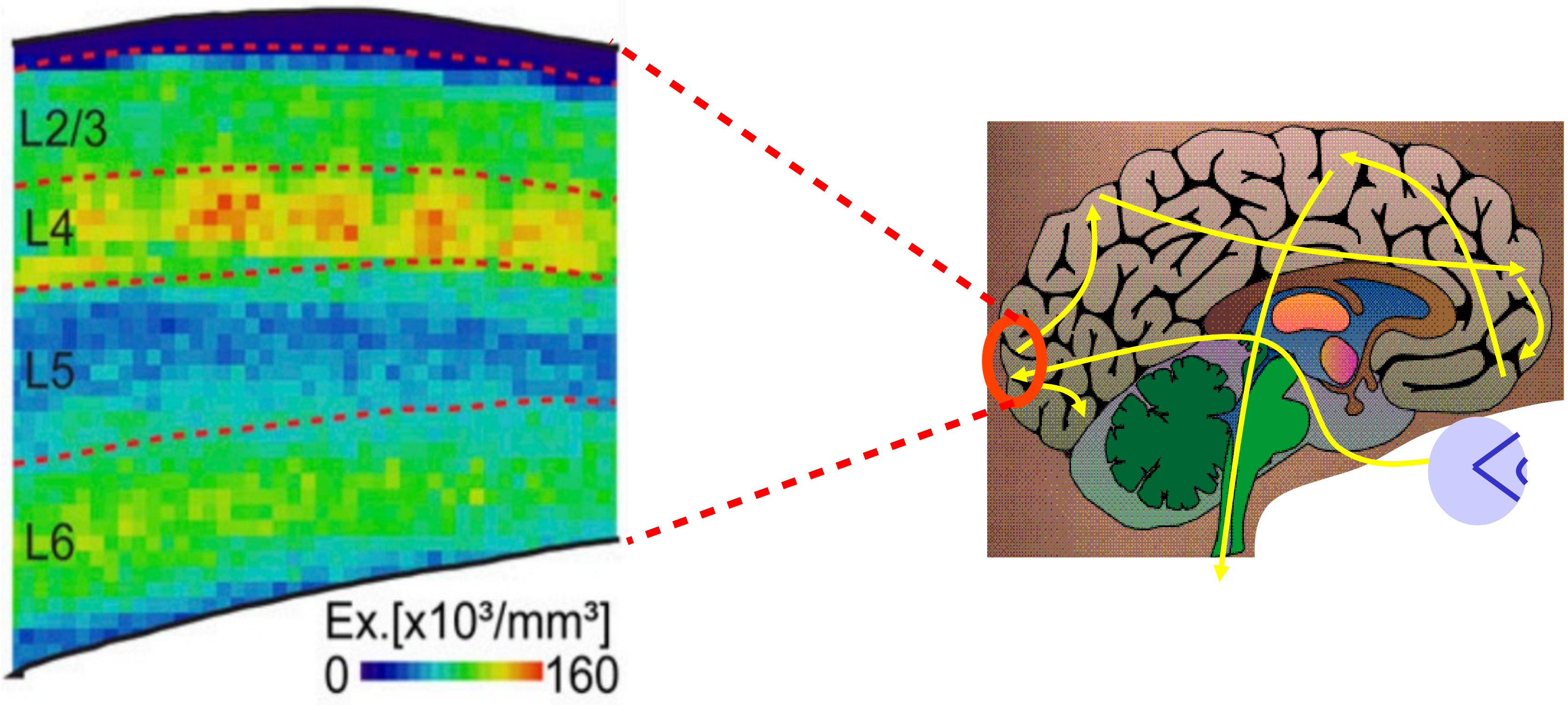
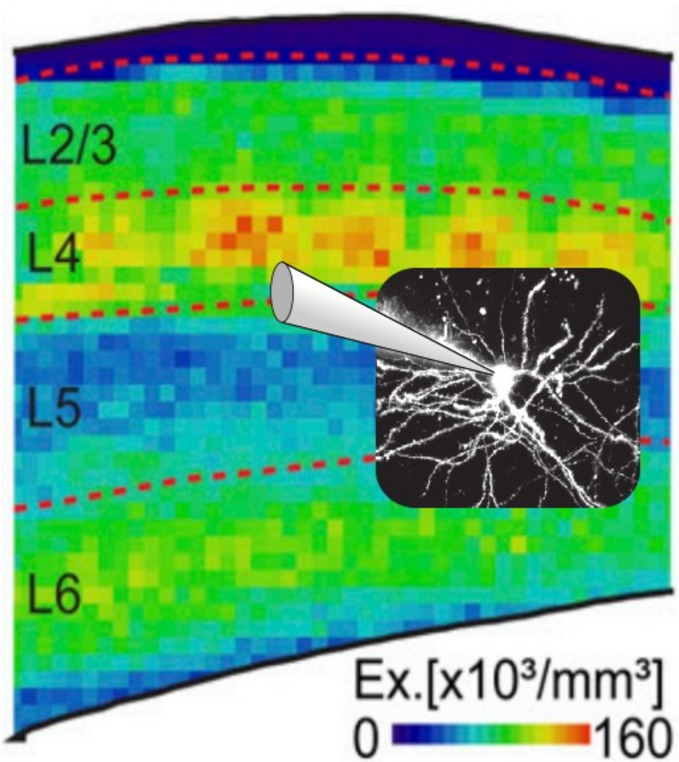
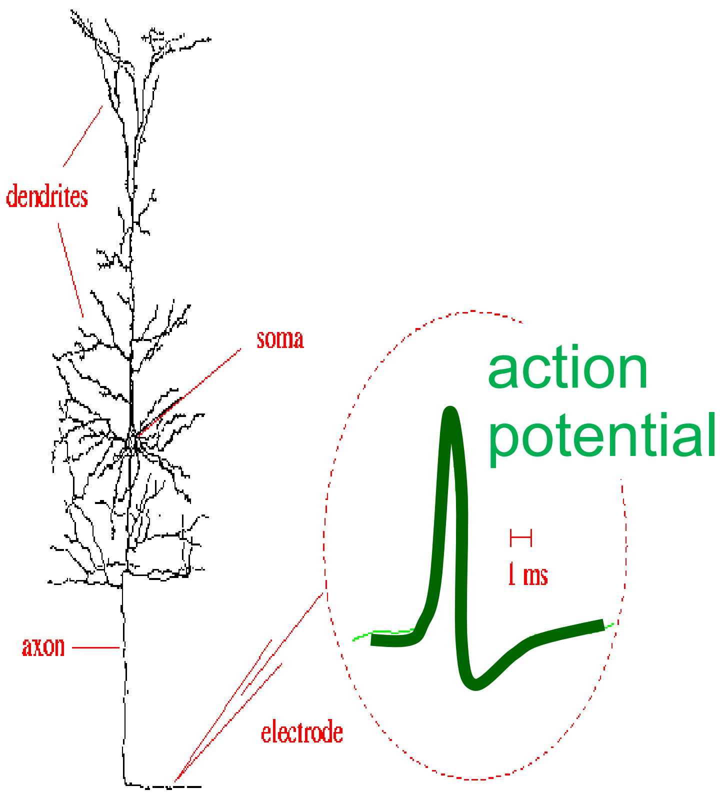
Modeling complex brain activity



Modeling complex brain activity

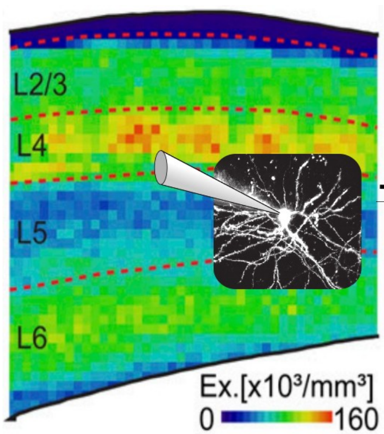


Signal:
action potential (spike)



How can we understand the collective neural activity (and computations) emerging from thousands of interacting neurons?

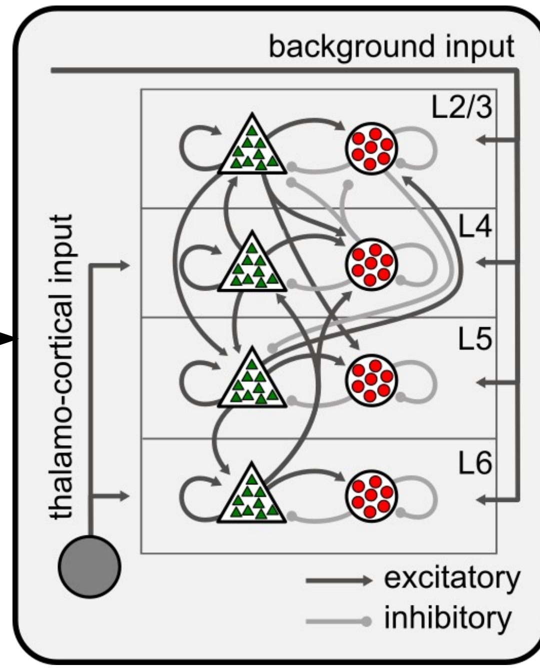
Microscopic model: individual neurons



Extract microscopic parameters

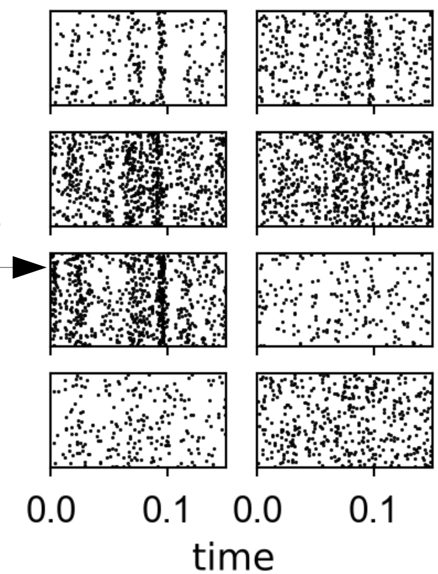
- Single neuron parameters
- Statistical connectivity
- Synaptic dynamics
- Number of neurons per population

Microscopic model of cortical microcircuit



simulate

Spiking activity



Potjans & Diesmann, Cereb Cortex (2014)

Markram et al. (2015), Cell
Billeh et al. (2020) Neuron
Schmidt M, et al. (2018) PloS Comput Biol
Izhikevich & Edelmann (2008), PNAS

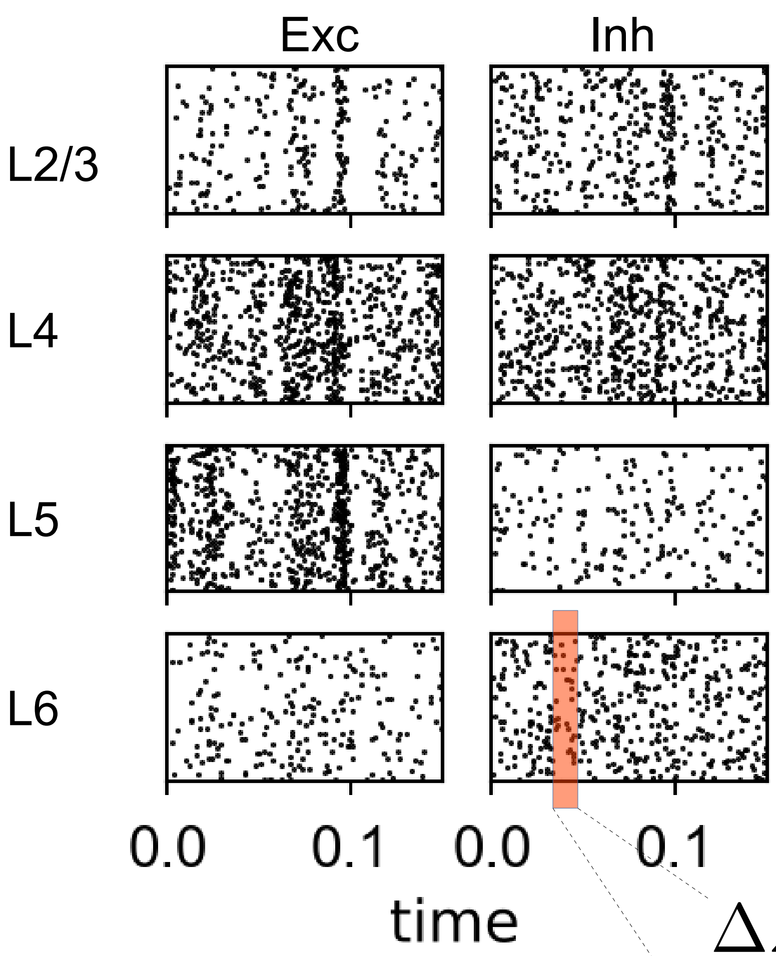
80'000 LIF neurons

Too complicated!

Coarse-graining: population activity

Spike trains

$$s_i^\alpha(t) = \sum_k \delta(t - t_{i,k}^\alpha)$$

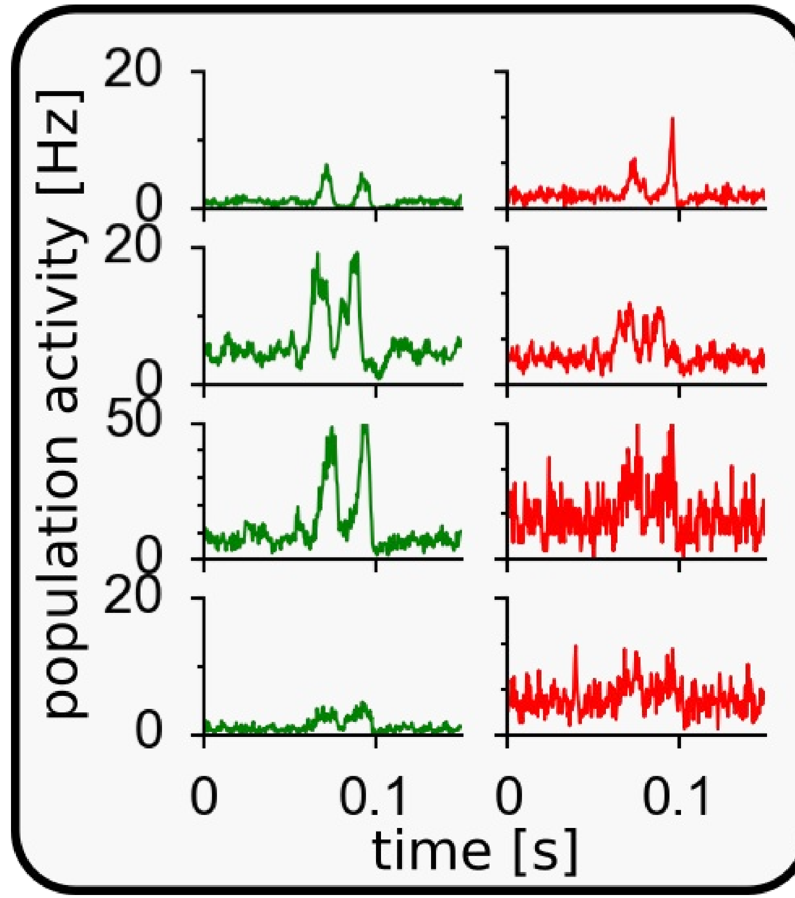


$\Delta Z^\alpha(t)$ spikes

in $[t, t+\Delta t)$

Population activities

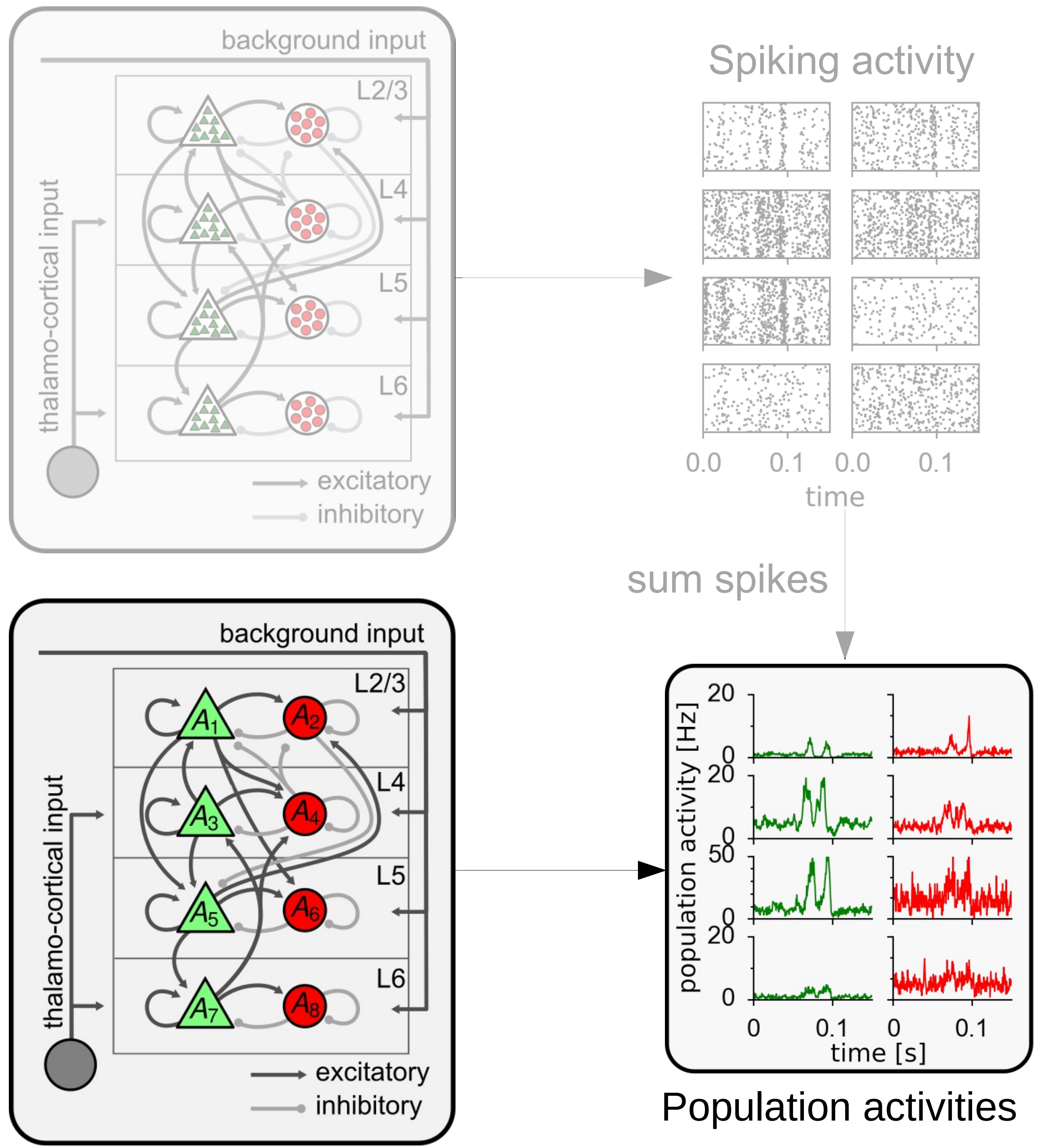
$$A_N^\alpha(t) = \frac{1}{N^\alpha} \sum_{i=1}^{N^\alpha} s_i^\alpha(t) \approx \frac{\Delta Z^\alpha(t)}{N^\alpha \Delta t}$$



Macro-/mesoscopic model: neural populations

Microscopic model
(network of 80 000 neurons)

Mesoscopic model
(network of 8 populations)



Firing-rate models

Wilson & Cowan, 1972

$$A^\alpha(t) = F_0^\alpha(h^\alpha(t))$$

$$\tau \frac{\mathrm{d}h^\alpha}{\mathrm{d}t} = -h^\alpha + \sum_\beta J^{\alpha\beta} A^\beta(t)$$

$F_0^\alpha(h)$ Steady-state transfer function (f-I curve)

τ Integration time constant

$J^{\alpha\beta}$ Synaptic efficacy from population β to α

- Low-dimensional, mathematically tractable
- Models, which „do something“
  - Perceptual decision making (Wong & Wang 2008, Moreno-Bote et al. 2007)
  - Visual processing (Ben-Yishai & Sompolinsky 1995, Ozeki et al. 2009, Rubin et al. 2015)
  - Working memory (Barak & Tsodyks 2007, Kilpatrick & Ermentrout 2013)
  - Associative memory (Hopfield 1984, Perreira & Brunel 2018)
  - Motor control (Zhang 1996, Hennequin et al. 2014)
  - Reservoir computing (Sussillo & Abbott 2009)

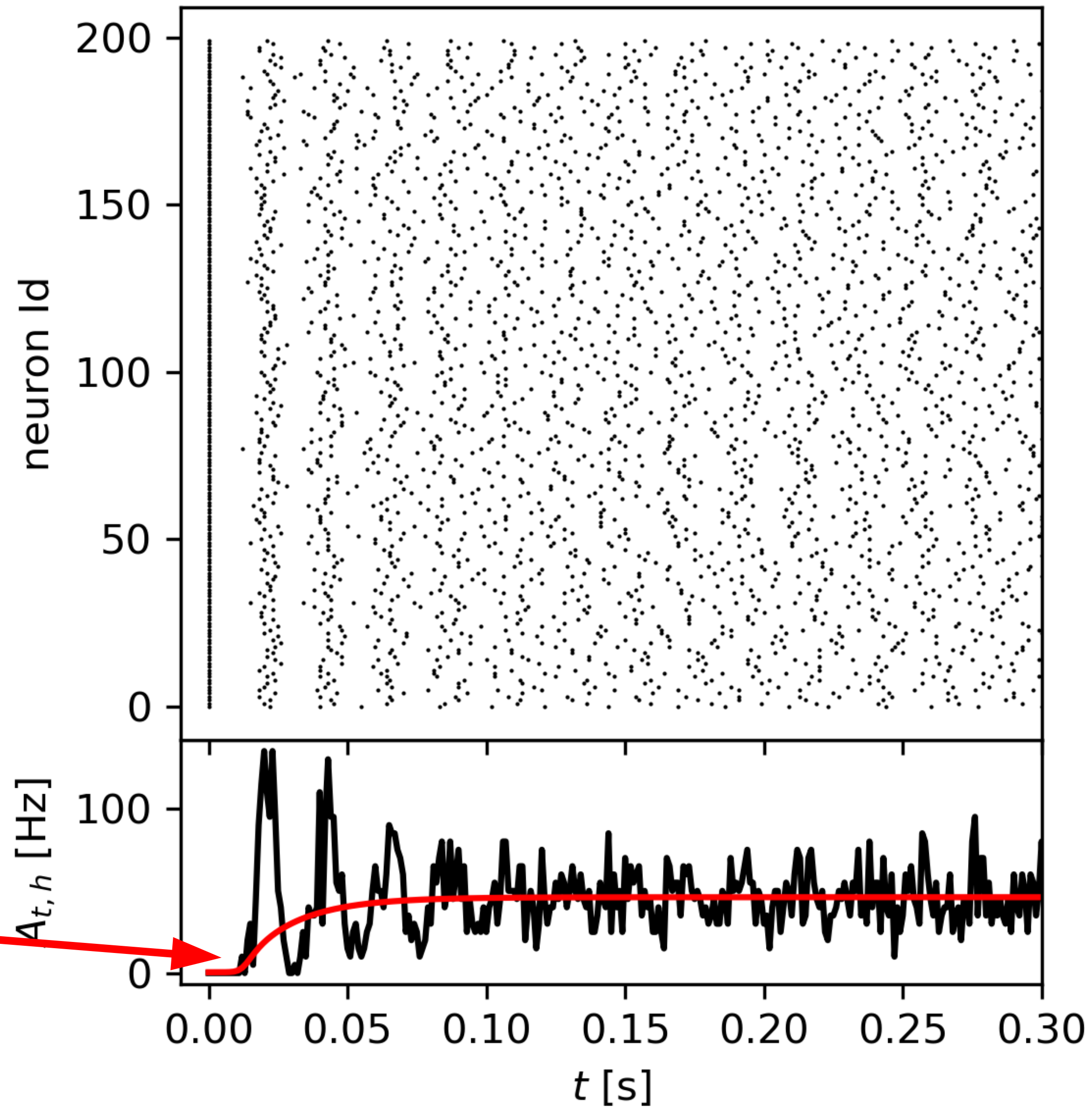
Challenges for firing-rate models

- Simulation of 200 leaky integrate-and-fire neurons synchronized at *t=0*
- Population activity:

$$A_{t,h} = \frac{\#\text{spikes in } (t, t+h]}{Nh}$$

- Heuristic firing-rate / neural-mass model

$$A_t = f(h_t)$$
$$\tau \frac{\mathrm{d}h_t}{\mathrm{d}t} = -h_t + I_t + JA_t$$



Too simple!

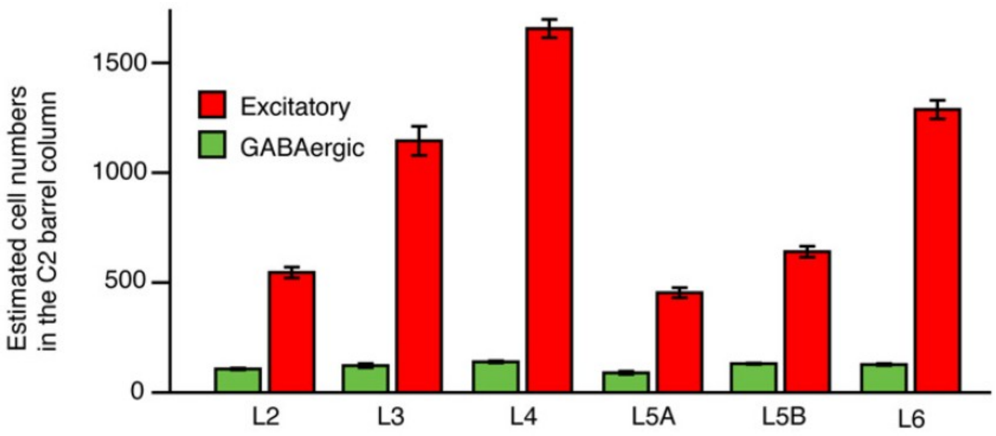
Problem: 1.Fast non-stationary (transient) dynamics (refractory effects)

2.Finite-size fluctuations

Challenge: Finite-size fluctuations

- Small population sizes

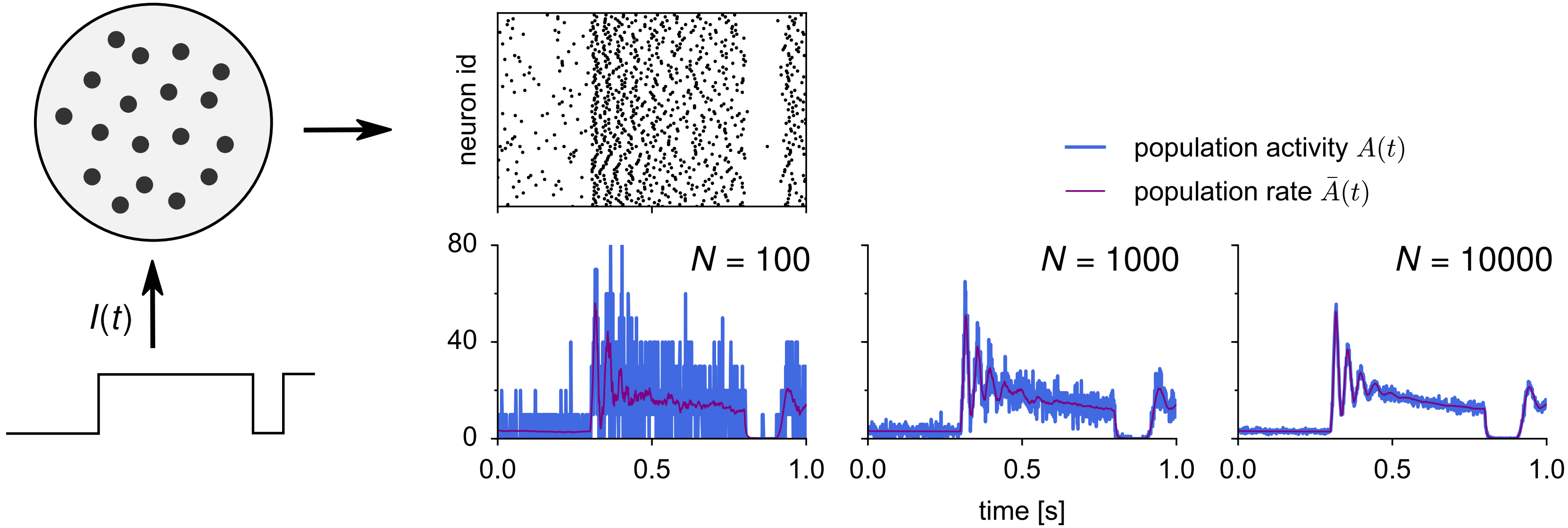
Lefort et al., Neuron (2009)



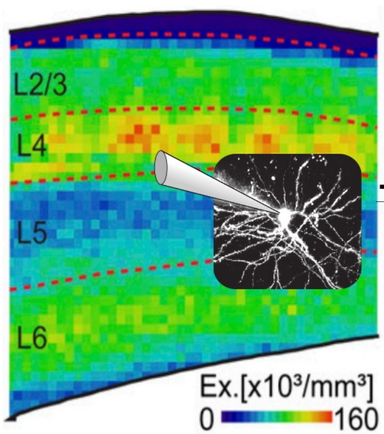
$\sim 10^2 - 10^3$ neurons / population

(„mesoscopic")

- Finite-size causes spiking noise in population activity



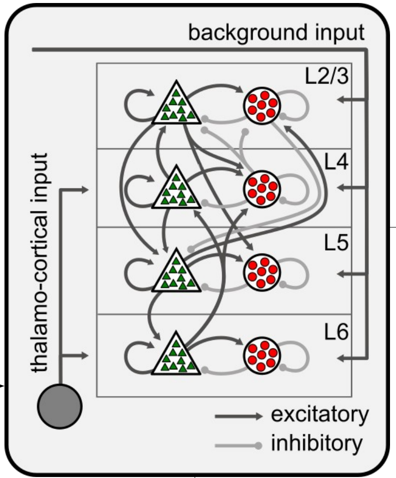
Goal: Consistent mesoscopic model



Microscopic parameters

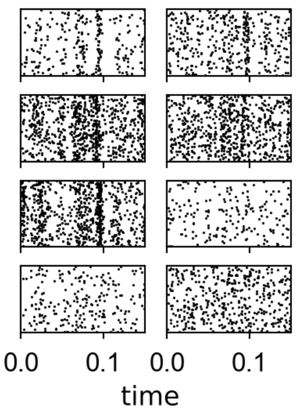
- Single neuron parameters
- Statistical connectivity
- Synaptic dynamics
- Number of neurons per population

Microscopic model



simulate

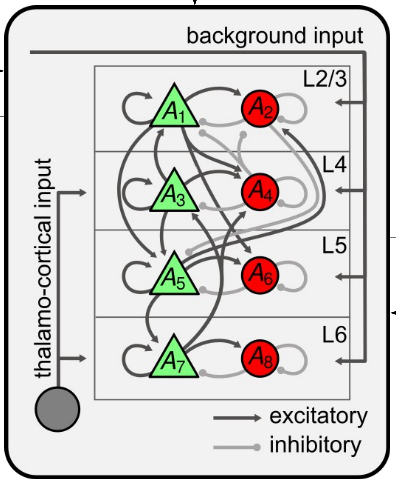
Spiking activity



sum spikes

Bottom-up theory:
(Quasi-)Renewal approach

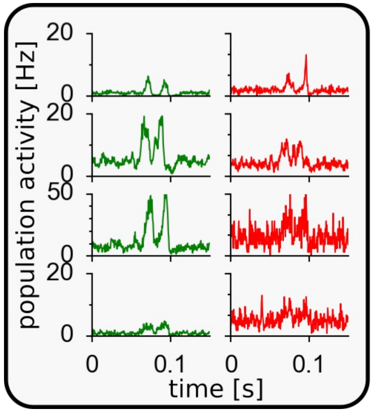
?



Mesoscopic model (firing-rate model)

simulate

infer



Population activities

Outline

1) Introduction

2) **Mesoscopic dynamics: Toy example without refractoriness**

3) Mesoscopic dynamics of integrate-and-fire neurons

4) Reduction to low-dimensional population dynamics

A tractable toy example

- Network of Linear-Nonlinear-Poisson neurons (nonlinear Hawkes process)

$$\frac{dh_i}{dt} = \frac{\mu(t) - h_i}{\tau} + \frac{J}{N}\sum_{j=1}^{N}\frac{dZ_j(t)}{dt}, \quad h_i(0) = 0$$

$$r_i(t) = f(h_i(t^-))$$

$$dZ_i(t) = \pi_i(dt, [0, r_i(t)]) \sim \text{Poisson}[r_i(t)dt], \qquad i = 1, \dots, N$$

- $s_i(t) = \frac{dZ_i(t)}{dt} = \sum_k \delta(t - t_{k,i})$ is spike train of neuron i with stochastic intensity $r_i(t)$
- Define population activity (coarse-graining): $A_N(t) := \frac{1}{N}\sum_{i=1}^{N} s_i(t)$
- $\{dZ_i(t)\}_{i=1,\dots,N}$ are conditionally independent Poisson given $h_i(t^-) =: h(t^-)$

$A_N(t)$?

$$\Rightarrow \quad dZ(t) := \sum_{i=1}^{N} dZ_i(t) \sim \text{Poisson}[Nf(h(t^-))dt]$$

- Equivalent mesoscopic model:

$$\frac{dh}{dt} = \frac{\mu(t) - h}{\tau} + JA_N(t), \quad h(0) = 0$$

$$A_N(t) = \frac{1}{N}\frac{dZ(t)}{dt}, \qquad dZ(t) \sim \text{Poisson}[Nf(h(t^-))dt]$$

Diffusion approximation for large N

- Equivalent mesoscopic model:

$$\frac{dh}{dt} = \frac{\mu(t) - h}{\tau} + JA_N(t), \quad h(0) = 0$$
$$r(t) = f(h(t^-))$$
$$A_N(t) = \frac{1}{N}\frac{dZ(t)}{dt}, \qquad dZ(t) \sim \text{Poisson}[Nr(t)dt]$$

- Central limit theorem:

$$dZ(t) \sim \mathcal{N}[Nr(t)dt, Nr(t)dt] \qquad N \to \infty$$

- Population activity:

$$A_N(t) = \frac{1}{N}\frac{dZ(t)}{dt} \approx r(t) + \sqrt{\frac{r(t)}{N}}\xi(t),$$

where $\xi(t)$ is Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$

Outline

1) Introduction

2) Mesoscopic dynamics: Toy example without refractoriness

3) **Mesoscopic dynamics of integrate-and-fire neurons**

4) Reduction to low-dimensional population dynamics

Wulfram Gerstner
(EPFL)

Moritz Deger
(EPFL)

Valentin Schmutz
(EPFL)

Eva Löcherbach
(Univ Paris 1)

Microscopic model

Network of leaky integrate-and-fire (LIF) neurons with escape noise:

$$dU_t^i = \frac{\mu_t - U_t^i}{\tau_{\mathrm{m}}} dt + \frac{J}{N} \sum_{j=1}^{N} dZ_t^j - U_{t-}^i \, dZ_t^i$$

$$dZ_t^i = \pi^i(dt, [0, f(U_{t-}^i)]),$$

- U_t^i membrane potential of neuron i, $i = 1, \ldots, N$
- Z_t^i spike count of neuron i firing with *stochastic intensity* $f(U_{t-}^i)$

Coarse-grained activity: $dZ_t := \frac{1}{N} \sum_{i=1}^{N} dZ_t^i(t)$

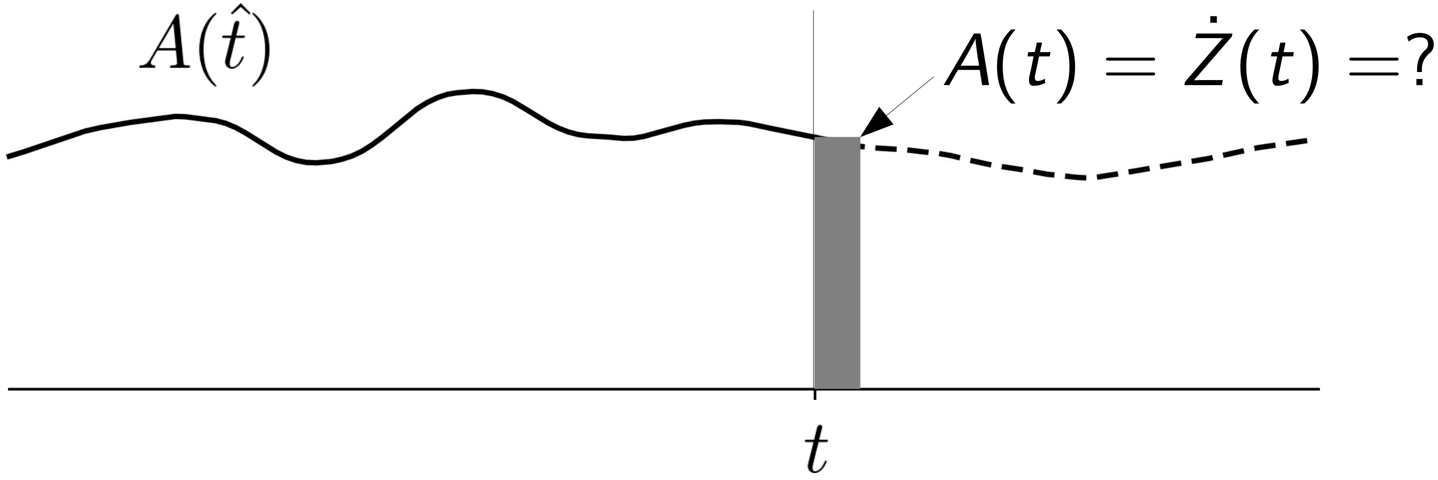
Membrane potentials differ because last spike times $\hat{t}_i$ are different:

$$U_t^i = \frac{1}{\tau} \int_{\hat{t}_i}^{t} e^{\frac{t-s}{\tau}} \left[\mu_s \, ds + J dZ_s\right] =: u^{\dot{Z}}(t|\hat{t}_i)$$

[Gerstner 2000, Galves & Löcherbach, 2016, Cormier et al. 2020]

Macroscopic population dynamics ($N \to \infty$): Integral equation

Wilson & Cowan, 1972
Gerstner, 1995, 2000
Cormier et al. 2020



Macroscopic population dynamics (N $\to \infty$): Integral equation

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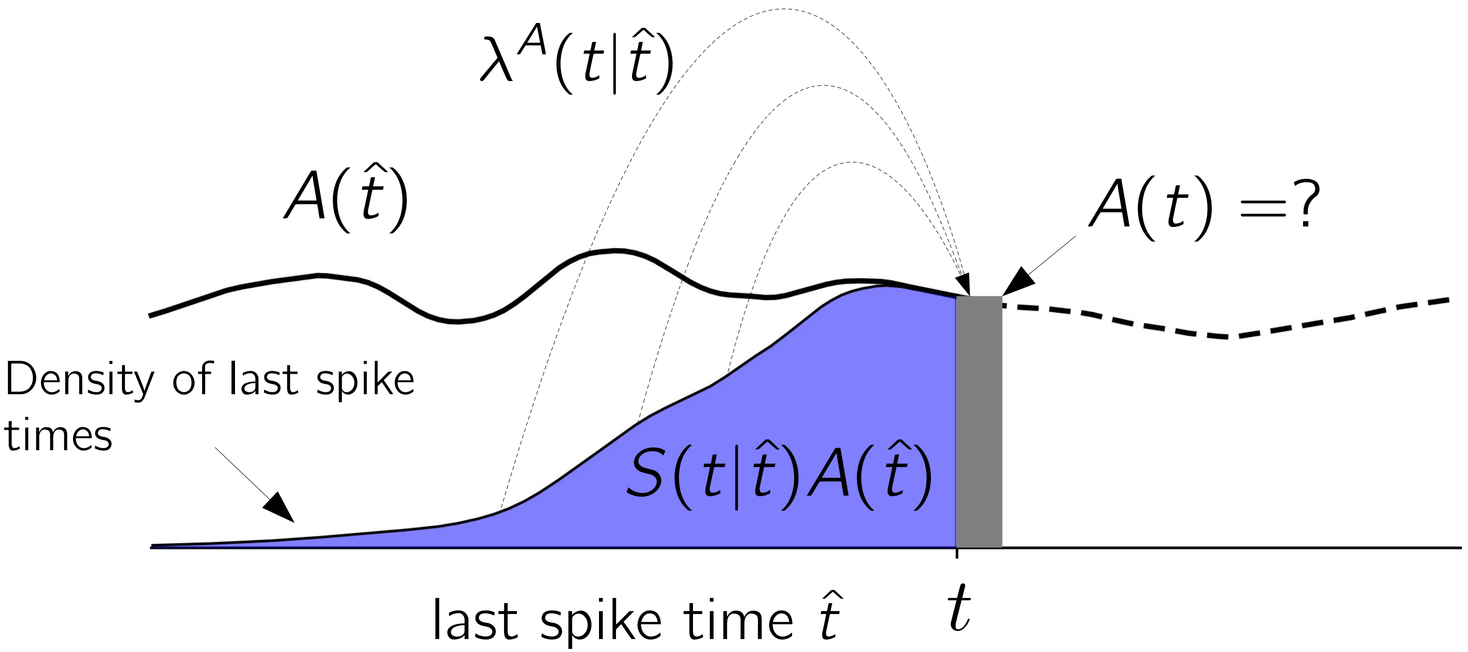


Conservation of neural mass:

$$A(t) = \int_{0^-}^{t} \lambda^A(t|\hat{t}) S^A(t|\hat{t}) A(\hat{t}) d\hat{t},$$

$$a_0 := \int_{0^-}^{t} S^A(t|\hat{t}) A(\hat{t}) d\hat{t} = 1$$

$$\lambda^A(t|\hat{t}) := f\left(u^A(t|\hat{t})\right)$$ hazard function

$$S^A(t|\hat{t}) := \exp\left(-\int_{\hat{t}}^{t} \lambda^A(r|\hat{t})\, dr\right)$$ survivor function

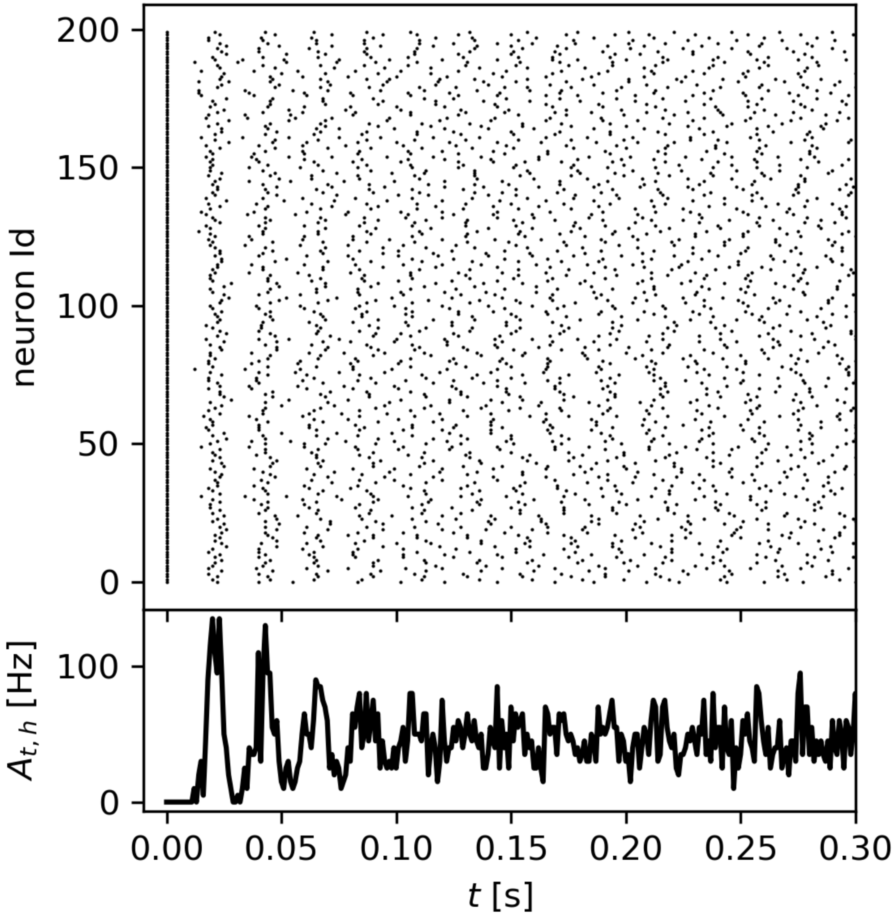
Naive finite-size extension of integral equation ("Λ=0")

$$r(t) = \left[\int_{0^-}^{t} \lambda^A(t|s) S^A(t|s) A_N(s)\, ds\right]_+$$

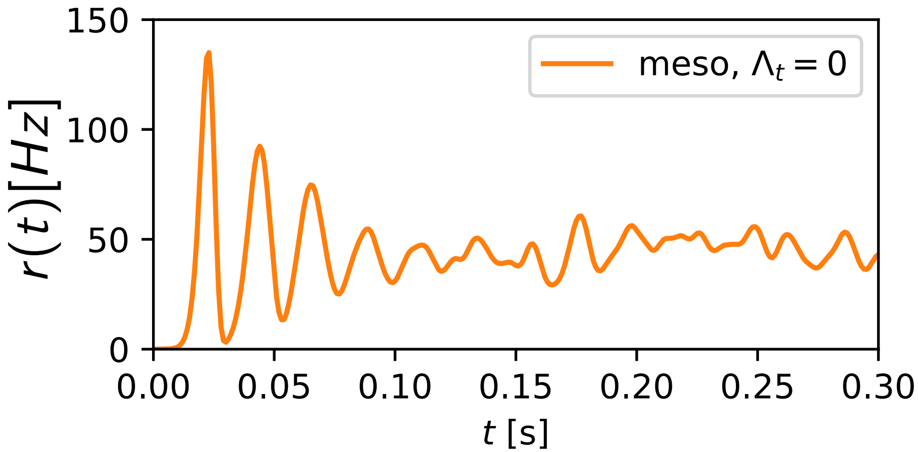
$$\lambda^A(t|\hat{t}) = f\left(u^A(t|\hat{t})\right), \quad S^A(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^{t} \lambda^A(t|\hat{t})\, dr\right)$$

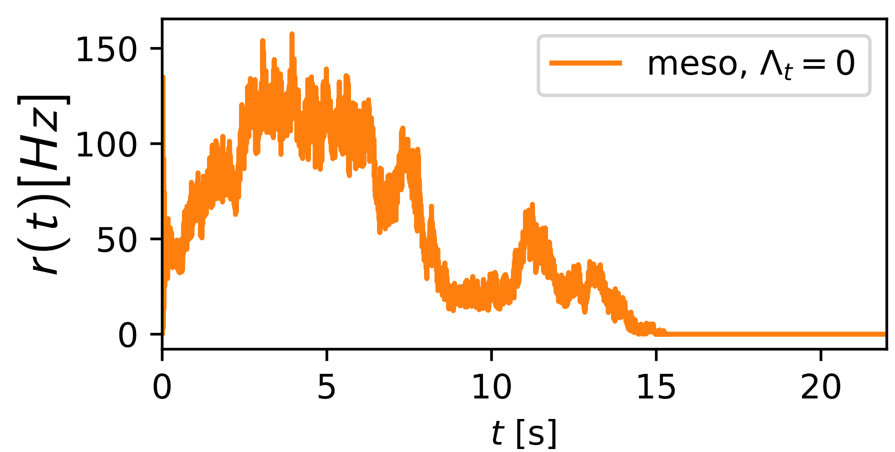
$$A_N(t) = r(t) + \sqrt{\frac{r(t)}{N}}\xi(t) \qquad \langle \xi(t)\xi(t')\rangle = \delta(t-t')$$

Microscopic model



Naive mesoscopic model





Activity dies out

Mesoscopic model for finite-size population: Stochastic integral equation

$$A_N(t) = r(t) + \sqrt{\frac{r(t)}{N}}\xi(t)$$

$$r(t) = \left[\int_{0^-}^{t} \lambda^A(t|s)S^A(t|s)A_N(s)\,ds + \Lambda_t^A\left(1 - \int_{0^-}^{t} S^A(t|s)A_N(s)ds\right)\right]_+$$

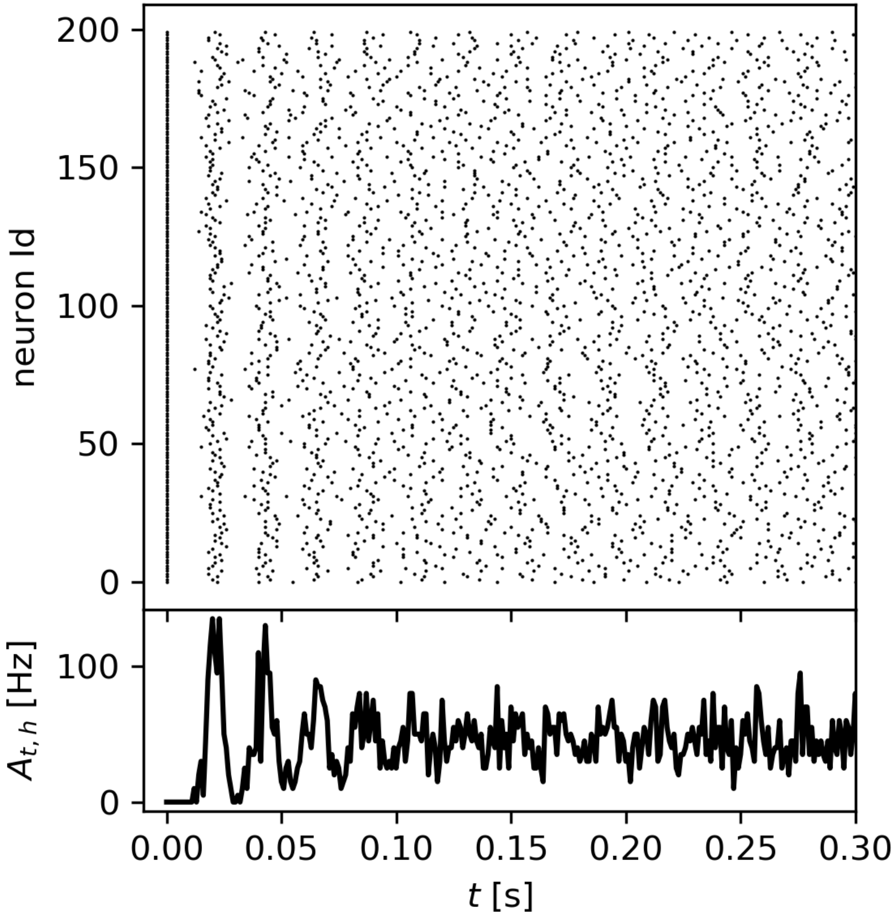
$$u^A(t|\hat{t}) = \frac{1}{\tau}\int_{\hat{t}}^{t} e^{\frac{t-s}{\tau}}\left[\mu(s) + JA_N(s)\right]\,ds$$

$$\lambda^A(t|\hat{t}) = f\left(u^A(t|\hat{t})\right), \quad S^A(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^{t}\lambda^A(r|\hat{t})\,dr\right)$$

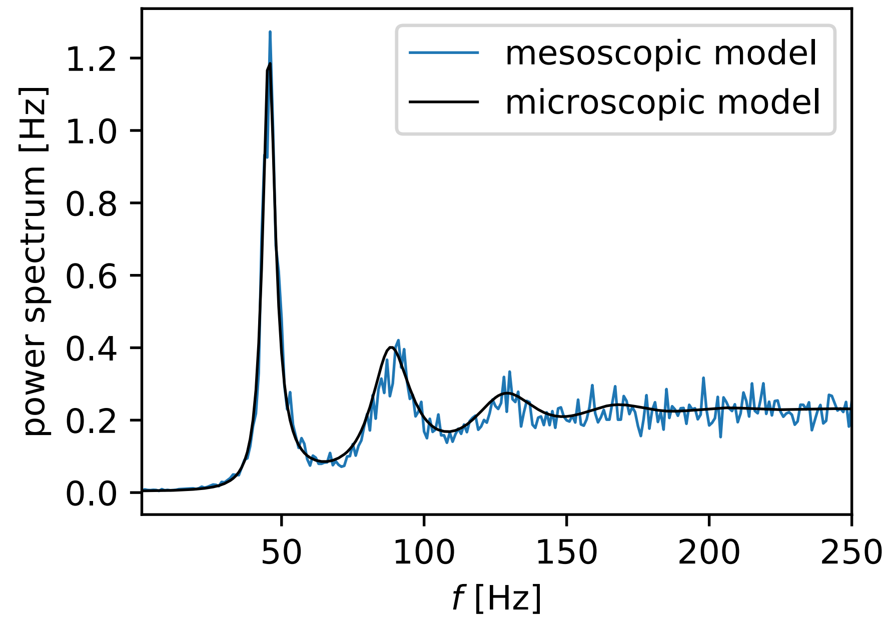
$$\Lambda_t^A = \frac{\int_{-\infty}^{t}\lambda^A(t|s)S^A(t|s)(1-S^A(t|s))A_N(s)\,ds}{\int_{-\infty}^{t}S^A(t|s)(1-S^A(t|s))A_N(s)\,ds}$$

Extension to models with adaptation (generalized integrate-and-fire model, nonlinear Hawkes process) available via quasi-renewal approximation

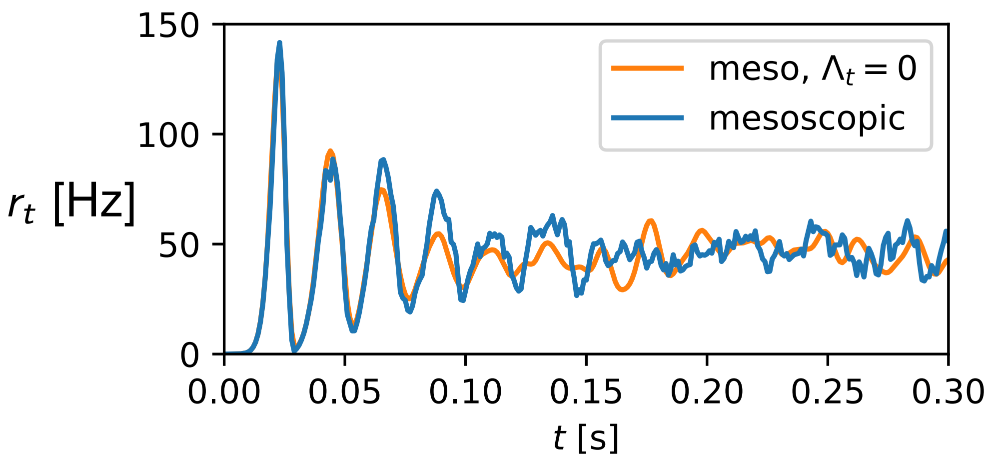
Schwalger et al., PloS Comput. Biol (2017)
Schmutz, Löcherbach, Schwalger, SIAM Appl. Dyn. Syst. (2023)

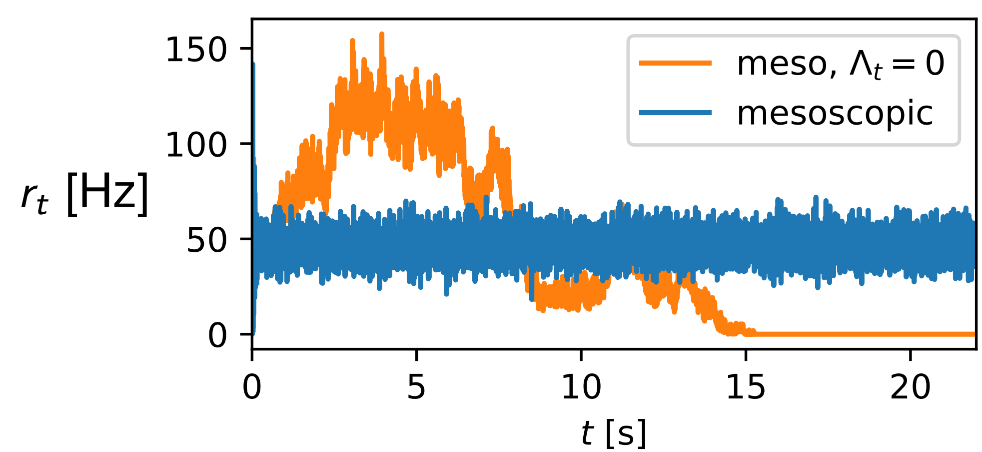


Microscopic model



Mesoscopic model





Schmutz, Löcherbach, Schwalger, SIAM Appl. Dyn. Syst. (2023)

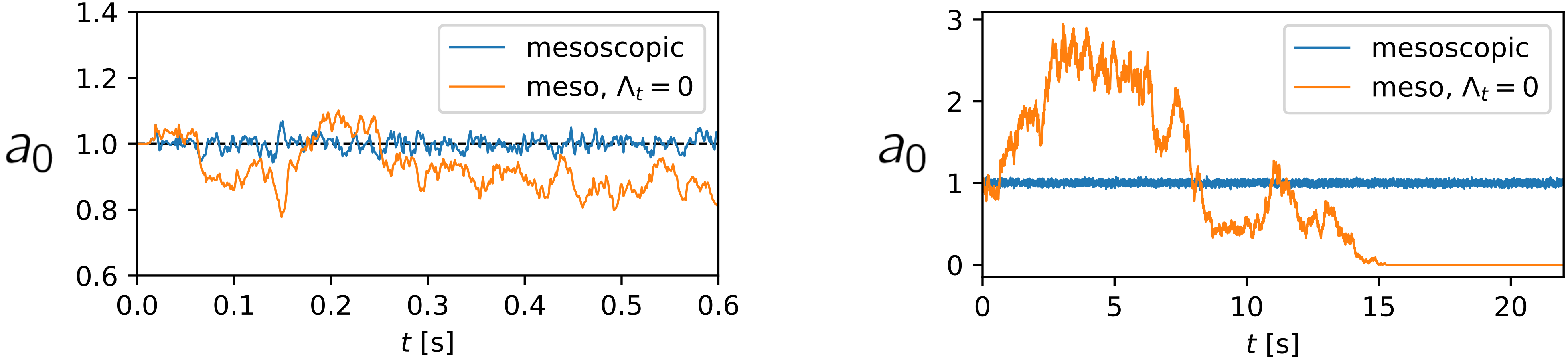
Activity is „stable“

Approximate conservation of neuronal mass

- Hitting time $\tau := \inf\{t > 0 : r(t) = 0\}$
- for $0 < t < \tau$, the neural mass $a_0(t) := \int_{-\infty}^{t} S^A(t|\hat{t})A(\hat{t})\, d\hat{t}$ obeys

$$\dot{a}_0 = \Lambda^A(1 - a_0) + \sqrt{\frac{r(t)}{N}}\xi(t)$$

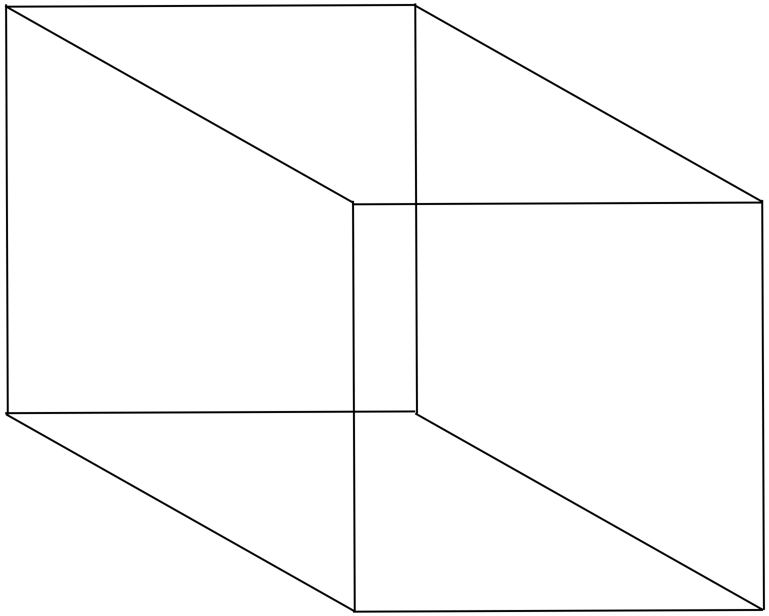
- if $\Lambda^A = 0$ then $a_0(t) = 0$ for all $t \geq \tau$
- simlar to Cox-Ingersoll-Ross process $\dot{a} = \Lambda(1-a)+\sigma\sqrt{a}\xi(t)$, which for $\Lambda = 0$ goes extinct with $P(a(t) = 0|a(0) = x) = e^{-\frac{x}{\sigma^2 t}}$



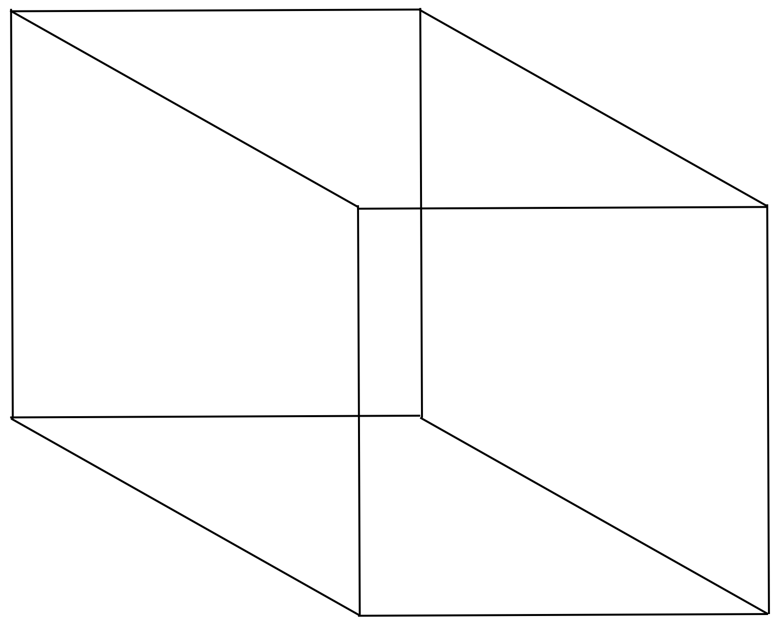
If $\Lambda^A > 0$, the mesoscopic model is well-posed and stable!

Schmutz, Löcherbach, Schwalger, SIAM Appl. Dyn. Syst. (2023)

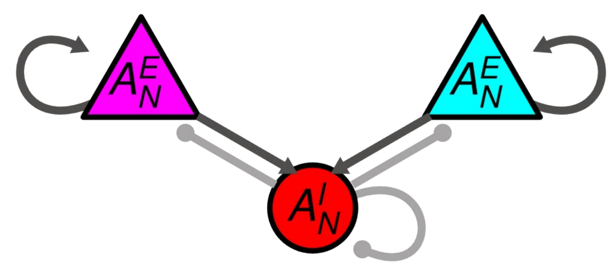
Finite-size noise causes hopping between attractors



Finite-size noise causes hopping between attractors

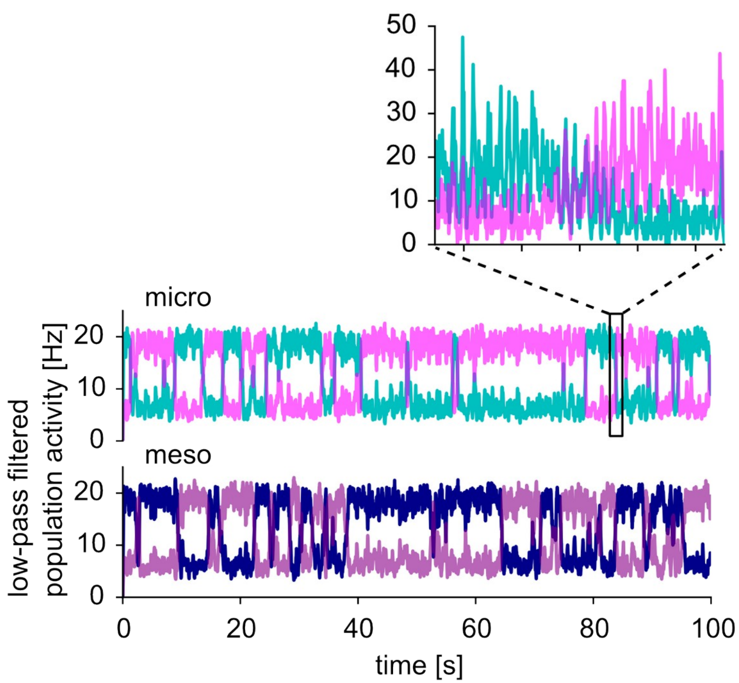


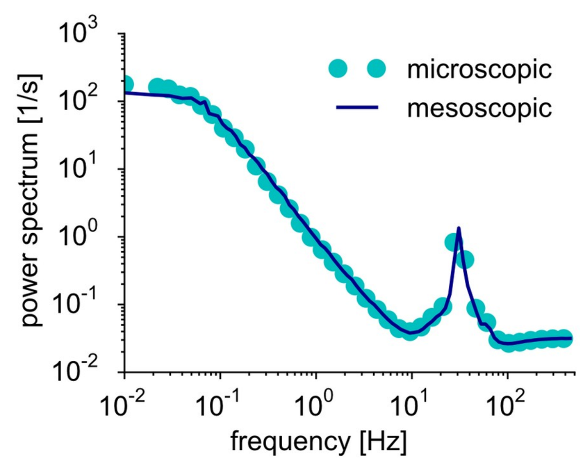
Schwalger, Deger, Gerstner,
PloS Comput Biol (2017)

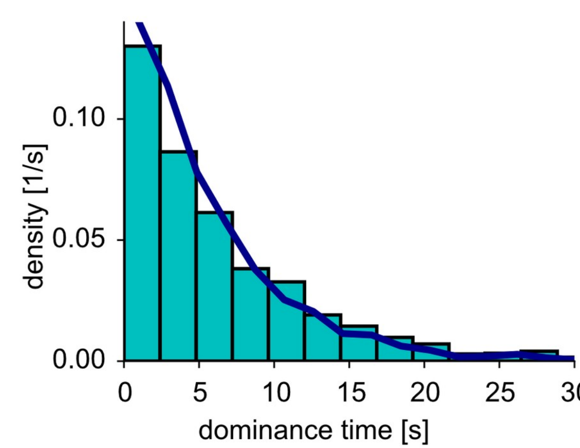


$$N^{E1} = N^{E2} = 400$$

$$N^I = 200$$





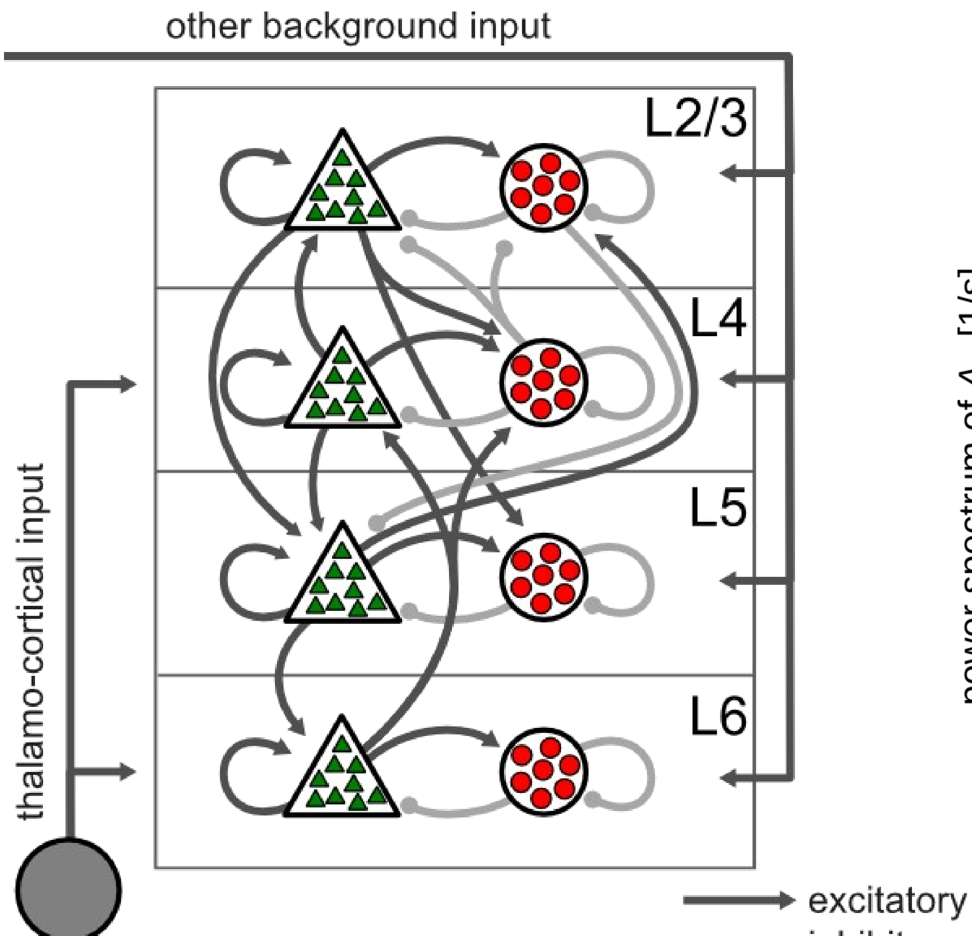


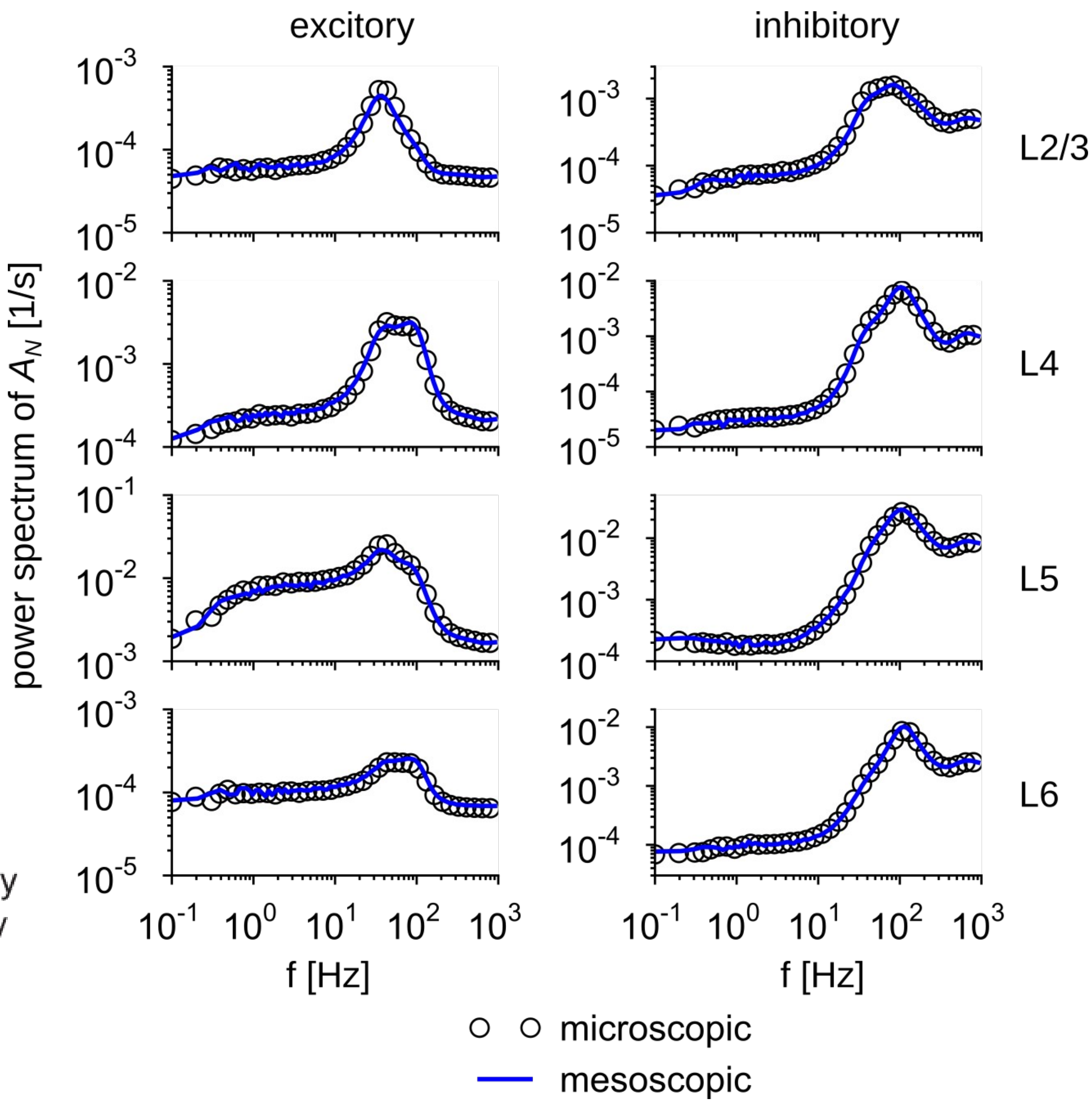
Model of a local cortical circuit

Potjans & Diesmann, Cereb Cortex (2014)

80'000 LIF neurons with adaptation

120x speed-up!





See also:
Bos et al. , Cain et al.
PloS Comput Biol 2016

Outline

1) Introduction

2) Mesoscopic dynamics: Toy example without refractoriness

3) Mesoscopic dynamics of integrate-and-fire neurons

4) **Reduction to low-dimensional population dynamics**

Bastian Pietras (TU Berlin)

Noe Gallice (EPFL)

Microscopic model

Network of leaky integrate-and-fire (LIF) neurons with escape noise:

$$\tau_{\mathrm{m}} dU_t^{i,N} = (h_t - U_t^{i,N})dt - \tau_{\mathrm{m}} U_{t^-}^{i,N} dZ_t^{i,N},$$

$$\tau_h dh_t = (-h_t + I_t)dt + \frac{J}{N}\sum_{j=1}^{N} dZ_t^{j,N}$$

$$Z_t^{i,N} = \int_{[0,t]\times\mathbb{R}_+} 1_{z \le f(U_{s^-}^{i,N})} \pi^i(ds, dz),$$

- $U_t^{i,N}$ membrane potential of neuron i, $i = 1, \dots, N$
- $Z_t^{i,N}$ spike count of neuron i firing with *stochastic intensity* $f(U_{t^-}^{i,N})$

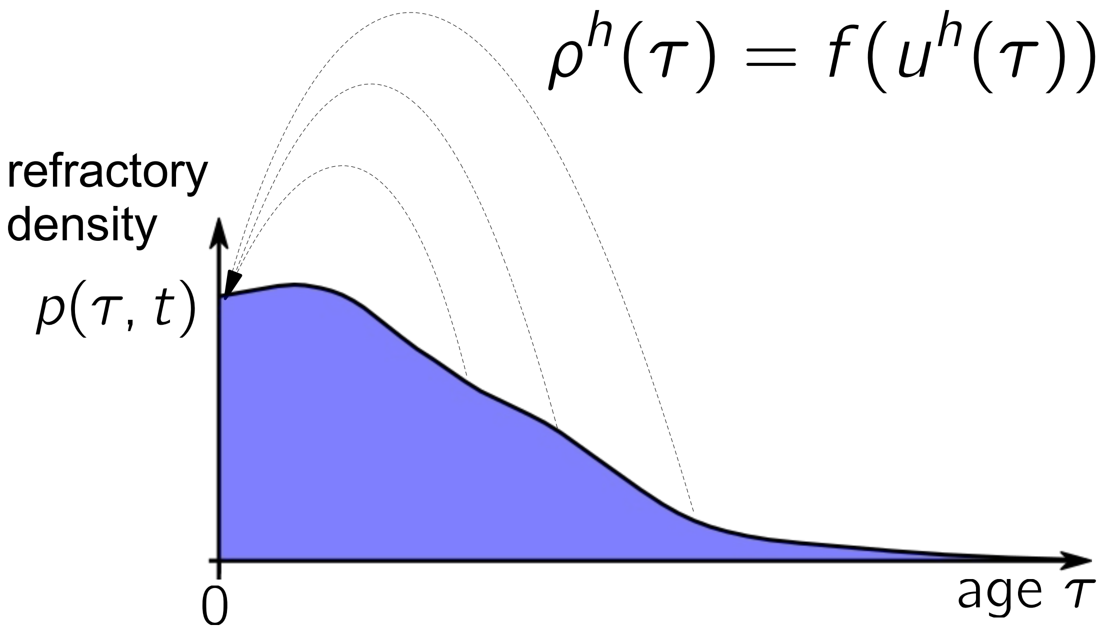
[Gerstner 2000, Galves & Locherbach, 2016, Cormier et al. 2020]

Macroscopic dynamics (N $\to \infty$): Refractory density equation

Time *since* the last spike: $\tau = t - \hat{t}$

(age-structured population dynamics)

$$\rho^h(\tau) = f(u^h(\tau))$$



Gerstner & Kistler 2001
Chizhov & Graham 2007, 2008
Pakdaman, Perthame, Salort, 2010
Dumont, Payeur, Longtin 2017
Schwalger & Chizhov 2019

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial \tau} = -\rho^h(\tau) p + A(t)\delta(\tau)$$

$$A(t) = \int_0^\infty \rho^h(\tau) p(\tau, t)\, d\tau$$

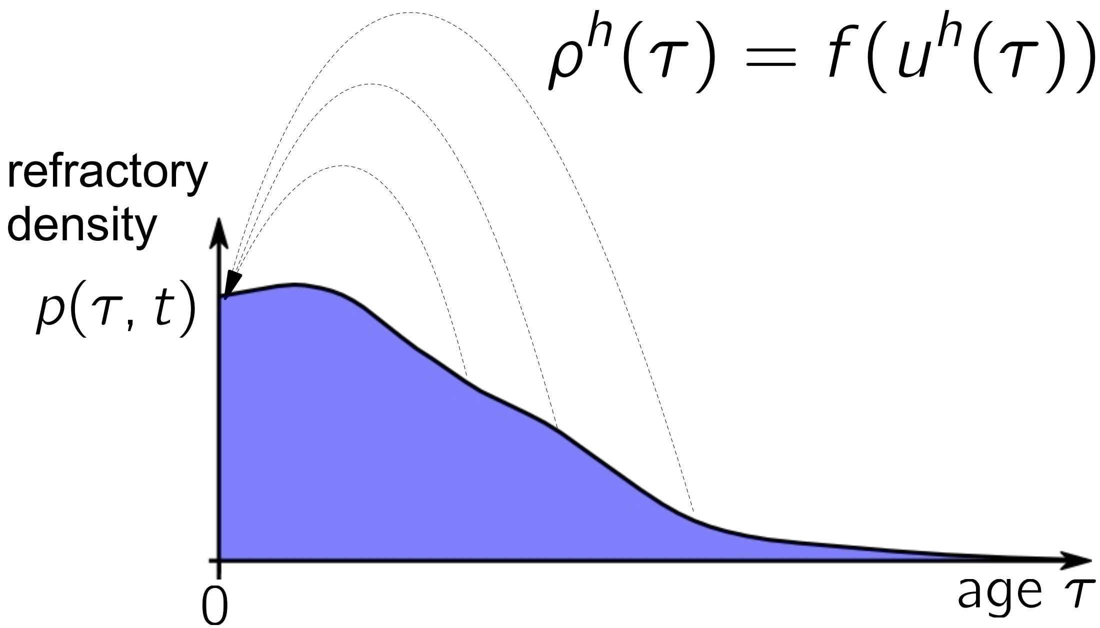
Conservation of neural mass:

$$a_0 := \int_0^\infty p(\tau, t)\, d\tau = 1$$

Macroscopic dynamics (N → ∞): Refractory density equation

(age-structured population dynamics)

$$\rho^h(\tau) = f(u^h(\tau))$$



Gerstner & Kistler 2001
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Pakdaman, Perthame, Salort, 2010
Dumont, Payeur, Longtin 2017
Schwalger & Chizhov 2019

$$\partial_t p = -[\partial_\tau + \rho(\tau, h)]p(\tau, t) =: \mathrm{L}_\tau(h)p(\tau, t)$$

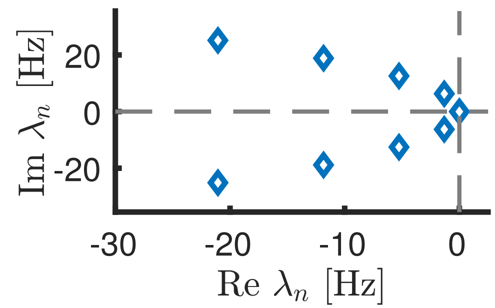
$$p(0, t) = A(t) \equiv \int_0^\infty \rho(\tau, h)p(\tau, t)\,d\tau$$

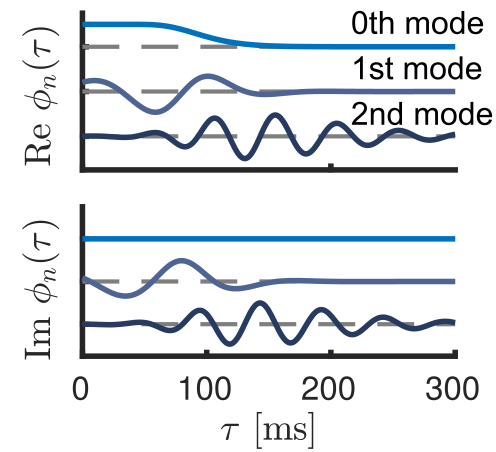
Eigenmode expansion for renewal models

- **Eigenmode expansion of refractory-density equation:**

  For any renewal model, the eigenvalues are the solutions of

$$P_L(\lambda) := \int_0^\infty P_{\mathrm{ISI}}(t)e^{-\lambda t}\,dt = 1$$





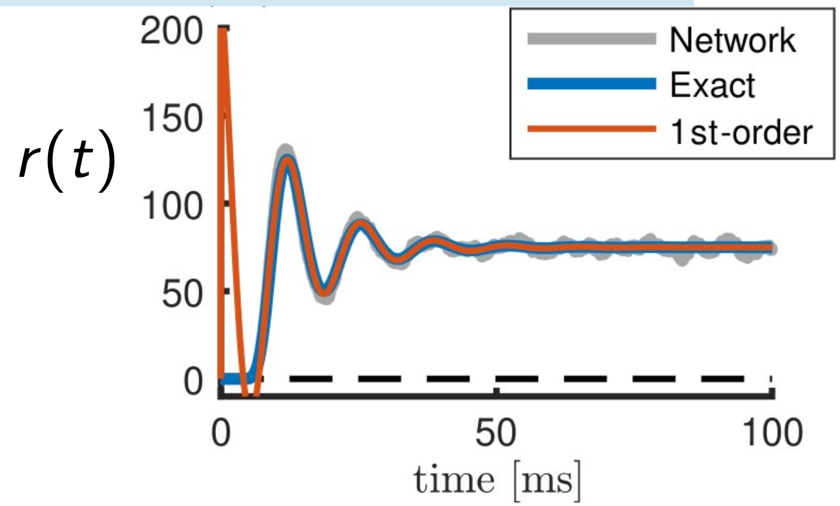
Same analysis for Fokker-Planck equation:
Knight et al. (1996)
Mattia & Del Giudice (2002)
Schaffer, Ostojic, Abbott (2013)
Augustin, et al. (2017)

- **Low-dimensional dynamics**

$$A(t) = F_0(h) - 2\sum_{n=1}^{M} \mathrm{Re}\left(\frac{a_n}{P_L'(\lambda_n)}\right)$$

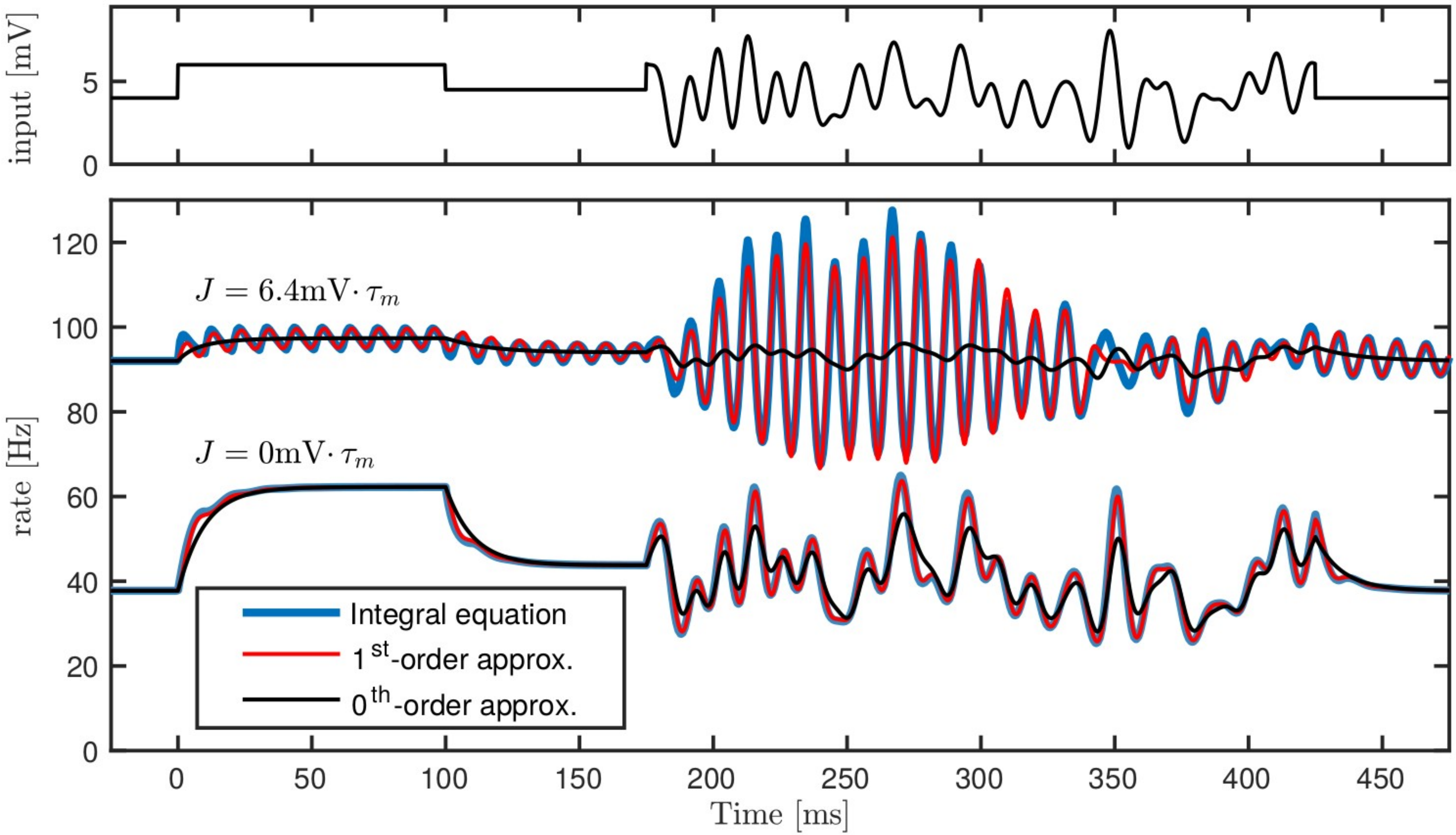
$$\tau_h \frac{dh}{dt} = -h(t) + I(t) + JA(t)$$

$$\frac{da_n}{dt} = \lambda_n a_n + \left\{c_0 + \sum_{m=1}^{M}[c_m a_m + \hat{c}_m a_m^*]\right\}\frac{dh}{dt}$$



Pietras, Gallice, Schwalger, Phys Rev E (2020)

Recurrent network



Pietras, Gallice, Schwalger, *Phys. Rev. E 2020*

Mesoscopic population equation (N < ∞)

Mesoscopic model:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial \tau} = -\rho^h(\tau)p(\tau,t) + A_N(t)\delta(\tau)$$

$$\bar{A}(t) = \left[\int_0^\infty \rho^h(\tau)p(\tau,t)\,d\tau + \underbrace{\Lambda^h(t)\left(1 - \int_0^\infty p(\tau,t)\,d\tau\right)}_{\text{„correction term“}}\right]_+$$

$$A_N(t) = \bar{A}(t) + \sqrt{\frac{\bar{A}(t)}{N}}\xi(t)$$

- $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$
- firing intensity $\rho^h(\tau) = f(u^h(\tau))$
- $\bar{A}(t)$: population firing rate
- $a_0 := \int_0^\infty p(\tau,t)\,d\tau \not\equiv 1$ is no longer the exact neural mass!

Schwalger, Deger, Gerstner, PloS Comput Biol (2017)

Langevin equation for $N < \infty$: Eigenmode expansion of mesoscopic equation

$$A_N(t) = r(t) + \sqrt{\frac{r(t)}{N}}\xi(t)$$

$$r(t) = \quad F_0(h) - 2\sum_{n=1}^{M} \mathrm{Re}\left(\frac{a_n}{P_L'(\lambda_n)}\right) \cdot$$

$$\tau_h \frac{\mathrm{d}h}{\mathrm{d}t} = -h + I(t) + JA_N(t)$$

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \lambda_n a_n + \left\{c_0 + \sum_{m=1}^{M}[c_m a_m + \hat{c}_m a_m^*]\right\}\frac{dh}{dt} + \Lambda(t)(1 - a_0) + \sqrt{\frac{r(t)}{N}}\xi(t)$$

$$\frac{da_0}{dt} = \Lambda(t)(1 - a_0) + \sqrt{\frac{r(t)}{N}}\xi(t)$$

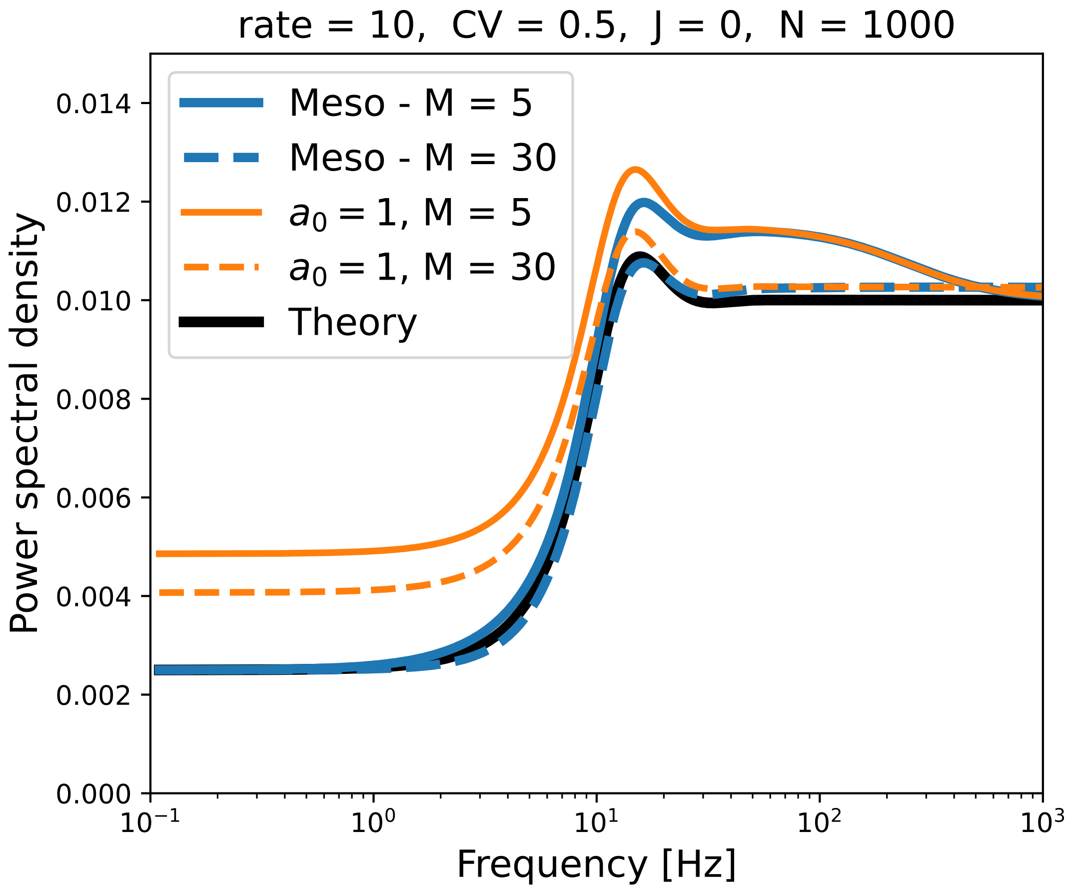
$$\Lambda(t) = \frac{1}{\sigma_{\mathrm{ISI}}(t)}$$

Power spectrum

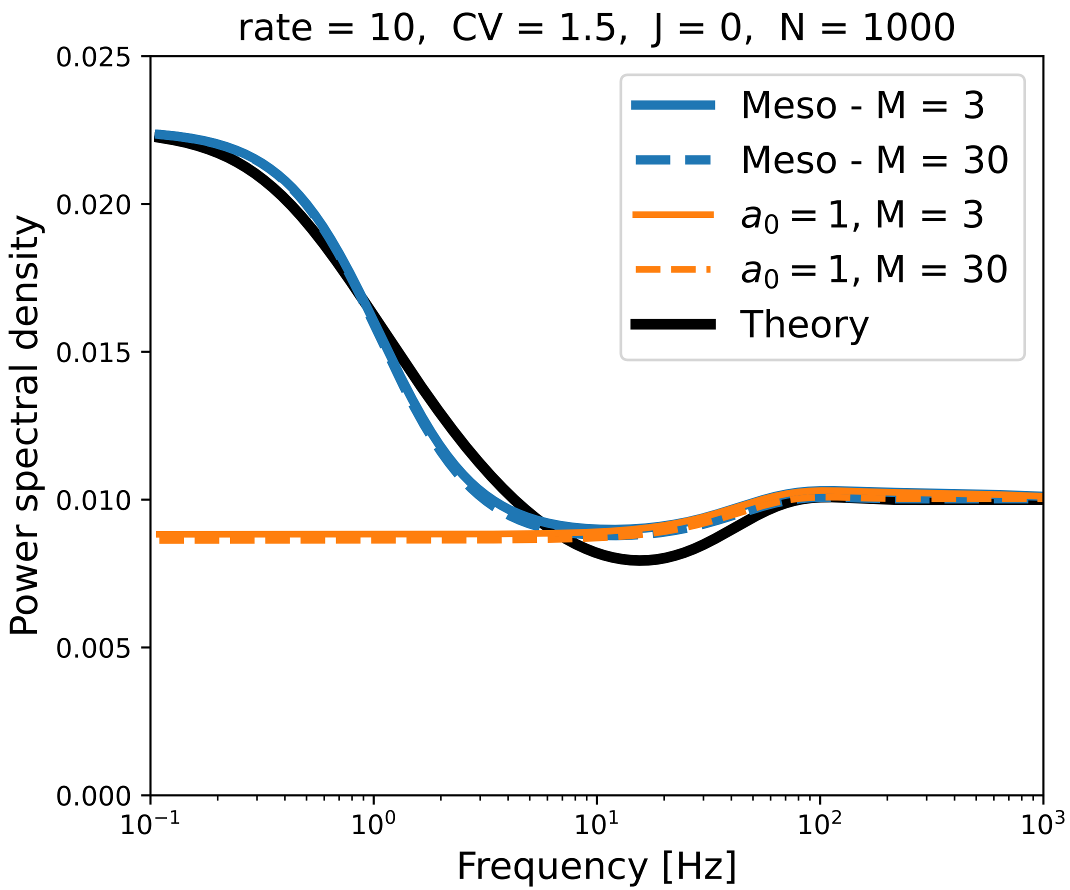
$$P_L(s) = \exp\left[\frac{1}{C_V^2}\left(1 - \sqrt{1 + \frac{2C_V^2 s}{r}}\right)\right].$$

Perfect integrate-and-fire model

$$\lambda_n = -2\pi^2 r C_V^2 n^2 + 2\pi r i n$$



Power spectrum



Summary

- Stochastic integral equation for mesoscopic population activity accurately describes finite-size integrate-and-fire network. Crucial point: Conservation of neural mass on average.
- The eigenmode expansion of the mesoscopic refractory density equation yields a low-dimensional firing-rate model that captures non-stationary dynamics and low-frequency fluctuations

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Noe Gallice (EPFL)

Eigenvalues and eigenfunctions of RDE

For arbitrary renewal models, eigenvalues given by:

$$P_L(\lambda_n) = 1 \quad \Leftrightarrow \quad S_L(\lambda_n) = 0$$

Laplace transforms of interspike interval density / survivor function

$$P_L(\lambda) \equiv \int_0^\infty P(\tau;h)e^{-\lambda\tau}\,d\tau \qquad S_L(\lambda) \equiv \int_0^\infty S(\tau;h)e^{-\lambda\tau}\,d\tau$$

- Eigenfunctions

$$\phi_n(\tau,h) = -\frac{S(\tau,h)\mathrm{e}^{-\lambda_n(h)\tau}}{P_L'\big(\lambda_n(h),h\big)}$$

$$\psi_n(\tau,h) = \frac{\mathrm{e}^{\lambda_n(h)\tau}}{S(\tau,h)}\left[1 - \int_0^\tau P(s,h)\mathrm{e}^{-\lambda_n(h)s}\,ds\right]$$

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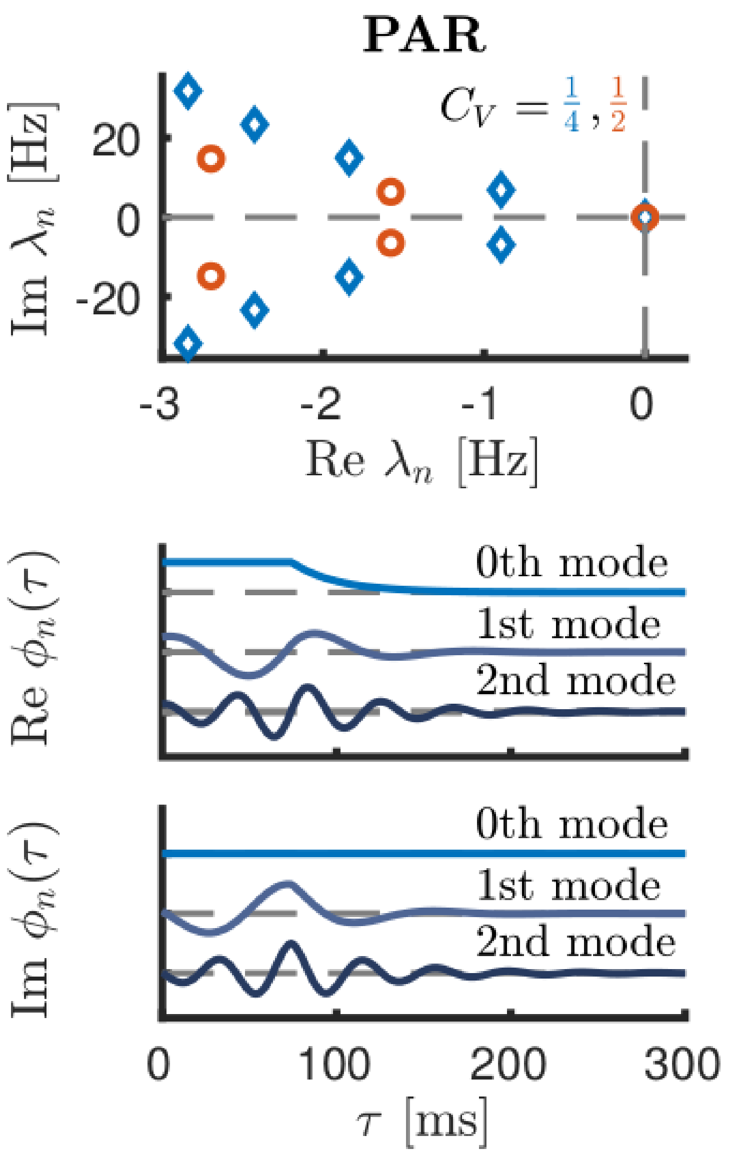


$$P_L(\lambda) = \frac{\nu}{\nu + \lambda} e^{-\lambda \Delta}$$

$$\lambda_n = \frac{1}{\Delta} W_n(\Delta \nu e^{\nu \Delta}) - \nu,$$

Eigenvalues: further examples

- Approximately Gaussian-distributed ISI densities

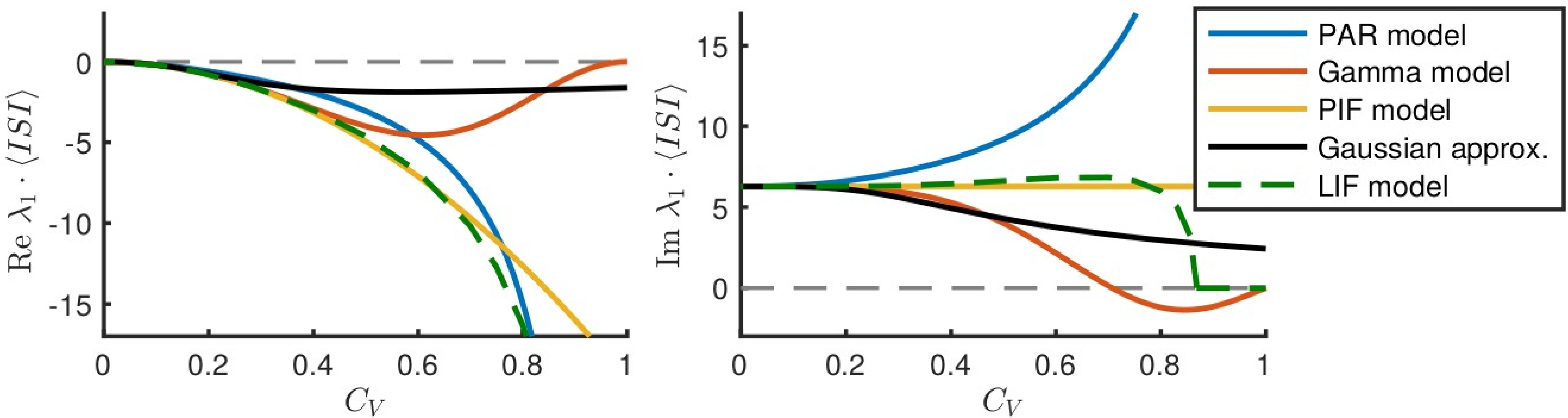
$$P_L(\lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!}\kappa_n\right) \approx \exp\Big(-\kappa_1\lambda + \frac{\kappa_2}{2}\lambda^2\Big)$$

$$\Rightarrow \quad \lambda_n = \frac{\kappa_1}{\kappa_2}\left(1 - \sqrt{1 + 4\pi i n \frac{\kappa_2}{\kappa_1^2}}\right)$$

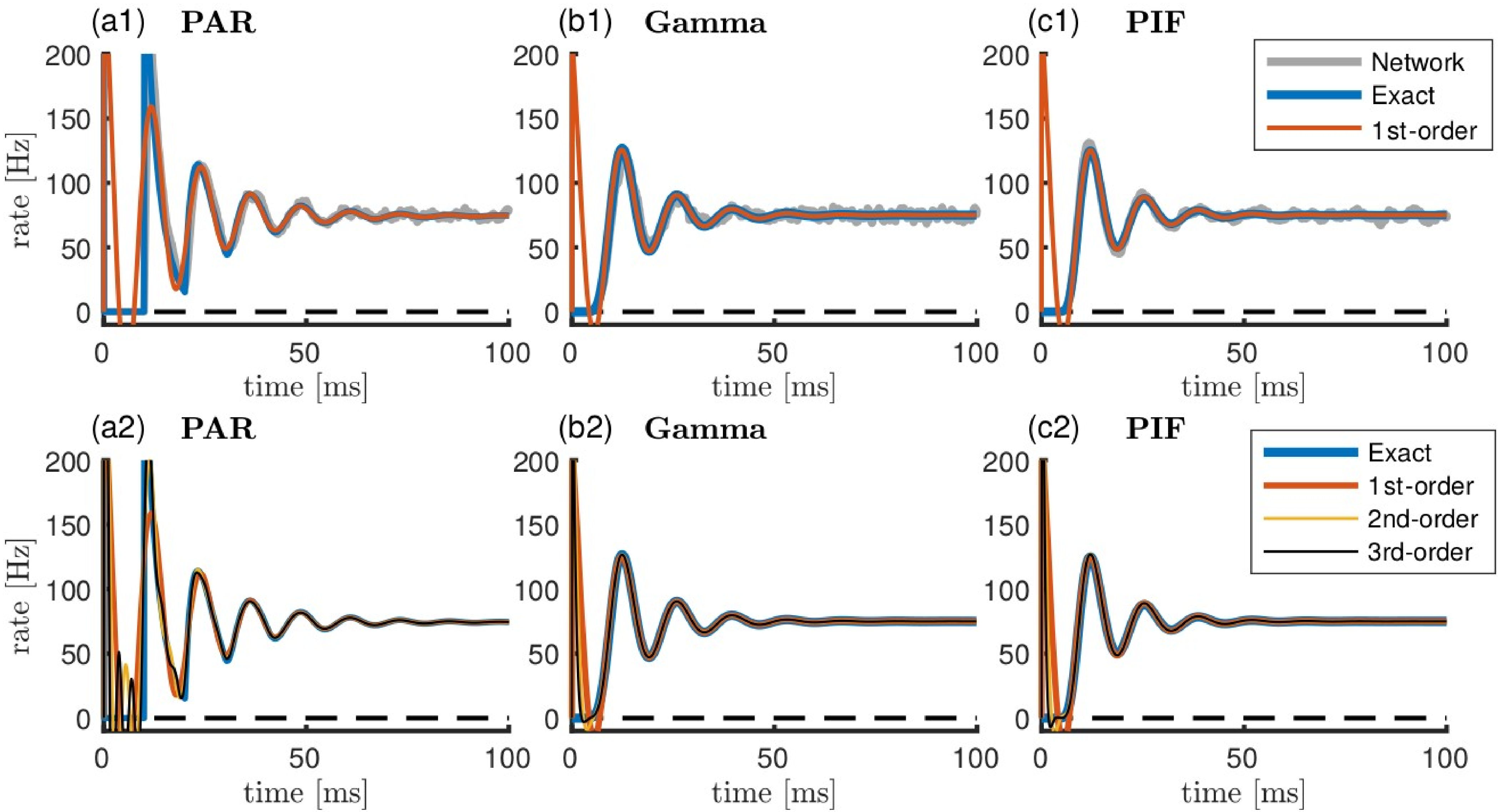
- Leaky integrate-and-fire model driven by Gaussian white noise

$$P_L(\lambda) = \mathrm{e}^{\delta}\, \frac{\mathcal{D}_{-\lambda}\left(\frac{\mu}{\sqrt{D}}\right)}{\mathcal{D}_{-\lambda}\left(\frac{\mu-1}{\sqrt{D}}\right)}, \quad \delta = \frac{1+2\mu}{4D},$$

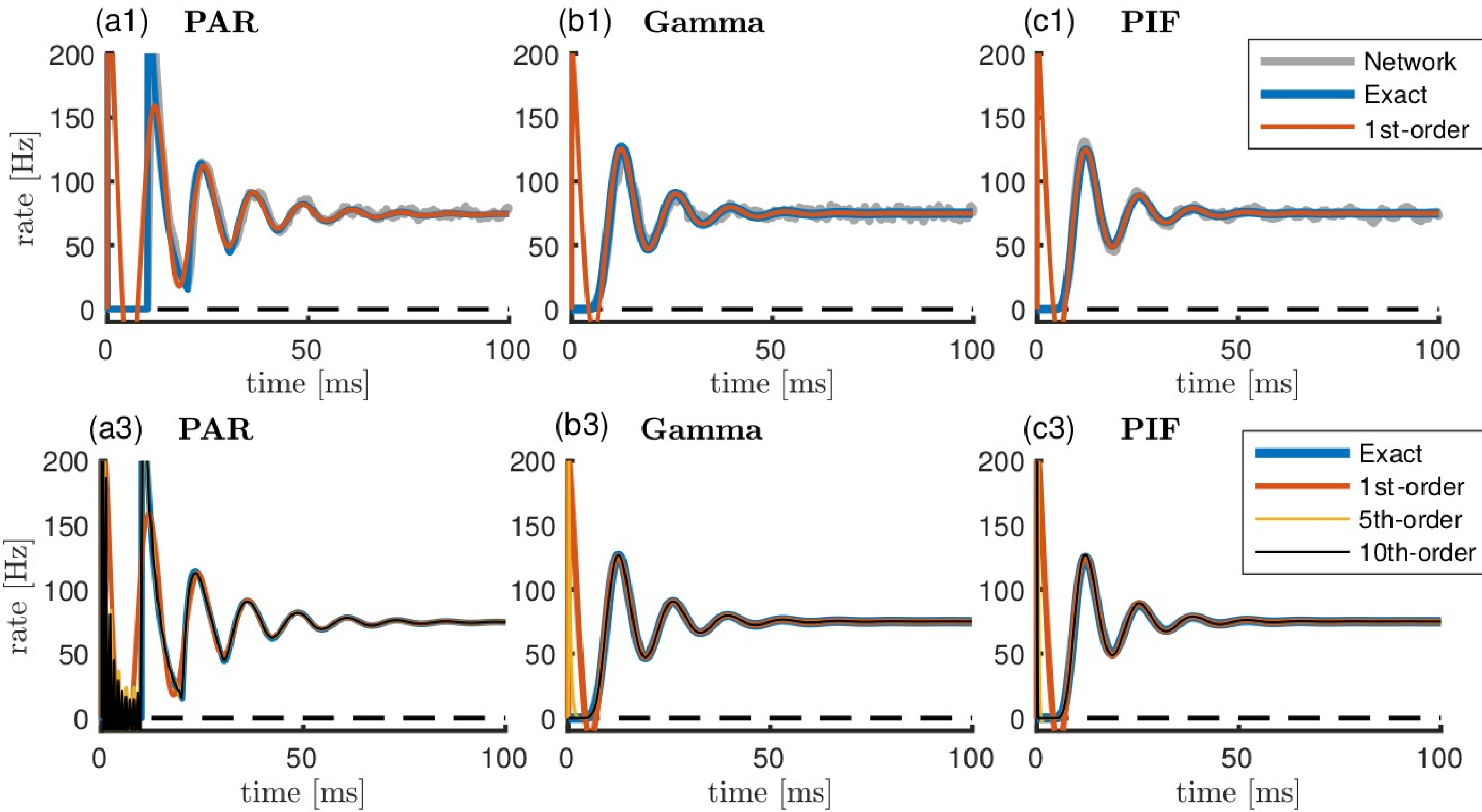
Dominant eigenvalues differ for different models with same rate and CV



Relaxation dynamics of population activity

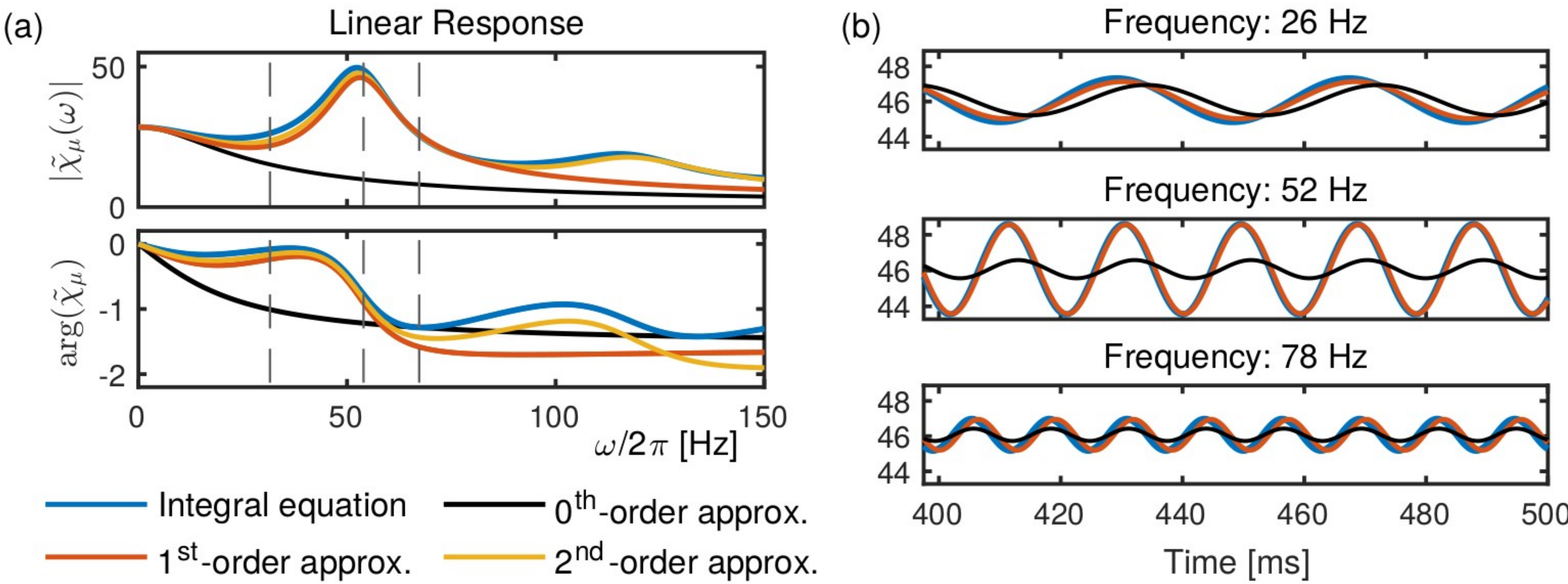


Relaxation dynamics of population activity

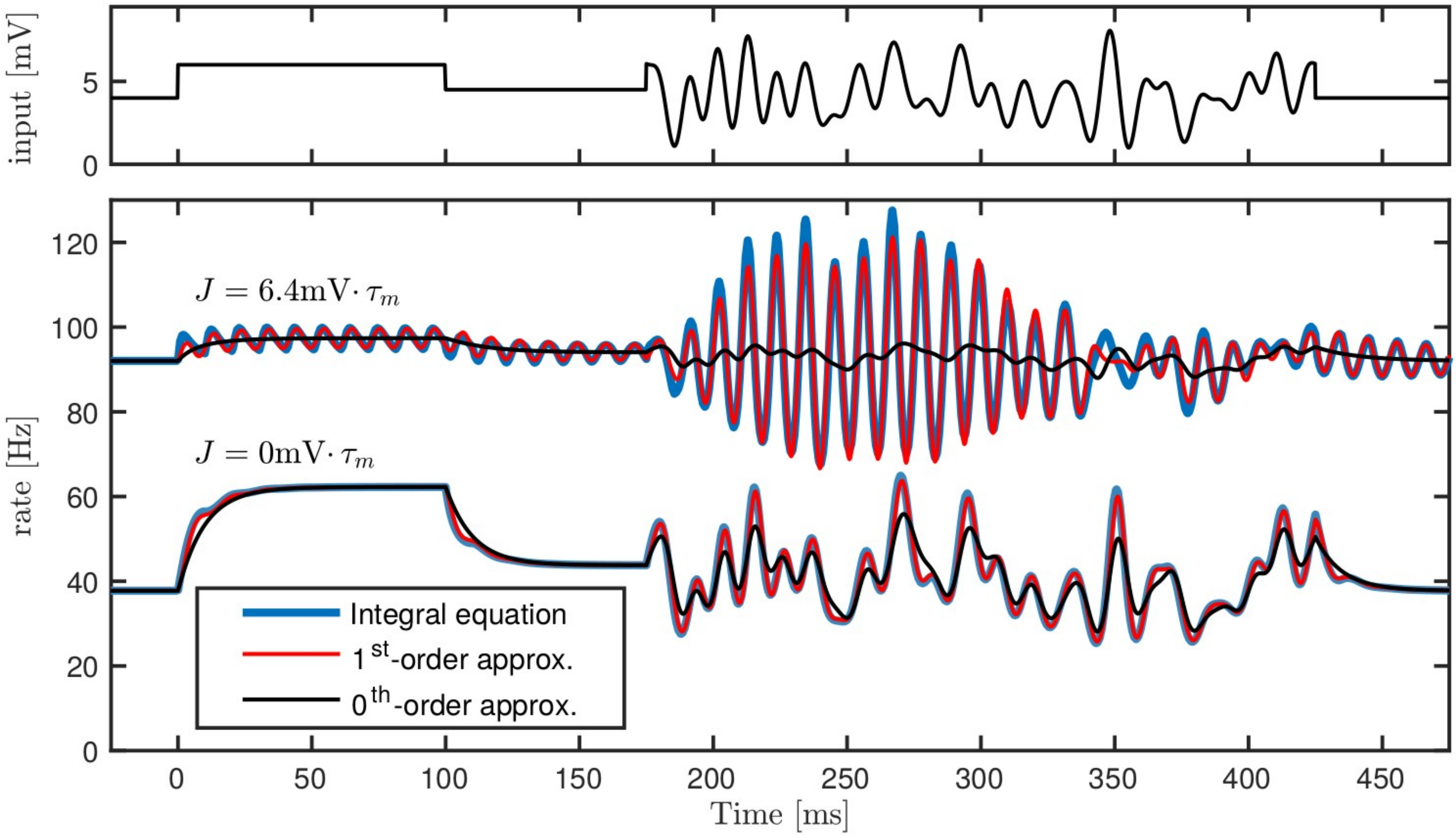


Linear response to current modulation

$$I(t) = I_0 + \epsilon \cos(\omega t)$$



Recurrent network



Pietras, Gallice, Schwalger, *Phys. Rev. E 2020*