Neural Networks and Biological Modeling

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Question set 7

Exercise 1: Dynamics of Hopfield model

Consider a network of $N = 20000$ neurons that has stored 4 patterns:

$\xi^1 = \{\xi_1^1, \dots \xi_N^1\}$,
$\xi^2 = \{\xi_1^2, \dots \xi_N^2\}$,
$\xi^3 = \{\xi_1^3 \dots \xi_N^3\}$,
$\xi^4 = \{\xi_1^4 \dots \xi_N^4\}$,

using the synaptic update rule $w_{ij} = (J/N) \sum_\mu \xi_i^\mu \xi_j^\mu$ where $J > 0$ is a parameter. **Each pattern has values $\xi_i^\mu = \pm 1$ so that exactly 50 percent of the neurons in a pattern have $\xi_i^\mu = +1$.**

Assume stochastic dynamics: neurons receive an input $h_i(t) = \sum_j w_{ij} S_j(t)$ where $S_j(t) = \pm 1$ is the state of neuron j. Neurons update their state

$$Prob\,\{S_i(t+1) = +1 | h_i(t)\} = 0.5[1 + g(h_i(t))] \tag{1}$$

where g is an **odd and monotonically increasing function**: $g(h) = 2h$ for $|h| < 0.5$ and $g(h) = 1$ for $h \geq 0.5$ and $g(h) = -1$ for $h \leq -0.5$.

1.1 Rewrite the righ-hand-side of equation (1) by introducting an overlap $m^\mu(t) = (1/N) \sum_j \xi_j^\mu S_j(t)$.

1.2 What is the significance of the overlap? Describe its meaning in one sentence; give examples if necessary.

1.3 Assume that the four patterns are orthogonal, i.e., $\sum_i \xi_i^\mu \xi_i^\nu = 0$ if $\mu \neq \nu$. **Assume that the overlap with pattern 4 at $t = 0$ has a value of 0.3 and $m^\mu(0) = 0$ for all other patterns.**

Suppose that neuron i is a neuron with $\xi_i^4 = -1$.

What is the probability that neuron i fires in time step 1? Give the formula for arbitrary J and evaluate then for $J = 1$.

What is the probability that another neuron k with $\xi_k^4 = +1$ fires in time step 1? Give the formula for arbitrary J and evaluate then for $J = 1$.

1.4 For the same assumptions as in 1.3, what is the expected overlap for $< m^4(t) >$ after the first time step.

1.5 For the same assumptions as in 1.3 and 1.4 , write the evolution of the overlap for $m^4(t)$ for an arbitrary time step and arbitrary J. **Assume that N is large ($N \to \infty$).** Because of the orthogonality of the patterns, you may also assume that $m^\mu(t) = 0$ for $\mu \neq 4$).

1.6 Can you relate the evolution of the overlap to the more general picture of mean-field analysis of coupled networks?

Exercise 2: Cable equation

The electrical properties of a neuronal process (axon or dendrite) can be described mathematically by the cable equation

$$\frac{\partial^2}{\partial x^2}\, u(t,x) + r_{\mathrm{L}}\, i_{\mathrm{ext}}(t,x) = c\, r_{\mathrm{L}}\, \frac{\partial}{\partial t}\, u(t,x) + r_{\mathrm{L}} \sum_{\mathrm{ion}} i_{\mathrm{ion}} \tag{2}$$

where $\sum_{\mathrm{ion}} i_{\mathrm{ion}}$ is the linear current density which flows through the ion channels of the membrane.

For dendrites, the membrane is usually assumed to be purely passive, i.e., $\sum_{\mathrm{ion}} i_{\mathrm{ion}} = u/r_T$ where r_T is the transverse resistance.

2.1 Multiply Eq.(2) by $\frac{r_T}{r_L}$ to obtain

$$\lambda^2\, \frac{\partial^2}{\partial x^2}\, u(t,x) = \tau\, \frac{\partial}{\partial t}\, u(t,x) + u(t,x) - r_{\mathrm{T}}\, i_{\mathrm{ext}}(t,x) \tag{3}$$

2.2 Change variables to obtain the canonical form

$$\frac{\partial}{\partial t}\, u(t,x) - \frac{\partial^2}{\partial x^2}\, u(t,x) = -u(t,x) + r_T i_{\mathrm{ext}}(t,x)\,, \tag{4}$$

2.3 Show that when $r_T i_{\mathrm{ext}}(t,x) = \delta(x)\delta(t)$, the solution is given by

$$G_\infty(t,x) = \frac{\Theta(t)}{\sqrt{4\pi\, t}}\, \exp\left[-t - \frac{x^2}{4\, t}\right]\,. \tag{5}$$

where $\Theta(t)$ is the Heaviside function, $\Theta(t) = 0$ if $t \leq 0$ and 1 if $t > 0$, with $d\Theta(t)/dt = \delta(t)$.

(i) What is the physical interpretation of this solution?

(ii) What is the difference with a purely diffusive process?

2.4 Optional homework

Construct a solution of the cable equation for a semi-infinite cable ($0 < x < \infty$) for a charge q injected at time $t = 0$ at position x_0.

To this end, impose a reflecting boundary condition at $x = 0$, that is, $\frac{du}{dx}\mid_{x=0}= 0$. The interpretation of this condition is that no current can flow through $x = 0$.

The condition can be implemented by using the method of images, which consists in adding an imaginary charge q at position $-x_0$.

(i) Discuss a possible biological interpretation if $x = 0$ represents the cell body of the neuron.

(ii) What would be, in this case, the temporal evolution of the somatic membrane potential following a current pulse in the dendrite?

2.5 Computer Excercise using NEURON

The aim of this exercise is the install and become comfortable with programming the NEURON simulation environment using Python.

NEURON installers are available for Windows, OSX and Linux-RPM at `www.neuron.yale.edu`. Follow the link "Download and Install", get the precompiled installer for your platform, and follow the installation instructions given there, with the following notes. For OSX, put the NEURON-7.0 folder under Applications. For Windows, install in the default path, `c:\nrn70`.

The caveat of this approach is that a Python has been *embedded* in NEURON, that is NEURON has its own executable and is disjunct from your standard Python environment.

If you would like to *import neuron* in IPython, the installation procedure is more tricky, but may be worth the effort. You have to get the source code and build NEURON on your machine. On Windows, this requires Cygwin, and on OSX a GNU-build environment such as Xcode. On Linux, a GNU-build environment is usually available by default. Detailed instructions for the build process are available as an appendix of this open-access article
`http://www.frontiersin.org/neuroinformatics/paper/10.3389/neuro.11/001.2009`.

For this exercise, I assume a precompiled installer was used to install NEURON.

Start on your platform as follows:
On Windows, start `NEURON 7.0->nrngui_python`.
On OSX, in a terminal run `/Applications/NEURON-7.0/nrn/umac/bin/nrngui -python`
On Linux, locate the `nrngui` binary and start as follows:

```
$ nrngui -python
>>>
```

Get the source for the toy example presented in class on the moodle: `neuron_example.py`

Assuming it is in your current working directory, execute as follows: `>>> execfile('neuron_example.py')`

(i) The output of the script is fairly uninteresting. Adjust the stimulus current amplitude and duration, and graph axes to see some spiking. For example:

```
>>> stim.amp = 1.0
>>> go()
```

(ii) Record simultaneously from both ends of the axon to determine the propagation speed of the action potential. Notice how the action potential takes its characteristic shape as it propagates along the axon.

(iii) Record on the dendrite at different distances from the soma and observe the attenuation of the "back-propagating" action potential. Extend the length of the dendrite and determine at what distance from the soma is the action potential only 1/10$^{\text{th}}$ its original height.