CHAOS, CRITICALITY, AND COMPUTATION IN RECURRENT NETWORKS

MORITZ HELIAS

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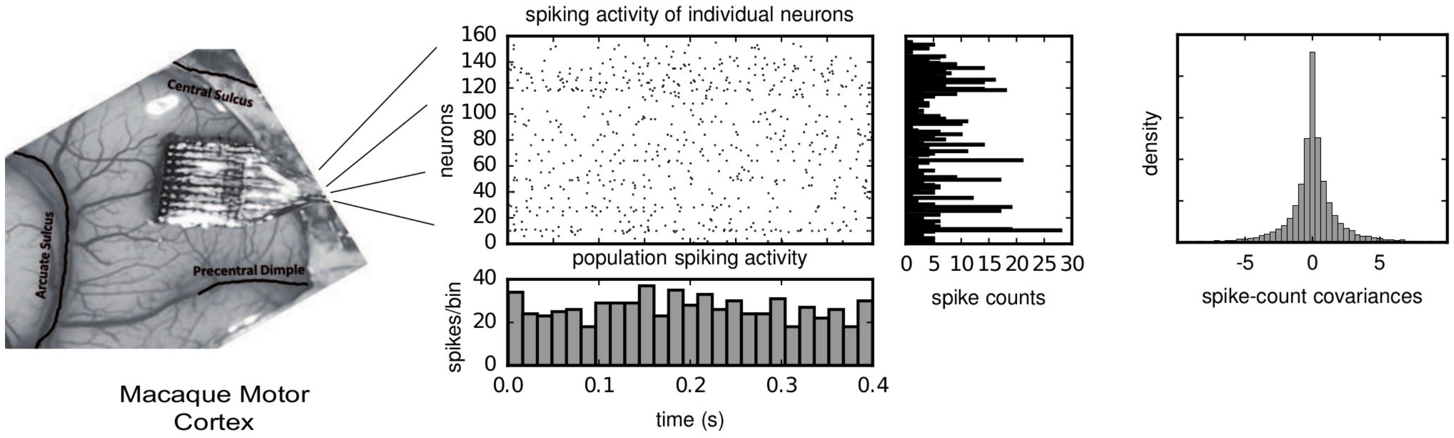
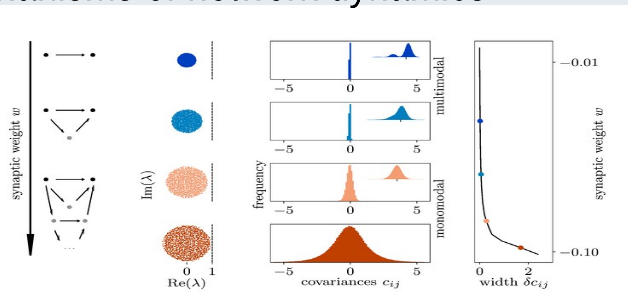
COMPUTATIONAL AND SYSTEMS NEUROSCIENCE (INM-6)
THEORETICAL NEUROSCIENCE (IAS-6)
FACULTY OF PHYSICS, RWTH AACHEN UNIVERSITY

INFERENCE OF COLLECTIVE NETWORK DYNAMICS FROM OBSERVED ACTIVITY



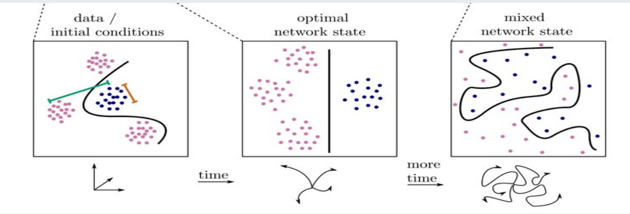
extract

mechanisms of network dynamics

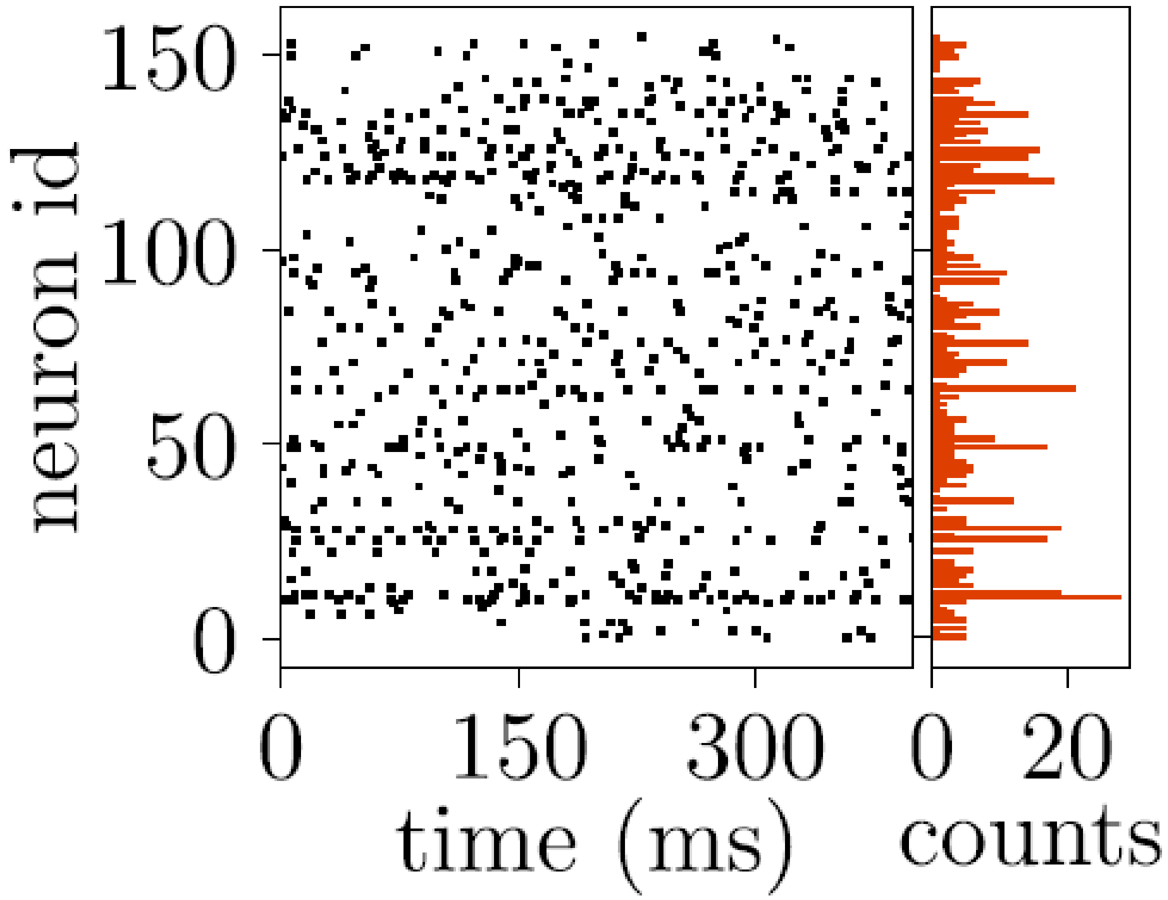


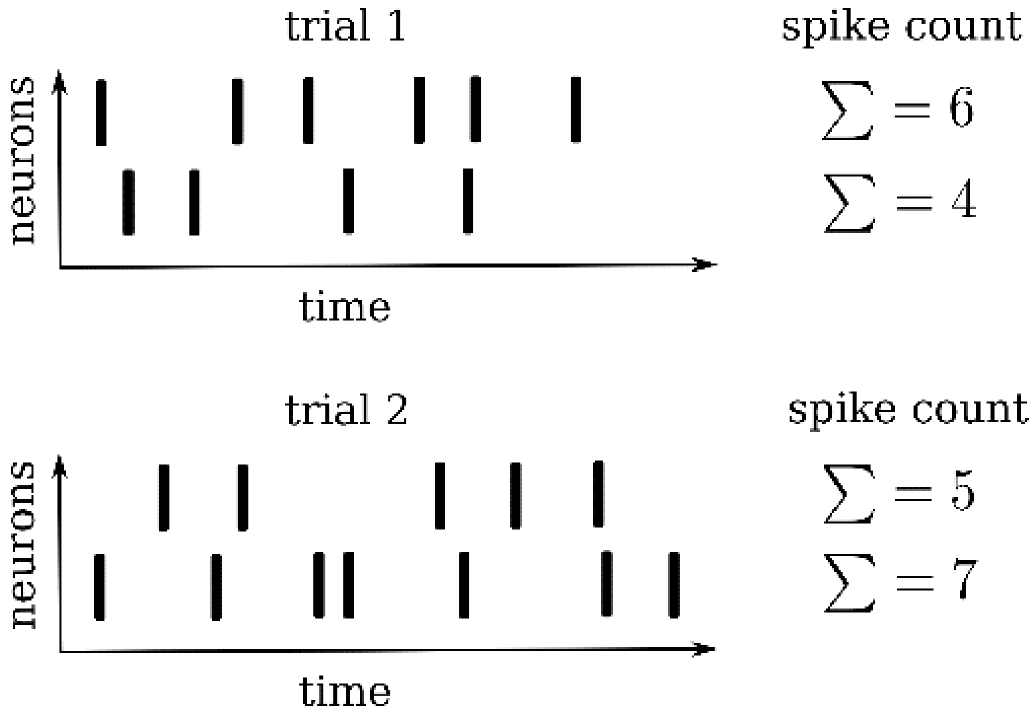
implement

neural information processing



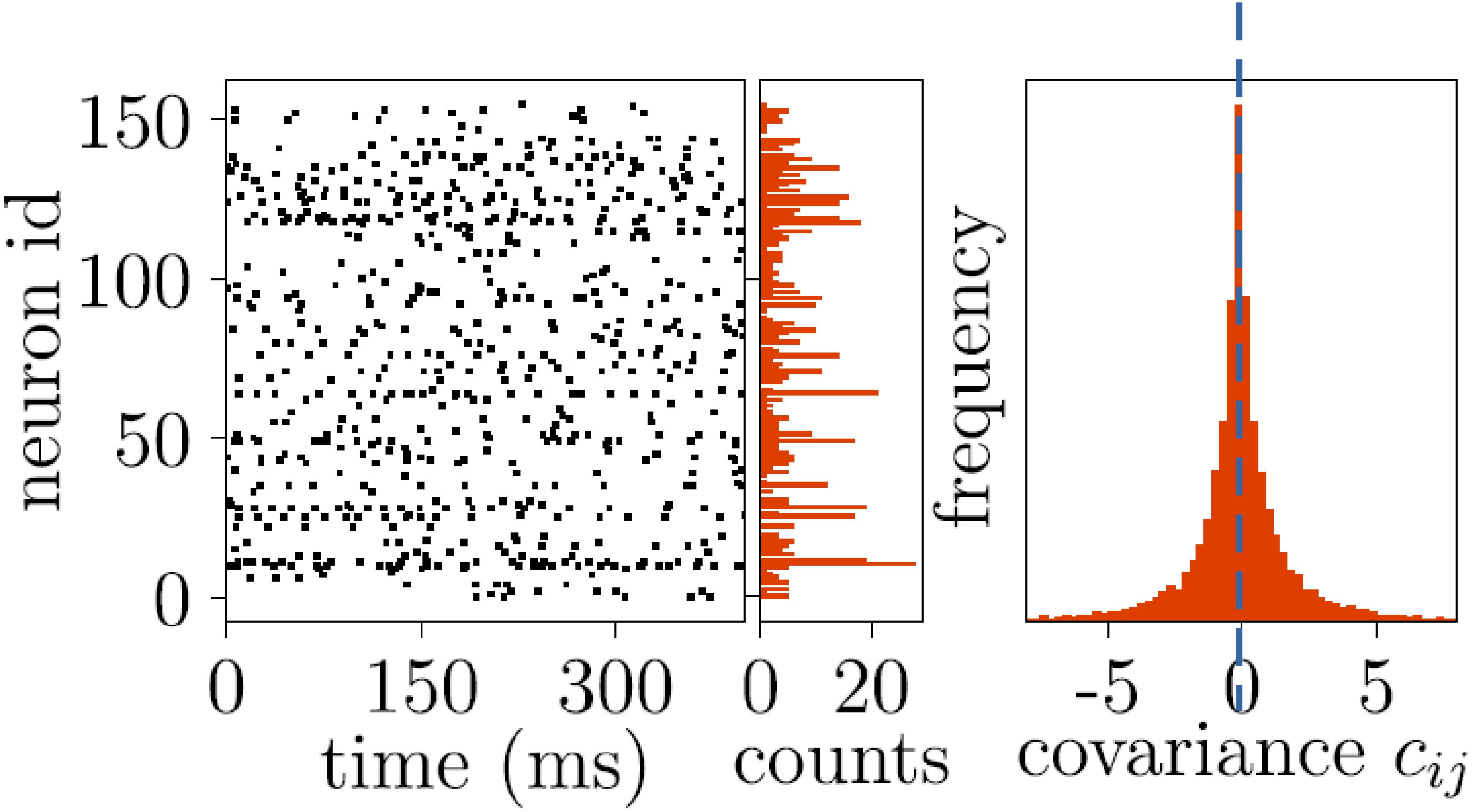
COLLECTIVE DYNAMICS - CORRELATIONS





(Brochier et al. 2018)

SMALL AVERAGE CORRELATIONS



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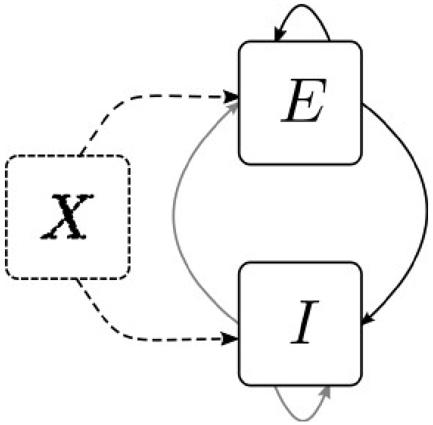
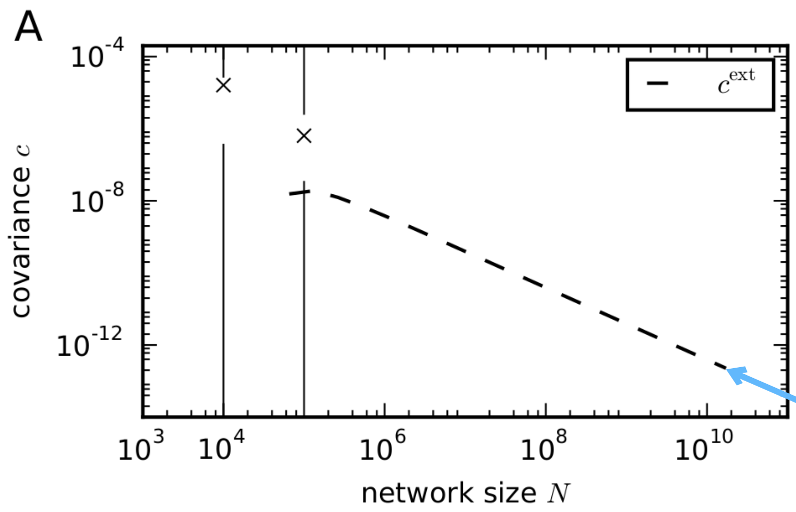
SMALL AVERAGE CORRELATIONS – BALANCED STATE





Finite size-theory of **average** fluctuations

Helias et al. 2014

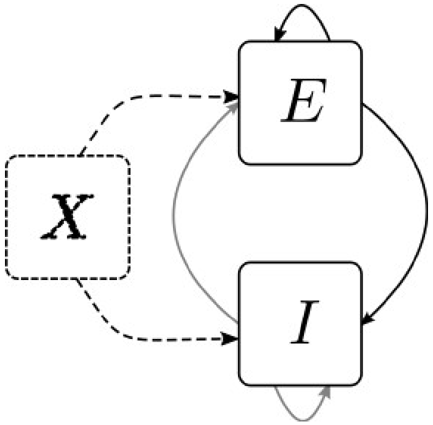
$$c_{\alpha\beta} = \langle c_{ij} \rangle_{i\in\alpha, j\in\beta} \qquad \alpha, \beta \in \{E, I, X\}$$

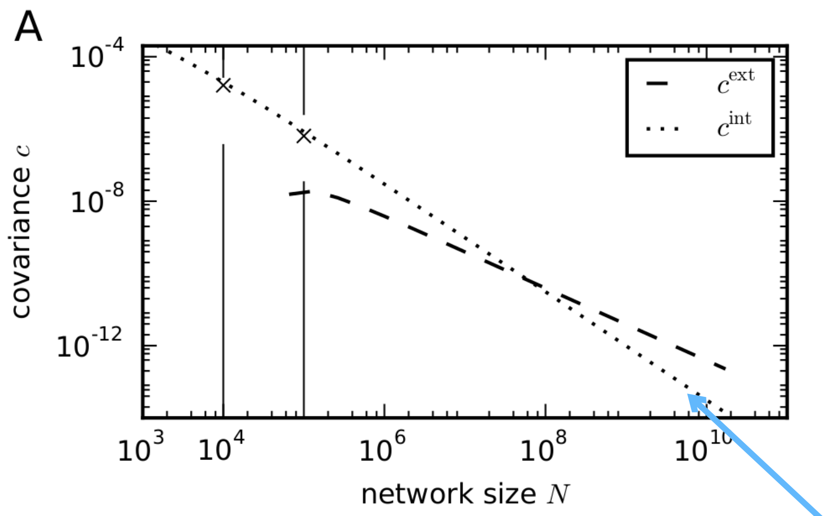
intrinsic externally-driven

$$\begin{pmatrix} c_{EE} \\ c_{EI} \\ c_{II} \end{pmatrix} = \mathbf{c}_{\text{int}}(a_E, a_I) + \mathbf{c}_{\text{ext}}(a_x)$$

Renart et al. 2010

SMALL AVERAGE CORRELATIONS – BALANCED STATE





$$c_{\text{ext}} \propto N^{-1}$$
$$c_{\text{int}} \propto N^{-\frac{3}{2}}$$

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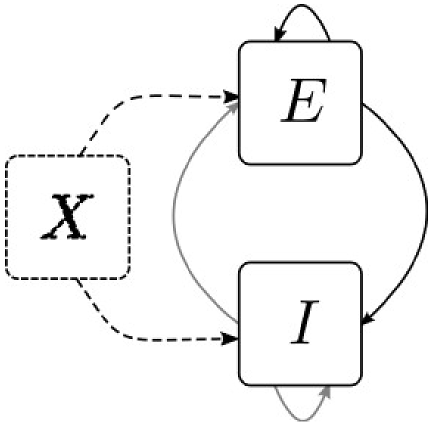
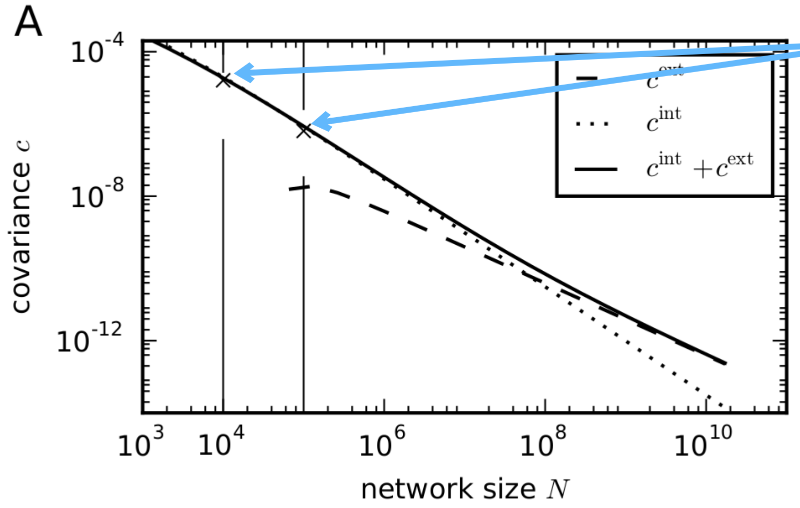
AVERAGE CORRELATIONS
- PREDOMINANTLY GENERATED INTRINSICALLY

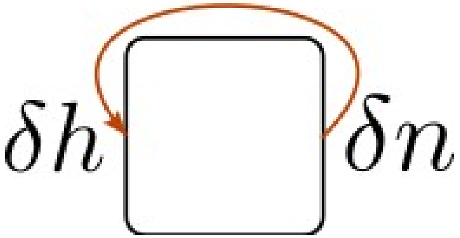




simulation

explanation:
negative feedback by inhibition

$$\mathrm{eig}(J) < 0$$



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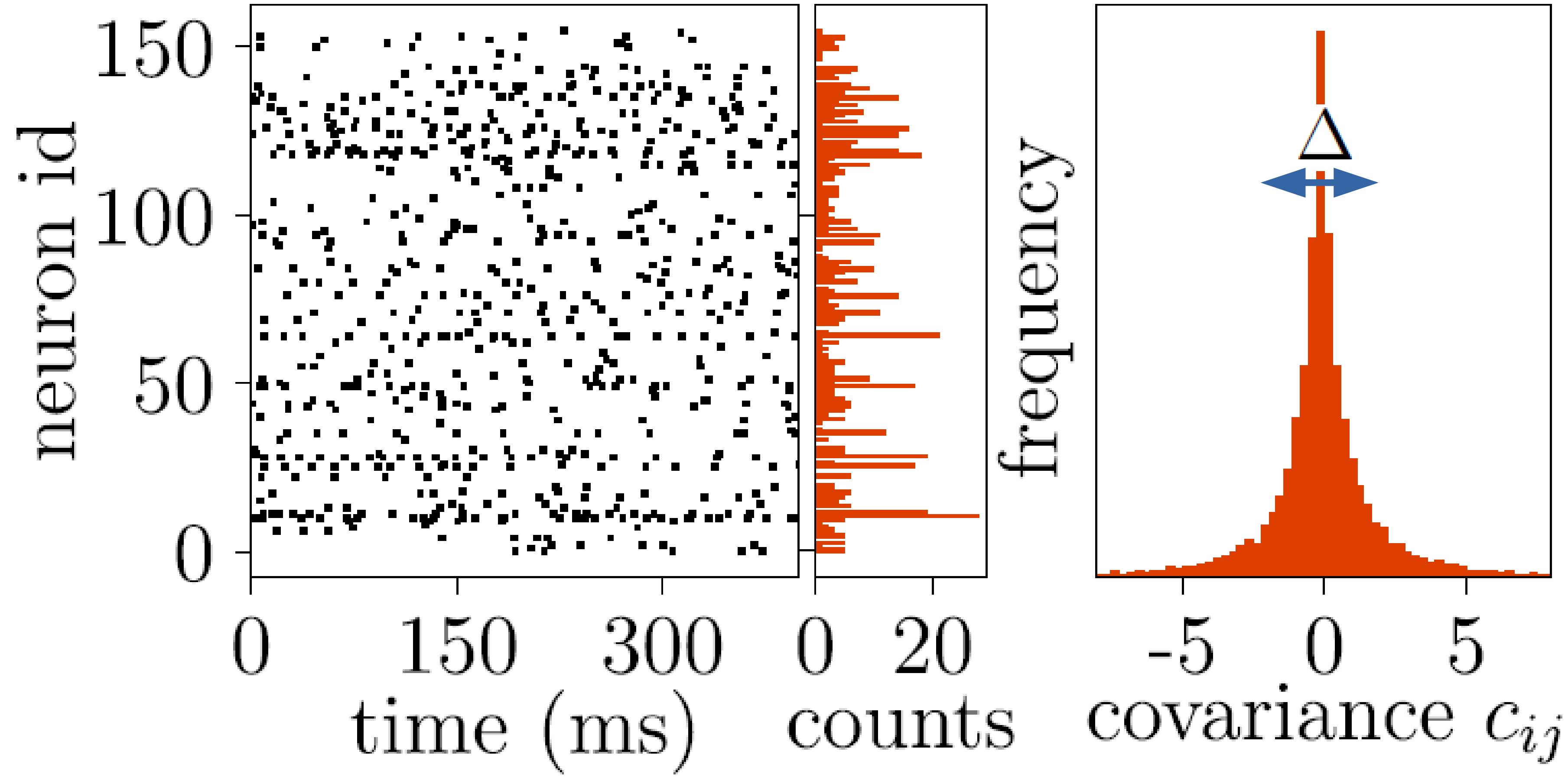
WIDE DISTRIBUTION



SIGNATURES OF CRITICAL STATES IN MOTOR CORTEX

DAVID DAHMEN

RWTH AACHEN UNIVERSITY

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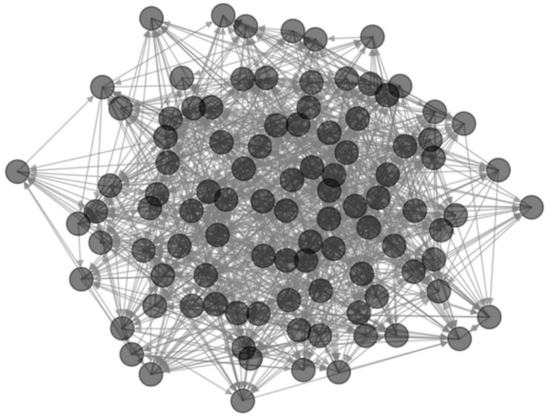
LINEAR NETWORK MODEL (LINEAR RESPONSE THEORY)



$$\tau \frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{N} W_{ij} x_j(t) + \xi_i(t)$$

exponential relaxation

i.i.d. Gaussian coupling weights

external white noise Var = D

- Linear response theory captures fluctuations in asynchronous irregular brain states (Lindner et al. 2006, Pernice et al. 2011, Trousdale et al. 2012, Grytskyy et al. 2014)

COVARIANCES ⟷ CONNECTIVITY

$$\boldsymbol{C} = [\mathbf{1} - \boldsymbol{W}]^{-1} D [\mathbf{1} - \boldsymbol{W}]^{-\mathrm{T}}$$

(Lindner et al. 2006, Pernice et al. 2011, Trousdale et al. 2012, Grytskyy et al. 2014)

COVARIANCES ⟷ CONNECTIVITY

$$C = [\mathbf{1} - W]^{-1} D [\mathbf{1} - W]^{-\mathrm{T}}$$

$$\left(\quad \right)_{N \times N}$$

matrix equation

(Lindner et al. 2006, Pernice et al. 2011, Trousdale et al. 2012, Grytskyy et al. 2014)

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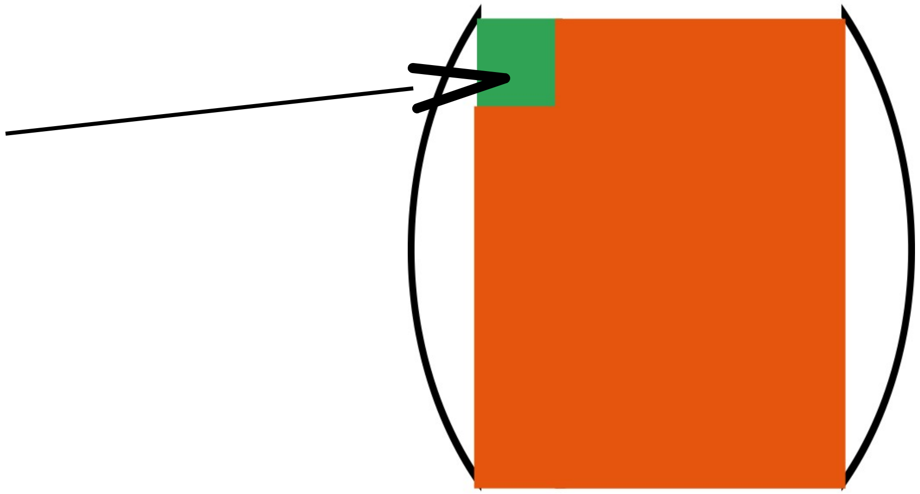


N x N

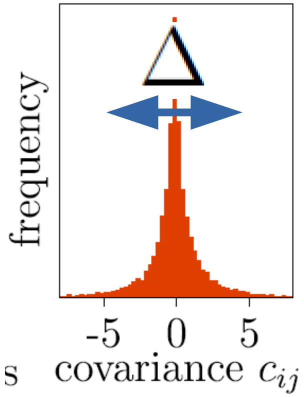
matrix equation

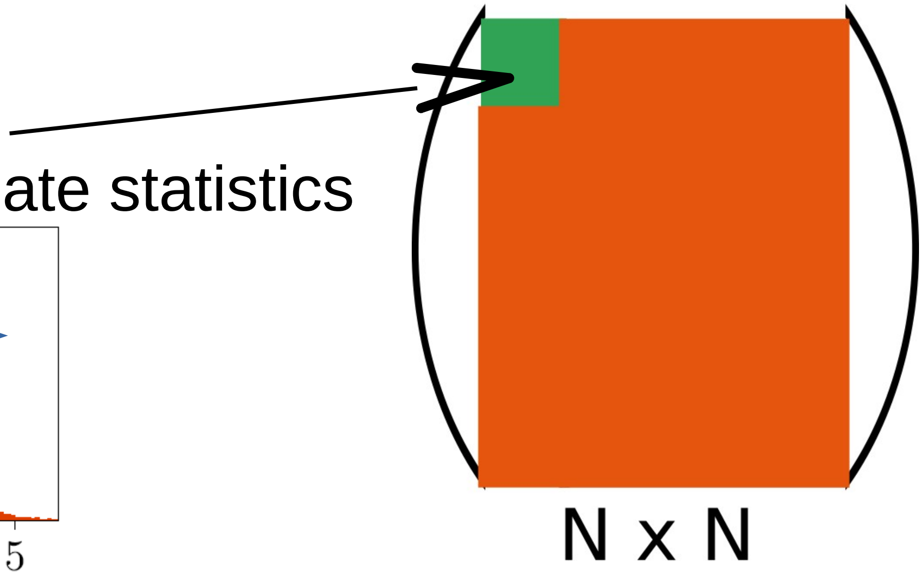
(Lindner et al. 2006, Pernice et al. 2011, Trousdale et al. 2012, Grytskyy et al. 2014)

COVARIANCES ⟷ CONNECTIVITY

$$C = [\mathbb{1} - W]^{-1} D [\mathbb{1} - W]^{-\mathrm{T}}$$

enough to estimate statistics





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FIELD THEORETIC FORMULATION

$$\boldsymbol{C} = [\mathbf{1} - \boldsymbol{W}]^{-1} D [\mathbf{1} - \boldsymbol{W}]^{-\mathrm{T}}$$

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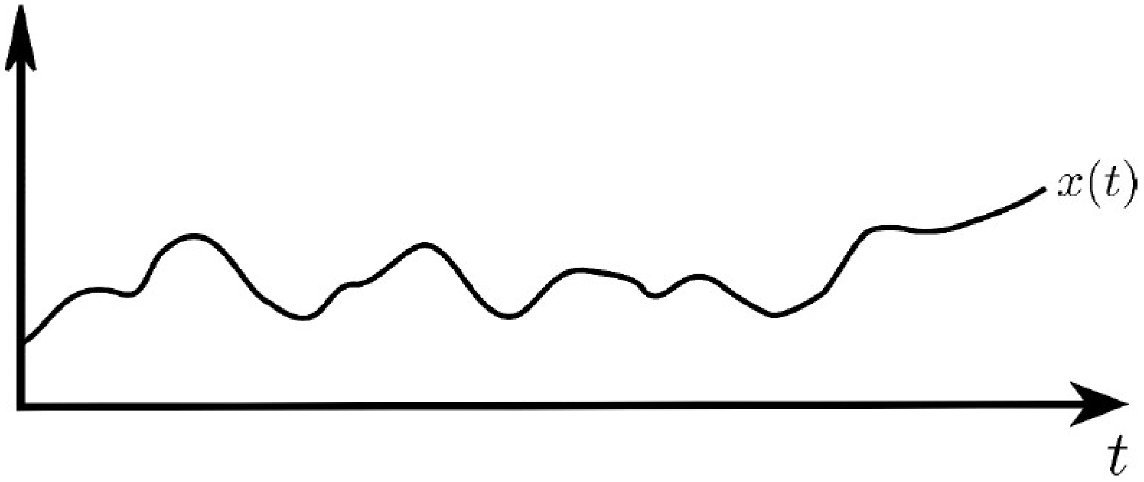
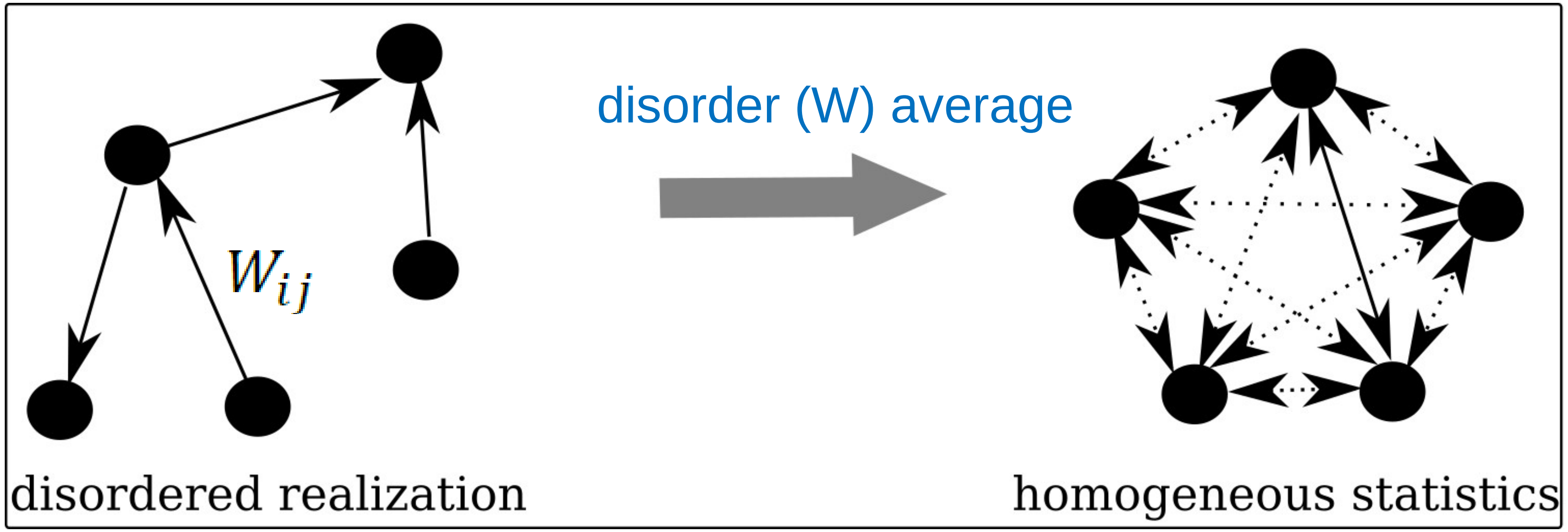


$$p[x(t)] = \int D\tilde{x}\, e^{S(\tilde{x},x|W)}$$

Martin Siggia Rose formalism
Martin et al. 1973, DeDominicis 1975, Janssen 1976

ENSEMBLES OF NETWORKS

$$p[x(t)] = \int D\tilde{x}\, e^{S(\tilde{x},x|W)}$$



$$S_0(\widetilde{X}, X) = \widetilde{X}^{\mathrm{T}}(1 - \mu\{\mathbf{1}\})X + \frac{D}{2}\widetilde{X}^{\mathrm{T}}\widetilde{X}$$

$$S_{\mathrm{int}}(\widetilde{X}, X) = \frac{\sigma^2}{2N}\widetilde{X}^{\mathrm{T}}\widetilde{X}X^{\mathrm{T}}X$$

BEYOND MEAN-FIELD THEORY

$$S_0(\tilde{X}, X) = \tilde{X}^{\mathrm{T}}(1 - \mu\{\mathbf{1}\})X + \frac{D}{2}\tilde{X}^{\mathrm{T}}\tilde{X}$$

$$S_{\mathrm{int}}(\tilde{X}, X) = \frac{\sigma^2}{2N}\tilde{X}^{\mathrm{T}}\tilde{X}X^{\mathrm{T}}X$$

mean + fluctuation corrections

Result:

variance of entries of W
spectral radius of connectivity W

$$R^2 = 1 - \sqrt{\frac{1}{1 + N\Delta}}$$

width of distribution of correlations

number of neurons

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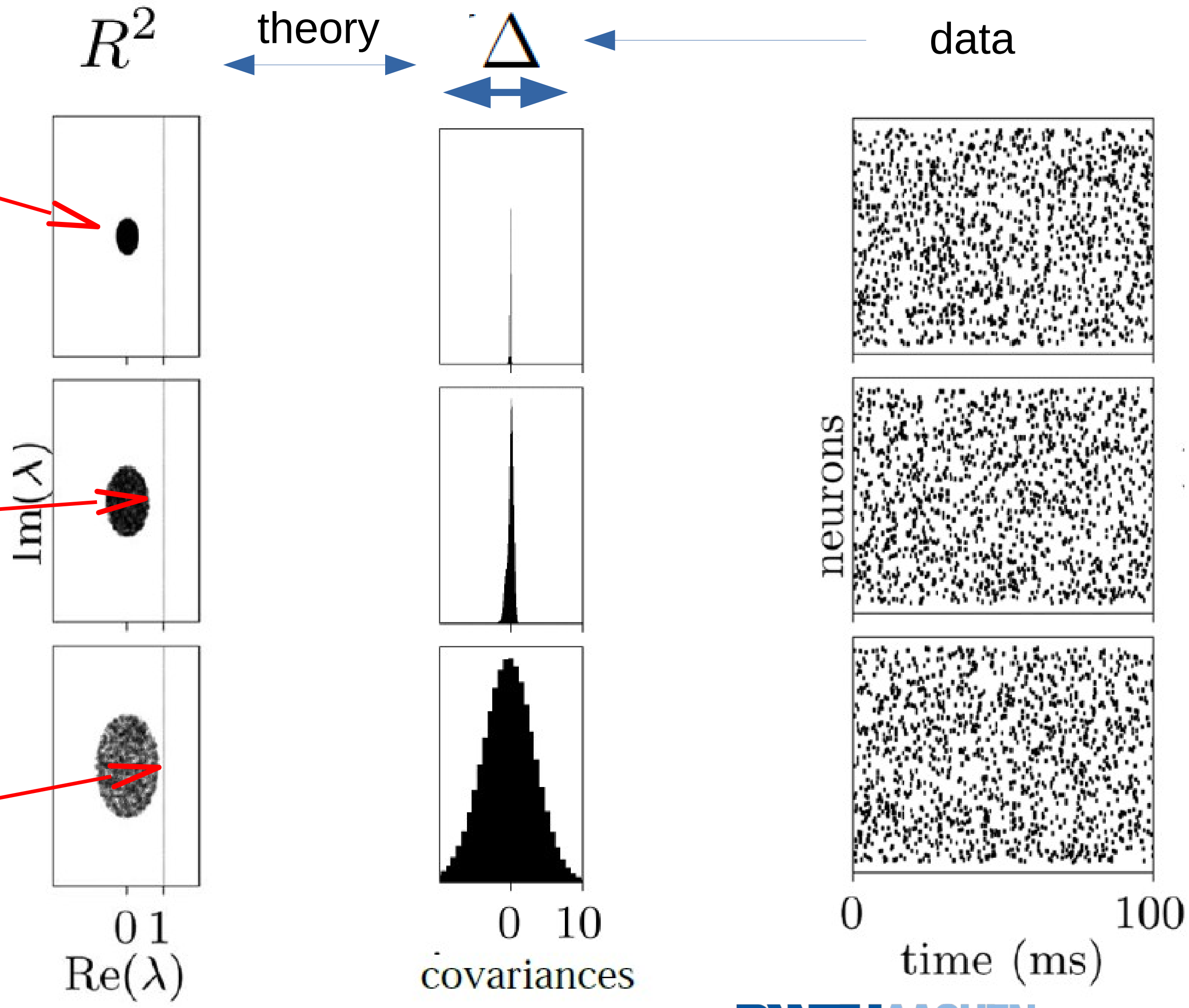
LARGE WIDTH IMPLIES CRITICALITY

spectrum of connectivity W

edge of spectrum λ_{max} determines stability of dynamics

loss of stability → critical point



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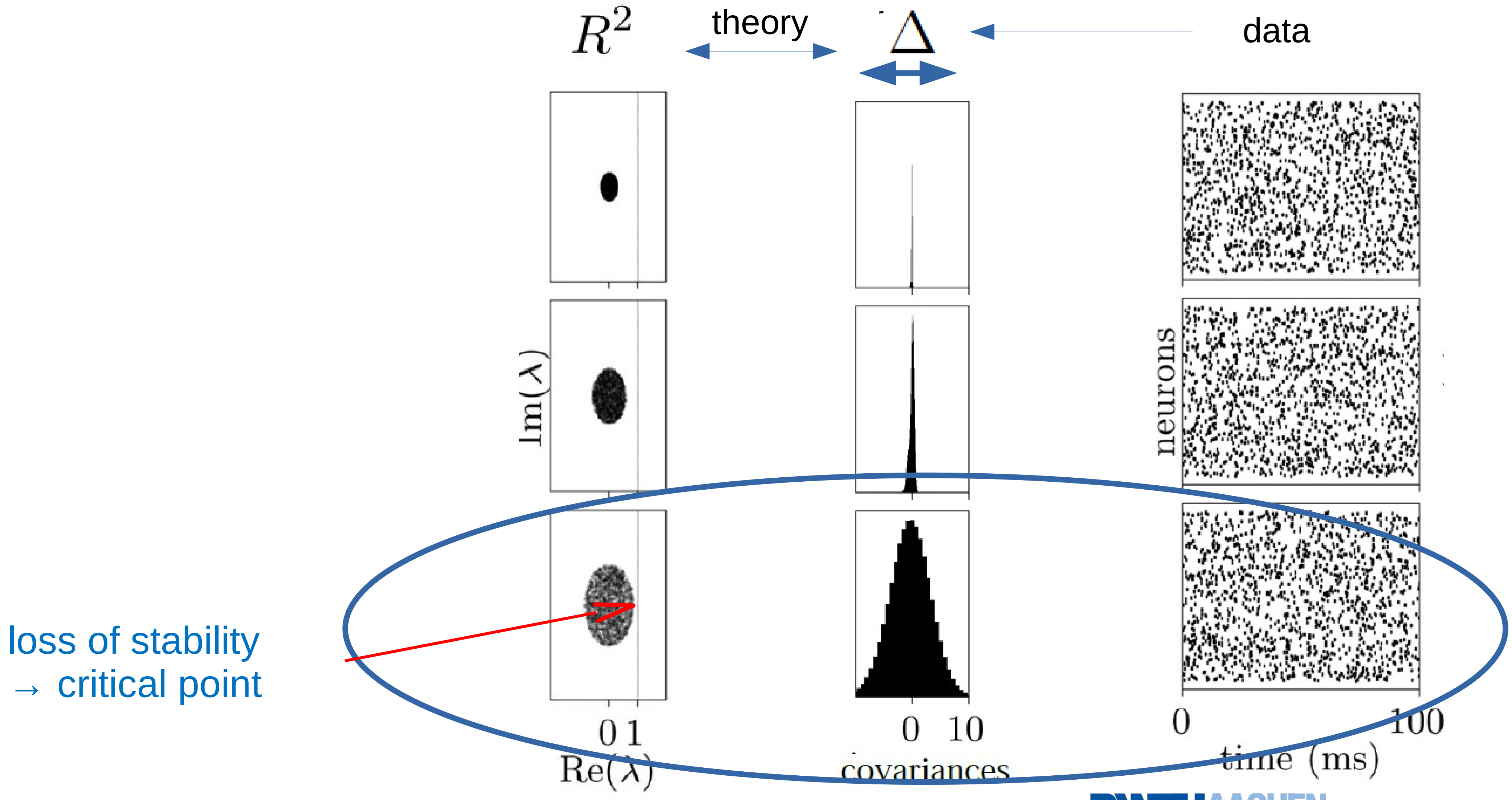


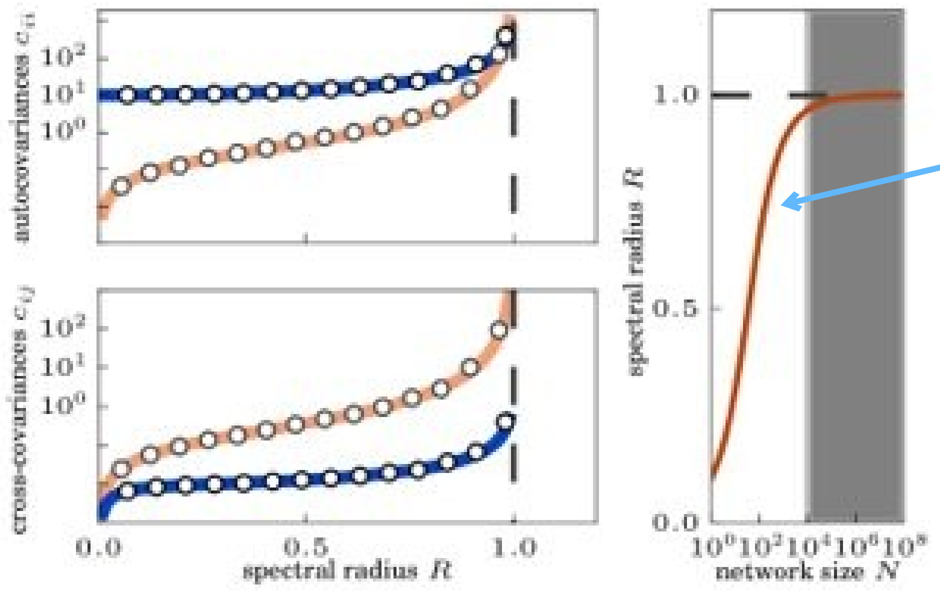
MOTOR CORTEX NEARLY UNSTABLE

theory

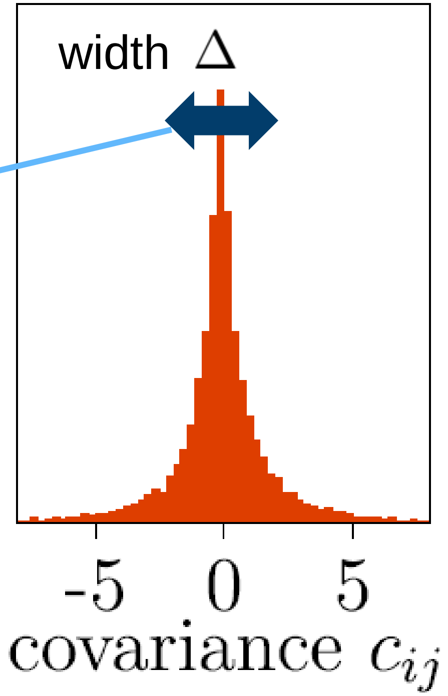
- spectral radius R of the coupling matrix

$$R^2 = 1 - \sqrt{\frac{1}{1 + N\,\Delta}}$$

width



data



➢ **Motor cortex is operating close to critical point of linear instability R=1 !**

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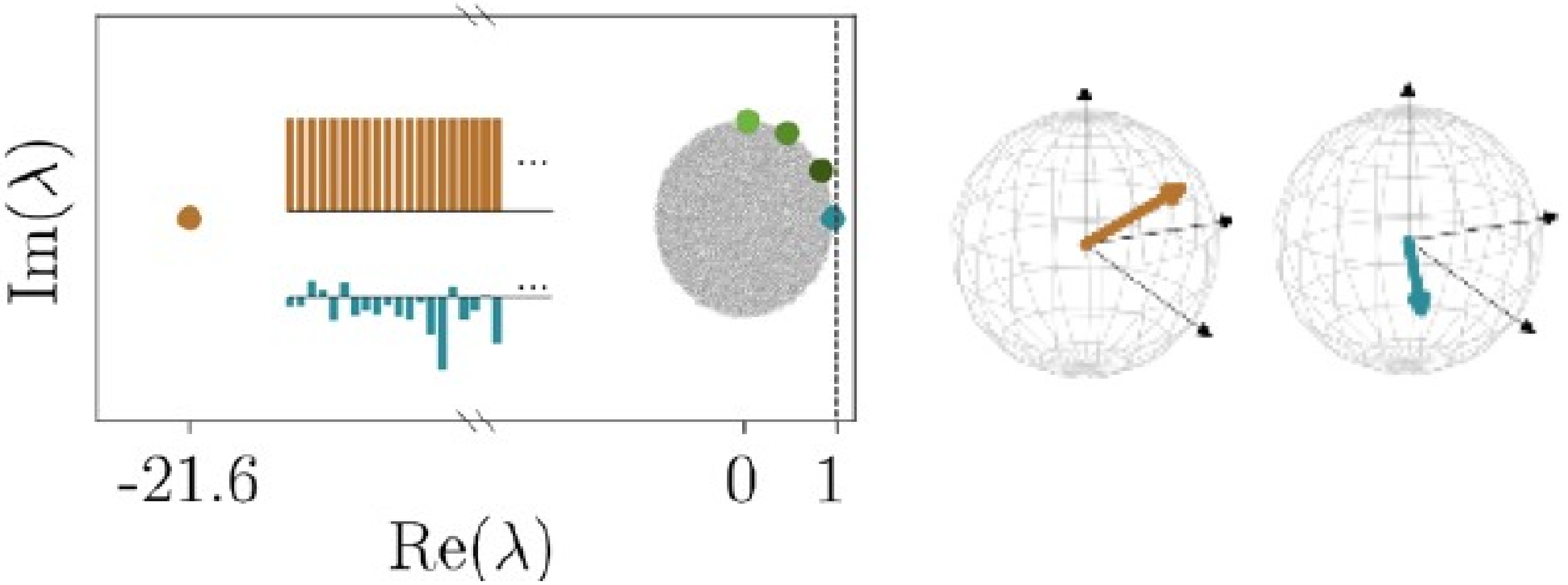
DYNAMICAL AND FUNCTIONAL CONSEQUENCES
- RICH REPERTOIRE OF DYNAMICAL MODES



Dahmen et al., Second type of criticality in the brain uncovers rich multiple-neuron dynamics, PNAS, 2019

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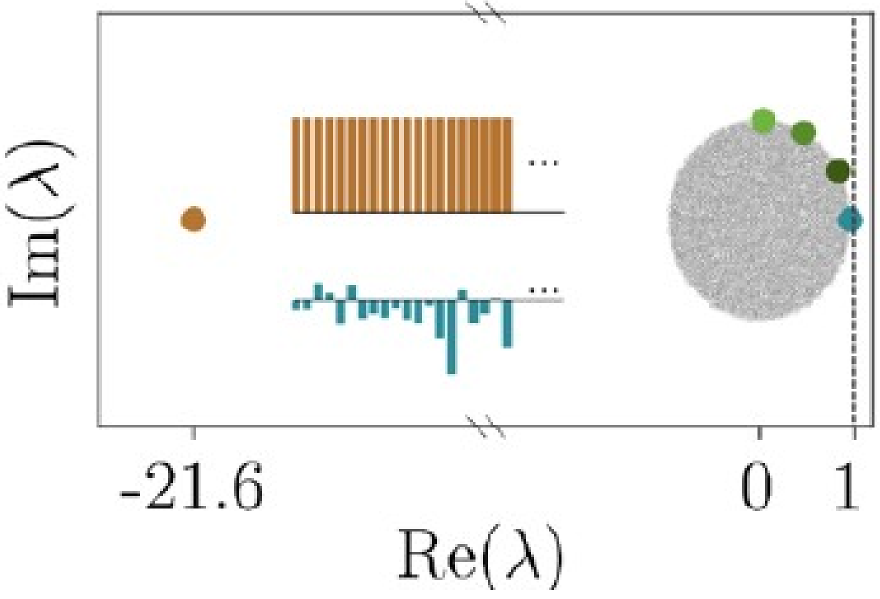
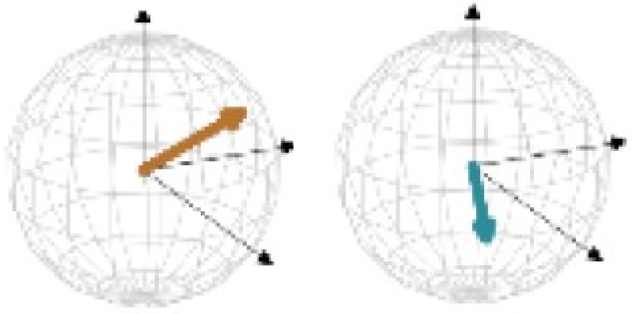
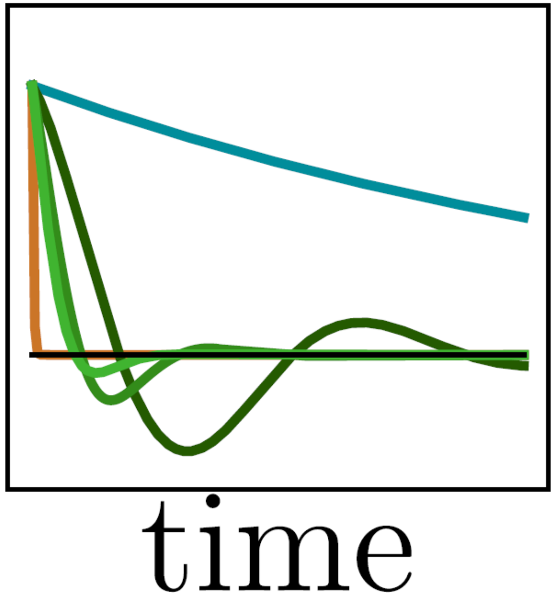




$$v_\alpha(t) \sim e^{-t/\frac{\tau}{1-\lambda_\alpha}}$$



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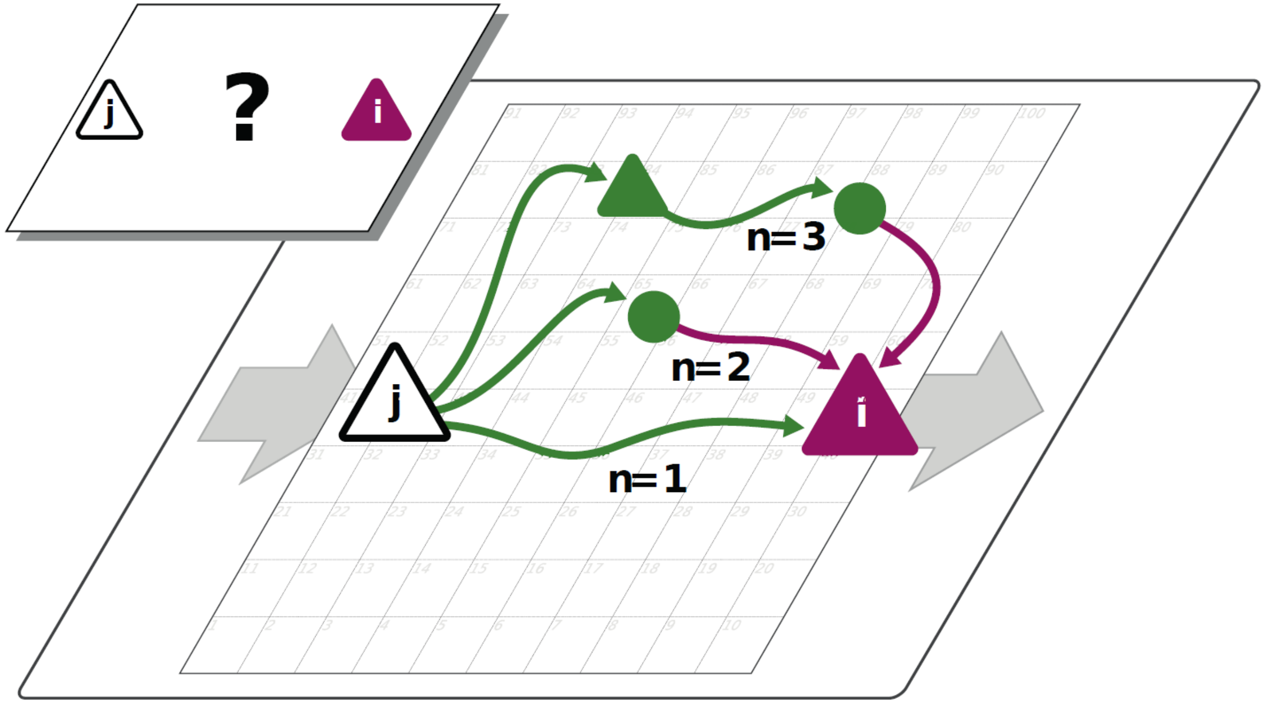
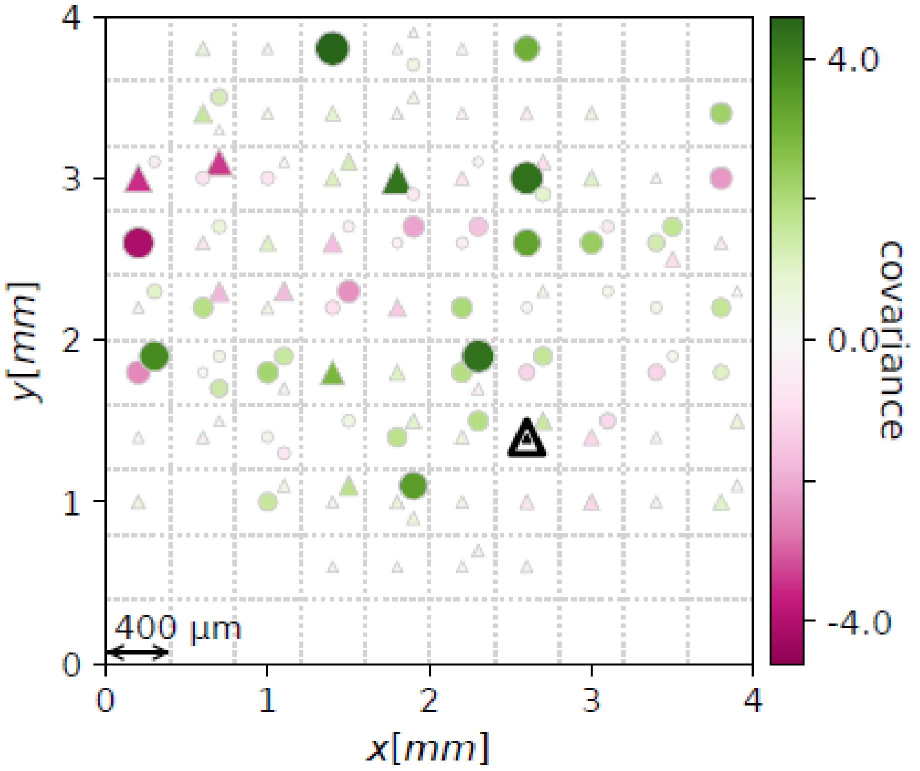
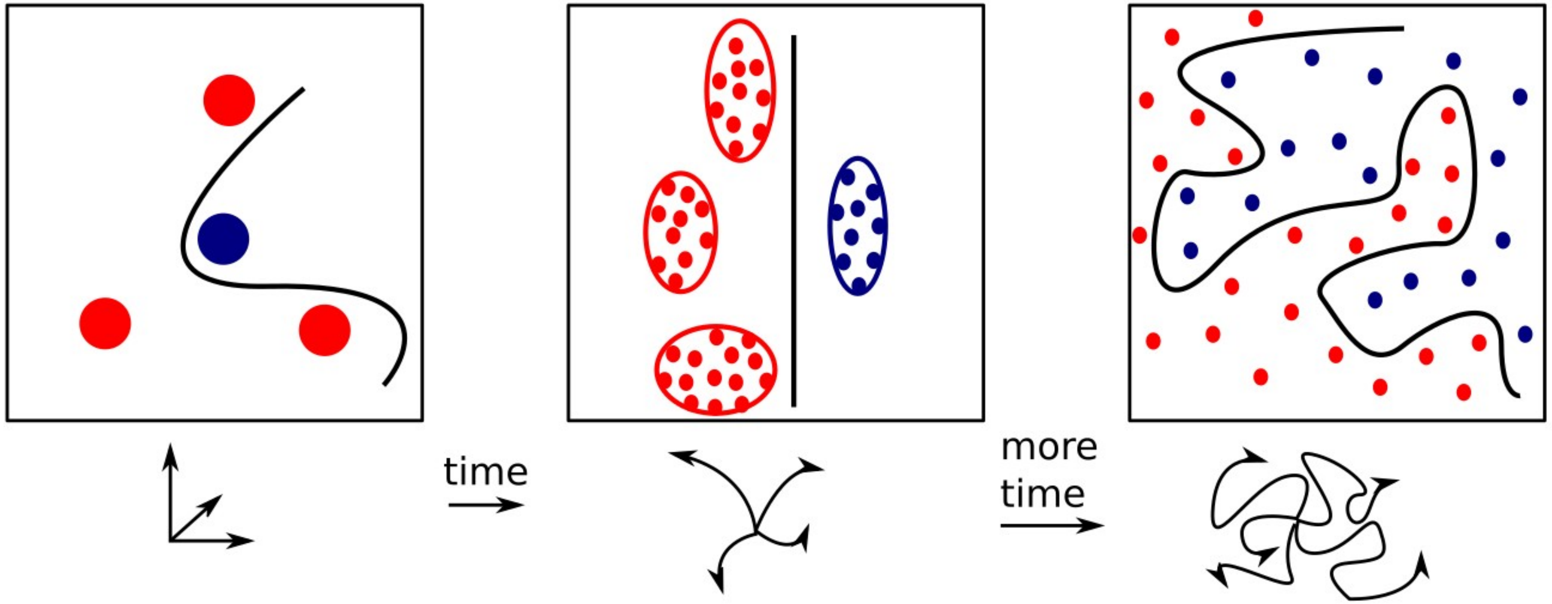
DYNAMICAL AND FUNCTIONAL CONSEQUENCES

-LONG-RANGE INTERACTIONS DESPITE SHORT-RANGE CONNECTIONS





Dahmen et al., Long-range coordination patterns in cortex change with behavioral context, elife, 2022




Christian
Keup


Tobias
Kühn

TRANSIENT CHAOTIC DIMENSIONALITY EXPANSION

CHRISTIAN KEUP, TOBIAS KÜHN, DAVID DAHMEN

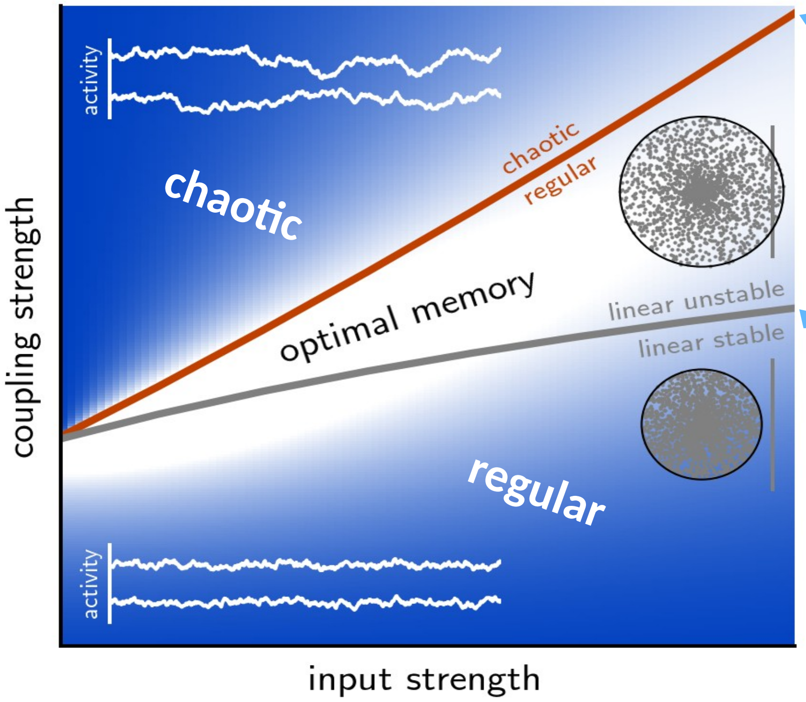
DRIVEN RANDOM RATE NETWORKS

- OPTIMAL MEMORY CLOSE TO CRITICALITY

nonlinear network:

$$\tau \frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{N} J_{ij}\phi(x_j(t)) + \xi_i(t)$$

coupling → J_{ij}; input → $\xi_i(t)$



transition to chaos   Var(input) > Var(neurons)

$$g^2\langle\phi^2\rangle > \langle x^2\rangle$$

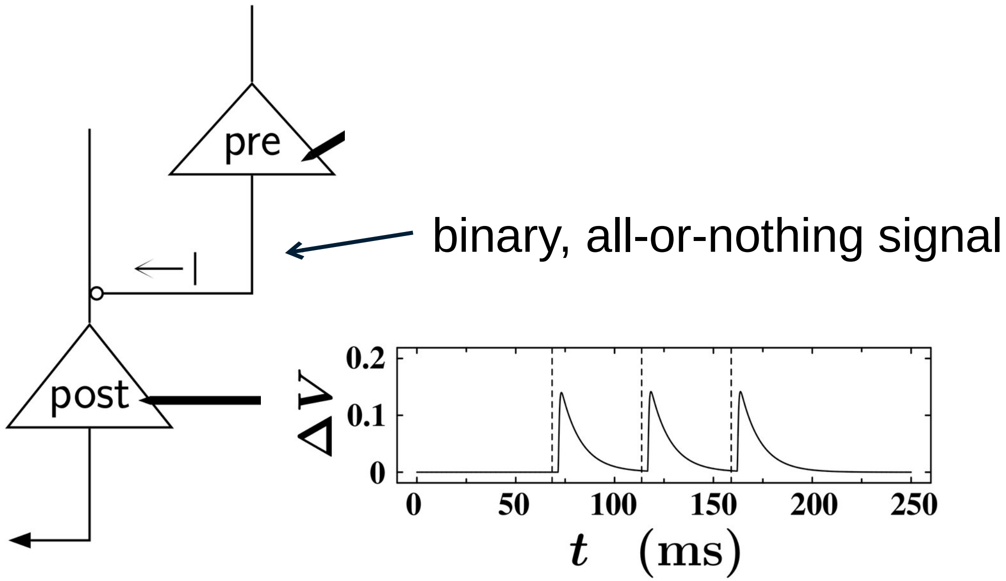
dynamical state between loss of linear stability and onset of chaos has optimal memory

linear instability $R^2 = g^2\langle\phi'^2\rangle > 1$

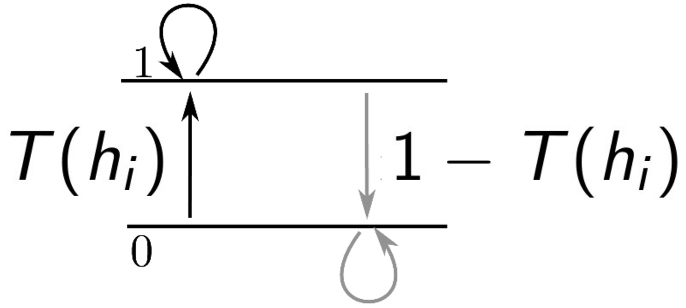
Schuecker et al., Optimal Sequence Memory in Driven Random Networks, PRX, 2018

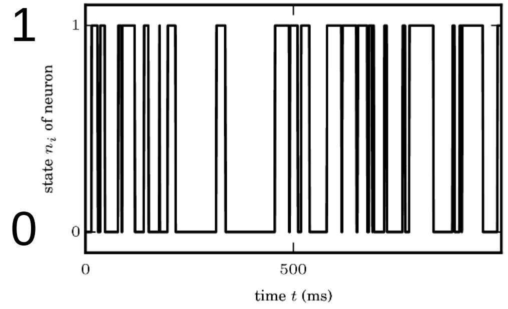
SPIKING INTERACTION: ABSTRACTION AS BINARY

Taking into account discrete coupling

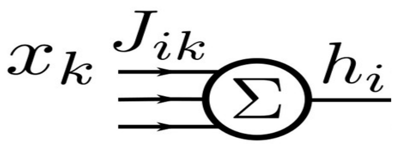


Binary neurons





DISCRETE COUPLING: BINARY NEURON



$$h_i = \sum_j J_{ij} x_j$$

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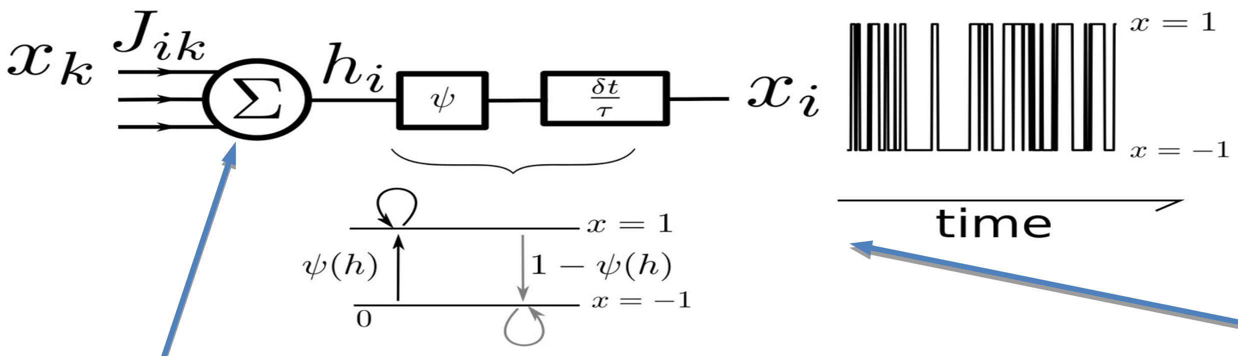
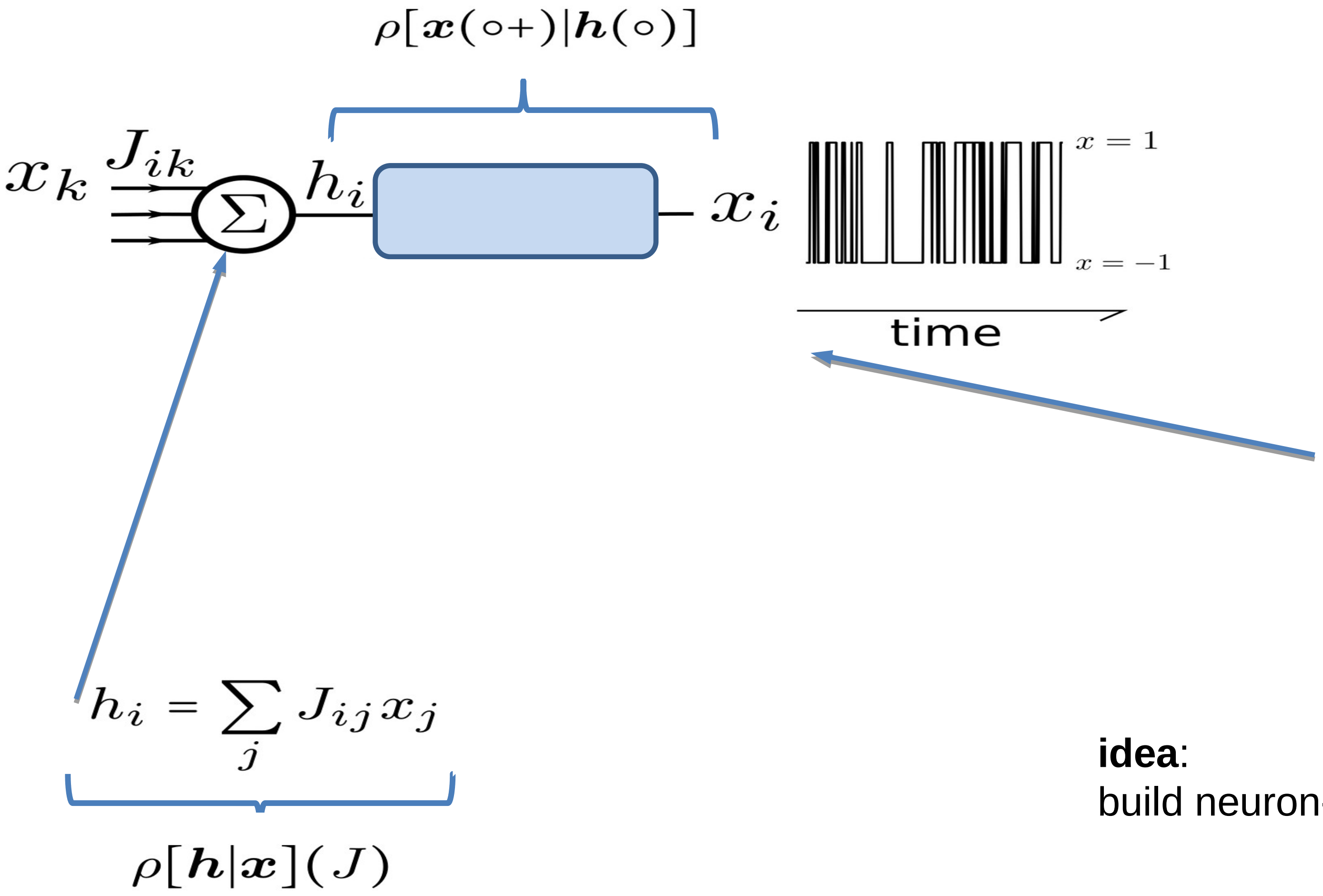


discrete signal sent to
other neurons

$$h_i = \sum_j J_{ij} x_j$$

$$t_u \sim \text{Poisson proc. } (\text{rate} = \tau^{-1})$$

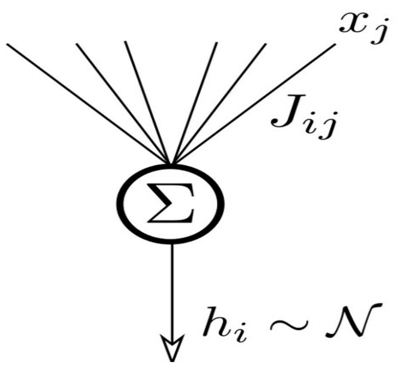
MODEL-INDEPENDENT FIELD THEORY



discrete signal sent to other neurons

idea:
build neuron-model independent field theory

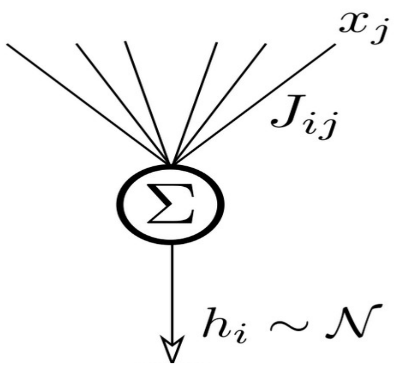
MODEL-INDEPENDENT FIELD THEORY



- instantaneous synaptic coupling

$$\rho[\boldsymbol{h}|\boldsymbol{x}](J) = \delta\left[\boldsymbol{h} - \boldsymbol{J}\boldsymbol{x}\right]$$

MODEL-INDEPENDENT FIELD THEORY



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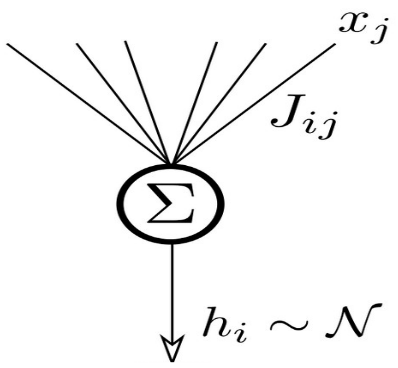
MODEL-INDEPENDENT FIELD THEORY

linear J in exponent



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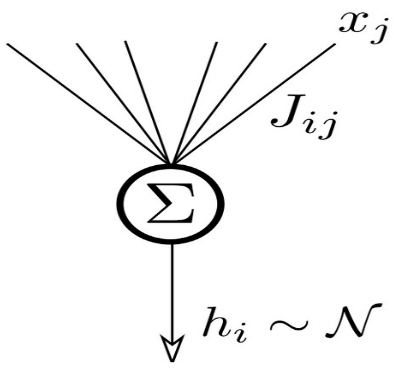

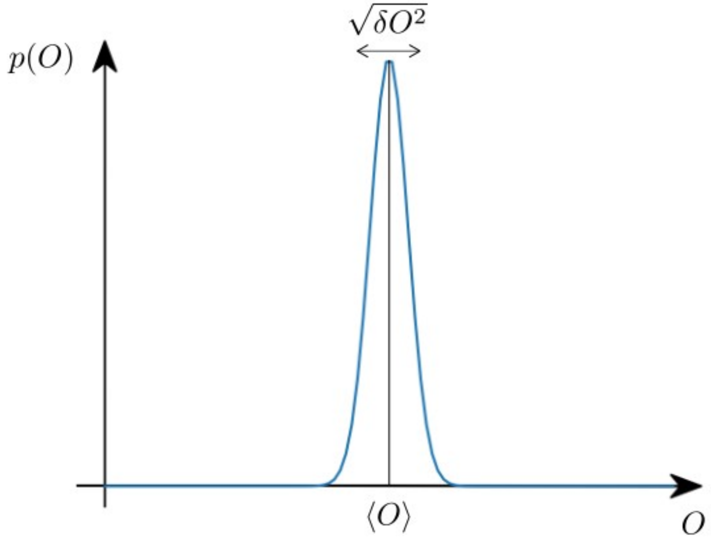

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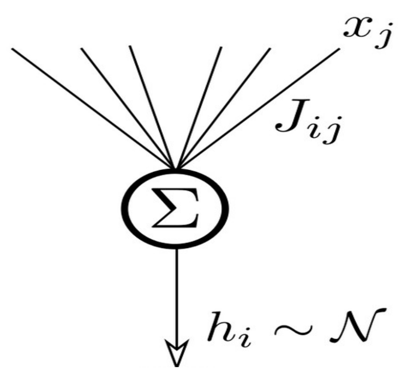

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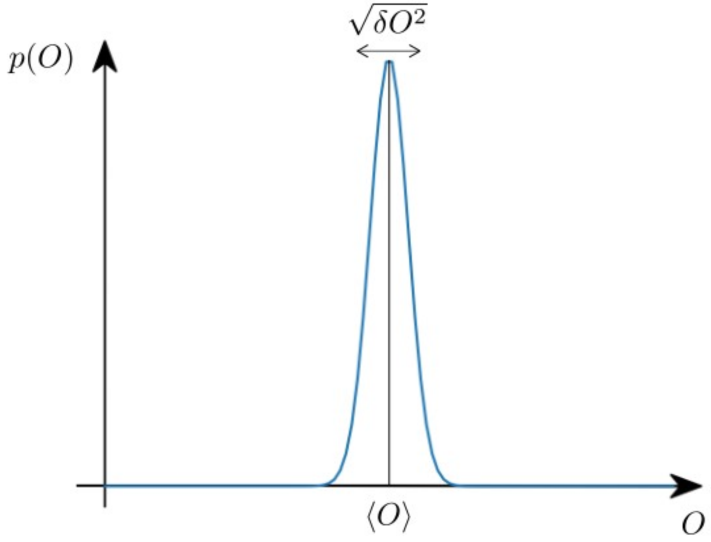
- only term affected: interaction

$$= \langle \exp(-\hat{h}^{\mathrm{T}} J\, x) \rangle_{J_{ij} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\frac{\bar{g}}{N}, \frac{g^2}{N})}$$
$$= \exp\left(-\frac{\bar{g}}{N}\hat{h}^{\mathrm{T}}\mathcal{R} + \frac{g^2}{2N}\hat{h}^{\mathrm{T}}\mathcal{Q}\hat{h}\right)$$

- auxiliary fields

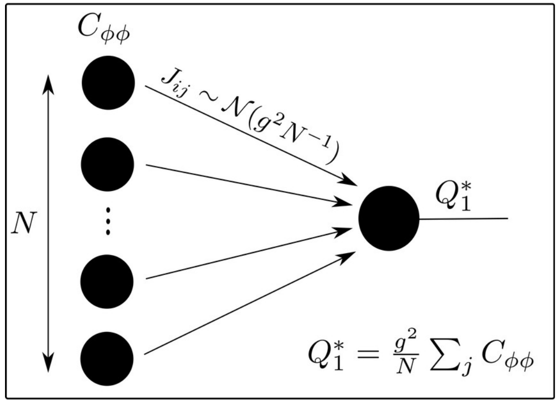
$$\mathcal{R}(t) = \frac{\bar{g}}{N}\sum_{j=1}^{N} x_j(t)$$
$$\mathcal{Q}(t,s) = \frac{g^2}{N}\sum_{j=1}^{N} x_j(t)x_j(s)$$



Macroscopic field theory

- auxiliary fields and conjugate fields $(\mathcal{R}, \mathcal{Q}, \hat{\mathcal{R}}, \hat{\mathcal{Q}}) \sim e^{N\,S[\mathcal{R},\mathcal{Q},\hat{\mathcal{R}},\hat{\mathcal{Q}}]}$

- saddle point approximation $\delta S/\delta R \overset{!}{=} 0, \ldots \rightarrow$



$$R(t) = \bar{g}\,\langle x(t)\rangle_{S(R,Q)}$$
$$Q(t,s) = g^2\,\langle x(t)x(s)\rangle_{S(R,Q)}$$

mean input to a neuron

variance of input

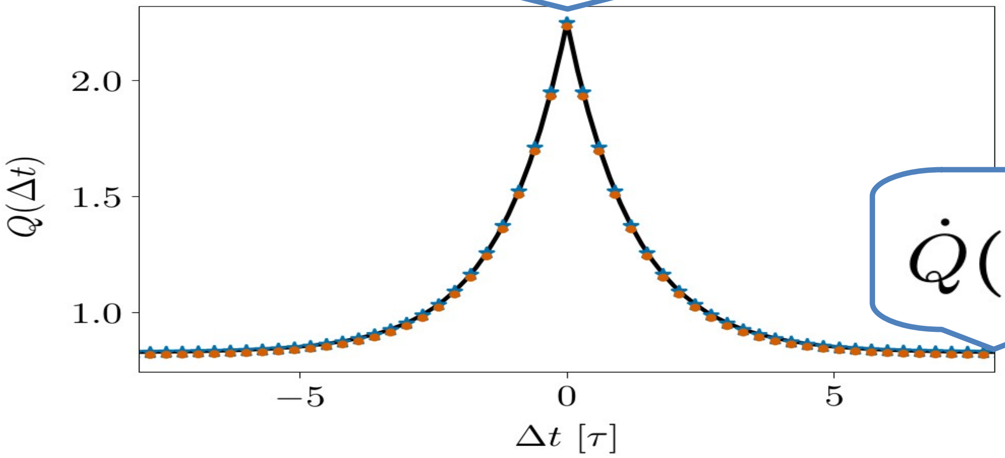
dynamical mean-field theory (DMFT)

Continuous and discrete coupling: same DMFT

binary: $Q(0) = g^2$

rate: $\dot{Q}(0+) = -\frac{D}{2}$

autocorrelation function



$$\dot{Q}(\infty) = 0$$

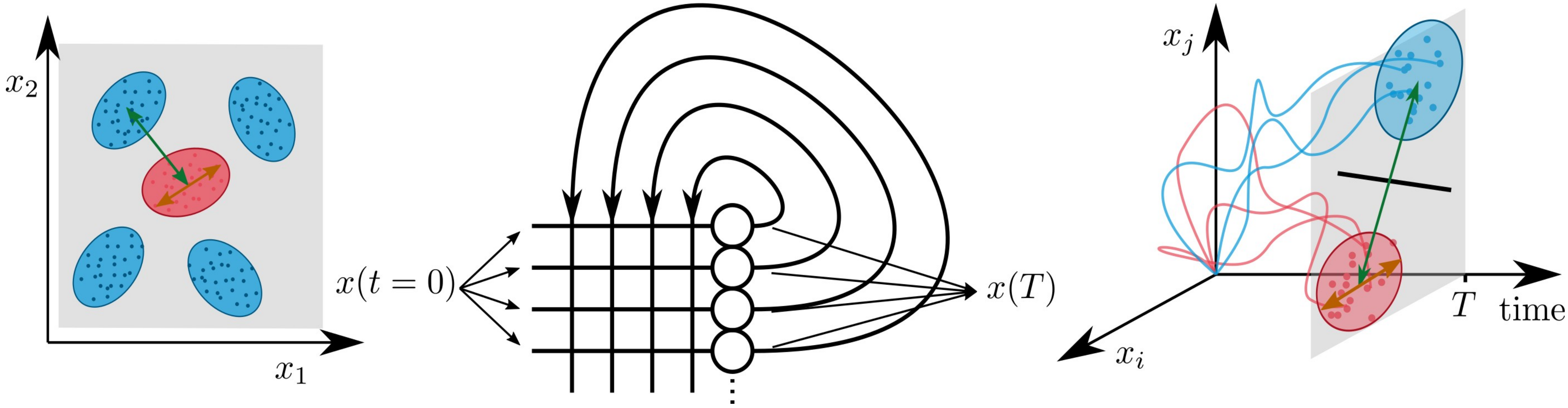
same dynamical e.o.m.

$$\tau^2 \ddot{Q}\left(\Delta t\right) = -V'_{Q(0)}\left(Q\left(\Delta t\right)\right)$$

- same activity statistics (mean and fluctuations)

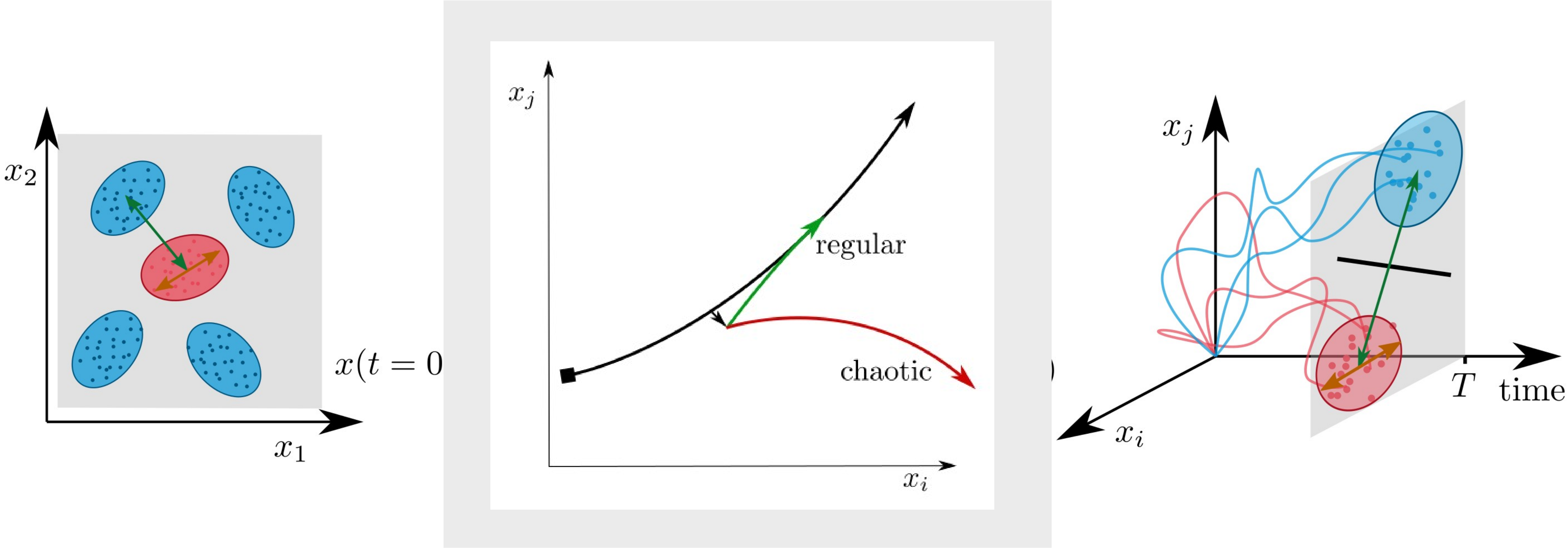
CLASSIFICATION OF INPUT PATTERNS

Reservoir computing setup

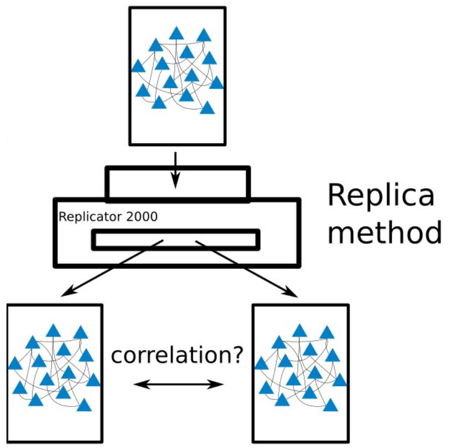


CLASSIFICATION OF INPUT PATTERNS

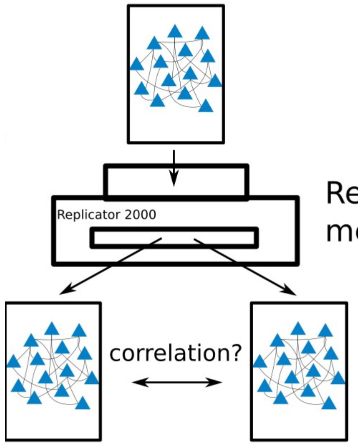
Reservoir computing setup



CHAOS AS CORRELATION TRANSMISSION

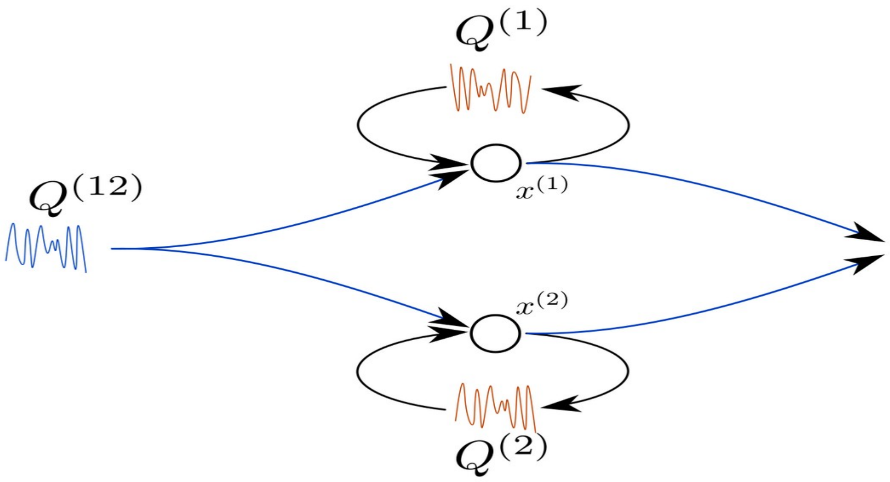


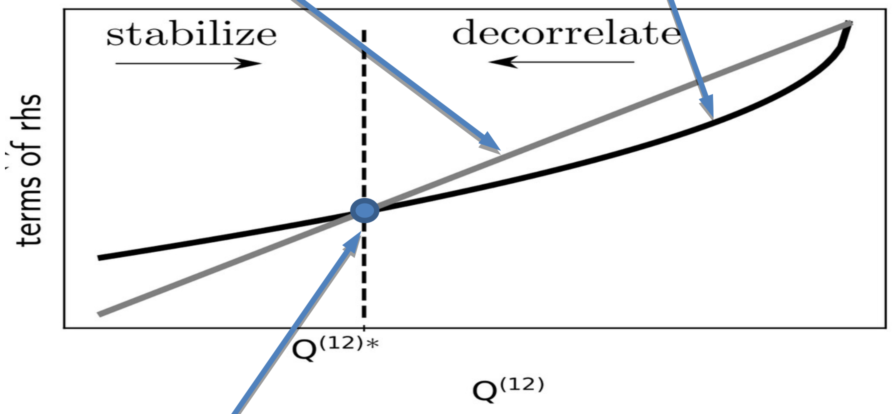
CHAOS AS CORRELATION TRANSMISSION



- correlation between replicas $Q^{(12)} = \frac{g^2}{N} \langle x^{(1)\mathrm{T}} x^{(2)} \rangle$

$$\tau \frac{d}{dt} Q^{(12)}(t) = -\underbrace{Q^{(12)}(t)}_{\text{correlation between outputs of replicas}} + \underbrace{g^2 \left(1 - \left\langle \left|\phi\left(h^{(1)}\right) - \phi\left(h^{(2)}\right)\right| \right\rangle\right)}_{\text{correlation between inputs}}$$



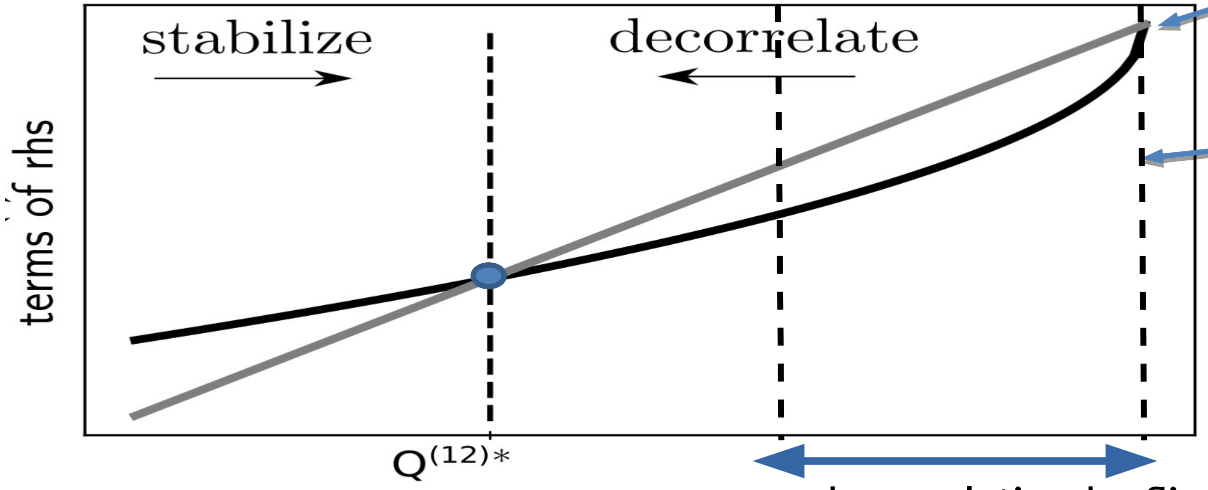


fixed point

infinite Lyapunov exponent
(van Vreeswijk & Sompolinsky 1996, 1998)

NETWORK-SIZE DEPENDENT TRANSITION

infinite
slope



perfect correlation of replicas

decorrelation by flipping single neuron

$$\Delta Q^{(12)}_{\min} = 2g^2/N$$

network size

correlates
-> regular

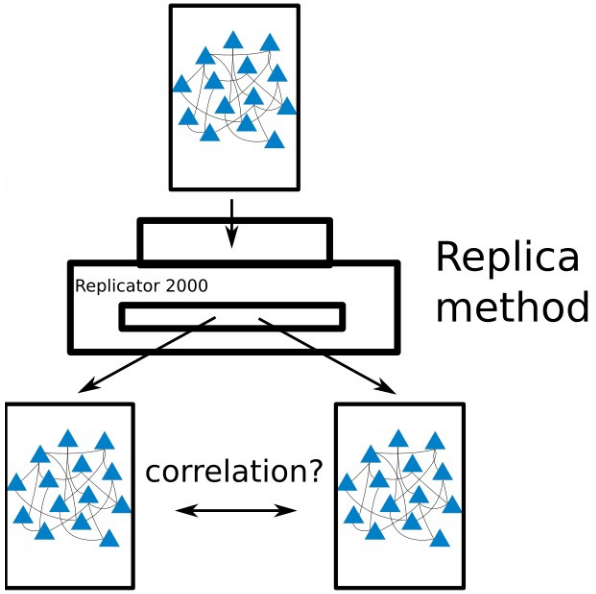
decorrelates
-> chaotic

N → infinity: always chaotic

Van Vreeswijk & Sompolinsky 1996

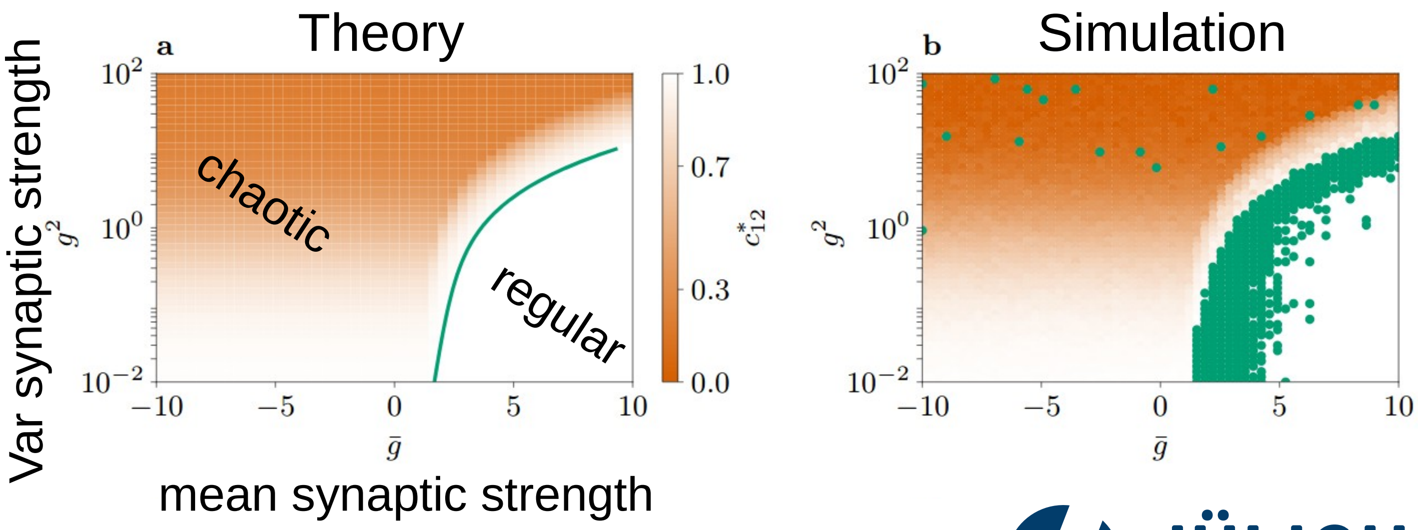
TRANSITION TO CHAOS IN BINARY NETWORKS

Replica decorrelation



condition for finite-size transition to chaos

$$1 \lesssim \sqrt{\frac{2}{\pi}} g \left\langle \mathrm{T}'(h) \right\rangle_h \sqrt{N}$$



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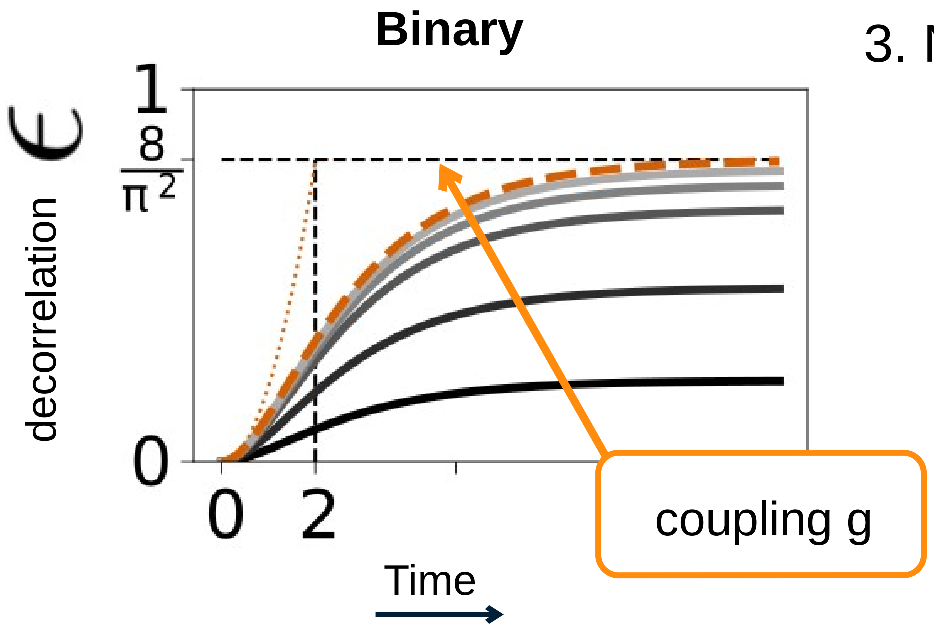
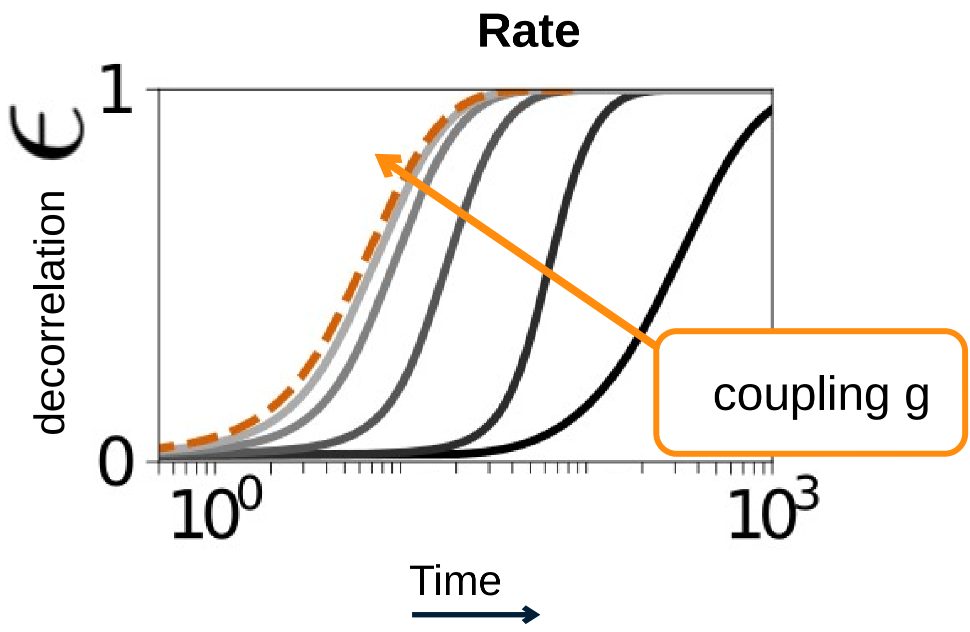
CHAOS IN BINARY NETWORKS

Differences to continuous rate networks

1. Mutually exclusive regimes.
2. Limited chaotic attractor.
3. No critical slowing down.





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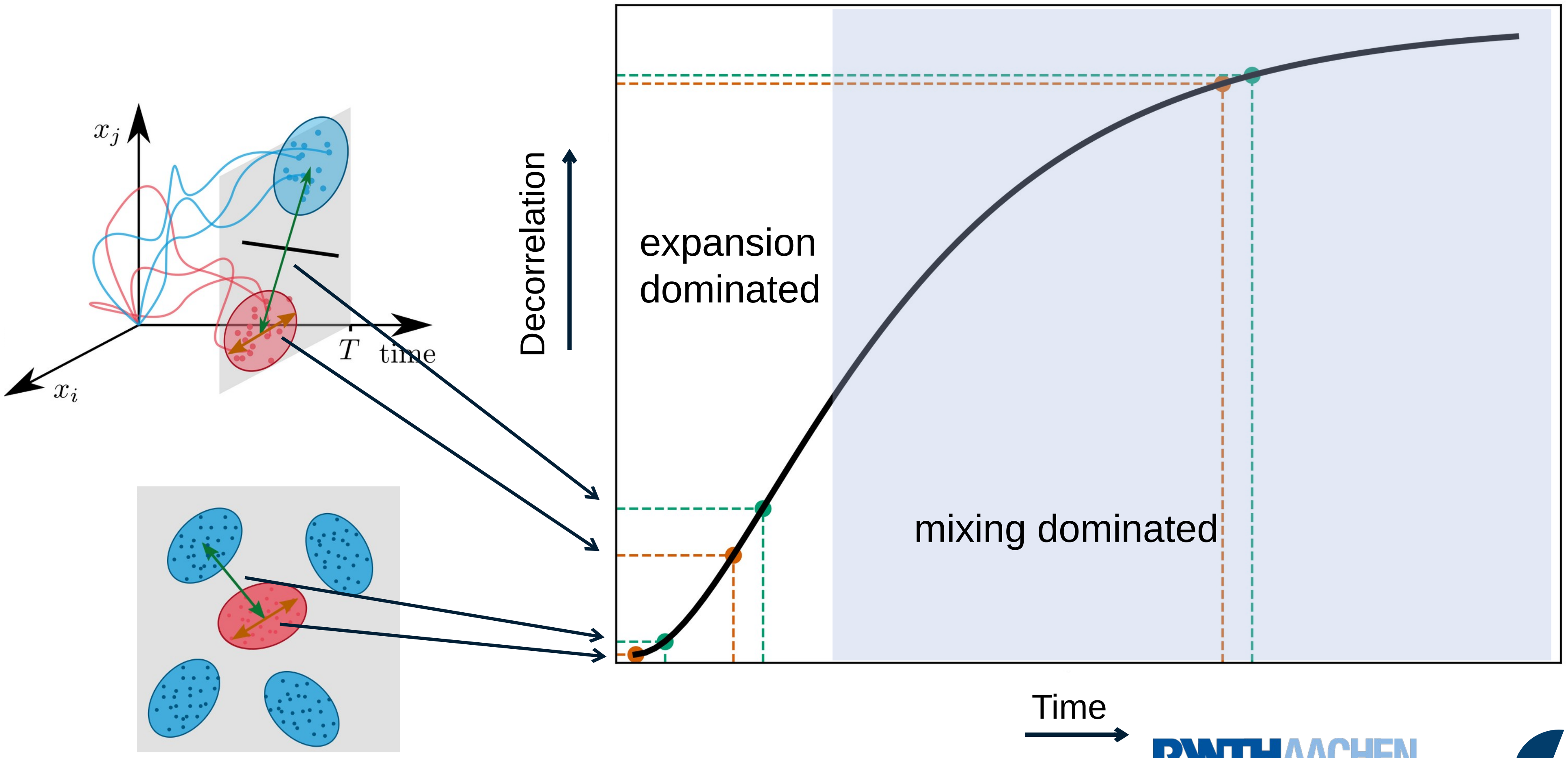
DECORRELATION CURVE

Inter-class distance increases compared to intra-class distance

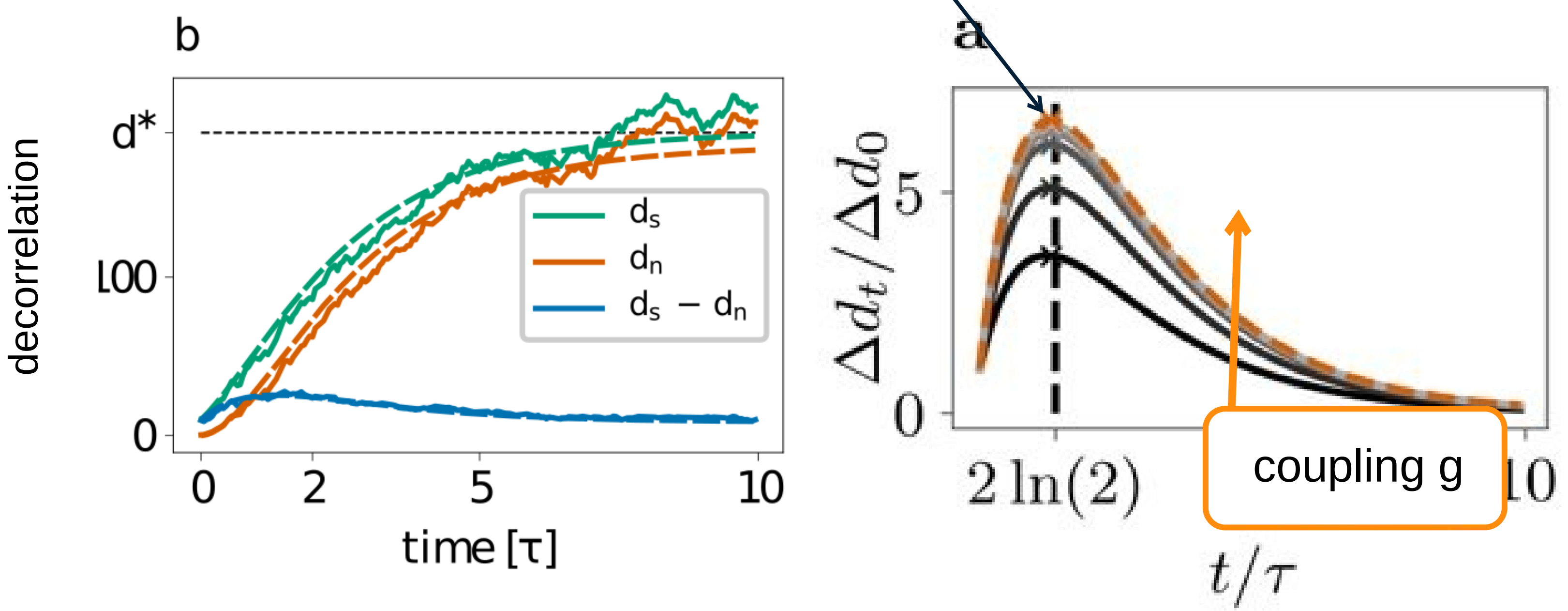


TRANSIENT CHAOTIC DIMENSIONALITY EXPANSION

Classification in chaotic binary networks

- Input data: 50 Gaussian classes in 8 dim. (not linearly separable)
- Linear readout accuracy peaks during expansion phase

optimal signal after
2 ln(2) ~ 1.5
activations per neuron



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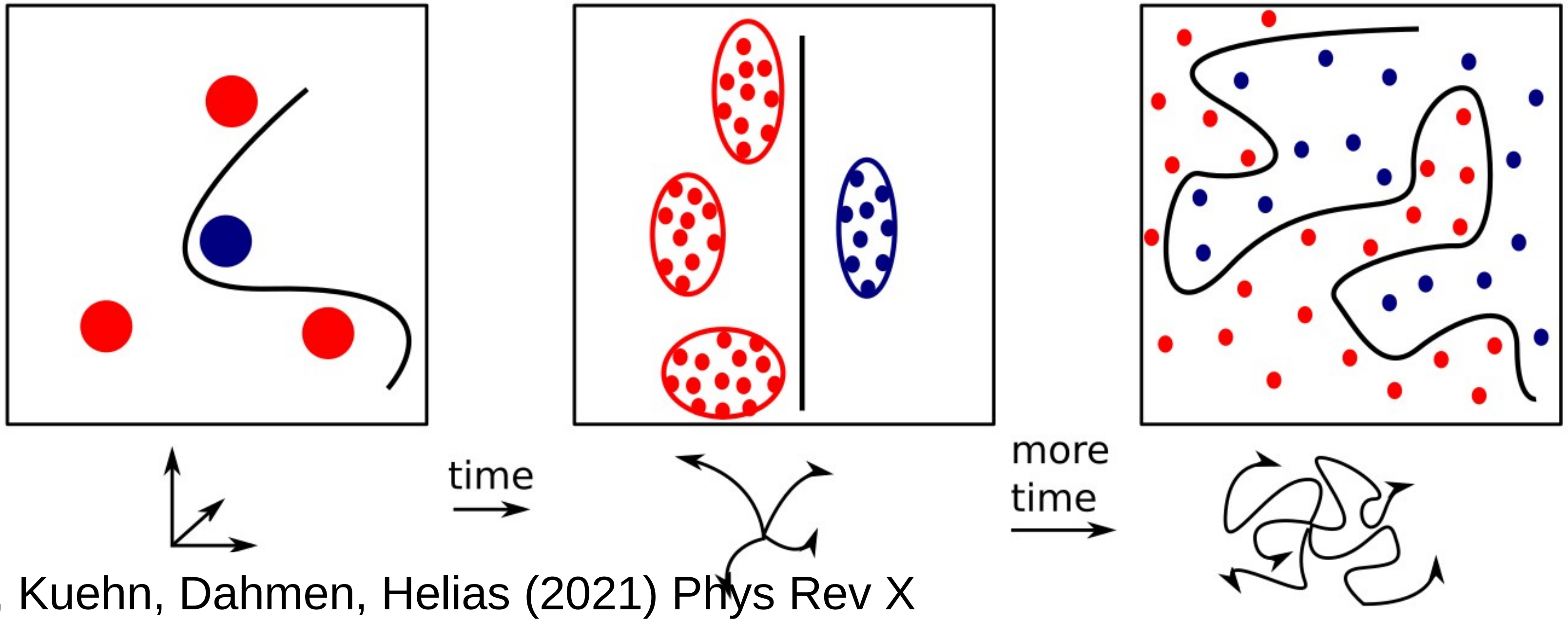
TRANSIENT CHAOTIC DIMENSIONALITY EXPANSION

Classification in chaotic binary networks

optimal signal after
2 ln(2) ~ 1.5
activations per neuron

- Input data: 50 Gaussian classes in 8 dim. (not linearly separable)
- Li



Keup, Kuehn, Dahmen, Helias (2021) Phys Rev X

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Acknowledgments

DFG Deutsche Forschungsgemeinschaft
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Federal Ministry of Education and Research

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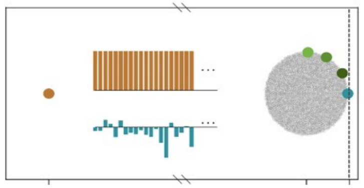
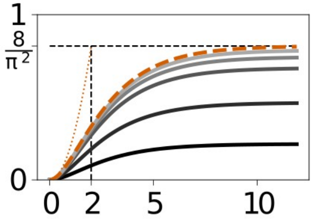
SUMMARY

- novel type of critical state
  - implied by wide distribution of correlations
  - dynamics close to linear instability and chaos
  - caused by disorder of connectivity



Dahmen, Gruen, Diesmann, Helias (2019) *PNAS*

- chaotic dynamics enhances separability
  - discrete coupling: stereotypical and fast
  - quick separation of signals by recurrent networks



Keup, Kuehn, Dahmen, Helias (2021) *Phys Rev X*