Neural Networks and Biological Modeling

Professor Wulfram Gerstner
Laboratory of Computational Neuroscience

Question set 10

Exercise 1: Flux across threshold

Assume a population of integrate-and-fire neurons with a known time varying voltage distribution $p(u,t)$.

1.1 Current pulse

At time t^*, all neurons receive a current pulse $I(t) = q\delta(t - t^*)$. What is the fraction of neurons in the population that is pushed across threshold by this pulse?

1.2 Current across the threshold

Assume that these pulses occur at a rate ν. What is the fraction of neurons that cross threshold per unit of time, due to the pulses?

Exercise 2: Ornstein-Uhlenbeck process

Consider the Ornstein-Uhlenbeck process with time-dependent mean and variance,

$$\tau \dot{u}(t) = -u(t) + \mu(t) + \sqrt{2\sigma^2\tau}\,\xi(t) \tag{1}$$

where $\mu(t) = RI(t)$ is the input and $\xi(t)$ is a Gaussian white noise, i.e. $\langle\, \xi(t)\,\rangle = 0$ and $\langle\, \xi(t)\xi(t')\,\rangle = \delta(t - t')$.

The probability distribution of the variable u obeys the Fokker-Planck equation

$$\begin{aligned}\tau \frac{\partial}{\partial t} p(u,t) &= \frac{\partial}{\partial u}\big((u - \mu(t))p(u,t)\big) + \sigma^2 \frac{\partial^2}{\partial u^2} p(u,t) \\ &= \frac{\partial}{\partial u}\left\{u - \mu(t) + \sigma^2 \frac{\partial}{\partial u}\right\} p(u,t)\,. \end{aligned} \tag{2}$$

Voltage distribution

Consider the Gaussian distribution

$$p(u,t) = \frac{1}{\sqrt{2\pi\Sigma^2(t)}} \exp\left(-\frac{(u - \bar{u}(t))^2}{2\Sigma^2(t)}\right) \tag{3}$$

where $\bar{u}(t)$ is the (deterministic) solution of equation

$$\tau \frac{d\overline{u}}{dt} = \mu(t) - \overline{u}\,, \qquad \overline{u}(0) = u_0$$

and

$$\Sigma^2(t) = \sigma^2[1 - e^{-\frac{2t}{\tau}}].$$

2.1 Show that for constant $\Sigma(t) = \sigma$, Eq.(3) solves the Fokker-Planck equation (2). Note that the initial condition is $p(u,0) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(u-u_0)^2}{2\sigma^2}\right)$. What is the physical interpretation of this initial condition?

2.2 Assume that all trajectories start at $u = u_0$. This corresponds to the initial condition $p(u,0) = \delta(u - u_0)$. Show that for constant $\mu(t) = \mu$, Eq.(3) solves the Fokker-Planck equation (2). What is the physical interpretation of the initial condition?

Exercise 3: Fokker-Planck equation with threshold

Take $\mu = 0$ and rewrite the Fokker-Planck equation in the form of a conservation law

$$\frac{\partial p(u,t)}{\partial t} = -\frac{\partial J(u,t)}{\partial u} \tag{4}$$

where $J(u,t)$ is the probability current. To incorporate the spike threshold, we impose an absorbing boundary condition at $u = \vartheta$, i.e.,

$$p(\vartheta, t) = 0 \,, \forall t \,. \tag{5}$$

Furthermore, the reset is described by adding a source term in the equation,

$$\frac{\partial p(u,t)}{\partial t} = -\frac{\partial J(u,t)}{\partial u} + \nu(t)\delta(u - u_r) \tag{6}$$

where $\nu(t) = J(u,t)\mid_{u=\theta}$ is the average firing frequency. This means that the probability current flowing through the threshold is reinjected at the reset potential, which is necessary to ensure the conservation of probability.

Stationary solution

We look for a stationary solution of (5-6), characterized by $\partial p/\partial t = 0$ and $\nu(t) = \nu = \text{cst}$. Integrating (6) once, we obtain

$$J(u,t) = \nu H(u - u_r) \,, \tag{7}$$

where H is the Heaviside function, and we have used the "natural" condition: $\lim_{u\to-\infty} J(u,t) = 0$.

3.1 Comparing Eq. 4 with equation 2, we see that the probability current is related to the stationary probability density according to: $J(u) = \left(-u - \sigma\frac{\partial}{\partial u}\right) p(u)$. With the results of exercise 2 show that the Gaussian distribution

$$p_1(u) = \frac{c_1}{\sigma} e^{-\frac{u^2}{2\sigma^2}} \tag{8}$$

is a solution of the Fokker-Planck equation (6-7) on the interval $[-\infty, u_r]$. Conclude that it is the desired solution for the interval $[-\infty, u_r]$.

3.2 Consider the "modified" Gaussian

$$p_2(u) = \frac{c_2}{\sigma} e^{-\frac{u^2}{2\sigma^2}} \int_u^{\vartheta} e^{\frac{x^2}{2\sigma^2}} dx \tag{9}$$

Show that it satisfies the solution for the current (7), the Fokker-Planck equation (6), and the boundary condition (5).

3.3 Set

$$p(u) = \begin{cases} p_1(u) & , u < u_r \\ p_2(u) & , u_r < u < \vartheta \,. \end{cases} \tag{10}$$

Find the relation between c_1 and c_2 which ensures the continuity of $p(u)$.

3.4 Normalize the probability density to find an expression for the constants.

3.5 What is the firing rate of this neuron $\nu = J(\vartheta)$.

3.6 Note that $p(u)$ can be written $p(u) = \bar{p}(u)q(u)$ where $\bar{p}(u)$ is the stationary solution of the problem *without* boundary condition (c.f. Ex.2). Sketch the form of $\bar{p}$, q, and p and give an interpretation.