# Studying Gravitational Waves with the PENCIL CODE

PhD student: Yutong He
Supervisor: Axel Brandenburg

17/05/2021

Nordita, KTH Royal Institute of Technology and Stockholm University

# PENCIL CODE as a GW solver

PENCIL CODE special modules for GWs:

- *gravitational_waves.f90*;
- *gravitational_waves_hij6.f90*;
- ✓ *gravitational_waves_hTXk.f90*.

Coupled to HD/MHD source module:

- *hydro.f90*
- *hydro_kinematic.f90*
- *magnetic.f90*
- *magnetic/maxwell.f90*
- ...

or another special modules (?)

# Studying GWs using the Pencil Code

- *The timestep constraint in solving the gravitational wave equations sourced by hydromagnetic turbulence* [Roper Pol, Brandenburg, et al. 2020]
- *Numerical simulations of gravitational waves from early-universe turbulence* [Roper Pol, Mandal, et al. 2020]
- *Circular polarization of gravitational waves from early-Universe helical turbulence* [Kahniashvili et al. 2021]
- *Relic gravitational waves from the chiral magnetic effect* [Brandenburg, He, et al. 2021]
- *Can we observe the QCD phase transition-generated gravitational waves through pulsar timing arrays?* [Brandenburg, Clarke, et al. 2021]
- *Spectrum of turbulence-sourced gravitational waves as a constraint on graviton mass* [He, Brandenburg, and Sinha 2021]
- *The scalar, vector, and tensor modes in gravitational wave turbulence simulations* [Brandenburg, Gogoberidze, et al. 2021]

# The gravitational wave (GW) equation

GW equation in terms of comoving $\boldsymbol{x}$, conformal $t$ and scaled source $T_{ij}$ in real space is[1]

$$(\partial_t^2 - \boldsymbol{\nabla}^2) h_{ij}(\boldsymbol{x}, t) = \mathcal{G}(t) T_{ij}(\boldsymbol{x}, t), \tag{1}$$

where $\mathcal{G}(t) = 6/t$, and in $\boldsymbol{k}$ space is

$$\ddot{\tilde{h}}_{ij}(\boldsymbol{k}, t) + \boldsymbol{k}^2 \tilde{h}_{ij}(\boldsymbol{k}, t) = \mathcal{G}(t) \tilde{T}_{ij}(\boldsymbol{k}, t). \tag{2}$$

Can evolve the wave equation as

$$\begin{pmatrix} \boldsymbol{k}\tilde{h} - \boldsymbol{k}^{-1}\mathcal{G}\tilde{T} \\ \dot{\tilde{h}} \end{pmatrix}_{+,\times}^{t+\delta t} = \begin{pmatrix} \cos \boldsymbol{k}\delta t & \sin \boldsymbol{k}\delta t \\ -\sin \boldsymbol{k}\delta t & \cos \boldsymbol{k}\delta t \end{pmatrix} \begin{pmatrix} \boldsymbol{k}\tilde{h} - \boldsymbol{k}^{-1}\mathcal{G}\tilde{T} \\ \dot{\tilde{h}} \end{pmatrix}_{+,\times}^{t}. \tag{3}$$

[1] Roper Pol, Brandenburg, et al. 2020.

# Turbulent sources

- Why?

  → Phase transitions during the early radiation era ⇒ inevitable turbulence.

- How?

  → Construct stress-energy tensor $\tilde{T}_{ij}$ for GWs

$$\tilde{T}_{ij} = \frac{4}{3}\gamma^2 \rho u_i u_j - B_i B_j + \cdots \tag{4}$$

  from turbulent $\boldsymbol{u}$, $\boldsymbol{B}$ fields.

# Turbulent sources: an example

The continuity, velocity and magnetic induction equations:

$$\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3}(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho) + \frac{1}{\rho}[\boldsymbol{u} \cdot (\boldsymbol{J} \cdot \boldsymbol{B}) + \eta \boldsymbol{J}^2] \tag{5}$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\boldsymbol{u}}{3}(\nabla \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \ln \rho) - \frac{\boldsymbol{u}}{\rho}[\boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) + \eta \boldsymbol{J}^2] - \frac{1}{4}\nabla \ln \rho + \frac{3}{4\rho}\boldsymbol{J} \times \boldsymbol{B} + \frac{2}{\rho}\nabla \cdot (\rho \nu \mathsf{S}) \tag{6}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{J} + \boldsymbol{\mathcal{F}}), \tag{7}$$

where $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ and $D/Dt = \partial/\partial t + \boldsymbol{u} \cdot \nabla$. The rate-of-strain tensor elements are $\mathsf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}u_{k,k}$. Also, relativistic EOS $p = \rho/3$ has been used.

# Turbulent sources: an example

The forcing to model generation of $\boldsymbol{B}$:

$$\mathcal{F}(\boldsymbol{x}, t) = \mathrm{Re}[\mathcal{N}\tilde{\boldsymbol{f}}(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{x} + i\varphi)]. \tag{8}$$

Each time step $t$:

- Random $\boldsymbol{k}$ and $\varphi$.
- $\tilde{\boldsymbol{f}}(\boldsymbol{k}) = (\boldsymbol{k} \times \boldsymbol{e})/\sqrt{\boldsymbol{k}^2 - (\boldsymbol{k}\cdot\boldsymbol{e})^2}$, where $\boldsymbol{e} \not\parallel \boldsymbol{k}$.
- Normalisation $\mathcal{N} = f_0/\delta t^{1/2}$, with $f_0$ varied by hand to achieve approximately the same $\mathcal{E}_{\mathrm{B}}^{\mathrm{max}} = \langle \boldsymbol{B}^2 \rangle/2$ for all runs (later).

# An application: massive GWs

- Modified action[2]

$$S = \int d^4x(\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{m_{\mathrm{g}}} + \mathcal{L}_{\mathrm{mat}}), \text{ where } \mathcal{L}_{m_{\mathrm{g}}} = \frac{1}{4}m_{\mathrm{g}}^2\sqrt{-g}\left(h_{\mu\nu}^2 - \frac{1}{2}h^2\right). \tag{9}$$

- Modified GW equation in $k$ space

$$\ddot{\tilde{h}}_{+,\times}(\boldsymbol{k}, t) + (\boldsymbol{k}^2 + m_{\mathrm{g}}^2)\tilde{h}_{+,\times}(\boldsymbol{k}, t) = \mathcal{G}(t)\tilde{T}_{+,\times}(\boldsymbol{k}, t). \tag{10}$$

- Nonlinear dispersion:

$$\omega = \sqrt{\boldsymbol{k}^2 + \omega_{\mathrm{cut}}^2}, \tag{11}$$

where $\omega_{\mathrm{cut}} = m_{\mathrm{g}}c^2/\hbar$ is an effective graviton mass term[3].

[2] Fierz and Pauli 1939.
[3] Lee 2013.

# An application: massive GWs

Spatial spectrum:

- $0.3 \le \omega_{\rm cut} \le 10$;
- $2.2 \times 10^{-23}{\rm eV} \lesssim m_{\rm g} \lesssim 7.4 \times 10^{-22}{\rm eV}$;
- $\Omega_{\rm GW} \sim k^1 \Rightarrow k^3$;
- $h_c \sim k^{-1/2} \Rightarrow k^{3/2}$;
- Consistent change for different $k_f$.

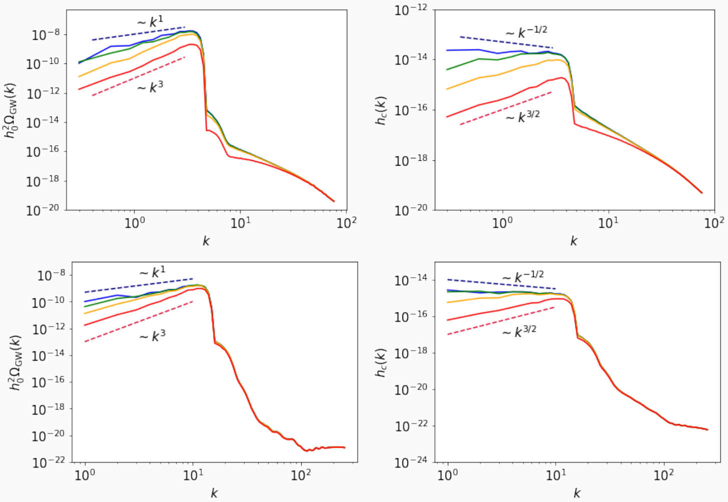


**Figure 1:** Spatial energy & strain spectra. $k_f = 2$ (up) & $k_f = 6$ (down).

# An application: massive GWs

Temporal spectrum:

- $\omega_{\mathrm{cut}} \in \{0, 1\}$,
- $m_g \approx 7.4 \times 10^{-23}\mathrm{eV}$;
- Clear cut-off;
- $f \sim 10\mathrm{nHz}$;
- $\Delta t_{\mathrm{phys}} \sim \mathrm{yr}$.

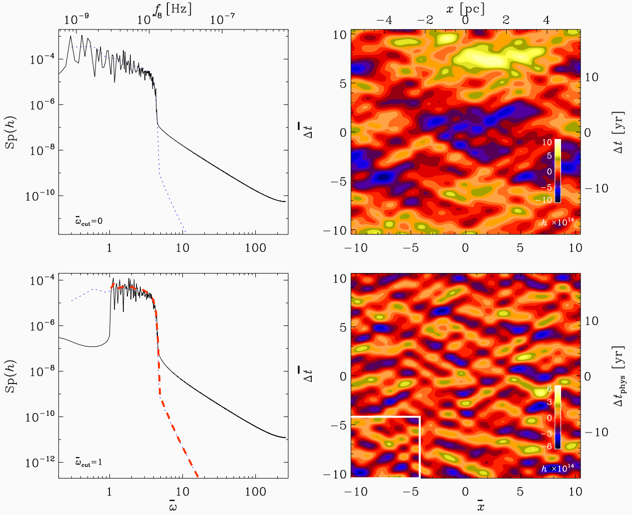


Figure 2: Temporal strain spectra. $\omega_{\mathrm{cut}} = 0$ (up) and $\omega_{\mathrm{cut}} = 1$ (down).

# Conclusions

Pencil Code for GW spectral analysis:

- Turbulent source: kinetic, (electro)magnetic...
- Early universe: EWPT, QCDPT...
- Modified gravity: massive gravity and more.

Questions?