Introduction to Dynamical Systems

Lecture 3

March 10, 2022

Outline

Dynamical System as a differential equation

A first order dynamical system (DS) is expressed as a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f(x,t), \quad x(0) = x_0, \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

$$x \in \mathbb{R}^n \qquad \textbf{State: } x = [x_1 \dots x_n]^\top$$

A second order DS

$$\ddot{x} = f(x, \dot{x}), \quad x, \dot{x} \in \mathbb{R}^n$$

Represented as two differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}y = z, \quad y(0) = y_0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}z = f(y,z), \quad z(0) = z_0$$

$$y,\, z \in \mathbb{R}^n \qquad \textbf{States: } y = [y_1 \dots y_n]^\top,\, z = [z_1 \dots z_n]^\top$$

Set of all possible y, z is called state space

In a control system the internal dynamics of the plant and the control effort $u(t)$ are distinguished. In this lecture we assume

- The solution of a DS is a path to be followed by a robot and that we can completely track this path with available controls.

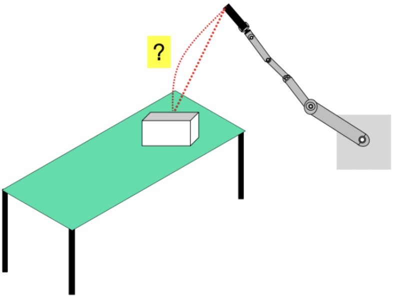

Figure 1: Robot moves towards box

- The time invariant case: $\frac{\mathrm{d}}{\mathrm{d}t}x = f(x)$, $x \in \mathbb{R}^n$

Coupled DS

Two DS can be coupled to achieve an objective

For example, to track a flying object, both the robot position and velocity must be coupled to that of the flying object

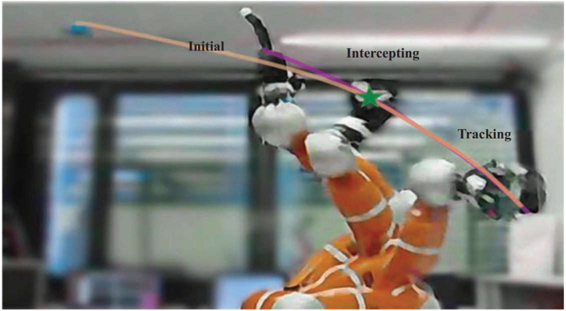

Figure 2: Coupled DS of robot and flying object such that they move together after the interception point

Representation

- Consider two DS: $\dot{x} = g(x)$, $\dot{y} = f(y)$
- In previous example the robot end effector $x(t)$, flying object $y(t)$
  - Objective was to modify $g(x)$ to $g(x, y)$ so that $\lim_{t\to\infty} x(t) - y(t) = 0$
- Coupled DS:

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = h(z) = \begin{pmatrix} g(x, y) \\ f(y) \end{pmatrix}$$

- Example of coupled linear DS:

$$\dot{x} = x - y \quad \dot{y} = -y + y_0$$

  Representation:

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}}_{A(z)} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{z} + \underbrace{\begin{pmatrix} 0 \\ y_0 \end{pmatrix}}_{b}$$

- We shall see in exercises how to choose $A(z)$ so that $\lim_{t\to\infty} x(t) - y(t) = 0$

First Order DS

$$\dot{x} = a(x)x, \quad a \in \mathbb{R} \to \mathbb{R}, \quad x(0) = 0$$
$$\text{Linear} : a(x) = c, \quad \text{Nonlinear} : a(x) = 1 - x$$

Second Order DS (Pendulum in $2D$)

$$y := \theta, \quad z = \dot{\theta}$$
$$\dot{y} = z, \quad y(0) = y_0$$
$$\dot{z} = -\frac{g}{l}\sin y - \frac{k}{m}z, \quad z(0) = z_0$$

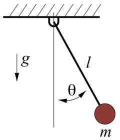

Figure 3: Second order DS

- Different initial conditions of the DS give different solutions
- $x(t)$, $t \in [0, \infty]$ in case of 1st order DS
  $[y(t) \quad z(t)]^\top$, $t \in [0, \infty]$ in case of a 2nd order DS .

Trajectory of a DS

Solution to ODE

Trajectory of a DS

Solution to ODE

First order DS: $\dot{x} = cx$

$$c \int_0^t \mathrm{d}t = \int_{x_0}^{x} \frac{\mathrm{d}x}{x}$$

Therefore,

$$\ln\left(\frac{x}{x_0}\right) = ct \implies \quad x(t) = e^{ct}x(0)$$

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Second Order DS: $\ddot{x} = 1$

State space representation:

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$\dot{z} = 1 \implies \quad z(t) = t + z(0)$$

$$\dot{y} = t + z(0) \implies y(t) = \frac{1}{2}t^2 + z(0)t + y(0)$$

Vector field of a DS

- Attach the vector $[z \quad f(y,z)]^\top$ at $[y \quad z]^\top$ on the state space
- Repeat the process at every point in the state space

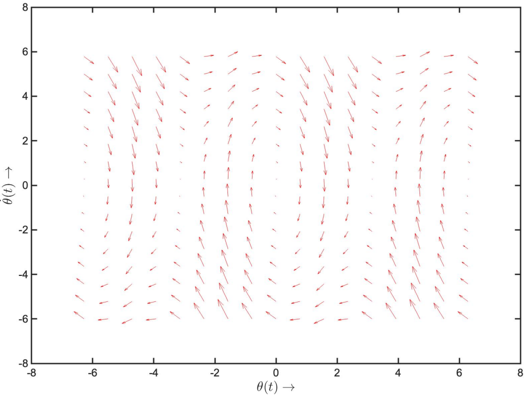


Figure 4: Vector field of pendulum DS: $\ddot{\theta} = -g\sin\theta - \dot{\theta}$ for $\theta \in [-2\pi, 2\pi]$, $\dot{\theta} \in [-6, 6]$

Path Integral

- Solution to the ODE of a DS integrated at some $x(0)$ is called a path integral

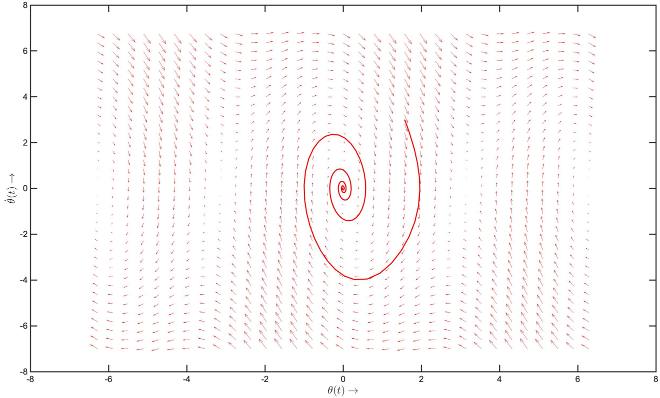


Figure 5: Path integral of pendulum DS for $[\theta(0) \quad \dot{\theta}(0)]^\top = (\pi/2, 3)$

Phase Plot of Pendulum DS (no damping)

All path integrals taken together generate a phase plot. Consider the DS

$$\ddot{\theta} = -g \sin \theta$$

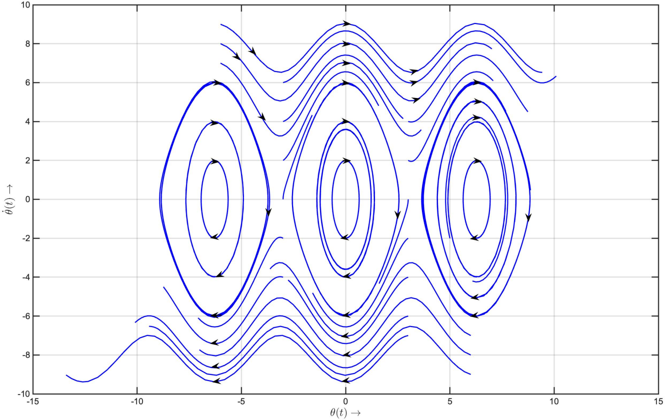


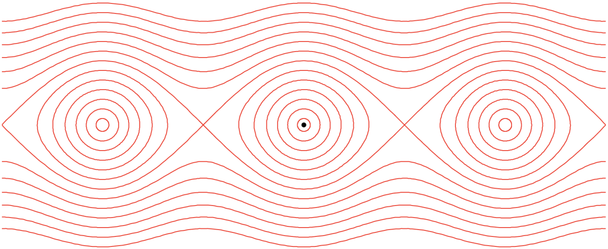
Figure 6: Phase Plot, Oscillations represented by closed curves

Phase Plot of Pendulum DS (no damping)

All path integrals taken together generate a phase plot. Consider the DS

$$\ddot{\theta} = -g \sin \theta$$

Low amplitude oscillations around $\theta = 0$:



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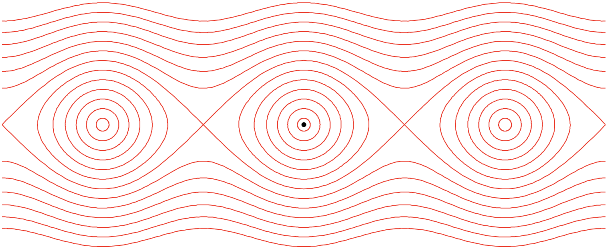

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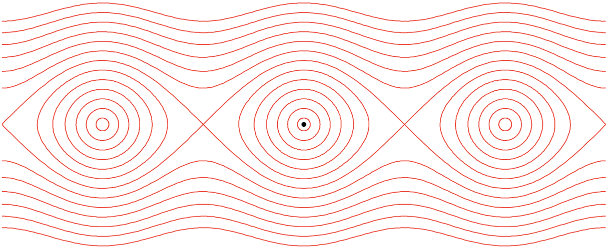
Clockwise motion:



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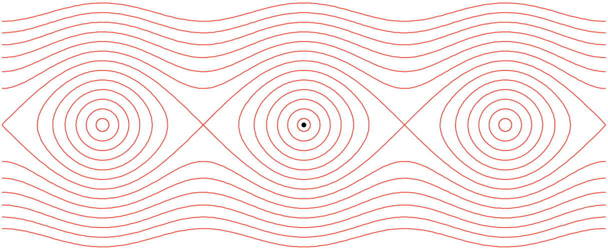

Phase Plot of Pendulum DS (no damping)

All path integrals taken together generate a phase plot. Consider the DS

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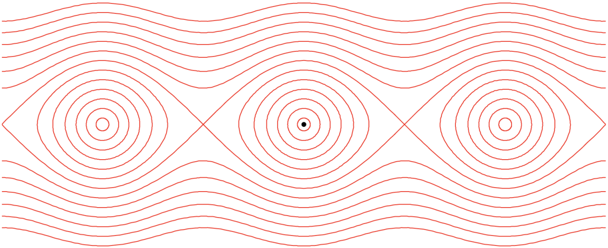
Counter clockwise motion:



Phase Plot of Pendulum DS (with damping)

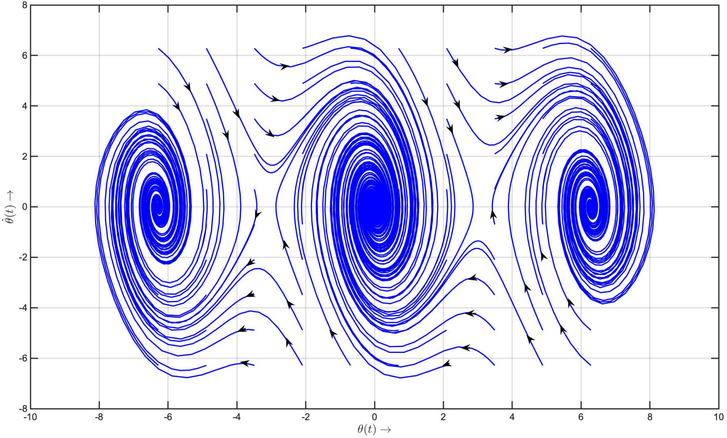


Figure 6: Pendulum DS with $m = l = k = 1$ is $\ddot{\theta} = -g \sin \theta - \dot{\theta}$

- The points from which the DS does not evolve further are *equilibrium points* or *fixed points* or *stationary points*
- If a DS is initialized at an equilibrium point the solution stays at the equilibrium point for all time.

Definition

The equilibrium points x^* of the DS: $\dot{x} = f(x)$ are those x which satisfy the equation $f(x) = 0$.

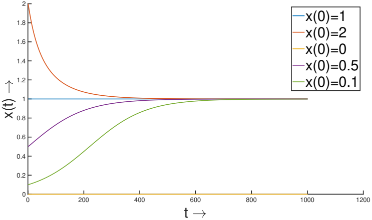


Figure 7: DS: $\dot{x} = x - x^2$, $x^* = \{0, 1\}$

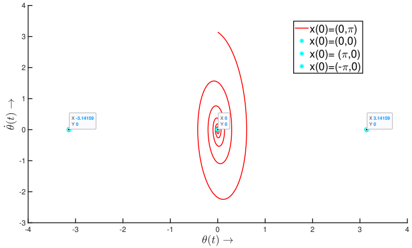


Figure 8: Pendulum DS with $x^* = (n\pi, 0)$, $n = 0, 1, 2, \ldots$

Vector field vanishes at equilibrium points

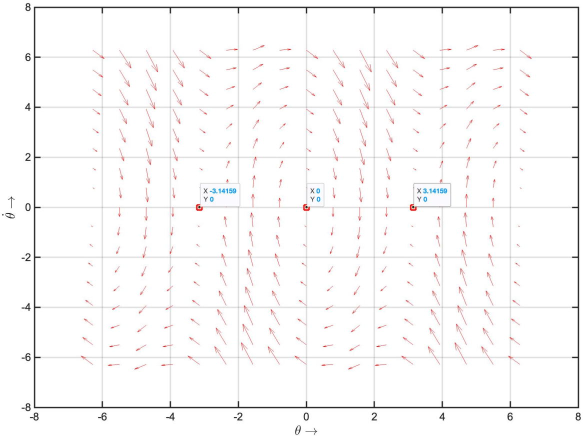


Figure 9: Vector field of damped pendulum DS: $\ddot{\theta} = -g \sin\theta - \dot{\theta}$ for $\theta \in [-2\pi, 2\pi]$, $\dot{\theta} \in [-2\pi, 2\pi]$

Equilibrium points can be isolated (as seen in examples above) or occur in clusters:

Linear DS:

$$\dot{x} = Ax, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} 1 & -0.5 \\ 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 - 0.5x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

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Equilibrium Points:
$\{(x_1, x_2) : x_1 = 0.5 * x_2 \}$

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Van der Pol Oscillator DS:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1 \quad \mu > 0$$

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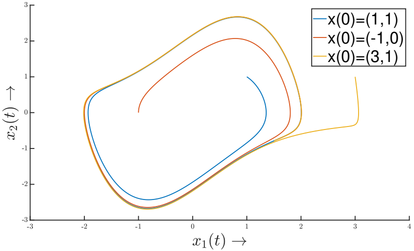

Figure 10: Stable limit cycle (an isolated periodic orbit)

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- Globally exponentially stable if the rate of convergence to x^* is exponentially fast from everywhere in the state space.

We make these notions mathematically precise after a study of stability in linear DS

Examination of stability using phase plot

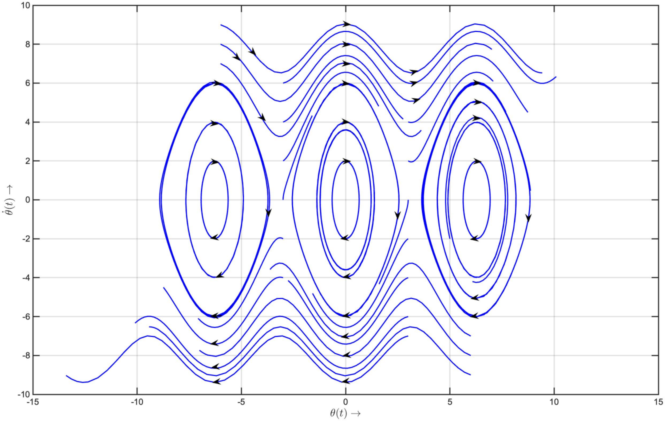

Figure 11: Pendulum DS with $m = l = 1$, $k = 0$ is $\ddot{\theta} = -g \sin \theta$

Observe that $(\theta, \dot{\theta}) = (0, 0)$ is stable and $(\theta, \dot{\theta}) \in \{(\pi, 0), (-\pi, 0)\}$ are unstable

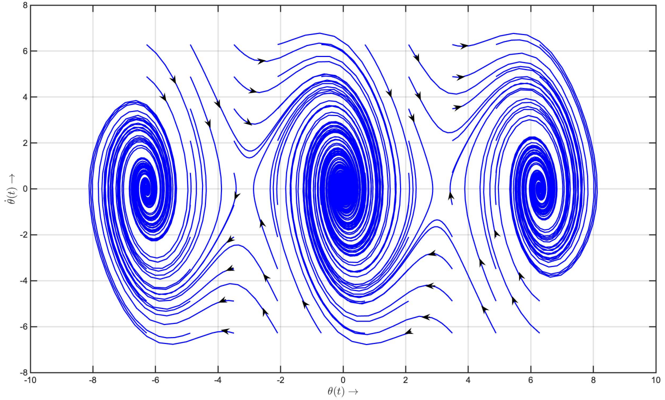

Figure 12: Pendulum DS with $m = l = k = 1$ is $\ddot{\theta} = -g \sin\theta - \dot{\theta}$

Observe that $(\theta, \dot{\theta}) = (0, 0)$ is asymptotically stable and $(\theta, \dot{\theta}) \in \{(\pi, 0), (-\pi, 0)\}$ are unstable

Linear DS in 2D

Consider the following DS:

$$\dot{x} = Ax, \quad x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Unique Equilibrium Point: $x^* = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$

Computation of matrix exponential

Steps:

- If $A = \mathbf{diag}(\lambda_1, \lambda_2)$
$$e^A = \mathbf{diag}(e^{\lambda_1}, e^{\lambda_2})$$
- If A has distinct non zero has eigen values λ_1, λ_2 then $\exists M \succ 0$ (i.e. M is positive definite) s.t. $MDM^{-1} = A$,

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$D = \mathbf{diag}(\lambda_1, \lambda_2)$, $M = \big(v_1, v_2\big)$, $Av_i = \lambda_i v_i$
$$e^A = Me^D M^{-1}$$

Solution to 2 dim linear DS:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Me^{Dt}M^{-1}x(0)$$

Assume that $A = \mathbf{diag}(\lambda_1, \lambda_2)$, the solution is:

$$x_1(t) = e^{\lambda_1 t} x_1(0) \quad \text{and} \quad x_2(t) = e^{\lambda_2 t} x_2(0)$$

Visualization of vector field:

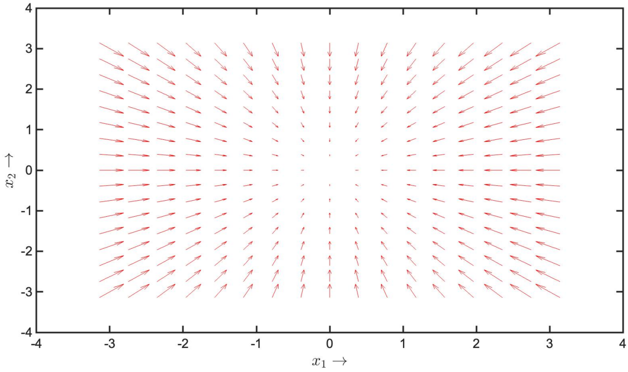

Figure 13: x^* is globally exponentially stable with $A = \mathbf{diag}(-1, -1)$

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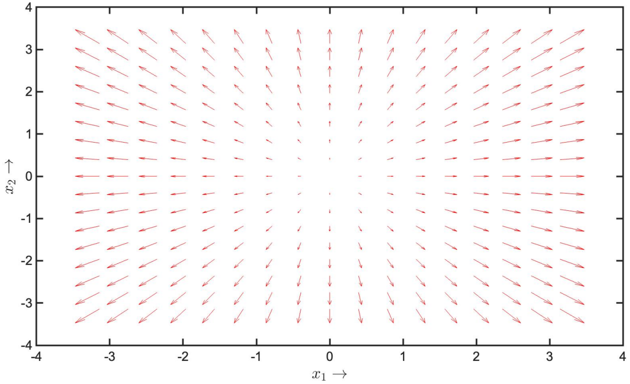

Figure 13: x^* is unstable with $A = \mathbf{diag}(1, 1)$

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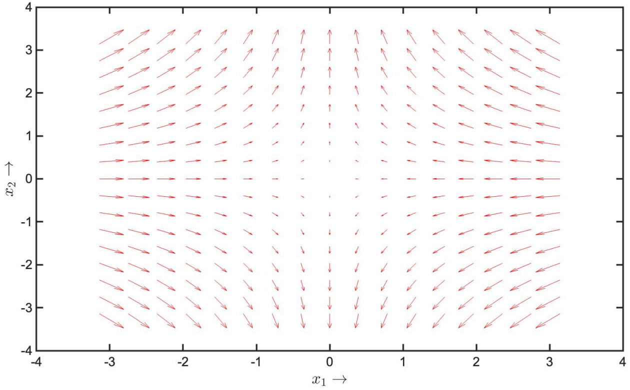

Figure 13: x^* is a saddle point with $A = \mathbf{diag}(-1, 1)$

Summary of results for Linear DS

Stability of a linear DS in 2 dimensions is easily verified

- If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, x^* is globally exponentially stable
- If $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$, x^* is unstable
- If $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, x^* is a saddle point

Questions:

- What about a higher dimensional linear DS?
- What about a nonlinear DS?

Stability of nonlinear DS

- Explicit solution to a nonlinear ODE is hard
- Hence a precise mathematical notion of stability is necessary

An equilibrium point x^* is

- Stable if for any $\epsilon > 0$, there exists a $\delta > 0$ s.t. for all $t > 0$,

$$\|x(0) - x^*\| < \delta \implies \|x(t) - x^*\| < \epsilon$$

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$$\|x(0) - x^*\| < \delta \implies \lim_{t \to \infty} \|x(t) - x^*\| = 0$$

- Exponentially stable if asymptotically stable and there exists $\delta, \alpha, \beta > 0$ s.t. for all $t > 0$

$$\|x(0) - x^*\| < \delta \implies \|x(t) - x^*\| \leq \alpha \|x(t) - x^*\| e^{-\beta t}$$

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$$\|x(0) - x^*\| < \delta \implies \|x(t) - x^*\| \leq \alpha \|x(t) - x^*\| e^{-\beta t}$$

- Unstable if not stable

Study of stability of x^* of a DS $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is simplified by the existence of a candidate Lyapunov function $V : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ s.t.

$$V(x^*) = 0, \quad V(x) > 0 \text{ for all } x \in \mathcal{D} - \{x^*\}$$

Lyapunov stability theorem

Given x^* if there exists a candidate V, x^* is

- Stable if $\frac{\mathrm{d}}{\mathrm{d}t}\{V(x)\} \leq 0$ for all $x \in \mathcal{D}$
- Asymptotically stable if $\frac{\mathrm{d}}{\mathrm{d}t}\{V(x)\} < 0$ for all $x \in \mathcal{D}$
- Exponentially stable if $\frac{\mathrm{d}}{\mathrm{d}t}\{V(x)\} \leq -\beta V(x)$ for all $x \in \mathcal{D}$ and a $\beta > 0$

For an asymptotically stable x^*

- $V(x)$ is an energy like function
- $\mathcal{D}$ defines the region of attraction

- General Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for a linear DS $\dot{x} = Ax$ is

$$V(x) = x^\top P x, \quad P \succ 0 \text{ (is positive definite)}$$

- Therefore,

$$\dot{V}(x) = x^\top P\dot{x} + \dot{x}^\top P x = x^\top (PA + A^\top P)x$$

- $x^* = 0$ is globally asymptotically stable if there exists $Q \succ 0$ s.t.

Lyapunov Equation

$$PA + A^\top P + Q = 0$$

- Closed form solution exists only if A has all negative eigen values

- General Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for a linear DS $\dot{x} = Ax$ is

$$V(x) = x^\top P x, \quad P \succ 0 \text{ (is positive definite)}$$

- Therefore,

$$\dot{V}(x) = x^\top P\dot{x} + \dot{x}^\top P x = x^\top (PA + A^\top P)x$$

- $x^* = 0$ is globally asymptotically stable if there exists $Q \succ 0$ s.t.

Lyapunov Equation

$$PA + A^\top P + Q = 0$$

- Closed form solution exists only if A has all negative eigen values

$$P = \int_0^\infty e^{A^\top t} Q e^{At} \mathrm{d}t, \quad Q \succ 0$$

Lyapunov functions

Linear DS with $A = \mathbf{diag}(-1, -1)$

- Choose $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V(x) = \frac{1}{2}x^\top P x = x_1^2 + x_2^2$
  - $\dot{V} = -2(x_1^2 + x_2^2) = -2V(x)$
  - $(0, 0)$ is globally exponentially stable

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  - $(0, 0)$ is globally exponentially stable

Pendulum DS $\dot{x}_1 = x_2$, $\dot{x}_2 = -g \sin x_1 - x_2$

- $V_1(x) = g(1 - \cos(x_1)) + 0.5x_2^2$
  - $V_1(\begin{pmatrix} 0 & 0 \end{pmatrix}^\top) = 0$ and $V(x) > 0$ for any $x \in \mathbb{R}^2 \setminus \{\begin{pmatrix} 0 & 0 \end{pmatrix}^\top\}$
  - $$\frac{\mathrm{d}}{\mathrm{d}t} V_1(x) = g \sin(x_1)\dot{x}_1 + x_2 \dot{x}_2 = -x_2^2 \leq 0$$
  - $(0, 0)$ is stable

Lyapunov functions

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  - $(0, 0)$ is stable
- $V_2(x) = \frac{1}{2} x^\top P x + g(1 - \cos(x_1))$, $P = \begin{pmatrix} b & b \\ b & 1 \end{pmatrix}$, $0 < b < 1$
  - P is positive definite as $Det(P) > 0$ and $Tr(P) > 0$
  - $$\frac{\mathrm{d}}{\mathrm{d}t} V_2(x) = -\frac{1}{2}\{g x_1 \sin(x_1) + x_2^2\} < 0 \text{ for all } -\pi < x_1 < \pi$$
  - Asymptotic stability in $\mathcal{D} = \{x \in \mathbb{R}^2 : |x_1| < \pi\}$

Level sets of Lyapunov function

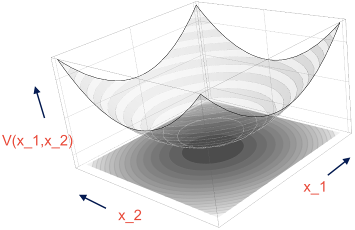

Figure 14: Level sets [1] of $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

- The condition $\dot{V}(x(t)) \leq 0 \quad \implies$ for some τ if $x(\tau) : V(x(\tau)) = c$, then for all $t > \tau$ we have $V(x(t)) \leq c$.
- When $V(x) < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c.
- As c decreases, the Lyapunov surface $V(x) = c$ shrinks to $V(x^*) = 0$ $\implies x(t) \to x^*$ as $t \to \infty$

[1] Source: https://www.ndsu.edu/pubweb/~novozhil/Teaching/48020Data/13.pdf

Invariant Set

- A set S is positively invariant w.r.t the dynamics if

$$x(0) \in S \implies x(t) \in S \quad \text{for all } t > 0$$

- The set of points $x \in \mathcal{D}$ for which $\frac{\mathrm{d}}{\mathrm{d}t}\{V(x)\} \leq 0$ is a positively invariant set
- In some cases if we have a candidate Lyapunov function $V(x)$ at a fixed point x^* satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\{V(x)\} \leq 0$, we can ensure asymptotic stability
- La Salle's Invariance Principle: If the only trajectory in $\{x : \dot{V}(x) = 0\}$ is $x^*(t)$, then x^* is asymptotically stable

Toy Example

$$\dot{x} = -(x - x^*), \quad V(x) = x^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = 2(x - x^*)\dot{x} = -2(x - x^*)^2 \leq 0$$

Observe that $\{x : \dot{V}(x) = 0\} = \{x^*\}$. Therefore x^* is asymptotically stable

Modulation of DS

In many applications modulating the behavior of a DS is essential

- Generate rich class of trajectories while preserving stability of fixed point
- To avoid either a single obstacle and converge asymptotically to a target

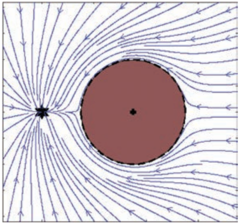

Figure 15: Single obstacle

Modulation of DS

- In many applications modulating the behavior of a DS is essential
- To avoid either a single obstacle and converge asymptotically to a target
- To avoid multiple obstacles and still converge to a target [2]

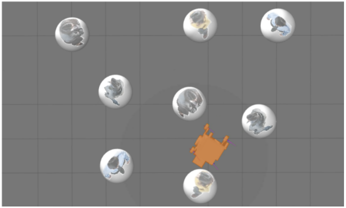

Figure 15: Wheelchair (orange) tries to avoid a human crowd (circles)

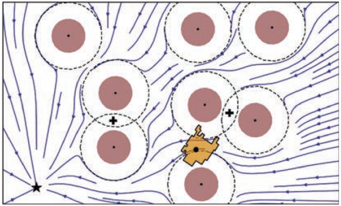

Figure 16: Multiple obstacles in phase plot

[2] Source: L. Huber et al, 'Avoidance of Convex and Concave Obstacles With Convergence Ensured Through Contraction'

Consider a DS in 2 dimensions asymptotically stable at x^* and a Lyapunov function $V(x) = (x - x^*)^\top (x - x^*)$

From Lyapunov theorem: $(x - x^*)^\top \dot{x} < 0$

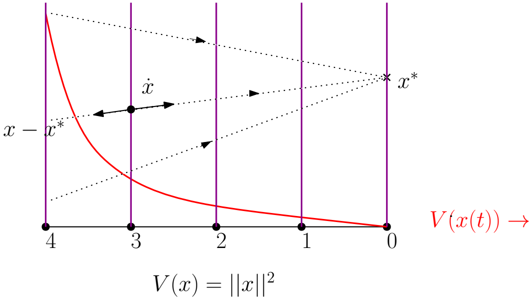

Figure 17: Linear DS asymptotically converging to x^*

Consider a DS in 2 dimensions asymptotically stable at x^* and a Lyapunov function $V(x) = (x - x^*)^\top (x - x^*)$

From Lyapunov theorem: $(x - x^*)^\top \dot{x} < 0$

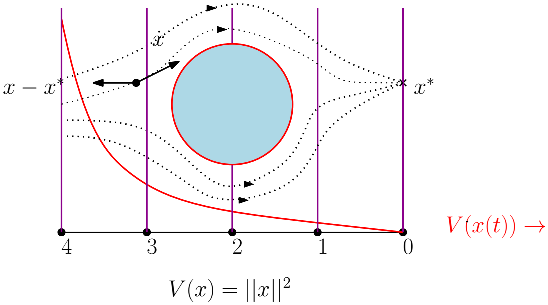

Figure 17: Convex obstacle avoidance with asymptotic stability (Lyapunov) at x^*

Consider a DS in 2 dimensions asymptotically stable at x^* and a Lyapunov function $V(x) = (x - x^*)^\top (x - x^*)$

From Lyapunov theorem: $(x - x^*)^\top \dot{x} < 0$

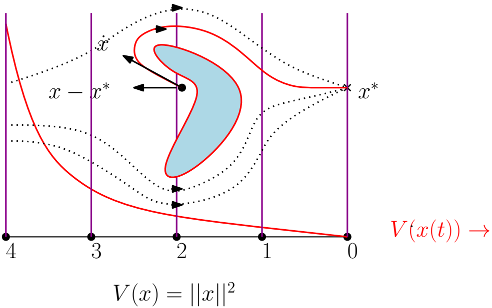

Figure 17: Concave obstacle avoidance with asymptotic stability at x^* (Lyapunov condition fails)

Contraction theory:

- To show red trajectory is 'close' to one of black trajectories
- To modulate the behavior of a DS by change of coordinates

Notation, Definitions

- Infinitesimal displacement from a trajectory $x(t)$ of the DS is denoted by $\delta x(t)$

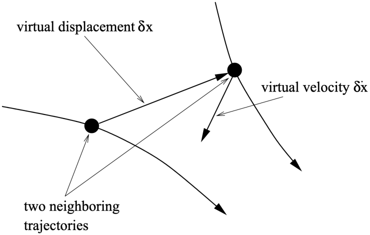

Figure 18: Visualization of $\delta x(t)$

- Rate of change of $\delta x(t)$-

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)+\delta x) - f(x(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\delta x = \frac{\partial f}{\partial x}\delta x$$

- Metric $M(x)$ is a positive definite, symmetric matrix
- Rate of change of distance $\delta x^\top M(x)\delta x$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^\top M(x)\delta x) = \delta x^\top \left[M(x)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top M(x) + \dot{M}(x)\right]\delta x$$

- Objective: To find the region $\mathcal{D}$ and conditions on $M(x)$ so that $(\delta x^\top M(x) \delta x)$ reduces exponentially

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- This means $\delta x(t)^\top M(x)\delta x(t) \to 0$ as $t \to \infty$
- If $\frac{\mathrm{d}}{\mathrm{d}t}y = -\beta y \implies \int_0^y \frac{\mathrm{d}y}{y} \leq \int_0^t \mathrm{d}t \implies \ln y \leq -\beta y$
  - Exponential function $t \to e^t$ is increasing so $t_1 \leq t_2 \implies e^{t_1} \leq e^{t_2}$
  - So, $y(t) \leq e^{-\beta} y(0)$

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- Now choose $y(t) = \delta x(t)^\top M(x)\delta x(t)$

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- Now choose $y(t) = \delta x(t)^\top M(x)\delta x(t)$

Contraction Region

A set $\mathcal{D} \subset \mathbb{R}^n$ where the following holds for all $x \in \mathcal{D}$

$$M(x)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top M(x) + \dot{M}(x) \leq -\beta M(x)$$

for some $\beta > 0$ is called a contraction region and $M(x)$ is called a contraction metric

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- Observe that the condition implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t)^\top M(x)\delta x(t) \leq -\beta(\delta x(t)^\top M(x)\delta x(t))$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x(t)^\top M(x)\delta x(t) \leq -\beta(\delta x(t)^\top M(x)\delta x(t))$$

An Equivalent Formulation

- As $M(x) \succ 0$ there exists an $N(x) \succ 0$ s.t. $M(x) = N^\top(x)N(x)$
- Metric formulated as change of coordinates
- Infinitesimal displacement δz is a coordinate change of δx defined as

$$\delta z = N\delta x$$

- Time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z = \left(\frac{\mathrm{d}}{\mathrm{d}t}N + N\frac{\partial f}{\partial x}\right) N^{-1}\delta z$$

- Ensure the time derivative evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z = -\delta z$$

by the following equivalent condition

Contraction Coordinate change

$N \succ 0$ defines a contraction region $\mathcal{D}$ if for all $x \in \mathcal{D}$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}N + N\frac{\partial f}{\partial x}\right) N^{-1} = -Q, \quad Q \succ 0$$

Contraction metric, Linear DS

- Consider the DS globally exponentially stable at $(0,0)$

$$\dot{x} = Ax, \quad A = \mathbf{diag}(-1,-1), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Consider transformed coordinates $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ given by

$$z = Nx, \quad N \succ 0,$$

Objective: To find N such that $z(t) \to (0,0)$ as $t \to \infty$

- Consider $N = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$, y_1, $y_2 > 0$ satisfying the coordinate change eqn. with $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is

$$\dot{N} + NA = -N \implies \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -y_2 \end{pmatrix}$$

$$\implies \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 e^{-t} \end{pmatrix}, c_1, c_2 \in \mathbb{R}, c_2 > 0$$

- Solution trajectory of DS is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} x_1(0) \\ c_2 e^{-t} x_2(0) \end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$
- Change of coordinates:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = N \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} x_1(0) \\ c_2 e^{-2t} x_2(0) \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}, c_2 > 0$$

- Clearly $z(t)$ converges to $(0, 0)$ and it is not a path integral of the DS.

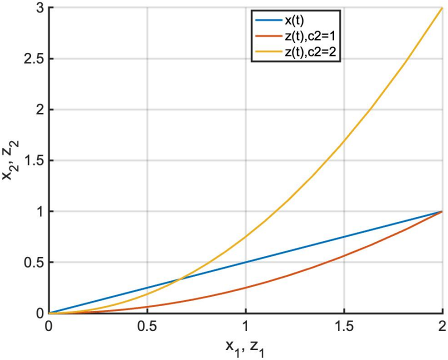

Figure 19: $z(t)$ in Yellow ($c_1 = 1$, $c_2 = 2$) and Red ($c_1 = 1$, $c_2 = 1$) contracting to Blue $x(t)$, path integral with $x(0) = (2, 1)$ in a globally contracting region $\mathbb{R}^2$

Contraction metric, nonlinear DS

- Consider the DS $\dot{x} = -sin(x)$
- Equation for N is

$$\frac{\partial N}{\partial x} f + N \frac{\partial f}{\partial x} = -\frac{\partial N}{\partial x} \sin x - N \cos x = -N$$

- Solution to PDE with $N(0) = 1$ is

$$N(x) = \frac{\tan(x/2)}{\sin(x)} \neq 0 \text{ for } x \in (2n\pi - \pi, 2n\pi + \pi)$$

- Contraction region is therefore $(2n\pi - \pi, 2n\pi + \pi)$
- Contraction metric is $M(x) = N(x)^2$

Comparison with Lyapunov theory

Consider a DS with an asymptotically stable fixed point x^*

Lyapunov Theory

- Existence of $V : \mathcal{D} \to \mathbb{R}$ s.t.

Contraction Theory

- Existence of metric $M(x)$ for all $x \in \mathcal{D}$ s.t. $(\delta x)^\top M(\delta x) \to 0$ as $t \to \infty$

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Consider a DS with an asymptotically stable fixed point x^*

Lyapunov Theory

- Existence of $V : \mathcal{D} \to \mathbb{R}$ s.t. $\|x(t) - x^*\| \to 0$ as $t \to \infty$
- Region of attraction $\mathcal{D}$

Contraction Theory

- Existence of metric $M(x)$ for all $x \in \mathcal{D}$ s.t. $(\delta x)^\top M(\delta x) \to 0$ as $t \to \infty$
- Contraction region $\mathcal{D}$

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- Existence of $V : \mathcal{D} \to \mathbb{R}$ s.t. $\|x(t) - x^*\| \to 0$ as $t \to \infty$
- Region of attraction $\mathcal{D}$
- Condition for asymptotic stability to x^* is related to existence of V s.t.

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x) < 0 \text{ for } x \in \mathcal{D}$$

Contraction Theory

- Existence of metric $M(x)$ for all $x \in \mathcal{D}$ s.t. $(\delta x)^\top M(\delta x) \to 0$ as $t \to \infty$
- Contraction region $\mathcal{D}$
- Condition for asymptotic stability to $x(t)$ is related to existence of M satisfying

$$M \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top M(x) + \dot{M} \leq -\beta M(x)$$

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- Existence of $V : \mathcal{D} \to \mathbb{R}$ s.t. $\|x(t) - x^*\| \to 0$ as $t \to \infty$
- Region of attraction $\mathcal{D}$
- Condition for asymptotic stability to x^* is related to existence of V s.t.

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x) < 0 \text{ for } x \in \mathcal{D}$$

- Trajectory always close to x^* w.r.t $\|.\|_2$

Contraction Theory

- Existence of metric $M(x)$ for all $x \in \mathcal{D}$ s.t. $(\delta x)^\top M(\delta x) \to 0$ as $t \to \infty$
- Contraction region $\mathcal{D}$
- Condition for asymptotic stability to $x(t)$ is related to existence of M satisfying

$$M \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top M(x) + \dot{M} \leq -\beta M(x)$$

- Trajectory not necessary close to x^* w.r.t $\|.\|_2$

Significance of a global contraction region

If the contraction region $\mathcal{D} = \mathbb{R}^n$ has a unique equilibirum point then all trajectories converge to it exponentially

- Consider a Lyapunov function

$$V(x) = f(x)^\top M(x) f(x)$$

  - Check that this is a valid Lyapunov function
- The rate of change of V-

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x) = f(x)^\top [M(x)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top M(x) + \dot{M}(x,t)] f(x) = -\beta V(x)$$

- Conversely, for any exponentially stable x^*, there exists a contraction metric $M(x)$.