Learning for Adaptive and Reactive Robot Control
Instructions for exercises of lecture 7

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Introduction

INTRO

This part of the course follows *exercises* 9.1 *to* 9.3 and *programming exercise* 9.1 of the book "*Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach. MIT Press, 2022*".

1 Theoretical exercises [1h]

Consider the nominal DS $\mathbf{f}(x) = Ax + b$ with $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $b = [-1, 1]^T$. We then construct a modulation to add a circular obstacle as follows:

$$\dot{x} = \mathbf{M}(x)\mathbf{f}(x) \quad \text{with} \quad \mathbf{M}(x) = \mathbf{E}(x)\mathbf{D}(x)\mathbf{E}(x)^{-1}, \tag{1}$$

where $M(x)$ is build through eigenvalue decomposition, using as the basis of the eigenvectors the normal $n(x)$ and tangents to the obstacle :

$$\mathbf{E}(x) = \begin{bmatrix} \mathbf{n}(x) & \mathbf{e}_1(x) \end{bmatrix}, \qquad \mathbf{D}(x) = \begin{bmatrix} \lambda_n(x) & 0 \\ 0 & \lambda_e(x) \end{bmatrix},$$

where the tangent $\mathbf{e}(x)$ forms an orthonormal basis to the gradient of the distance function $d\Gamma(x)/dx$.

We then set the eigenvalues to cancel the flow in the normal direction once it reaches the boundary of the obstacle:

$$\lambda_n(x) = 1 - \frac{1}{\Gamma(x)} \quad \text{and} \quad \lambda_e(x) = 1 + \frac{1}{\Gamma(x)}$$

With $\Gamma(x)$ our distance function constructed so that :

- $\Gamma(x) > 1$ outside the obstacle
- $\Gamma(x) = 1$ at the obstacle boundary
- $\Gamma(x) < 1$ inside the obstacle

Since we want to model a circular obstacle, we first recall the circle equation :

$$d(x, x_o)^2 = r^2,$$

where x_o is the center of the circle, $d(x, x_o)$ is the euclidean distance between x and x_o and r is the radius of the circle. We can implicitly embed the circle equation into $\Gamma(x)$ to create a circular obstacle, while still keeping the properties cited above by defining $\Gamma(x)$ as :

$$\Gamma(x) = d(x, x_o)^2 - r^2 + 1.$$

This definition ensures $\Gamma(x)$ follows the prerequisites to implement obstacle avoidance in our DS.

1.1

The modulation generated by the eigenvalues results in a change of magnitude along the various basis directions, as described earlier. The increase in velocity along the tangent direction is bounded.

- What is its upper bound?
  Solution: The upper bound when the eigenvalue reaches the maximum. The highest eigenvalue we can obtain is in tangent direction, with $\lambda_e(x) = 2$. Hence, the maximum speed up is a factor of 2.

- Where does it occur?
  Solution: It occurs on the surface of the obstacle, i.e. and the velocity being parallel to the tangent, i.e.

$$\dot{x} = \lambda_e(x)\mathbf{f}(x) = 2\,\mathbf{f}(x) \qquad \frac{\dot{x}^T\mathbf{f}(x)}{\|x\|\,\|\mathbf{f}(x)\|} = 1,\, \Gamma(x) \approx 1$$

1.2

In certain scenarios, the conservation of the magnitude of the velocity is required.

1. How can you change the eigenvalues such that the modulated velocity is always slower than the initial velocity, i.e., $\|\dot{x}\| \leq \mathrm{f}(x)$?
   Solution: The diagonal matrix can be changed to

$$\mathbf{D}(x) = \begin{bmatrix} \frac{\lambda_n(x)}{\lambda_e(x)} & 0 \\ 0 & 1 \end{bmatrix},$$

2. How can you change the eigenvalues such that the modulated velocity is always faster than the initial velocity, i.e., $\|\dot{x}\| \geq \mathrm{f}(x)$?
   Solution: The diagonal matrix can be changed to

$$\mathbf{D}(x) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\lambda_e(x)}{\lambda_n(x)} \end{bmatrix},$$

   Note, that when getting close to the obstacle there are high accelerations. Furthermore we assume to never reach the obstacle, i.e. $\Gamma(x) > 1 \Rightarrow \lambda_n(x) > 0$.

3. Can you set the eigenvalues such that the norm of the velocity is preserved around the obstacle?
   Solution: Just changing the eigenvalues of the diagonal matrix $\mathbf{D}(x)$ does not allow to conserve the velocity. But the normalization step can be applied after the modulation, i.e.

$$\dot{\bar{x}} = \frac{\|\mathrm{f}(x)\|}{\|\dot{\bar{x}}\|}\dot{\bar{x}} \qquad \text{with} \quad \dot{\bar{x}} = \mathbf{M}(x)\mathrm{f}(x)$$

1.3

How do you need to modify the modulation in Equation (1), such that it is evaluated with respect to a linearly, moving obstacle? Assume that we know the obstacle's linear velocity v^o.
Solution: The new equation is given as:

$$\dot{x} = \mathbf{M}(x)\left(\mathbf{f}(x) - v^o\right) + v^o$$

1.4

Optional Can you create a dynamical system inspired by the obstacle avoidance description, which leads to a local spurious attractor, while remaining stable at the origin?
Solution: Let us choose a spurious attractor at x^a. We construct the modulation matrix as follows

$$\mathbf{M}(x) = \mathbf{E}(x)\mathbf{D}(x)\mathbf{E}(x)^T \quad \text{with } \mathbf{E}(x) = \begin{bmatrix} n^a(x) & t^a(x) \end{bmatrix}, \; \mathbf{D}(x) = \text{diag}\left(\lambda^n(x), \; \lambda^t(x)\right)$$

where $n^a(x) = (x - x^a)/\|x - x^a\|$ is the normal direction to the attractor, and $t^a(x)$ is the tangent direction (which is orthonormal to the normal).
Additionally $\lambda^n(x) = \min(1, \|x - x^a\|/r)$ is the eigenvalue in normal direction, and $\lambda^t(x) = \min(1, \|x - x^a\|/r)^p$ is the eigenvalue in tangent direction with $p \in \mathbb{R}{>}\ 1$. The influence radius being chosen such that the attractor at the origin does not get affected, i.e. $r < \|x^a\|$.
Note that this is not a stable attractor.