Learning for Adaptive and Reactive Robot Control
Solutions for exercises of lecture 3

Professor: Aude Billard
Contact: aude.billard@epfl.ch

1 Exercise 1 - Dynamical Systems and Stability

1.1 Exercise 1.1

Consider a 2 dimensional linear DS, $\dot{x} = Ax$, with $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We wish to introduce a modulation matrix M to modify the dynamics as follows: $\dot{x} = MAx$. Given $M = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$,

1. Find a diagonal matrix $A = \mathbf{diag}(a_1, a_2)$, with $a_1 \neq a_2$ for which the system converges to $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. Compute the path integral of the modulated DS

Solution

1. We only need to find a matrix A such that the eigenvalues of MA are with negative real parts. Let $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, then eigenvalues of MA are (a_1, a_2). Therefore, any diagonal matrix A with negative diagonal elements will make the system converge to $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$, then the corresponding DS is $\dot{x} = MAx = \begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix} x$, where MA can be diagonalized as,

$$\begin{pmatrix} -2 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = PDP^{-1}$$

then according to Slides 19, the path integral can be written as,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = Pe^{(Dt)}P^{-1}x(0)$$

1.2 Exercise 1.2

Consider two variables x and y coupled with the following dynamics

$$\dot{x} = \beta x, \quad \beta \in \mathbb{R}$$
$$\dot{y} = -y + \alpha x, \quad \alpha \in \mathbb{R}$$

Answer the following:

1. Does this system have a fixed point? What is it?
2. For what values of α and β is the system stable at the fixed point?
3. For what values of α and β is the system unstable at the fixed point?

Solution

1. Solving the equation $\begin{cases} \beta x = 0 \\ -y + \alpha x = 0 \end{cases}$, yields $\begin{cases} x = 0 \\ y = 0 \end{cases}$. Therefore, $(x, y) = (0, 0)$ is the fixed point.

2, 3. Notice that the eigenvalues of the corresponding linear system are $(\beta, -1)$, therefore, the system is stable if $\beta < 0$ and unstable if $\beta > 0$, irrespective of the value of α.

1.3 Exercise 1.3

Consider a Lyapunov function $V(x) = x_1^2 + x_2^2$ for the following DS

$$\dot{x}_1 = -x_1 + x_1 x_2, \quad \dot{x}_2 = -x_2$$

1. Find the fixed point
2. Find a region of attraction and show that the fixed point is asymptotically stable

Solution

1. Solve the equation $\begin{cases} -x_1 + x_1 x_2 = 0 \\ -x_2 = 0 \end{cases}$, yields $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$. Therefore, $(x_1, x_2) = (0, 0)$ is the fixed point.
2. Taking the time derivative of the Lyapunov function,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= 2x_1(-x_1 + x_1 x_2) + 2x_2(-x_2) \\ &= -2x_1^2(1 - x_2) - 2x_2^2 \end{aligned}$$

If $x_2 < 1$, then $\dot{V} < 0$. Therefore, $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 < 1\}$ is a region of attraction. Furthermore, the fixed point $(x_1, x_2) = (0, 0)$ is contained in the above region of attraction, $\dot{V}(0, 0) = 0$, therefore, the fixed point is asymptotically stable.

1.4 Exercise 1.4 (Bonus)

Consider the pendulum DS without friction

$$\ddot{\theta} = -g \sin(\theta)$$

1. Write down a state space representation using variable $x = (x_1, x_2)$.

2. As $x = (0,0)$ is a Lyapunov-stable fixed point, there exists a $V(x)$ such that:

$$V(0,0) = 0$$
$$V(x) > 0, \dot{V}(x) \leq 0 \quad \forall x \neq (0,0)$$

   Furthermore, from mechanical intuition, we knew that the pendulum DS without friction is energy conservative, therefore, we hypothesis that there exists $V(x)$ with $\dot{V} \equiv 0$.

   (a) Expand $\dot{V}(x(t))$ and obtain a partial differential equation (PDE) in x_1 and x_2 that satisfies $\dot{V}(x) = 0$,

   (b) Solve the PDE to find $V(x)$.

Solution

1. Let $x_1 = \theta, x_2 = \dot{\theta}$, the state space representation can be written as,

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -g\sin(x_1) \end{pmatrix}$$

2. (a) $\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{\partial V}{\partial x_1}x_2 + \frac{\partial V}{\partial x_2}(-g\sin(x_1)) = 0$, we have

$$\frac{\partial V}{\partial x_1}x_2 = g\frac{\partial V}{\partial x_2}\sin(x_1)$$

   (b) Let $\begin{cases} \dfrac{\partial V}{\partial x_1} = g\sin(x_1) \\ \dfrac{\partial V}{\partial x_2} = x_2 \end{cases}$, noticed that the PDE is decoupled from x_1 amd x_2, then a simple $V(x)$ can be,

$$V(x_1, x_2) = g(1 - \cos(x_1)) + \frac{x_2^2}{2}$$

   which satisfies Lyapunov condition and hence certifies that the system is Lyapunov-stable.

Now consider the pendulum DS with friction

$$\ddot{\theta} = -g\sin(\theta) - \dot{\theta}$$

1. Conclude that $(0,0)$ is stable with the previously obtained $V(x)$.
2. Show that the only trajectory of the DS in the set $S = \{x : \dot{V}(x) = 0\}$ is $x(t) = (0,0)$ for all t and conclude that $(0,0)$ is asymptotically stable by La Salle's Invariance principle.

Solution

1. $\dot{V} = gx_2\sin(x_1) + x_2(-g\sin(x_1) - x_2) = -x_2^2 \leq 0$, hence $(0,0)$ is stable.
2. $\dot{V}(x_1, x_2) = -x_2^2 = 0 \implies x_2 = 0$, hence the invariant set is $\mathcal{S} = \{(x_1, x_2) \,|\, x_2 = 0\}$. Then

$$x_2(t) = 0, \forall t \implies \frac{d}{dt}x_2(t) = 0 \implies g\sin(x_1(t)) = 0$$

   So, if the region of attraction $\mathcal{D} = \{(x_1, x_2) \,|\, |x_1| < \pi\}$, then

$$g\sin(x_1(t)) = 0 \implies x_1(t) = 0$$

References

[1] Aude Billard, Sina Mirrazavi, and Nadia Figueroa. *Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach*. MIT press, 2022.