# Compressible Test Field Method

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# Mean-field electrodynamics

decompose: $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$

represent mean EMF $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$ by $\overline{\boldsymbol{B}}$: $\overline{\boldsymbol{\mathcal{E}}} = \alpha \cdot \overline{\boldsymbol{B}} + \ldots$

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in general:

split into parts existing/vanishing for vanishing $\overline{\boldsymbol{B}}$:

$$\boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(B)}, \quad \overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{\mathcal{E}}}^{(0)} + \overline{\boldsymbol{\mathcal{E}}}^{(B)}$$

$$\begin{aligned}\overline{\boldsymbol{\mathcal{E}}}^{(B)} = \overline{(\boldsymbol{u} \times \boldsymbol{b})^{(B)}} &= \overline{\boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(B)}} + \overline{\boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(B)}} + \overline{\boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)}} \\ &= \underbrace{\overline{\boldsymbol{u} \times \boldsymbol{b}^{(B)}}}_{\text{covered by QKIN}} + \underbrace{\overline{\boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)}}}_{\text{not covered}}\end{aligned}$$

# TFM for isothermal compressible MHD

## Basic equations

$$\begin{aligned}
\mathcal{D}^A \boldsymbol{A} &= \boldsymbol{U} \times \boldsymbol{B} + \eta \boldsymbol{\nabla}^2 \boldsymbol{A} + \boldsymbol{F}_{\rm M}, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \\
\mathcal{D}^U \boldsymbol{U} &= -\boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{U} - c_{\rm s}^2 \boldsymbol{\nabla} \ln \rho + \boldsymbol{J} \times \boldsymbol{B}/\rho + \boldsymbol{\nabla} \cdot (2\nu\rho \mathbf{S})/\rho + \boldsymbol{F}_{\rm K}, \\
\mathcal{D} \ln \rho &= -\boldsymbol{U} \cdot \boldsymbol{\nabla} \ln \rho - \boldsymbol{\nabla} \cdot \boldsymbol{U}
\end{aligned}$$

shear-temporal derivative operators:

$\mathcal{D}^A \boldsymbol{A} = \mathcal{D}\boldsymbol{A} + SA_y \hat{\boldsymbol{x}}, \quad \mathcal{D}^U \boldsymbol{U} = \mathcal{D}\boldsymbol{U} + SU_x \hat{\boldsymbol{y}}, \quad \mathcal{D} = \partial_t + Sx\partial_y$

$\boldsymbol{F}_{\rm M,K}$ external forcings

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to avoid triple correlations: $\rho \to \rho_{\rm ref} =$ const. or $\rho = \overline{\rho}$ in Lorentz force

# TFM for isothermal compressible MHD

## Equations for fluctuations

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**Equations for fluctuations**

$\ln \rho \equiv H = \overline{H} + h$

$$\begin{aligned}
\mathcal{D}^A \boldsymbol{a} &= \overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})' + \eta \boldsymbol{\nabla}^2 \boldsymbol{a} + \boldsymbol{f}_{\mathrm{M}}, \\
\mathcal{D}^U \boldsymbol{u} &= -c_{\mathrm{s}}^2 \boldsymbol{\nabla} h + \left( \overline{\boldsymbol{J}} \times \boldsymbol{b} + \boldsymbol{j} \times \overline{\boldsymbol{B}} + (\boldsymbol{j} \times \boldsymbol{b})' \right) / \rho_{\mathrm{ref}} \\
&\quad - \overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{U}} - (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})' \\
&\quad + \nu \left( \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{u}/3 \right) \\
&\quad + 2\nu \left( \overline{\mathbf{S}} \cdot \nabla h + \mathbf{s} \cdot \nabla \overline{H} + (\mathbf{s} \cdot \nabla h)' \right) + \boldsymbol{f}_{\mathrm{K}} \\
\mathcal{D} h &= -\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} h - \boldsymbol{u} \cdot \boldsymbol{\nabla} \overline{H} - (\boldsymbol{u} \cdot \boldsymbol{\nabla} h)' - \boldsymbol{\nabla} \cdot \boldsymbol{u}
\end{aligned}$$

$(\cdot)'$ = extraction of fluctuating part

# TFM for isothermal compressible MHD

## Background turbulence

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\begin{aligned}
\mathcal{D}^A \boldsymbol{a}^{(0)} &= \overline{\boldsymbol{U}} \times \boldsymbol{b}^{(0)} + \left(\boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)}\right)' + \eta \boldsymbol{\nabla}^2 \boldsymbol{a}^{(0)} + \boldsymbol{f}_{\rm M} \\
\mathcal{D}^U \boldsymbol{u}^{(0)} &= -c_{\rm s}^2 \boldsymbol{\nabla} h^{(0)} + \left(\boldsymbol{j}^{(0)} \times \boldsymbol{b}^{(0)}\right)' / \rho_{\rm ref} \\
&\quad -\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} - \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{U}} - \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)}\right)' \\
&\quad +\nu \left(\boldsymbol{\nabla}^2 \boldsymbol{u}^{(0)} + \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)}/3\right) \\
&\quad +2\nu \left(\overline{\mathsf{S}} \cdot \nabla h^{(0)} + \mathbf{s}^{(\mathbf{0})} \cdot \nabla \overline{H} + \left(\mathbf{s}^{(\mathbf{0})} \cdot \nabla h^{(0)}\right)'\right) + \boldsymbol{f}_{\rm K} \\
\mathcal{D} h^{(0)} &= -\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} h^{(0)} - \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \overline{H} - \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(0)}\right)' - \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)}.
\end{aligned}
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\mathcal{D}^U \boldsymbol{u}^{(0)} &= -c_{\mathrm{s}}^2 \boldsymbol{\nabla} h^{(0)} + \left(\boldsymbol{j}^{(0)} \times \boldsymbol{b}^{(0)}\right)' / \rho_{\mathrm{ref}} \\
&\quad -\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} - \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{U}} - \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)}\right)' \\
&\quad +\nu \left(\boldsymbol{\nabla}^2 \boldsymbol{u}^{(0)} + \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)}/3\right) \\
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\mathcal{D} h^{(0)} &= -\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} h^{(0)} - \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \overline{H} - \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(0)}\right)' - \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)}.
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full nonlinear problem; $\overline{\boldsymbol{U}}$, $\boldsymbol{\nabla}\overline{H}$ – external parameters

# TFM for isothermal compressible MHD

## $\overline{\boldsymbol{B}}$ dependent problems

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$$\mathcal{D}^A \boldsymbol{a}^{(\boldsymbol{B})} - \overline{\boldsymbol{U}} \times \boldsymbol{b}^{(B)} - \left(\boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(B)}\right)' - \eta \boldsymbol{\nabla}^2 \boldsymbol{a}^{(\boldsymbol{B})}$$
$$= \boldsymbol{u} \times \overline{\boldsymbol{B}}$$

$$\mathcal{D}^U \boldsymbol{u}^{(B)} + c_{\mathrm{s}}^2 \boldsymbol{\nabla} h^{(\boldsymbol{B})} - \left(\boldsymbol{j}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(0)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(B)}\right)' / \rho_{\mathrm{ref}}$$
$$+ \overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{U}} + \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)}\right)'$$
$$- \nu \Big[ \boldsymbol{\nabla}^2 \boldsymbol{u}^{(B)} + \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(B)} / 3 + 2 \overline{\mathbf{S}} \cdot \nabla h^{(\boldsymbol{B})} + \mathbf{s}^{(\boldsymbol{B})} \cdot \nabla \overline{H}$$
$$+ 2 \left(\mathbf{s}^{(\mathbf{0})} \cdot \nabla h^{(\boldsymbol{B})} + \mathbf{s}^{(\boldsymbol{B})} \cdot \nabla h^{(0)} + \mathbf{s}^{(\boldsymbol{B})} \cdot \nabla h^{(\boldsymbol{B})}\right)' \Big]$$
$$= \left(\overline{\boldsymbol{J}} \times \boldsymbol{b} + \boldsymbol{j} \times \overline{\boldsymbol{B}}\right) / \rho_{\mathrm{ref}}$$

$$\mathcal{D} h^{(\boldsymbol{B})} + \overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \overline{H} + \left(\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h^{(0)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})}\right)'$$
$$= - \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(B)}$$

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## $\overline{\boldsymbol{B}}$ dependent problems

$$\mathcal{D}^A \boldsymbol{a}^{(\boldsymbol{B})} - \overline{\boldsymbol{U}} \times \boldsymbol{b}^{(B)} - \left( \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(B)} \right)' - \eta \boldsymbol{\nabla}^2 \boldsymbol{a}^{(\boldsymbol{B})}$$
$$= \boldsymbol{u} \times \overline{\boldsymbol{B}}$$

$$\mathcal{D}^U \boldsymbol{u}^{(B)} + c_\mathrm{s}^2 \boldsymbol{\nabla} h^{(\boldsymbol{B})} - \left( \boldsymbol{j}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(0)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(B)} \right)' / \rho_\mathrm{ref}$$
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$$- \nu \Big[ \boldsymbol{\nabla}^2 \boldsymbol{u}^{(B)} + \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(B)} / 3 + 2 \overline{\mathbf{S}} \cdot \nabla h^{(\boldsymbol{B})} + \mathbf{s}^{(\boldsymbol{B})} \cdot \nabla \overline{H}$$
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$$\mathcal{D} h^{(\boldsymbol{B})} + \overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \overline{H} + \left( \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h^{(0)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} \right)'$$
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  $\longrightarrow$ take over $\boldsymbol{u}$, $\boldsymbol{b}$, $h$ from the main run

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**reformulation into *test problems***

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  $\longrightarrow$ somehow "linearize" quadratic terms
- for nonlinearity (quenching): effect of $\overline{\boldsymbol{B}}$ in main run needed
  $\longrightarrow$ take over $\boldsymbol{u}, \boldsymbol{b}, h$ from the main run
- replace $\overline{\boldsymbol{B}}$ in turn by one of the four test fields

$$\overline{\boldsymbol{B}}^{(1)} = (\cos kz, 0, 0), \quad \overline{\boldsymbol{B}}^{(2)} = (\sin kz, 0, 0),$$
$$\overline{\boldsymbol{B}}^{(3)} = (0, \cos kz, 0), \quad \overline{\boldsymbol{B}}^{(4)} = (0, \sin kz, 0),$$

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## “Linearization” of nonlinear terms

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**"Linearization" of nonlinear terms**

e.g.

$$\boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)} + \underbrace{\boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(B)}}_{\text{nonlinear !}}$$

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formally linear when writing as

$$\boldsymbol{u} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)} \quad \text{or} \quad \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}$$

and identifying $\boldsymbol{u}, \boldsymbol{b}$ with fluctuations from main run

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and identifying $\boldsymbol{u}, \boldsymbol{b}$ with fluctuations from main run

likewise for

$$\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} + \underbrace{\boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)}}$$

$$\longrightarrow \quad \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u} \quad \text{or} \quad \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(B)} + \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)}$$

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5 nonlinear terms $\longrightarrow$ 32 variants of the test problems

# TFM for isothermal compressible MHD

**Algorithm (nlCOMP)**

along with main run

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1. solve test problems for test fields $\overline{\boldsymbol{B}}^{(i)}$, $i = 1, \ldots, 4$
2. form mean EMF, mean ponderomotive force and mean mass source for each test solution:

$$\overline{\mathcal{E}}^{(i)} = \overline{\boldsymbol{u} \times \boldsymbol{b}^{(B)} + \boldsymbol{u}^{(B)} \times \boldsymbol{b}^{(0)}} \qquad \text{(or variant)}$$

$$\overline{\mathcal{F}}^{(i)} = \overline{\boldsymbol{j} \times \boldsymbol{b}^{(B)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(0)}} + \ldots \qquad \text{–"–}$$

$$\overline{\mathcal{Q}}^{(i)} = \overline{-\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} - \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h} \qquad \text{–"–}$$

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$$\overline{\mathcal{Q}}^{(i)} = \overline{-\boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} h^{(\boldsymbol{B})} - \boldsymbol{u}^{(B)} \cdot \boldsymbol{\nabla} h} \qquad \text{–"–}$$

3. invert

$$\overline{\boldsymbol{\mathcal{E}}}^{(i)} = \boldsymbol{\alpha} \cdot \overline{\boldsymbol{B}}^{(i)} + \boldsymbol{\eta} \cdot \overline{\boldsymbol{J}}^{(i)}$$

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# TFM for isothermal compressible MHD

## Algorithm (nlCOMP)

along with main run

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2. form mean EMF, mean ponderomotive force and mean mass source for each test solution:

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$$\overline{\mathcal{F}}^{(i)} = \overline{\boldsymbol{j} \times \boldsymbol{b}^{(B)} + \boldsymbol{j}^{(B)} \times \boldsymbol{b}^{(0)}} + \ldots \qquad \text{–"–}$$

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for transport coefficient tensors $\boldsymbol{\alpha}, \boldsymbol{\eta}, \boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{\sigma}, \boldsymbol{\tau}$

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**Remarks**

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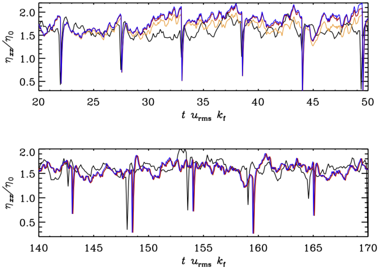


comparison of TFM variants

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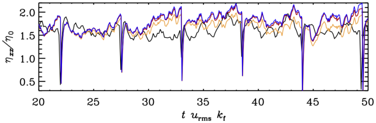


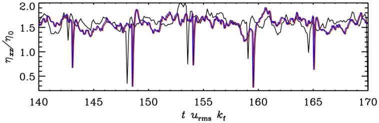


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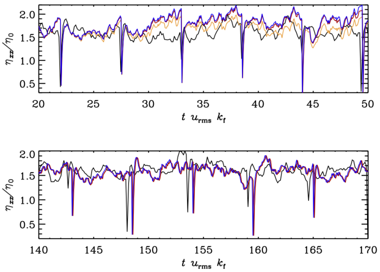


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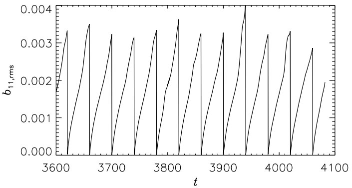


unstable test solution

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- in turbulence:
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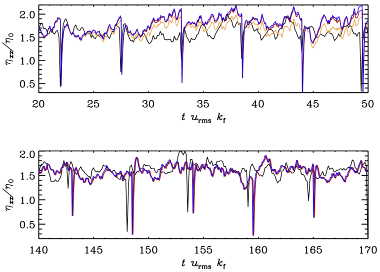
  in time and space

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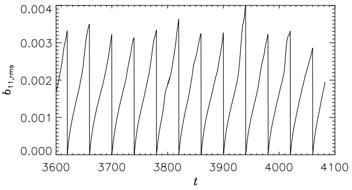


unstable test solution

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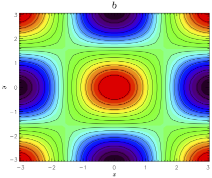

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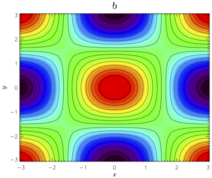
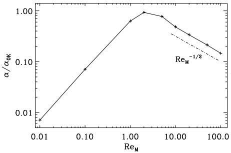
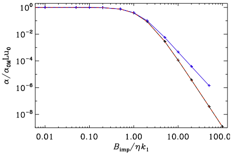



dependence on




$\mathrm{Re_M}$

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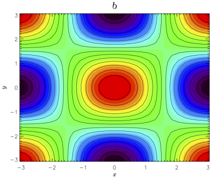


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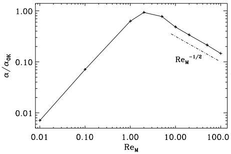


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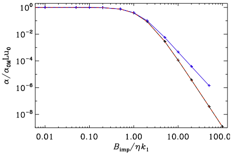


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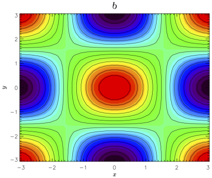


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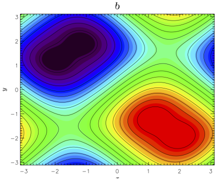


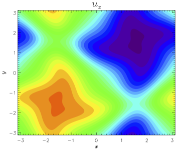


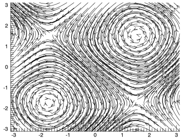

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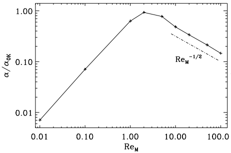


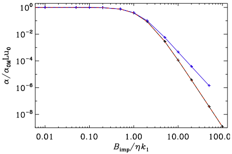


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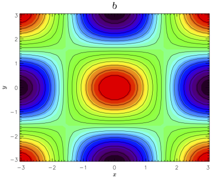


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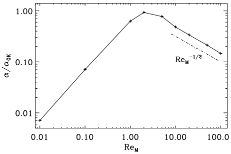


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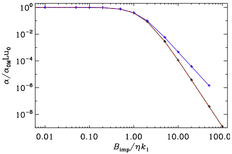


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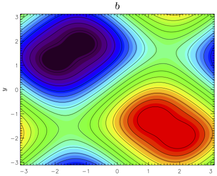


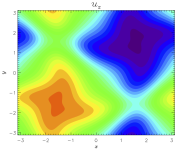


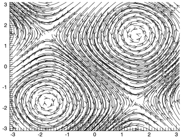

$\alpha_{xx} \neq \alpha_{yy}$ !