On the shear–current effect: toward understanding why theories and simulations have mutually and separately conflicted

Hongzhe Zhou (Nordita), Eric Blackman (UR)  arXiv:2104.11112

Large-scale dynamo (LSD): Amplification of magnetic energy beyond the energy-dominant scale of turbulence

Shear-current effect (SC/SCE): Non-helical turbulence + shear flow lead to LSD

 Might operate in weakly stratified flows, planetary cores (source of helicity unclear); may coexist with other dynamo drivers ($\Omega \times J$, incoherent $\alpha$, etc.)

Basic setup in all theories and most simulations:
Cartesian geometry; mean fields defined by $xy$-average; forced isotropic turbulence perturbed by a shear flow

Compare with: MRI turbulence

PCUM 2021
Shear-current effect:

Mean electromotive force $\varepsilon_i = \langle u \times b \rangle_i = \alpha_{ij}B_j - \beta_{ij}J_j$ with negative $\beta_{21}$

+ Background shear flow $U = -Sx\hat{y}$

Amplification of mean magnetic energy: large-scale dynamo

Main question:
What are the respective contributions to $\beta_{21}$ from turbulent velocity and magnetic fields?

$$\beta_{21} = \frac{\tau}{3} \beta^u_{21} \langle u^2 \rangle + \frac{\tau}{3} \beta^b_{21} \langle b^2 \rangle \propto \langle b^2 \rangle$$, magnetic SCE

$$\propto \langle u^2 \rangle$$, kinetic SCE
<table>
<thead>
<tr>
<th>Theory</th>
<th>Simulation</th>
<th>Kinetic/Magnetic SCE?</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>†Rogachevskii &amp; Kleedorin (2003)</td>
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<td>Y/</td>
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<td>†Rogachevskii &amp; Kleedorin (2004)</td>
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<td>Y/Y</td>
<td></td>
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<tr>
<td>Rädler &amp; Stepanov (2006)</td>
<td></td>
<td>N/</td>
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<td>Rüdiger &amp; Kitchatinov (2006)</td>
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<td>N/</td>
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<tr>
<td>†Pipin (2008)</td>
<td></td>
<td>N/</td>
<td>Re = $\mathcal{O}(1)$, $\text{Rm} \lesssim \mathcal{O}(100)$</td>
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<tr>
<td>Sridhar &amp; Subramanian (2009)</td>
<td></td>
<td>N/</td>
<td>Quasi-linear in shearing frame</td>
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<td>Squire &amp; Bhattacharjee (2015a)</td>
<td></td>
<td>N/Y</td>
<td></td>
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<tr>
<td>Squire &amp; Bhattacharjee (2015b)</td>
<td></td>
<td>N/Y</td>
<td>Re = $\text{Rm} \simeq 5$</td>
</tr>
<tr>
<td>Singh &amp; Jingade (2015)</td>
<td></td>
<td>N/</td>
<td>$\text{min} {\text{Re}, \text{Rm}} &lt; 1$</td>
</tr>
<tr>
<td>*Squire &amp; Bhattacharjee (2016)</td>
<td></td>
<td>N/Y</td>
<td>$8\text{Re} = \text{Rm} \lesssim 5 \langle u^2 \rangle / \langle B^2 \rangle$</td>
</tr>
<tr>
<td>Käpylä et al. (2020)</td>
<td></td>
<td>N/N</td>
<td>MHD burgulence, $\text{Re} &lt; 1$, $\text{Rm} &lt; 15$</td>
</tr>
</tbody>
</table>

†theory = using $\tau$ closure ($\sim$ applies for $\text{Re} & \text{Rm} \gg 1$)
otherwise using SOCA or quasi-linear ($\sim$ applies for $\text{Re} & \text{Rm} < 1$)

*simulation = using the projection method
otherwise using test field methods
<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>τ closure</td>
</tr>
<tr>
<td>Kinetic SC</td>
<td>Yes</td>
</tr>
<tr>
<td>Magnetic SC</td>
<td>Yes</td>
</tr>
</tbody>
</table>

(Wissing+2021: MRI with SPH, no SCE in ustr+nf, ustr+znf, or str+nf; contradicts Shi+2016)

Theories and simulations have mutually and separately conflicted
\[
\partial_t\langle u \rangle = \langle uu \rangle + \cdots \\
\partial_t\langle uu \rangle = \langle uuu \rangle + \cdots \\
\cdots
\]

**Second-order-correlation-approximation (SOCA):**
Drops nonlinear terms
\[ \rightarrow \partial_t\langle uu \rangle = \cdots + \nu \nabla^2 \langle uu \rangle \]
Justified at low Reynolds numbers or low Strouhal number (=correlation time/eddy turnover time)

**Spectral-τ or minimal-τ:**
Replaces order $\geq 3$ terms by a damping term
\[ \rightarrow \partial_t\langle uu \rangle = \cdots - \langle uu \rangle / \tau(k) \] (in Fourier space)
A closure at high Reynolds numbers
(hence small dissipation terms)
Collectively we can write \( \partial_t \langle ub \rangle = \text{linear terms} + ck^\lambda \langle ub \rangle \)

where \( ck^\lambda = \begin{cases} -\nu k^2 & \text{for SOCA} \\ -\tau k^{q-1} & \text{for spectral } \tau \end{cases} \)

**ZB21:** to the \( \mathcal{O}(\text{Sh}) \) order, \( \beta_{21}^u < 0 \) only if \( \lambda < 1 \), thus:

- In SOCA, \( \beta_{21}^u > 0 \) always \( \rightarrow \) **no kinetic SC**
- In spectral-\( \tau \), \( \beta_{21}^u < 0 \) if \( q < 2 \) \hspace{1cm} (Rogachevskii & Kleeorin 2003, 2004)

In both theories magnetic SC always works (\( \beta_{21}^b < 0 \)), regardless of the spectral index \( q \)

<table>
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<tr>
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<th>Test field</th>
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<tbody>
<tr>
<td>Kinetic SC</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
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<td>Magnetic SC</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
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</table>
Simulations not finding KSC: too low Re and Rm?

- Squire & Bhattacharjee (2015c)  N/Y  \( \text{Re} = \text{Rm} \approx 5 \)
- Squire & Bhattacharjee (2015b)  N/Y  \( \text{Re} = \text{Rm} \lesssim 15 \)
- Singh & Jingade (2015)  N/  \( \min \{\text{Re}, \text{Rm}\} < 1 \)
- *Squire & Bhattacharjee (2016)  N/Y  \( 8\text{Re} = \text{Rm} \lesssim 5 \langle u^2 \rangle / \langle B^2 \rangle \)
- Käpylä et al. (2020)  N/N  MHD burgulence, \( \text{Re} < 1, \text{Rm} < 15 \)

Kinematic test field method (no magnetic background)

→ evidence of transiting from positive to negative values (but with large error bars)

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<th>SOCA/Quasi-linear</th>
<th>Test field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinetic SC</td>
<td>Yes</td>
<td>No</td>
<td>No at small Re</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Yes at large Re?</td>
</tr>
<tr>
<td>Magnetic SC</td>
<td>Yes</td>
<td>Yes</td>
<td>No at Re,Rm&lt;15</td>
</tr>
</tbody>
</table>
Why no magnetic SC detected in test field methods even though both theories predicted it?

\[ \beta_{21} = \frac{\tau}{3} \beta^{u}_{21} \langle u^2 \rangle + \frac{\tau}{3} \beta^{b}_{21} \langle b^2 \rangle \]

Kinetic SC depends more sensitively on spectral index
→ for steep spectra (as at low Re & Rm),
\[ | \beta^{u}_{21} | \langle u^2 \rangle > | \beta^{b}_{21} | \langle b^2 \rangle \text{ even if } \langle u^2 \rangle \lesssim \langle b^2 \rangle \]
The full solution of $\beta_{ij}$ for the pure shear case (Cases of shear + rotation / pure rotation / burgulence in ZB21)

1. Comparable $\beta_{12}$ and $\beta_{21} \rightarrow$ may invalidate setting $\beta_{12} = 0$ in the projection method
2. No dynamo cycle period $\rightarrow$ SCE at best subdominant in simulations, unless $\beta_{ij}$ itself is periodic (requires nonlinear theory)
Subtlety 1: The spectral index

The energy-dominant modes are $k = 4, 5, 6$, but:

1. $q(5, 6)$ very large: a sudden steep slope because of forcing
   → destructive to SCE even if $q < 2$ inside the inertial range

2. $q(4, 5) < 0$: positive slope at $k < k_f$
   → constructive to SCE, but not considered in theories

The combined effect is...?
Subtlety 2: The boundary condition

Theories: normally period boundary conditions
\[ f(x + L, y) = f(x, y) \]

Shearing box simulations: shear periodic boundary condition
\[ f(x + L, y - StL) = f(x, y) \]

In all theories, \( \hat{P}_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \) is found crucial to produce SCE, but does using different b.c. change the solution because of \( \partial^{-2} \)?

A possible solution: shearing coordinates (Sridhar, Subramanian, Singh ...), but
1. no calculation for magnetic SCE yet
2. How to incorporate a closure at large Re & Rm?
Summary:
Assuming using normally or shear periodic b.c. does not qualitatively change the story:

**Different powers of $k$ in the closure term;**
Need large enough Re & Rm, and shallow enough spectrum

<table>
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<th>Projection</th>
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<tr>
<td>Kinetic SC</td>
<td>Yes</td>
<td>No</td>
<td>No at Rm&lt;~15</td>
<td>No at Rm&lt;~15</td>
</tr>
<tr>
<td>Magnetic SC</td>
<td>Yes</td>
<td>Yes</td>
<td>No at Rm&lt;~15</td>
<td>Yes(?) at Rm&lt;~15</td>
</tr>
</tbody>
</table>

Too steep spectrum of $u^2$?

Some unresolved issues:
1. More careful justification of the slope dependence
2. Any influence from different boundary conditions?
3. Physical picture compatible the slope dependence
\[
q_s(k) = -\frac{k}{E(k)} \frac{E(k+1) - E(k)}{(k+1) - k}
\]

\[
\overline{q}_s(k) = \frac{\sum_{k \geq k_f} q_s(k)E(k)}{\sum_{k \geq k_f} E(k)}
\]

\((\Delta k = 1)\)

For all \(Pm\), fitted critical exponent is \(\{x \rightarrow 3.15362\}\)

\begin{center}
\begin{tabular}{ c c c }
Stat & P-Value \\
\hline
Pearson Correlation & 0.70625 & 0.000500877 \\
\end{tabular}
\end{center}

For \(Pm=1\), fitted critical exponent is \(\{x \rightarrow 3.30541\}\)

\begin{center}
\begin{tabular}{ c c c }
Stat & P-Value \\
\hline
Pearson Correlation & 0.925254 & 0.000986349 \\
\end{tabular}
\end{center}

For \(Pm=0.2\), fitted critical exponent is \(\{x \rightarrow 2.29412\}\)

\begin{center}
\begin{tabular}{ c c c }
Stat & P-Value \\
\hline
Pearson Correlation & 0.844132 & 0.0345487 \\
\end{tabular}
\end{center}

For \(Pm=5\), fitted critical exponent is \(\{x \rightarrow 3.46678\}\)

\begin{center}
\begin{tabular}{ c c c }
Stat & P-Value \\
\hline
Pearson Correlation & 0.773489 & 0.07115 \\
\end{tabular}
\end{center}
Subtlety: Role of the pressure gradient term

\[ \partial_t \mathbf{u} = - \nabla p + \text{other terms} \]

If \( \mathbf{u} \) is incompressible, then \( p \) is not dynamical but instead determined by other fields and boundary conditions:

\[ \partial_t \mathbf{u} = \hat{\mathbf{P}}(\text{other terms}), \quad \hat{P}_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \]

Theory: \( \hat{\mathbf{P}} \) is necessary for both kinetic and magnetic SCEs (ZB21, Squire & Bhattacharjee 2016)

Simulation: not the case? (Käpylä+2020)
Käpylä+2020:

<table>
<thead>
<tr>
<th>Run</th>
<th>$\text{Re}_M$</th>
<th>$\lambda/\eta_0 k_f^2$</th>
<th>$\eta_{xx}/\eta_0$</th>
<th>$\eta_{yy}/\eta_0$</th>
<th>$\eta_{yx}/\eta_0$</th>
</tr>
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<tr>
<td>FK1a</td>
<td>2.1</td>
<td>-0.0354</td>
<td>0.557 ± 0.006</td>
<td>0.547 ± 0.007</td>
<td>0.048 ± 0.001</td>
</tr>
<tr>
<td>FK1b</td>
<td>11.9</td>
<td>0.0140</td>
<td>0.608 ± 0.015</td>
<td>0.598 ± 0.014</td>
<td>0.023 ± 0.001</td>
</tr>
<tr>
<td>FK8a</td>
<td>2.1</td>
<td>-0.0008</td>
<td>0.572 ± 0.010</td>
<td>0.563 ± 0.011</td>
<td>0.044 ± 0.002</td>
</tr>
<tr>
<td>FK8b</td>
<td>12.7</td>
<td>0.0166</td>
<td>0.641 ± 0.019</td>
<td>0.634 ± 0.017</td>
<td>0.023 ± 0.001</td>
</tr>
<tr>
<td>SK1a</td>
<td>2.0</td>
<td>0.0006</td>
<td>0.367 ± 0.001</td>
<td>0.393 ± 0.002</td>
<td>-0.003 ± 0.000</td>
</tr>
<tr>
<td>SK1b</td>
<td>12.3</td>
<td>0.0183</td>
<td>0.440 ± 0.004</td>
<td>0.412 ± 0.001</td>
<td>-0.011 ± 0.002</td>
</tr>
<tr>
<td>SK4a</td>
<td>2.1</td>
<td>-0.0042</td>
<td>0.367 ± 0.003</td>
<td>0.390 ± 0.003</td>
<td>-0.004 ± 0.000</td>
</tr>
<tr>
<td>SK4b</td>
<td>13.3</td>
<td>0.0185</td>
<td>0.334 ± 0.037</td>
<td>0.339 ± 0.044</td>
<td>-0.004 ± 0.005</td>
</tr>
<tr>
<td>SK8a</td>
<td>2.1</td>
<td>0.0033</td>
<td>0.367 ± 0.003</td>
<td>0.390 ± 0.004</td>
<td>-0.003 ± 0.000</td>
</tr>
<tr>
<td>SK8b</td>
<td>12.8</td>
<td>0.0192</td>
<td>0.401 ± 0.005</td>
<td>0.424 ± 0.005</td>
<td>-0.015 ± 0.000</td>
</tr>
<tr>
<td>SKM1a</td>
<td>1.9</td>
<td>...</td>
<td>1.794 ± 0.039</td>
<td>1.278 ± 0.045</td>
<td>0.200 ± 0.025</td>
</tr>
<tr>
<td>SKM4a</td>
<td>2.1</td>
<td>...</td>
<td>2.012 ± 0.179</td>
<td>1.191 ± 0.014</td>
<td>0.221 ± 0.012</td>
</tr>
<tr>
<td>SKM8a</td>
<td>1.8</td>
<td>...</td>
<td>3.054 ± 0.625</td>
<td>1.481 ± 0.131</td>
<td>0.338 ± 0.064</td>
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<tr>
<td>SKM16a</td>
<td>2.0</td>
<td>...</td>
<td>2.238 ± 0.552</td>
<td>1.215 ± 0.010</td>
<td>0.249 ± 0.062</td>
</tr>
<tr>
<td>SKM1ad</td>
<td>2.1</td>
<td>0.0103</td>
<td>1.228 ± 0.214</td>
<td>1.326 ± 0.074</td>
<td>0.247 ± 0.043</td>
</tr>
<tr>
<td>SKM4ad</td>
<td>1.9</td>
<td>0.0315</td>
<td>1.279 ± 0.150</td>
<td>1.455 ± 0.066</td>
<td>0.222 ± 0.022</td>
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<td>SKM8ad</td>
<td>1.5</td>
<td>0.0948</td>
<td>1.688 ± 0.165</td>
<td>2.040 ± 0.150</td>
<td>0.516 ± 0.061</td>
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<tr>
<td>SKM16ad</td>
<td>1.9</td>
<td>0.0344</td>
<td>1.231 ± 0.070</td>
<td>1.589 ± 0.019</td>
<td>0.364 ± 0.116</td>
</tr>
</tbody>
</table>

MHD burgulence:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u}$$

But...

$$\mathbf{J} \times \mathbf{B} = -\nabla \frac{\mathbf{B}^2}{2} + \mathbf{B} \cdot \nabla \mathbf{B}$$