Numerical Viscosity and Diffusion in Finite Difference Eulerian Codes

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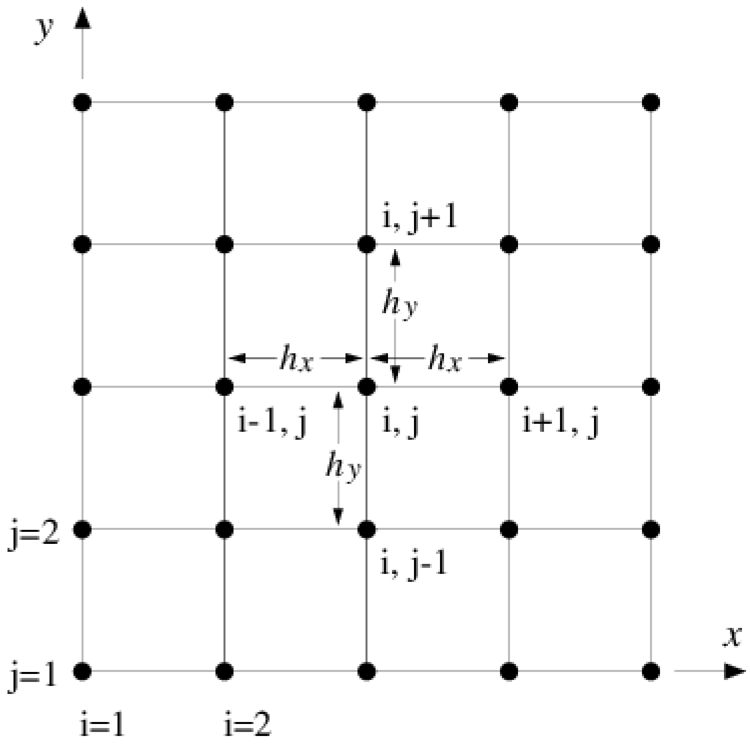

What is Numerical Diffusion?

Everyone is talking about it,
but no one knows what is really is.

Numerical Experiments

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On the Measurements of Numerical Viscosity and Resistivity in Eulerian MHD Codes

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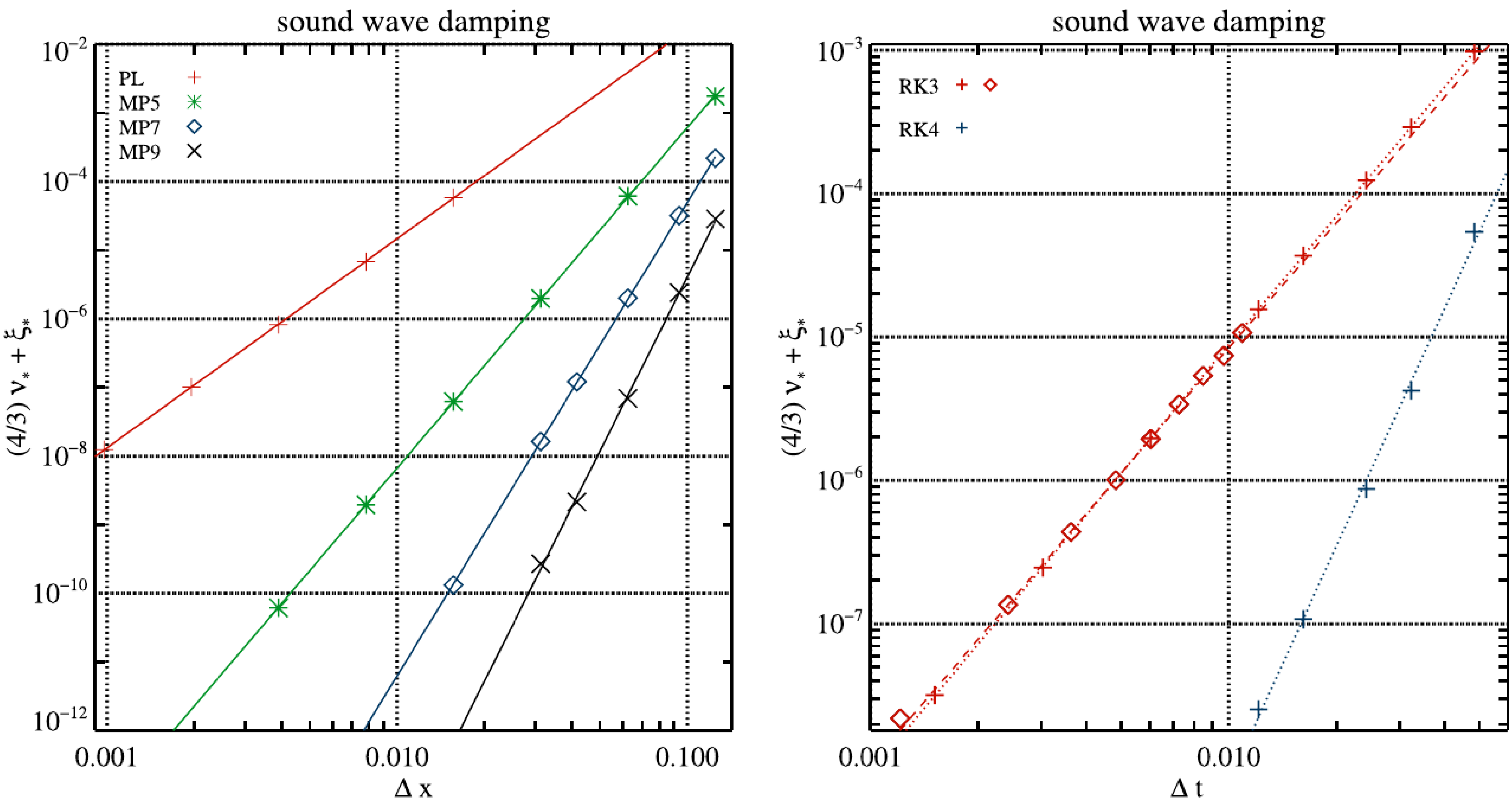
Table 1
Wave Damping Simulations I

Series	Wave	Reco	Riemann	Time	CFL	Resolution	$\mathfrak{N}^{\Delta x}_{\rm tot}$	r	$\mathfrak{N}^{\Delta t}_{\rm tot}$	q
#S1	sound	PL	HLL	RK4	0.01	64...1028	14.3 ± 0.7	3.049 ± 0.009	...	...
#S2	sound	MP5	LF	RK4	0.01	8...256	42.9 ± 2.3	4.957 ± 0.013	...	...
#S3	sound	MP5	HLL	RK4	0.01	8...256	43.4 ± 2.5	4.961 ± 0.014	...	...
#S4	sound	MP5	HLLD	RK4	0.01	8...256	42.7 ± 2.2	4.956 ± 0.013	...	...
#S5	sound	MP7	HLL	RK4	0.01	8...64	302 ± 20	6.897 ± 0.021	...	...
#S6	sound	MP9	HLL	RK4	0.01	8...32	830 ± 340	8.42 ± 0.15	...	...
#S7	sound	MP9	HLL	RK3	0.5	8...256	...	...	1.492 ± 0.013	2.985 ± 0.002
#S8	sound	MP9	HLL	RK3	0.1...0.9	64	...	...	2.45 ± 0.17	2.95 ± 0.01
#S9	sound	MP9	HLL	RK4	0.5	8...32	...	...	71 ± 32	5.5 ± 0.2
#A1	Alfvén	MP5	LF	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	...	...
#A2	Alfvén	MP5	HLL	RK4	0.01	8...256	42.6 ± 2.1	4.96 ± 0.01	...	...
#A3	Alfvén	MP5	HLLD	RK4	0.01	8...256	42 ± 3	4.95 ± 0.02	...	...
#A4	Alfvén	MP7	HLL	RK4	0.01	8 ...128	44 ± 53	6.19 ± 0.03	...	...
#A5	Alfvén	MP9	HLL	RK4	0.01	8...64	1190 ± 190	8.57 ± 0.06	...	...
#A6	Alfvén	MP9	HLL	RK3	0.8	16...128	...	...	0.86 ± 0.08	2.949 ± 0.022
#A7	Alfvén	MP9	HLL	RK4	0.8	8...64	...	...	7.6 ± 2.5	5.18 ± 0.10
#A8	Alfvén	MP5	HLL	RK3	0.5	5...1024	...	...	...	...
#MS1	magnetosonic	MP5	HLL	RK4	0.01	8...128	40 ± 3	4.95 ± 0.02	...	...
#MS2	magnetosonic	MP7	HLL	RK4	0.01	8...64	288 ± 20	6.903 ± 0.023	...	...
#MS3	magnetosonic	MP9	HLL	RK4	0.01	8...32	1970 ± 160	8.82 ± 0.03	...	...
#MS4	magnetosonic	MP9	HLL	RK3	0.1...0.9	64	...	...	1.77 ± 0.06	2.977 ± 0.007
#MS5	magnetosonic	MP9	HLL	RK4	0.2...0.9	64	...	...	4.3 ± 0.8	4.834 ± 0.013

Wave Damping



Analytical Approach

Numerical Methods

University of Glasgow

VIA VERITAS VITA

Radostin Simitev

July 23, 2019

Local Truncation Error

discretized exact $\mathcal{L}[\hat{u}] = 0$

Definition 7.4. *The quantity*

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = \mathcal{A}_{(h)}[\hat{u}] = A_{(h)}\hat{u} - F_{(h)}.$$

is called the local truncation error (local residual) of the numerical scheme $\mathcal{A}_{(h)}[] = 0$.

Example 7.5. *Find the local truncation error of the numerical scheme*

$$\mathcal{A}_{(h)}[u] = \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} - f_k = 0$$

for the solution of

$$u'' - f = 0.$$

This is solved exactly.

Solution. Now

$$\mathcal{A}_{(h)}[\hat{u}] = \frac{\hat{u}_{k-1} - 2\hat{u}_k + \hat{u}_{k+1}}{h^2} - f_k = (\hat{u}_k'' + O(h^2)) - f_k,$$

but

$$\mathcal{L}[\hat{u}] = \hat{u}_k'' - f_k = 0,$$

so using the definition directly

$$\tau_{(h)} = \mathcal{A}_{(h)}[\hat{u}] - \mathcal{L}[\hat{u}] = O(h^2).$$

Numerical Diffusion

PDEs: $\mathcal{L}[\hat{u}] = 0$

In the PencilCode do we have $\mathcal{A}_{(h)}[\hat{u}] = c\partial_{xx}\hat{u} + \ldots$?

What is c ?

Approach:
1. Discretize PDEs.
2. Apply method of lines to get set of coupled ODEs.
3. Construct the Runge-Kutta intermediate steps.
4. Eliminate off-center values using the Taylor expansion.
5. Eliminate intermediate time steps using time Taylor expansion.

$$f_{i\pm 1} = f_i \pm \mathrm{d}x f_i' + \frac{\mathrm{d}x^2}{2} f_i'' \pm \frac{\mathrm{d}x^3}{6} f_i''' + \ldots$$

$$\frac{f_{i+1} - f_{i-1}}{2\mathrm{d}x} = f_i' + \frac{\mathrm{d}x^2}{6} f_i''' + \ldots$$

Inviscid Navier-Stokes 3d

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - c_{\mathrm{s}}^2 \nabla \ln(\rho)$$

second order space
second order Runge-Kutta

$$\frac{\partial \ln(\rho)}{\partial t} = -\mathbf{u} \cdot \nabla \ln(\rho) - \nabla \cdot \mathbf{u}$$

Truncation errors with $\partial_{xx} u_x$:

$$-\frac{c_{\mathrm{s}}^2 \mathrm{d}t^2 \mathrm{d}x^2 \ln(\rho)_{xxx} u_x}{24} - \frac{c_{\mathrm{s}}^2 \mathrm{d}t^2 \mathrm{d}x^2 \ln(\rho)_x u_{x,xx}}{8} + \ldots$$

Similar for $\partial_{yy} u_x$ and $\partial_{zz} u_x$.

Proper diffusion terms: $\partial_t u_x = \partial_{xx} u_x + \partial_{yy} u_x + \partial_{zz} u_x$

Conclusions

- Numerical viscosity and diffusion can be calculated analytically.
- Need to find proper interpretation of the terms.
- Next: higher order space and time discretization, MHD.

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