Learning Control Laws

Stable Estimator of Dynamical Systems (SEDS)
Global Asymptotic Stability of Autonomous Dynamical System (DS)

Lyapunov’s Theorem for Global Asymptotic Stability

**Theorem:** A DS is **globally asymptotically stable** at \( x^* \in \mathbb{R}^N \) iff there exists a Lyapunov candidate function \( V(x): \mathbb{R}^N \rightarrow \mathbb{R}^N \) that is radially unbounded; i.e. \( V(x) \rightarrow \infty \) as \( ||x|| \rightarrow \infty \), \( C^1 \) and satisfies the following conditions:

1. \( V(x^*) = 0 \),
2. \( V(x) > 0 \) \( \forall x \in \mathbb{R}^N \setminus x = x^* \),
3. \( \dot{V}(x^*) = 0 \),
4. \( \dot{V}(x) < 0 \) \( \forall x \in \mathbb{R}^N \setminus x = x^* \)

\[
\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0
\]

\( V \) should be non-increasing along all trajectories of \( f(x) \)

Lyapunov Function \( V(x) \) ~ Energy-like Function
Global Asymptotic Stability of Autonomous Dynamical System (DS)

Lyapunov’s Theorem for Global Asymptotic Stability

Theorem A DS is globally asymptotically stable at $x^* \in \mathbb{R}^N$ iff there exists a Lyapunov candidate function $V(x): \mathbb{R}^N \to \mathbb{R}$ $C^1$ that is radially unbounded; i.e. $V(x) \to \infty$ as $\|x\| \Rightarrow \infty$ and satisfies the following conditions:

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$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$

$V$ should be non-increasing along all trajectories of $f(x)$
Stability of a Linear Autonomous Dynamical System (DS)

Stable Linear DS $\dot{x} = Ax + b$

Quadratic Lyapunov Function (QLF)

$V(x) = (x - x^*)^T (x - x^*)$

How to ensure $\dot{V}(x)$ is always negative?

$A^T + A < 0$

Enforce the eigenvalues to be negative!

$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$
What if $f(x)$ is non-linear?

- Not easy to assess whether the system is stable.
- Traditionally, the following has been done:
  - local linearization;
  - numerical estimation of stability;
  - analytical solution in special cases.
Stable Estimator of Dynamical Systems (SEDS)


Mohi Khansari
How to model this non-linear dynamical system?
DATA: set of $M$ reference trajectories
\[
\{ X, \dot{X} \} = \left\{ \{ x_t^i, \dot{x}_t^i \}_{t=1}^{T_m} \right\}_{m=1}^{M}
\]
$T_m$: Length of each trajectory

Start with sampled trajectories from a nonlinear DS
Model the data with a probabilistic model: \( p(\dot{x}, x; \Theta) \)

\( \Theta \): Model's parameters

\[
p(\dot{x}, x; \Theta) = \sum_{k=1}^{K} \pi_k \cdot p(\dot{x}, x; \mu^k, \Sigma^k), \quad \text{with} \quad p(\dot{x}, x; \mu^k, \Sigma^k) = N(\mu^k, \Sigma^k), \quad 0 < \pi_k \leq 1
\]

\( \Theta = \{\pi_k, \mu^k, \Sigma^k\}_{k=1}^{K} \): priors, means and covariance matrices of the \( K \) Gauss functions
Generate an estimate of the DS: \( \dot{x} = f(x; \Theta) := E\{p(\dot{x} | x; \Theta)\} \)

Nonlinearity comes from

\[
\gamma_k(x) = \frac{\pi_k \cdot p(x; \mu^k_x, \Sigma^k_x)}{\sum_{k=1}^{K} \alpha_k \cdot p(x; \mu^k_x, \Sigma^k_x)}
\]

Gaussian Mixture Regression:

\[
\dot{x} = \sum_{k=1}^{K} \gamma_k(x) \left( \frac{1}{A^k} \left( \sum_{k}^{k} \left( \frac{1}{\sum_{xx}^{k}} \right)^{-1} x + \mu^k_x - \sum_{k}^{k} \left( \frac{1}{\sum_{xx}^{k}} \right)^{-1} \mu^k_x \right) \right) = \sum_{k=1}^{K} \gamma_k(x) (A^k + b^k)
\]

K linear DS
**SEDS as a mixture of linear DS**

**Mixing function**

\[ \gamma_k(x) : \mathbb{R}^N \to \mathbb{R} \]

\[ 0 < \gamma_k(x) < 1 \]

\[ \sum_{k=1}^{K} \gamma_k(x) = 1 \]

\[ f_k(x) = A_k x + b_k \]

\[ \gamma_k(x) \]

1st eigenvector of each \( A^k \) matrix gives direction of velocity of DS locally.

\[ \dot{x} = \sum_{k=1}^{K} \gamma_k(x) \left( A^k + b^k \right) \]

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Conditions for SEDS stability

Model is parameterized only by the $A^k$ matrices and $b^k$ vectors.

This can be guaranteed through 2 conditions:

a) $b^k = -A^k x^*$ - Stability at attractor

b) $A^k + (A^k)^T < 0 \ \forall k$ - Energy decreases

(Proof: see book)

Need to guarantee stability at the attractor $x^*$.

$f(x) = 0$
Two possible objective functions:

a) Maximum likelihood → fits at best the entire density

b) Mean-square error → fits at best the state space trajectories and velocities

Need to ensure that the flow is aligned as closely as possible to demonstrated trajectories.
Optimization of SEDS

Maximum likelihood

$$\min_{\Theta_{\text{GMR}}} J(\Theta_{\text{GMR}}) = -\frac{1}{L} \sum_{m=1}^{M} \sum_{t=0}^{T_m} \log p \left( x^{t,m}, \dot{x}^{t,m} | \Theta_{\text{GMR}} \right)$$

Mean-square error

$$\min_{\Theta_{\text{GMR}}} J(\Theta_{\text{GMR}}) = \frac{1}{2L} \sum_{m=1}^{M} \sum_{t=0}^{T_m} \| f(x^{t,m}) - \dot{x}^{t,m} \|^2.$$ 

Set of constraints

(a) \( b^k = -A^k x^* \)
(b) \( A^k + (A^k)^T < 0 \)
(c) \( \sum_k > 0 \) \( \forall \) \( k = 1, \ldots, K \)
(d) \( 0 < \pi_k \leq 1 \)
(e) \( \sum_{k=1}^{K} \pi_k = 1 \)

Nonlinear optimization

\[
\Sigma^k = \begin{bmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy}
\end{bmatrix}
\]
When trained with mean-square error (MSE) as objective function, the Gauss function no longer need to fit the distribution of the data.
Prior to training SEDS, the user must make a number of choices that will influence the quality of the learned model.

The choices are:

- **Type of objective function**
  → This will affect the placement of the Gauss functions.

- **Number of Gauss functions**
  → This can be automated by using the Bayesian Information Criterion (BIC) – BIC finds a balance between improved quality of the fit and increase in number of parameters.
LASA Handwriting Dataset - Benchmark

https://www.epfl.ch/labs/lasa/datasets/
LASA Handwriting Dataset - Benchmark

Fit with maximum likelihood

Accurate fit except for regions with high nonlinearities

Demonstrated trajectories
Reproduction from same initial position
Learning and adaptive control for robots

LASA Handwriting Dataset - Benchmark

SEDs $J(\theta_{GMR}) = \text{MSE}$

Fit with MSE

Same issue
Learning and adaptive control for robots

LASA Handwriting Dataset - Benchmark

Fit with maximum likelihood

Extreme case – due to a violation of 2\textsuperscript{nd} Lyapunov constraint for simple quadratic energy function
Extreme case – due to a violation of 2nd Lyapunov constraint for simple quadratic energy function
Kinesthetic teaching
Learning and adaptive control for robots

Trajectory of reproductions

Velocity profile of reproductions

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Learning and adaptive control for robots

Reproduction
Applying Disturbance During Reproduction
(a) Trajectory of reproductions

- Target
- Saucer: Adapted Trajectory
- Saucer: Original Trajectory
- Cup: Adapted Trajectory
- Cup: Original Trajectory

Change in the target position at $t = 2$

(b) Velocity profile for the saucer task

(c) Velocity profile for the cup task
Robustness to Perturbations
Automatically estimate *globally asymptotically stable* dynamical systems from sampled trajectories

- Extension of Gaussian Mixture Model
  - Uses same objective function (maximum likelihood)
  - Add new set of constraints to enforce stability

- Stability is guaranteed through Lyapunov stability constraints
  - Assumes a quadratic Lyapunov function

- High accuracy for a large number of nonlinear dynamics

- Limitations:
  - Non convex optimization
  - Poor accuracy for highly nonlinear dynamics (high curvature)
# Extensions to SEDS

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