A brief introduction to error analysis and propagation

Georg Fantner

February 2011

Contents

1 Acknowledgements

This text is based on texts written by Vern Lindberg and others.

2 Random and systematic errors

No measurement made is ever exact. The accuracy (correctness) and precision (number of significant figures) of a measurement are always limited by the degree of refinement of the apparatus used, by the skill of the observer, and by the basic physics in the experiment. In doing experiments we are trying to establish the best values for certain quantities, or trying to validate a theory. We must also give **a range of possible true values based on our limited number of measurements**.
Why should repeated measurements of a single quantity give different values? Mistakes on the part of the experimenter are possible, but we do not include these in our discussion. A careful researcher should not make mistakes! (Or at least she or he should recognize them and correct the mistakes.) We use the synonymous terms uncertainty, error, or deviation to represent the variation in measured data. Two types of errors are possible. *Systematic error* is the result of a mis-calibrated device, or a measuring technique which always makes the measured value larger (or smaller) than the "true" value. An example would be using a steel ruler at liquid nitrogen temperature to measure the length of a rod. The ruler will contract at low temperatures and therefore overestimate the true length. Careful design of an experiment will allow us to eliminate or to correct for systematic errors. Even when systematic errors are eliminated there will remain a second type of variation in measured values of a single quantity. These remaining deviations will be classed as *random errors*, and can be dealt with in a statistical manner. This document does not teach statistics in any formal sense, but it should help you to develop a working methodology for treating errors.

3 Determining random errors

How can we estimate the uncertainty of a measured quantity? Several approaches can be used, depending on the application.

3.1 Instrument Limit of Error (ILE) and Least Count

The least count is the smallest division that is marked on the instrument. Thus a meter stick will have a least count of 1.0 mm, a digital stop watch might have a least count of 0.01 sec. The instrument limit of error, ILE for short, **is the precision to which a measuring device can be read**, and is always equal to or smaller than the least count. Very good measuring tools are calibrated against standards maintained by the National Institute of Standards and Technology. The Instrument Limit of Error is generally taken

to be the least count or some fraction (1/2, 1/5, 1/10) of the least count). You may wonder which to choose, the least count or half the least count, or something else. No hard and fast rules are possible, instead you must be guided by common sense. If the space between the scale divisions is large, you may be comfortable in estimating to 1/5 or 1/10 of the least count. If the scale divisions are closer together, you may only be able to estimate to the nearest 1/2 of the least count, and if the scale divisions are very close you may only be able to estimate to the least count.

For some devices the ILE is given as a tolerance or a percentage. Resistors may be specified as having a tolerance of 5%, meaning that the ILE is 5% of the resistor's value.

3.2   Estimated Uncertainty

Often other uncertainties are larger than the ILE. We may try to balance a simple beam balance with masses that have an ILE of 0.01 grams, but find that we can vary the mass on one pan by as much as 3 grams without seeing a change in the indicator. We would use half of this as the estimated uncertainty, thus getting uncertainty of 1.5 grams. Another good example is determining the focal length of a lens by measuring the distance from the lens to the screen. The ILE may be 0.1 cm, however the depth of field may be such that the image remains in focus while we move the screen by 1.6 cm. In this case the estimated uncertainty would be half the range or 0.8 cm.

3.3   Average Deviation: Estimated Uncertainty by Repeated Measurements

The statistical method for finding a value with its uncertainty is to **repeat the measurement several times**, find the average, and find either the average deviation or the standard deviation. Suppose we repeat a measurement several times and record the different values. We can then find the average value, here denoted by a symbol between angle brackets, $< t >$, and use it as our best estimate of the reading. How can we determine the uncertainty? Let us use the following data in table 1 as an example. Column 1 shows a time in seconds.

A simple average of the times is the sum of all values (7.4+8.1+7.9+7.0) divided by the number of readings (4), which is 7.6 sec. We will use angular brackets around a symbol to indicate average; an alternate notation uses a bar is placed over the symbol.

Column 2 of Table 1 shows the deviation of each time from the average, $(t- < t >)$. A simple average of these is zero, and does not give any new information.

To get a non-zero estimate of deviation we take the average of the absolute values of the deviations, as shown in Column 3 of Table 1. We will call this the average deviation, Dt.

Column 4 has the squares of the deviations from Column 2, making the answers all positive. The sum of the squares is divided by 3, (one less than the number of readings), and the square root is taken to produce the sample standard deviation. An explanation of why we

3

Table 1: Values showing the determination of average, average deviation, and standard deviation in a measurement of time. Notice that to get a non-zero average deviation we must take the absolute value of the deviation.

Time t/s	$(t-<t>)/s$	$\|(t-<t>)\|/s$	$(t-<t>)^2/s^2$
7.4	-0.2	0.2	0.04
8.1	0.5	0.5	0.25
7.9	0.3	0.3	0.09
7.0	-0.6	0.6	0.36
$<t>=$	$<t-<t>>=$	$<\|t-<t>\|>=$	$\left\langle (t-<t>)^2 \right\rangle =$
7.6	0	0.4	0.247
		standard deviation	$\sqrt{\left\langle (t-<t>)^2 \right\rangle} =$
			0.5

divide by (N-1) rather than N is found below. The sample standard deviation is slightly different than the average deviation, but either one gives a measure of the variation in the data.

For a second example, consider a measurement of length shown in Table 2. The average and average deviation are shown at the bottom of the table.

Table 2: Example of finding an average length and an average deviation in length. The values in the table have an excess of significant figures. Results should be rounded as explained in the text. Results can be reported as (15.5 ± 0.1) m or (15.47 ± 0.13) m. If you use standard deviation the length is (15.5 ± 0.2) m or (15.47 ± 0.18) m.

Length, x, m	$\|x-<x>\|$, m	$(x-<x>)^2/m^2$
15.4	0.06667	0.004445
15.2	0.26667	0.071112
15.67	0.133337	0.017777
15.77	0.23333	0.054443
15.5	0.03333	0.001111
15.4	0.06667	0.004445
Average: 15.46667 m	0.133333 m	St. dev. 0.17512

3.4 How to Compute the Standsrd Deviation

How to calculate the standard deviation

1. Compute the square of the difference between each value and the sample mean.

2. Add those values up.

3. Divide the sum by n-1. This is called the variance.

4. Take the square root to obtain the Standard Deviation.

3.4.1 Why n-1?

Why divide by n-1 rather than n in the third step above? In step 1, you compute the difference between each value and the mean of those values. You don't know the true mean of the population; all you know is the mean of your sample. Except for the rare cases where the sample mean happens to equal the population mean, the data will be closer to the sample mean than it will be to the true population mean. So the value you compute in step 2 will probably be a bit smaller (and can't be larger) than what it would be if you used the true population mean in step 1. To make up for this, divide by n-1 rather than n.

But why n-1? If you knew the sample mean, and all but one of the values, you could calculate what that last value must be. Statisticians say there are n-1 degrees of freedom.

3.4.2 When should the SD be computed with a denominator of n?

Statistics books often show two equations to compute the SD, one using n, and the other using n-1, in the denominator. Some calculators have two buttons.

The n-1 equation is used in the common situation where you are analyzing a sample of data and wish to make more general conclusions. The SD computed this way (with n-1 in the denominator) is your best guess for the value of the SD in the overall population.

If you simply want to quantify the variation in a particular set of data, and don't plan to extrapolate to make wider conclusions, then you can compute the SD using n in the denominator. The resulting SD is the SD of those particular values. It makes no sense to compute the SD this way if you want to estimate the SD of the population from which those points were drawn. It only makes sense to use n in the denominator when there is no sampling from a population, there is no desire to make general conclusions.

The goal of science is always to generalize, so the equation with n in the denominator

should not be used. The only example I can think of where it might make sense is in quantifying the variation among exam scores. But much better would be to show a scatterplot of every score, or a frequency distribution histogram.

3.5   Conflicting Uncertainty Numbers

In some cases we will get an ILE, an estimated uncertainty, and an average deviation and we will find different values for each of these. We will be pessimistic and take the largest of the three values as our uncertainty. [When you take a statistics course you should learn a more correct approach involving adding the variances.] For example we might measure a mass required to produce standing waves in a string with an ILE of 0.01 grams and an estimated uncertainty of 2 grams. We use 2 grams as our uncertainty.

The proper way to write the answer is

1. Choose the largest of (i) ILE, (ii) estimated uncertainty, and (iii) average or standard deviation

2. Round off the uncertainty to 1 or 2 significant figures.

3. Round off the answer so it has the same number of digits before or after the decimal point as the answer.

4. Put the answer and its uncertainty in parentheses, then put the power of 10 and unit outside the parentheses.

3.6   Why make many measurements? Standard Error in the Mean.

We know that by making several measurements (4 or 5) we should be more likely to get a good average value for what we are measuring. Is there any point to measuring a quantity more often than this? When you take a statistics course you will learn that the standard error in the mean is affected by the number of measurements made.

The standard error in the mean in the simplest case is defined as the standard deviation divided by the square root of the number of measurements.

The following example illustrates this in its simplest form. I am measuring the length of an object. Notice that the average and standard deviation do not change much as the number of measurements change, but that the standard error does dramatically decrease as N increases.

Table 3: Influence of the number of samples on the standard deviation and standard error.

Number of Measurements, N	Average	Standard Deviation	Standard Error
5	15.52 cm	1.33 cm	0.59 cm
25	15.46 cm	1.28 cm	0.26 cm
625	15.49 cm	1.31 cm	0.05 cm
10000	15.49 cm	1.31 cm	0.013 cm

3.6.1 When to use the standard error and when the standard deviation?

We can consider the difference between standard deviation and standard error like this:

- The standard deviation (SD) is how spread out THINGS in the population are, and this is calculated (somehow) from the data in your sample. It is useful in describing the population itself.

- The standard error (SE) is how spread out the SAMPLE MEAN will be around the true population mean. It is useful in describing how close your results will be to the right answer.

As a simple rule we can decide on if we use the standard deviation or the standard error by deciding if we are measuring one value multiple times (use standard error), or if we are measuring one quantity in multiple cases (use standard deviation).Another way to decide if you imagine you could make a perfect measurement, would you always get the same number, then use the standard error.

This means that when we want to describe for example a population of cells, by measuring their length, we will calculate the mean and the standard deviation, because there is no "right length". When we want to measure the temperature in our incubator, we will calculate the average temperature and the standard error, because there is a "right temperature".

3.6.2 How are Standard Error and Standard Deviation Related?

It so happens that there is a very simple relationship between SD and SE. You can calculate SE by the following formula:

$$SE = \frac{SD}{\sqrt{n}}$$

Now this is really quite a simple and beautiful little formula.

- It starts out with the way the world IS (that's SD - how spread out the data are, and there is virtually NOTHING you can do about it).

- It then talks about how hard you WORK (that's the sample size "n"), and you ARE in control of that. Please note that is how hard you work, not how smart).

- It then tells you HOW GOOD your average is likely to be with that amount of effort (the Standard Error).

4   Relative vs. Absolute Errors

When stating the error of a measurement, we can either state it as the absolute value of the error we calculated (absolute error), for example $R = (33 \pm 1.65)k\Omega$. Or we could state our error percentage $R = (33 \pm 5\%)k\Omega$. The relative error is the fractional uncertainty. Percentage error is the fractional error multiplied by 100%. While strictly speaking the relative error and the percentage error are different things, they are often used synonymously. In practice, either the percentage error or the absolute error may be provided. Thus in machining an engine part the tolerance is usually given as an absolute error, while electronic components are usually given with a percentage tolerance.

5   Propagation of Errors

Suppose two measured quantities x and y have uncertainties, Dx and Dy, determined by procedures described in previous sections: we would report (x ± Dx), and (y ± Dy). From the measured quantities a new quantity, z, is calculated from x and y. What is the uncertainty, Dz, in z? There are two ways to get an estimate for the error of z. In the simplified version the guiding principle in all cases is to consider the most pessimistic situation. In this case we add the individual uncertainties. This certainly gives us the *safe limit* of our estimate, but sometimes we want to be more restrictive in our answers. In the proper statistical treatment of error propagation we use the standard deviations to calculate the resulting uncertainty
The examples included in this section also show the proper rounding of answers. The examples use the propagation of errors using average deviations.

5.1   Addition and Subtraction: z=x+y or z=x-y

Derivation: We will assume that the uncertainties are arranged so as to make z as far from its true value as possible.

Average deviations $Dz = |Dx| + |Dy|$ in both cases. With more than two numbers added or subtracted we continue to add the uncertainties.

Using average errors	Using standard deviations
$\Delta z = |\Delta x| + |\Delta y| + \ldots$	$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2 + \ldots}$

Example: $w = (4.52 \pm 0.02)cm, x = (2.0 \pm 0.2)cm, y = (3.0 \pm 0.6)cm$. Find $z = x + y - w$ and its uncertainty.

$$z = x + y - w = 2.0 + 3.0 - 4.5 = 0.5cm$$

For the simplified method we get:

$$\Delta z = \Delta x + \Delta y + \Delta w = 0.2 + 0.6 + 0.02 = 0.82$$

rounding to 0.8 cm, So $z = (0.5 \pm 0.8)cm$
When using the standard deviation we get:

$$\Delta z = \sqrt{0.2^2 + 0.6^2 + 0.02^2} = 0.633$$

So $z = (0.5 \pm 0.6)cm$.

5.2 Multiplication by an exact number

When multiplying a measurement value with an exact number, multiply the uncertainty also with the exact number.
Example: The radius of a circle is $r = (3.0 \pm 0.2)cm$. Find the circumference and its uncertainty.

$$\begin{aligned} C &= 2\pi r = 18.850cm \\ \Delta C &= 2\pi \Delta r = 1.257cm \text{ (The factors of 2 and } \pi \text{ are exact)} \\ C &= (18.8 \pm 1.3)cm \end{aligned}$$

We round the uncertainty to two figures since it starts with a 1, and round the answer to match.

5.3 Multiplication and Division: z = x y or z = x/y

Derivation: We can derive the relation for multiplication easily. Take the largest values for x and y, that is:

$$z + \Delta z = (x + \Delta x)(y + \Delta y) = xy + x\Delta y + y\Delta x + \Delta x \Delta y$$

Usually $\Delta x << x$ and $\Delta y << y$ so that the last term is much smaller than the other terms and can be neglected. Therefore:

$$z = xy,$$
$$\Delta z = y\Delta x + x\Delta y$$

which we write more compactly by forming the relative error, that is the ratio of $\Delta z/z$, namely:

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \dots$$

Using average errors	Using standard deviations
$\frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \dots$	$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \dots}$

Example: $w = (4.52 \pm 0.02)cm, x = (2.0 \pm 0.2)cm$. Find $z = w \cdot x$ and its uncertainty.

$$z = w \cdot x = (4.52)(2.0) = 9.04cm^2$$
$$\frac{\Delta z}{9.04cm^2} = \frac{0.02cm}{4.52cm} + \frac{0.2cm}{2.0cm} = 0.1044$$
$$\Delta z = 0.1044 \cdot (9.04cm^2) = 0.944cm^2 \Rightarrow 0.9cm^2$$

Using the standard deviations:

$$\frac{\Delta z}{9.04cm^2} = \sqrt{\left(\frac{0.02cm}{4.52cm}\right)^2 + \left(\frac{0.2cm}{2.0cm}\right)^2} = 0.1$$
$$\Delta z = 0.9cm^2$$

therefore

$$z = (9.0 \pm 0.9)cm^2$$

5.4 Products of Powers $z = x^m + y^n$

Using average errors	Using standard deviations
$\frac{\Delta z}{z} = \|m\|\frac{\Delta x}{x} + \|n\|\frac{\Delta y}{y} + \dots$	$\frac{\Delta z}{z} = \sqrt{\left(\frac{m\Delta x}{x}\right)^2 + \left(\frac{n\Delta y}{y}\right)^2 + \dots}$

5.5 Mixtures of multiplication, division, addition, subtraction, and powers.

If z is a function which involves several terms added or subtracted we must apply the above rules carefully. This is best explained by means of an example.

Example: $w = (4.52 \cdot 0.02)cm, x = (2.0 \cdot 0.2)cm, y = (3.0 \cdot 0.6)cm$. Find $z = wx + y^2$

First we compute $v = wx$ to get $v = (9.0 \pm 0.9)cm^2$
Next we compute $\Delta(y^2)$:

$$
\begin{aligned}
\frac{\Delta(y^2)}{y^2} &= \frac{2\Delta y}{y} = \frac{2 \cdot 0.6cm}{3.0cm} = 0.40 \\
\Delta(y^2) &= 0.40(9.00cm^2) = 3.6cm^2
\end{aligned}
$$

finally we compute

$$
\begin{aligned}
\Delta z &= \Delta v + \Delta(y^2) = 0.9 + 3.6 = 4.5cm^2 \Rightarrow 4cm^2 \\
z &= (18 \pm 4)cm^2
\end{aligned}
$$

6 Significant Digits

The rules for propagation of errors hold true for cases when we are in the lab, but doing propagation of errors is time consuming. The rules for significant figures allow a much quicker method to get results that are approximately correct even when we have no uncertainty values. A significant figure is any digit 1 to 9 and any zero which is not a place holder. Thus, in 1.350 there are 4 significant figures since the zero is not needed to make sense of the number. In a number like 0.00320 there are 3 significant figures –the first three zeros are just place holders. However the number 1350 is ambiguous. You cannot tell if there are 3 significant figures –the 0 is only used to hold the units place –or if there are 4 significant figures and the zero in the units place was actually measured to be zero. How do we resolve ambiguities that arise with zeros when we need to use zero as a place holder as well as a significant figure? Suppose we measure a length to three significant figures as 8000 cm. Written this way we cannot tell if there are 1, 2, 3, or 4 significant figures. To

make the number of significant figures apparent we use scientific notation, $8 \times 10^3 cm$ cm (which has one significant figure), or $8.00 \times 10^3 cm$ (which has three significant figures), or whatever is correct under the circumstances. We start then with numbers each with their own number of significant figures and compute a new quantity. How many significant figures should be in the final answer? In doing running computations we maintain numbers to many figures, but we must report the answer only to the proper number of significant figures.

In the case of addition and subtraction we can best explain with an example. Suppose one object is measured to have a mass of 9.9 gm and a second object is measured on a different balance to have a mass of 0.3163 gm. What is the total mass? We write the numbers with question marks at places where we lack information. Thus 9.9???? gm and 0.3163? gm. Adding them with the decimal points lined up we see

$$
\begin{array}{r}
09.9???? \\
00.3163? \\
\hline
10.2???? \\
\end{array} +
$$

$$= \quad 10.2gm$$

In the case of multiplication or division we can use the same idea of unknown digits. Thus the product of 3.413? and 2.3? can be written in long hand as

$$
\begin{array}{r}
3.413? \\
2.3? \\
\hline
????? \\
10219?0 \\
6816?00 \\
\hline
7.8????? \\
\end{array} \times
$$

$$= \quad 7.8$$

The short rule for multiplication and division is that the answer will contain a number of significant figures equal to the number of significant figures in the entering number having the least number of significant figures. In the above example 2.3 had 2 significant figures while 3.413 had 4, so the answer is given to 2 significant figures. It is important to keep these concepts in mind as you use calculators with 8 or 10 digit displays if you are to avoid mistakes in your answers and to avoid the wrath of physics instructors everywhere. A good procedure to use is to use use all digits (significant or not) throughout calculations, and only round off the answers to appropriate "sig fig."

7 Rounding off answers in regular and scientific notation

In theexamples we were careful to round the answers to an appropriate number of significant figures. The uncertainty should be rounded off to one or two significant figures. If the leading figure in the uncertainty is a 1, we use two significant figures, otherwise we use one significant figure. Then the answer should be rounded to match.