Proposal for the measurement of the time correlation of the helicity fluctuations in the shear dynamo simulations

Naveen Jingade
(with Nishant K. Singh)

Pencil-code meeting

17-21 May 2021

Problem

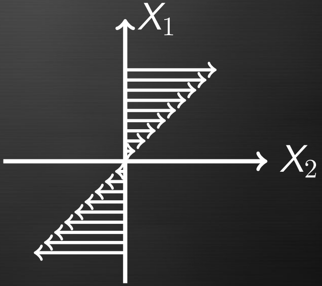
- In standard theory, Kinetic Helicity leads to Dynamo action. e.g. In the solar convection zone.

- Rotation + Convection $\Rightarrow \alpha-$ effect (generation of helicity)

- In Galaxy: Differentially rotating disc with turbulence powered by Supernovae explosions. α is subcritical to trigger the Dynamo.

- Question: Can non-helical turbulence in a differentially rotating disc lead to Dynamo Action.

The shear dynamo problem: the model system



Spacetime coordinates: $\boldsymbol{X} = (X_1, X_2, X_3)\,; \quad \text{time} = \tau$

Velocity field: $\boldsymbol{V} = S X_1 \boldsymbol{e}_2 + \boldsymbol{v}(\boldsymbol{X}, \tau)$

Fluctuations incompressible with zero mean: $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0\,; \quad \langle \boldsymbol{v} \rangle = \mathbf{0}$

Induction equation for the total magnetic field $\boldsymbol{B}'(\boldsymbol{X}, \tau)$:

$$\left(\frac{\partial}{\partial \tau} + S X_1 \frac{\partial}{\partial X_2}\right) \boldsymbol{B}' - S B_1' \boldsymbol{e}_2 = \boldsymbol{\nabla} \times \left(\boldsymbol{v} \times \boldsymbol{B}'\right) + \eta \boldsymbol{\nabla}^2 \boldsymbol{B}'$$

Navier-Stokes equation for the velocity field $\boldsymbol{v}(\boldsymbol{X}, \tau)$:

$$\left(\frac{\partial}{\partial t} + S x_1 \frac{\partial}{\partial x_2}\right) \boldsymbol{v} + S v_1 \boldsymbol{e}_2 + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} P + \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho} + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}$$

Turbulence is set up by a forcing function $\boldsymbol{f}$ in simulation which is taken to be *homogeneous*, *isotropic* and *delta–correlated–in–time*.

PENCIL-CODE: weakly compressible MHD code.

Motivation to perform simulation at low Re

(i) Dynamos due to non–helical *flows* ; Absence of alpha (at low Re ?)

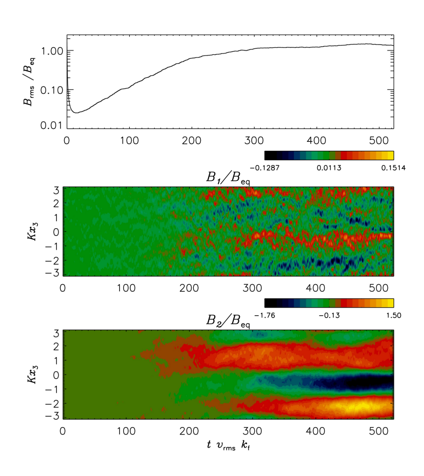
(ii) Authors[1] have rigorously proved that non–helical forcing gives rise to non–helical flows *in the limit of low fluid Reynolds number (*Re*)*

(iii) Low Re situation is analytically more tractable problem than high Re

(iv) Solution for Navier-Stokes equation can be obtained rigorously for low Re without the Lorentz force in it.

[1] N.K.Singh and S.Sridhar, Phys. Rev. E, **89**, 056309(2011)

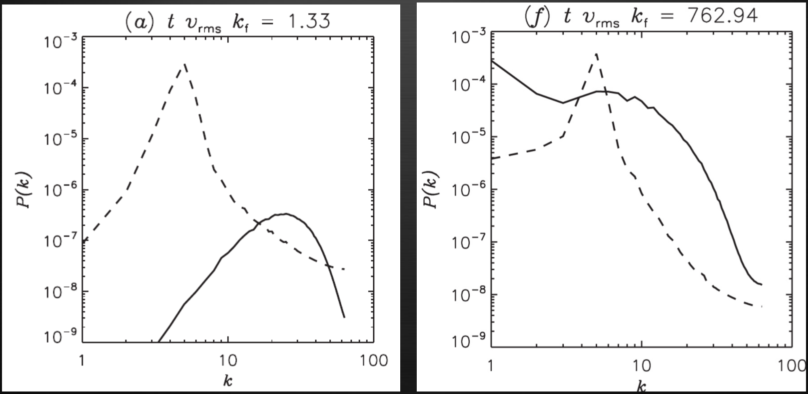
Dynamo action and Spacetime diagrams



$\mathrm{Re} \approx 0.641\,,\ \mathrm{Rm} \approx 32.039\,,\ \mathrm{Pr} \approx 50.0\,,\ k_f/K = 5.09\,,\ \mathrm{S_h} \approx -0.60$

Power spectrum at low Re

Nishant K. Singh & Naveen, Jingade, ApJ 2015, 806, 118



Dynamo is insensitive to the kinetic spectrum

Delta-correlated helicity fluctuations

Scale separation

$$\ell_0 \ll \ell_\alpha \ll L; \qquad \tau_0 \ll \tau_\alpha \ll T,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\alpha \, \boldsymbol{B}) + \eta_T \nabla^2 \boldsymbol{B}$$

- Helicity fluctuation scale assumed to be larger than velocity variation scale.
- Split the mean field as $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$
- Kraichnan (1976) derived the mean-field equation by averaging over the scale ℓ_α and took the delta correlated time limit.

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = (\eta_T - \alpha_{\mathrm{rms}}^2 \tau_\alpha) \nabla^2 \overline{\boldsymbol{B}} : \qquad \overline{\alpha(\boldsymbol{x}, t)} = 0$$

Growth if there is negative diffusion.

Single scale helical wave

Naveen Jingade & Nishant K.Singh MNRAS, 495, 4557, 2020

- Can we understand what is happening in simulation by considering some model?

$$\left(\frac{\partial}{\partial \tau} + S X_1 \frac{\partial}{\partial X_2}\right) \boldsymbol{v} + S v_1 \boldsymbol{e}_2 + \cancel{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}} = -\boldsymbol{\nabla} p + \nu \cancel{\nabla^2 \boldsymbol{v}}$$

- Solution: Shearing helical Waves

$$\boldsymbol{v}(\boldsymbol{X}, t) = \boldsymbol{A}(t, \boldsymbol{q}) \sin(\boldsymbol{Q}(t) \cdot \boldsymbol{X} + \psi) + h\, \mathbf{C}(\mathbf{t}, \mathbf{q}) \cos(\mathbf{Q}(\mathbf{t}) \cdot \mathbf{X} + \psi)$$

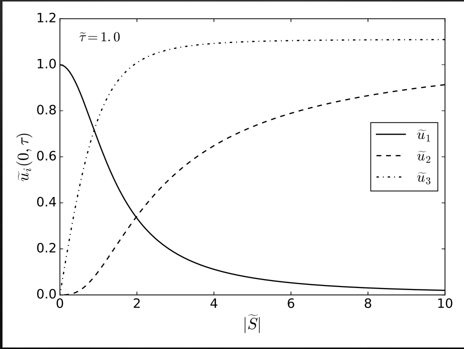
$$\boldsymbol{Q} = (q_1 - S\, q_2\, t, q_2, q_3)$$

$$\boldsymbol{Q}(t) \cdot \boldsymbol{A}(t) = 0; \quad \text{and} \quad \boldsymbol{Q}(t) \cdot \boldsymbol{C}(t) = 0;$$

$$\text{Helicity:} \quad H = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = h\, \boldsymbol{C}(t) \cdot (\boldsymbol{Q}(t) \times \boldsymbol{A}(t))$$

Shearing Amplitudes

$$A_1(\boldsymbol{q}, t) = \frac{q^2}{(q_1 - Stq_2)^2 + q_2^2 + q_3^2}\, a_1 \,, \quad \boldsymbol{A}(\boldsymbol{q}, 0) = \boldsymbol{a}$$



Renovating flows

- τ is renovation time
- Velocity field in different time interval are statistically independent realizations drawn from the PDF.

velocity ensemble construction

- ψ is averaged from $[0,2\pi]$
- $\boldsymbol{q}$ is averaged over the sphere of radius q
- initial amplitudes $(\boldsymbol{a}, \boldsymbol{c})$ are averaged over circle $\perp$ to $\boldsymbol{q}$.

Renovating flows

$$\boldsymbol{U}(\boldsymbol{X}, t) = SX_1\boldsymbol{e}_2 + \boldsymbol{u}(\boldsymbol{X}, t)$$

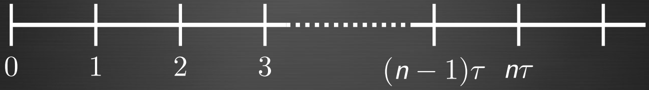
$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{V}$$

Cachy's Solution

$$B_i(\boldsymbol{X}, n\tau) = \frac{\partial X_i}{\partial X_{0j}} B_j(\boldsymbol{X}_0, (n-1)\tau)$$

where Lagrangian co-ordinate is given by

$$X_i = X_{0i} + \int_0^t V_i(\boldsymbol{X}_0, s)\,\mathrm{d}s$$



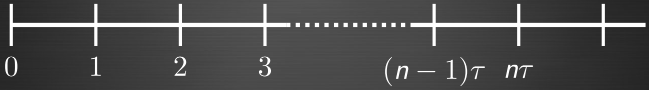
Fourier Transform of average field

$$\widetilde{B}_i(\boldsymbol{k}, t) = \int \langle B_i(\boldsymbol{x}, t) \rangle \exp(-i\boldsymbol{k} \cdot \boldsymbol{x})\, \mathrm{d}^3 x$$

$$B_i(\boldsymbol{k}, n\tau) = G_{ij}(\boldsymbol{k}) B_j(\boldsymbol{k}, (n-1)\tau)$$

$$B_i(\boldsymbol{k}, n\tau) = \sigma^n \times B_j(\boldsymbol{k}, 0)$$

$B_j(\boldsymbol{k}, 0)$ and σ are eigenvector and eigenvalue of G_{ij}



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$B_j(\boldsymbol{k}, 0)$ and σ are eigenvector and eigenvalue of G_{ij}

Dispersion relation: $\lambda = \frac{1}{\tau} \ln \sigma = \gamma + i\omega$ growth if $|\sigma| > 1$

$$\text{Growth rate} \quad \gamma = \frac{1}{\tau} \ln |\sigma|$$

$$\text{Cycle period} \quad P_{\mathrm{cyc}} = \frac{2\pi}{\omega}$$

Expression for response tensor

$$G_{ij} = \gamma_{ik}(\tau)\sigma_{kj}(\boldsymbol{k}) \qquad (\gamma_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ St & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

After averaging over ψ from 0 to 2π

$$\sigma_{kj}(\boldsymbol{k}) = \left\langle \delta_{kj} J_0(\Delta) - i\, h\, q_j \frac{[\boldsymbol{k}\times(\tilde{a}\times\tilde{c})]_k}{\Delta} J_1(\Delta) \right\rangle$$

where

$$\Delta = \sqrt{(\boldsymbol{k}\cdot\tilde{\boldsymbol{a}})^2 + h^2(\boldsymbol{k}\cdot\tilde{\boldsymbol{c}})^2}$$

Case 1 : $h = 0$, $\sigma < 1$ because $J_0(\Delta) < 1$

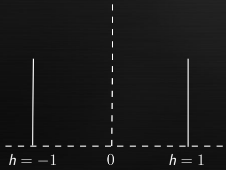
Case 2 : fluctuation in h is symmetric; $\sigma < 1$

Expression for response tensor

Case 3 : helicity correlation time $\geq 2\times$velocity correlation time;

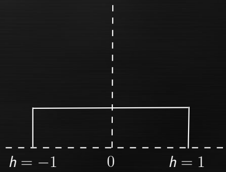
$$\hat{B}(\boldsymbol{k}, n\tau) = \underbrace{\left\langle \hat{G} \dots \hat{G} \right\rangle_h}_{m \ \text{times}} \hat{B}(\boldsymbol{k}, (n-m)\tau)$$

$\tau_h = m\tau$ where τ_h is renovation time of h



spiked

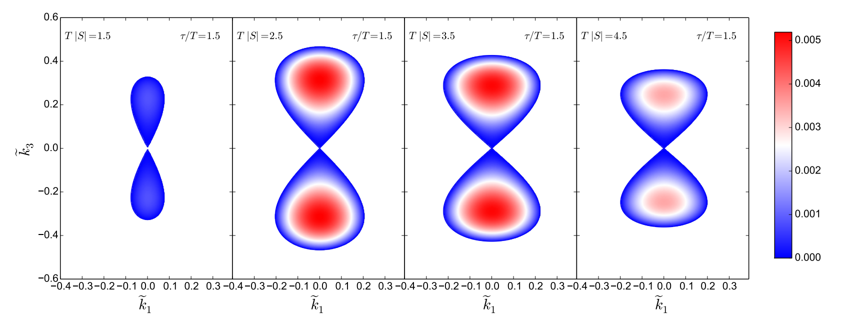
$\langle h^2 \rangle = 1$



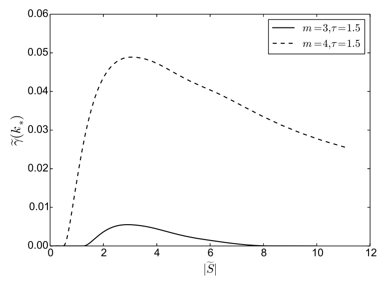
Uniform

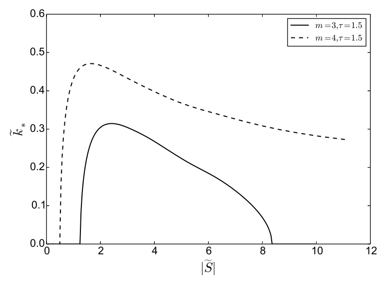
$\langle h^2 \rangle = 1/3$

Axisymmetric growing modes($k_2 = 0$), $m = 3$



Growth rates and maximum growing modes

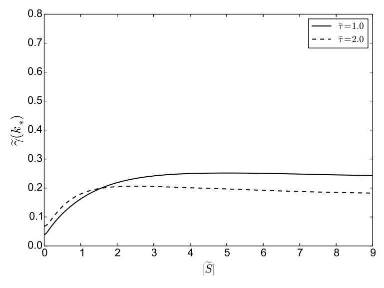


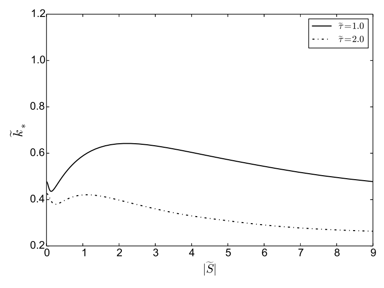


(Left) Maximum growing mode as a function of shear. (Right) Growth rates at maximum growing mode as function of shear.

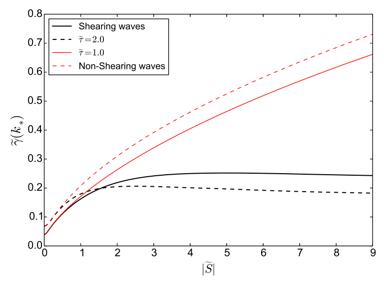
$$\widetilde{k}^* = \frac{k}{q} \quad \widetilde{|S|} = \frac{S}{q\,a}$$

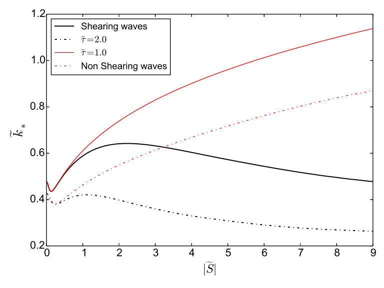
Contrast with fixed helicity $h = 1$





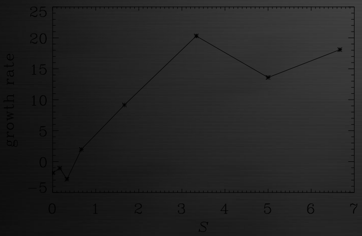
Contrast with fixed helicity $h = 1$

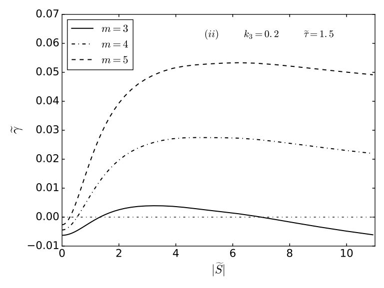




$$\boldsymbol{v}(\boldsymbol{X}, t) = \boldsymbol{a}\sin(\boldsymbol{q}\cdot\boldsymbol{x} + \psi) + h\,\mathbf{c}\cos(\boldsymbol{q}\cdot\mathbf{x} + \psi)$$

Comparison with simulation(fluctuating helicity)





(Left) Plot of growth rate versus shear take from Hughes & proctor (2009). (Right) Plot of growth rate as a function of shear strength for fixed value of wavenumber.

When can there be a scale separation between helicity and velocity fluctuations?

Pencil code

- Velocity correlator: $\langle \boldsymbol{v}(\boldsymbol{x},t)\cdot \boldsymbol{v}(\boldsymbol{x},t')\rangle = C_v(\boldsymbol{x}-\boldsymbol{x}', t-t')$
- Helicity correlator: $\langle H(\boldsymbol{x},t)H(\boldsymbol{x},t')\rangle = C_h(\boldsymbol{x}-\boldsymbol{x}', t-t')$

Calculations at low *re*

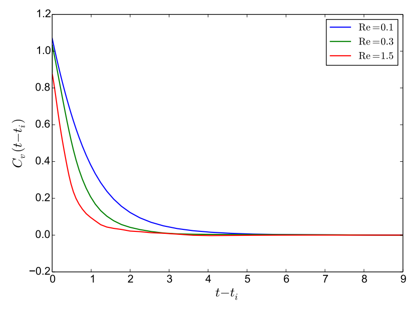
$$\left(\frac{\partial}{\partial t} + Sx\frac{\partial}{\partial y}\right)\boldsymbol{v} + Sv_1\boldsymbol{e}_2 + \cancel{(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}$$

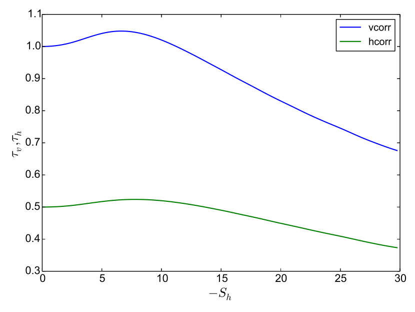
$\boldsymbol{f}$ is delta-correlated in time and forced at a single wavenumber K_F.

- Velocity correlator: $\boldsymbol{v}(\mathbf{0},t)\cdot\boldsymbol{v}(\mathbf{0},t') = C_v(t-t')$
- Helicity correlator: $H(\mathbf{0},t)H(\mathbf{0},t') = C_h(t-t')$

where $H = \boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})$

Correlation times





(Left) Velocity correlator as a function of time. (Right) Comparison of helicity and velocity correlation time.

Rotating turbulence

Dallas,V & Tobias, Steve 2018, *Rotationally induced coherence in turbulent kinematic dynamos*

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}$$

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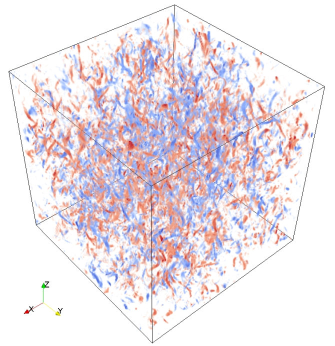
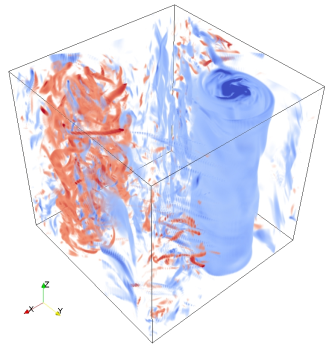
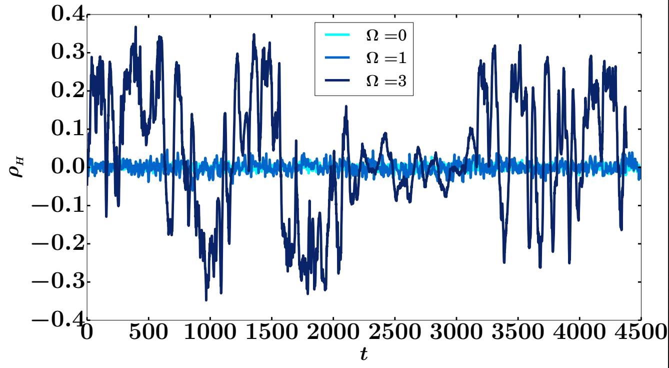
$\Omega = 0$



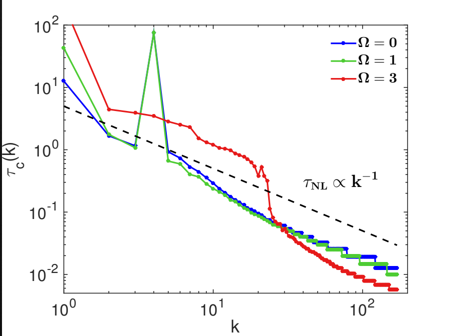
$\Omega = 3$



Average relative helicity

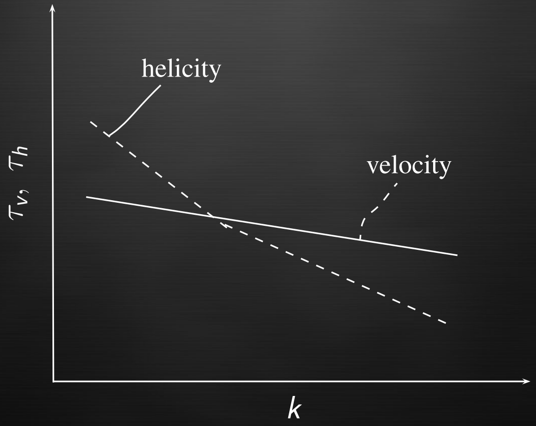


Scale dependent correlation time of velocity



$$R(k,\tau) = \frac{\langle \hat{\mathbf{u}}(\mathbf{k},t)\hat{\mathbf{u}}^*(\mathbf{k},t+\tau)\rangle}{\langle \hat{\mathbf{u}}(\mathbf{k},t)\hat{\mathbf{u}}^*(\mathbf{k},t)\rangle}$$

Proposal



Other origin for $\alpha-$fluctuations

THE ASTROPHYSICAL JOURNAL, 332:857–871, 1988 September 15

TURBULENT TRANSPORT OF MAGNETIC FIELDS. III. STOCHASTIC EXCITATION OF GLOBAL MAGNETIC MODES

P. HOYNG
Laboratory for Space Research, Beneluxlaan 21, 3527 HS Utrecht, The Netherlands
Received 1988 January 11; accepted 1988 March 2

INDUCTION EQUATION

SPACE OR TIME AVERAGE (TWO-SCALE METHOD)	ENSEMBLE AVERAGE
$\partial_t \langle \underline{B} \rangle = \underline{\underline{\mathcal{D}}} \langle \underline{B} \rangle + \underline{F}$	$\partial_t \langle \underline{B} \rangle = \underline{\underline{\mathcal{D}}} \langle \underline{B} \rangle$
<u>FLUCTUATIONS</u>: RANDOM FORCING TERM $\underline{F}$	<u>FLUCTUATIONS</u>: $\langle \underline{B} \rangle \to 0$ DUE TO PHASE MIXING