Neural Networks and Biological Modeling

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Question set 3

Exercise 1: Separation of time scales

A. One-dimensional system

Consider the following differential equation

$$\tau \frac{dx}{dt} = -x + c\,. \tag{1}$$

1.1 Find the fixed point of this system.

1.2 Show that the fixed point is a stable one, and that the solution of (1) converges exponentially towards the fixed point with a time constant τ.

1.3 Consider the case where c is time-dependent, namely,

$$c \equiv c(t) = \begin{cases} 0 & \text{for } t < 0 \\ c_0 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } t > 1\,. \end{cases}$$

Calculate the solution $x(t)$ with initial condition $x(t = -10) = 0$.

B. Separation of time scales

Consider the following system of equations:

$$\begin{aligned} \frac{du}{dt} &= f(u) - m \\ \epsilon \frac{dm}{dt} &= -m + c(u)\,. \end{aligned}$$

1.4 Exploit the fact that $\epsilon \ll 1$ and reduce the system to one equation (note the similarity between the m equation and Eq.(1)).

1.5 Set $f(u) = -au + b$ where $a > 0$, $b \in \mathbb{R}$ and $c(u) = \tanh(u)$. Discuss the stability of the fixed points with respect to a and b.

Exercise 2: Phase plane stability analysis

2.1 Linear system

Consider the following linear system:

$$\begin{cases} \dfrac{du}{dt} &= \alpha u - w \\ \dfrac{dw}{dt} &= \beta u - w\,. \end{cases}$$

These equations can be written in matrix form as $\frac{d}{dt}x = Ax$ where $x = \begin{pmatrix} v \\ w \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & -1 \\ \beta & -1 \end{pmatrix}$.
Determine the conditions for stability of the point $(u = 0, w = 0)$ in the case $\beta > \alpha$ by studying the eigenvalues of the above matrix.

2.2 Piecewise linear Fitzhugh-Nagumo model

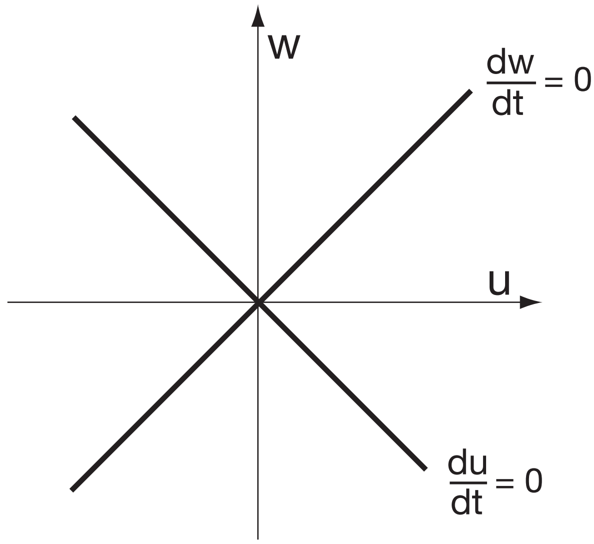
The Fitzhugh-Nagumo model is defined by the equations

$$\begin{cases} \dfrac{du}{dt} &= F(u,w) = f(u) - w + I \\ \dfrac{dw}{dt} &= G(u,w) = bu - \gamma w \end{cases}$$

Here, $u(t)$ is the membrane potential and $w(t)$ is a second, time-dependent variable. I stands for the injected current. A simplified model is obtained by considering a piecewise linear $f(u)$:

$$f(u) = \begin{cases} -u & \text{si } u < 1 \\ \dfrac{u-1}{a} - 1 & \text{si } 1 \leq u < 1 + 2a \\ 2(1+a) - u & \text{si } u > 1 + 2a \end{cases}$$

with $\gamma = 1$, $a < 1$ and $b > 1/a$.



(i) Sketch the "nullclines" $du/dt = 0$ and $dw/dt = 0$ for the case $I = 0$. How does the fixed point move as I is varied? Sketch the form of the flow (i.e., the vector $(du/dt, dw/dt)$) along the nullclines and deduce qualitatively the shape of the trajectories.

(ii) Calculate the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial w} \\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial w} \end{pmatrix}$$

Determine, by studying the eigenvalues of J, the linear stability of the fixed point as a function of I. What happens when the fixed point destabilizes?

Exercise 3: Type I or Type II?

For this computer exercise, download TypeX.py, TypeY.py, tools.py, fvsI.py from the course moodle page (week 3). One of TypeX.py or TypeY.py is a FitzHugh-Nagumo Type II model, and the other is a Morris-Lecar Type I model. The objective of this exercise is to find out which is which.

As last week, start pylab (Enthought pylab icon on windows, `ipython -pylab` in a command shell on Linux or OSX) in the directory containing the downloaded files. You should then be able to import the two models:

```
>> import TypeX
>> import TypeY
```

3.1 What is the threshold current for repetitive firing for TypeX, TypeY?

To this end, use the TypeX.PlotStep or TypeY.PlotStep to plot the response to a step current which starts after 100ms (to let the system equilibrate) and lasting 1000ms (to detect repetitive firing with a long period):

For example using:

```
>> TypeY.PlotStep(I_amp=0.5,Step_tstart=100.0,Step_tend=1000.0,tend=1000.0)
```

Exploring various values of `I_amp`, find the range in which the threshold occurs, to a precision of 0.01.

Already from the voltage response near threshold you might have an idea which is type I or II, but let's investigate further ...

3.2 Plot on one axis the response to short pulses near threshold, and interpret the results. Which is Type I, II ?

Example:

```
>> figure()
>> t,v,w,I = TypeY.Step(I_amp=1.05,Step_tstart=100.0,Step_tend=110.0,tend=300.0)
>> plot(t,v)
>> t,v,w,I = TypeY.Step(I_amp=1.1,Step_tstart=100.0,Step_tend=110.0,tend=300.0)
>> plot (t,v)
>> t,v,w,I = TypeY.Step(I_amp=1.15,Step_tstart=100.0,Step_tend=110.0,tend=300.0)
>> plot (t,v)
```

N.B. - Refer to the lecture notes on the moodle to recall the differences one should observe between Type I and Type II.

3.3 Plot f-vs-I curves for each TypeX, TypeY

Provided in tools.py is a function to determine the spike times from t,v:

```
>> import tools
>> t,v,w,I = TypeX.Step(I_amp=0.40,Step_tstart=100.0,Step_tend=1000.0,tend=1000.0)

>> st = tools.spiketimes(t,v)
>> print st
[ 102.96  146.7   189.09]
```

Using this function one can calculate an estimate of the firing rate:

First calculate the inter-spike intervals (time difference between spikes) using this elegant indexing idiom:

```
>> isi = st[1:]-st[:-1]
```

Then find the mean and take the reciprocal, converting from 1/ms to Hz, to yield the firing-rate:

```
>> f = 1000.0/mean(isi)
```

This function is provided in `tools.py` as the function `tools.f`

Now let's use it to plot an f-vs-I curve for each TypeX/TypeY

To save you some typing, this simple script to plot an f-vs-I curve is provided on the moodle as fvsI.py

```
I = arange(0.0,1.0,0.1)
f = []

for x in I:
   f.append(tools.f(TypeX,x))
```

```
figure()
plot(I,f)
```

Change TypeX to TypeY, and change the `I` range to zoom in near the threshold.

Which is Type I and which is Type II?