Robust Control

Roland Longchamp

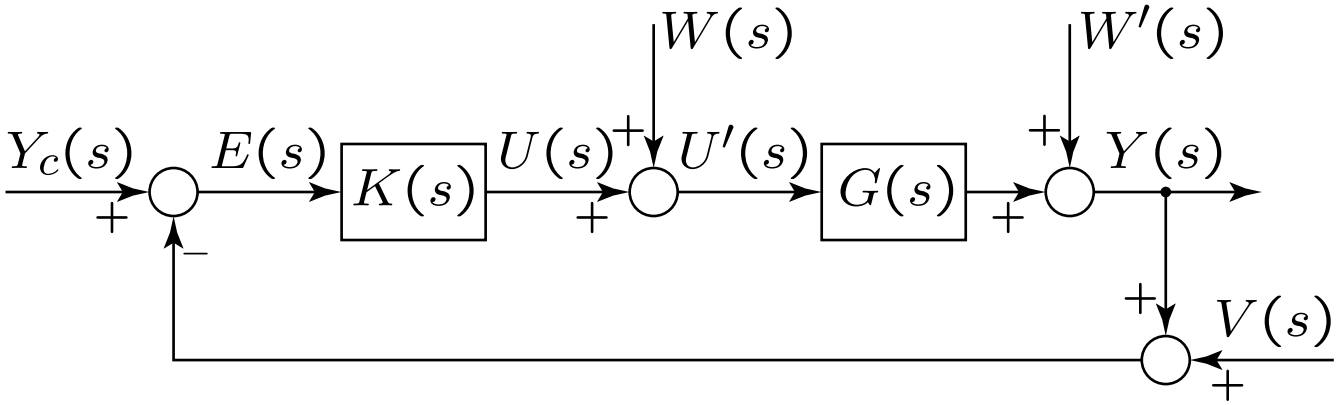
Laboratoire d'automatique
École polytechnique fédérale de Lausanne
1015 Lausanne
Switzerland
e-mail: roland.longchamp@epfl.ch

Outline

1. Basic Concepts
2. Plant Uncertainty
3. Robust Stability
4. Robust Performance
5. Design Constraints
6. Standard Loopshaping
7. Concluding Remarks

Basic Concepts

SISO, continuous-time systems



Loop transfer function $L(s) = K(s)\,G(s)$

Sensitivity function

$$S(s) = \frac{1}{1 + L(s)} = \frac{E(s)}{Y_c(s)} = -\frac{E(s)}{V(s)} = \frac{U'(s)}{W(s)}$$

Complementary sensitivity function

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{Y(s)}{Y_c(s)} = -\frac{Y(s)}{V(s)} = -\frac{U(s)}{W(s)}$$

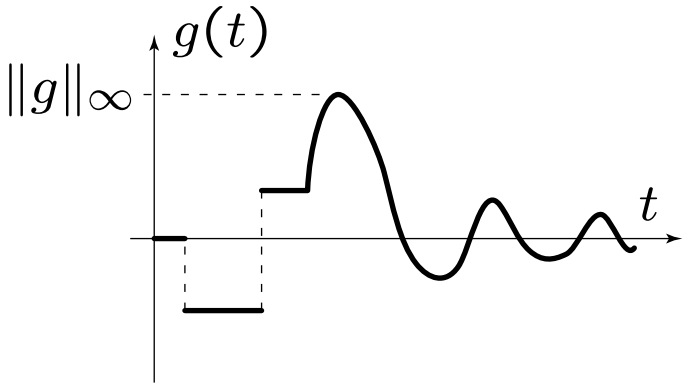
$$S(s) + T(s) = 1 \qquad S(G) = \frac{\dfrac{\mathrm{d}T(G)}{\mathrm{d}G}}{\dfrac{T(G)}{G}}$$

Infinity-Norms

∞-signal norm

$$g : \mathbb{R} \to \mathbb{R}, t \mapsto g(t)$$

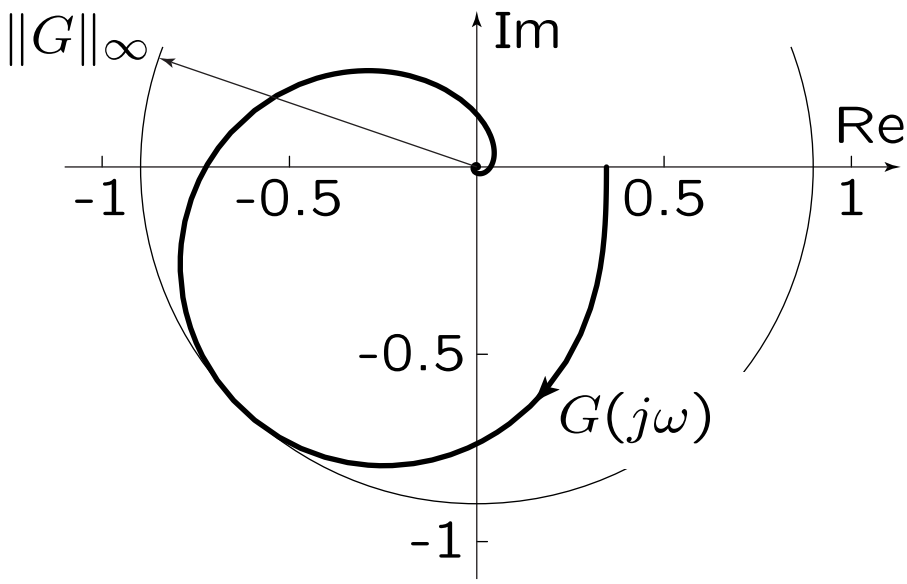
$$\|g\|_\infty = \sup_t |g(t)|$$



∞-system norm

$$G : \mathbb{C} \to \mathbb{C}, s \mapsto G(s)$$

$\|G\|_\infty = \sup_\omega |G(j\omega)|$: Exists iff $G(s)$ is proper and has no poles on the imaginary axis



∞-system norm is submultiplicative

$$\|K\,G\|_\infty \leq \|K\|_\infty \, \|G\|_\infty$$

Nominal Performance

$$|S(j\omega)| < \sigma \qquad \omega \in [0, \omega_b],\ \sigma \in \mathbb{R}$$

ω_b: Closed-loop bandwidth

$$\max_{\omega \in [0,\omega_b]} \frac{1}{\sigma} |S(j\omega)| < 1$$

$$W_1(j\omega) = \begin{cases} \dfrac{1}{\sigma} & \text{if } \omega \in [0, \omega_b] \\ 0 & \text{if } \omega \notin [0, \omega_b] \end{cases}$$

$$\sup_{\omega} |W_1(j\omega)\, S(j\omega)| < 1$$

$|W_1(j\omega)|$ is a roughly decreasing function of ω:
Performance decreases with increasing frequency.

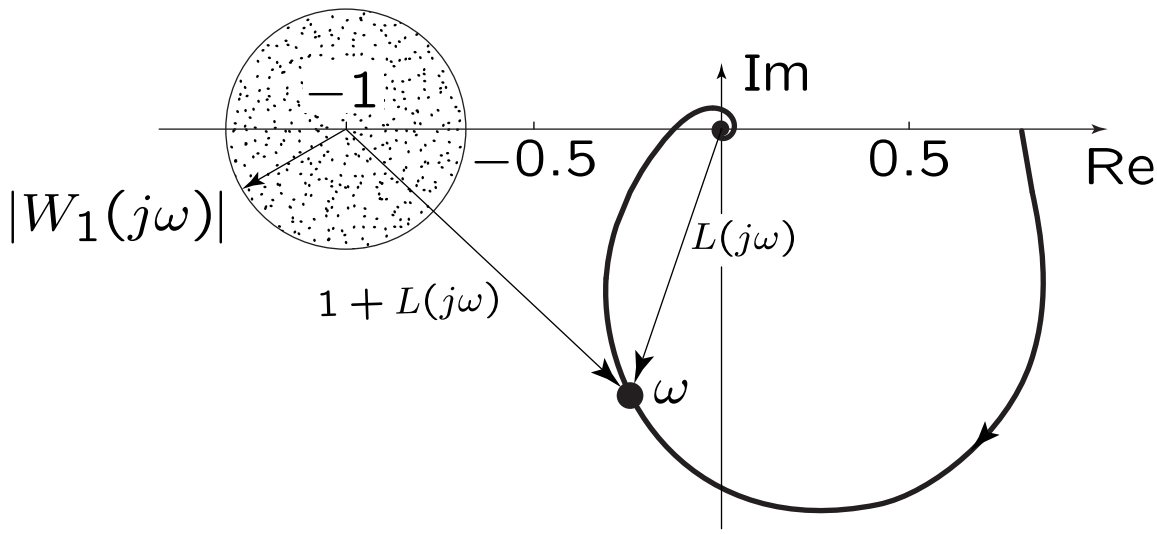
Nominal Performance (cont.)

Nominal performance

$$\|W_1\, S\|_\infty < 1 \qquad W_1(s): \text{ Weight}$$

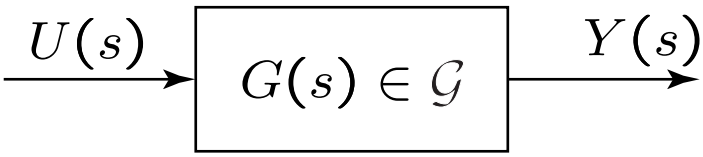
$$\Leftrightarrow \left|W_1(j\omega)\,\frac{1}{1+L(j\omega)}\right| < 1 \qquad \forall\omega$$

$$\Leftrightarrow |W_1(j\omega)| < |1 + L(j\omega)| \qquad \forall\omega$$



At each frequency, the point $L(j\omega)$ on the Nyquist plot lies outside the disk of center -1, radius $|W_1(j\omega)|$.

Plant Uncertainty



Parametric uncertainty

$$\mathcal{G} = \left\{ \frac{B(s)}{s^n + a_1 s^{n-1} + \ldots + a_n} : \begin{array}{l} \underline{a}_i \leq a_i \leq \bar{a}_i, \\ i = 1, 2, \ldots, n \end{array} \right\}$$

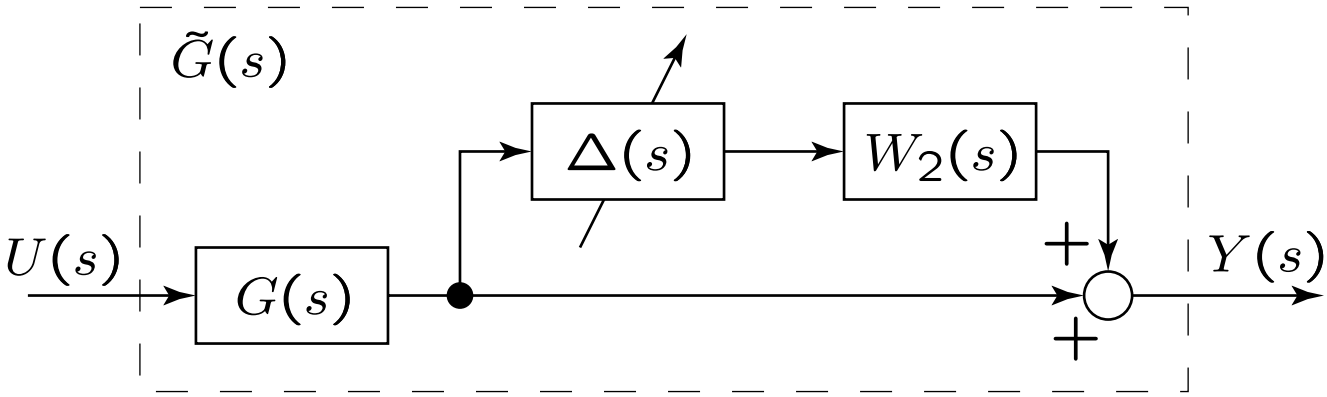
Finite set of models

$$\mathcal{G} = \{G_1(s), G_2(s), \ldots, G_n(s)\}$$

Unstructured uncertainty (multiplicative form)

$$\mathcal{G} = \left\{ \tilde{G}(s) = (1 + \Delta(s)\, W_2(s))\; G(s) : \|\Delta\|_\infty \leq 1 \right\}$$

$G(s)$	:	Nominal plant transfer function
$W_2(s)$	:	Fixed stable transfer function (weight)
$\Delta(s)$	:	Variable stable transfer function

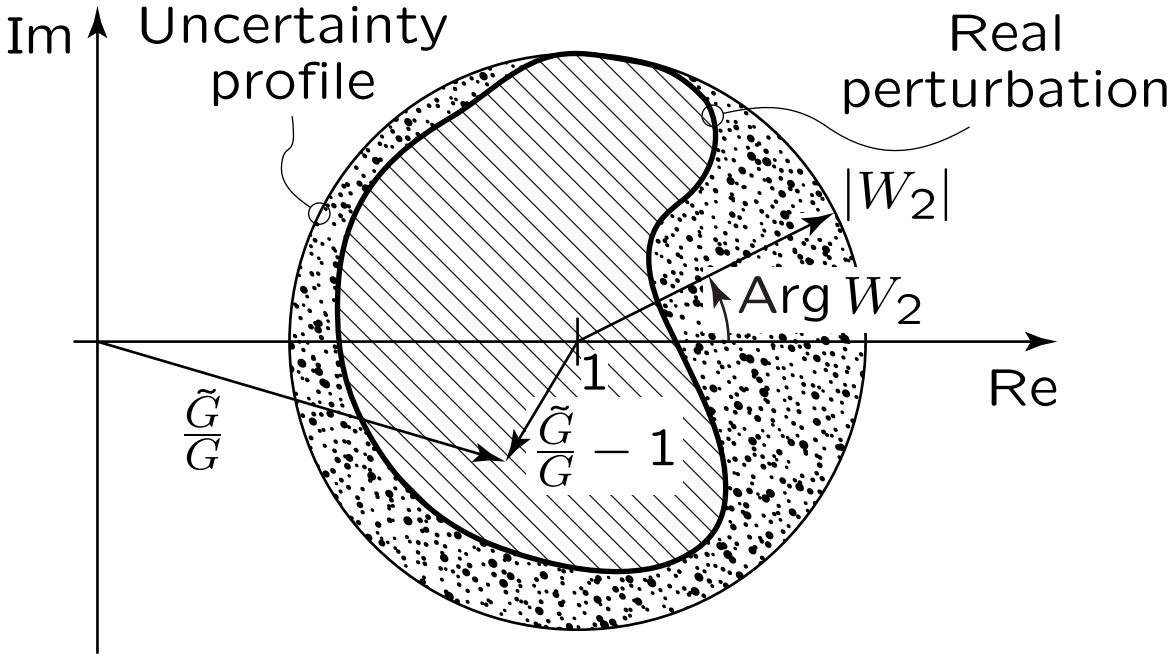


Multiplicative Perturbation

$s = j\omega$, ω given

$$\frac{\tilde{G}}{G} - 1 = \Delta W_2$$

$$\Leftrightarrow \begin{cases} \left|\frac{\tilde{G}}{G} - 1\right| &= |\Delta|\,|W_2| & \in [0, |W_2|] \\ \text{Arg}\left(\frac{\tilde{G}}{G} - 1\right) &= \text{Arg}\Delta + \text{Arg}W_2 & \in (-\pi, \pi] \end{cases}$$



1 : No perturbation ($\tilde{G} = G$)

$\frac{\tilde{G}}{G} - 1$: Can be everywhere in the disk with center 1, radius $|W_2|$: $\left|\frac{\tilde{G}}{G} - 1\right| \leq |W_2|$

$|W_2(j\omega)|$ is a roughly increasing function of ω: Uncertainty increases with increasing frequency.

Example: Time Constant

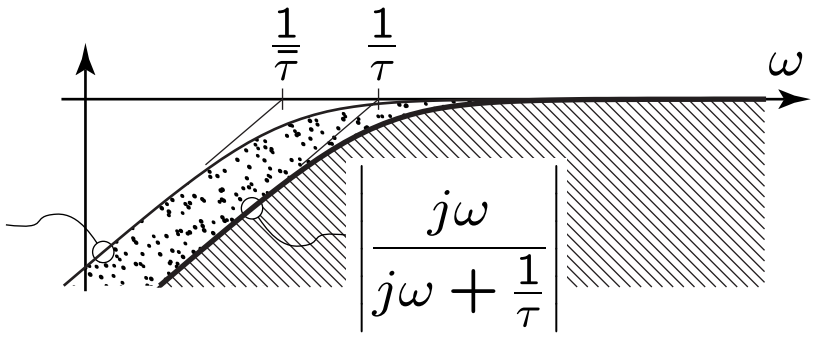
Neglected time constant (*e.g.* the inductance in an electrical drive)

$$\tilde{G}(s) = G(s)\,\frac{1}{\tau s + 1} \qquad\qquad \tau \in [0, \bar{\tau}]$$

$$\frac{\tilde{G}(s)}{G(s)} - 1 = \frac{1}{\tau s + 1} - 1 = -\frac{s}{s + \frac{1}{\tau}} = \Delta(s)\,W_2(s)$$

$$\left|\frac{j\omega}{j\omega + \frac{1}{\tau}}\right| \leq |W_2(j\omega)| \qquad\qquad \forall \omega, \tau$$

$$\left|\frac{j\omega}{j\omega + \frac{1}{\bar{\tau}}}\right| = |W_2(j\omega)|$$



$W_2(s) = \frac{\bar{\tau} s}{\bar{\tau} s + 1}$: $|W_2(j\omega)|$ increases with increasing frequency

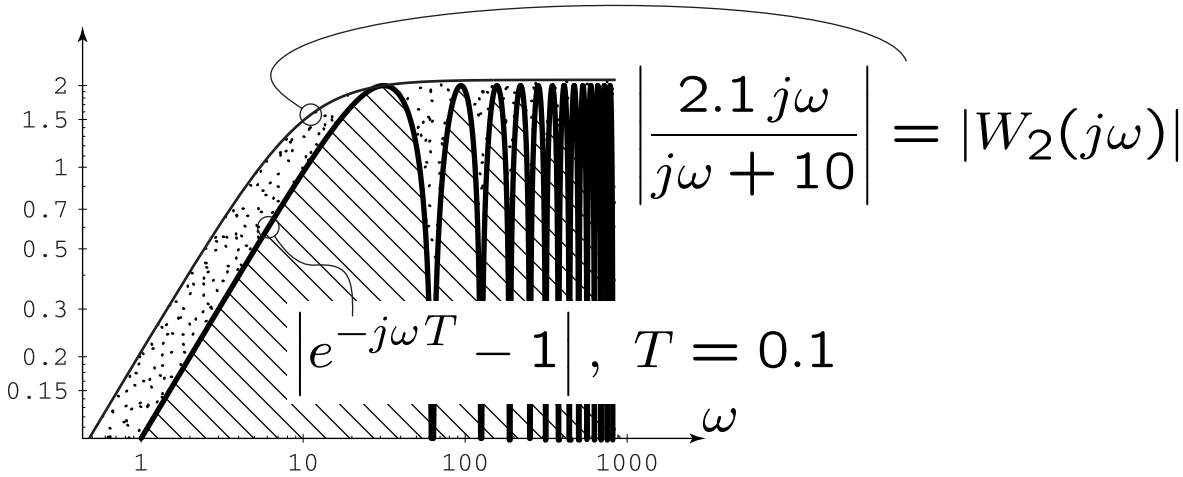
$$\left\{\frac{\tilde{G}(j\omega)}{G(j\omega)} : 0 \leq \tau \leq \bar{\tau}\right\} \subset \{s \in \mathbb{C} : |s - 1| \leq |W_2(j\omega)|\} :$$

Conservatism

Example: Time Delay

Neglected time delay (*e.g.* in power electronics)

$$\tilde{G}(s) = G(s)\, e^{-sT} \qquad T \in [0, 0.1]$$
$$\frac{\tilde{G}(s)}{G(s)} - 1 = e^{-sT} - 1 = \Delta(s)\, W_2(s)$$
$$\left| e^{-j\omega T} - 1 \right| \leq |W_2(j\omega)| \qquad \forall \omega, T$$



$W_2(s) = \frac{2.1\, s}{s+10}$: $|W_2(j\omega)|$ increases with increasing frequency

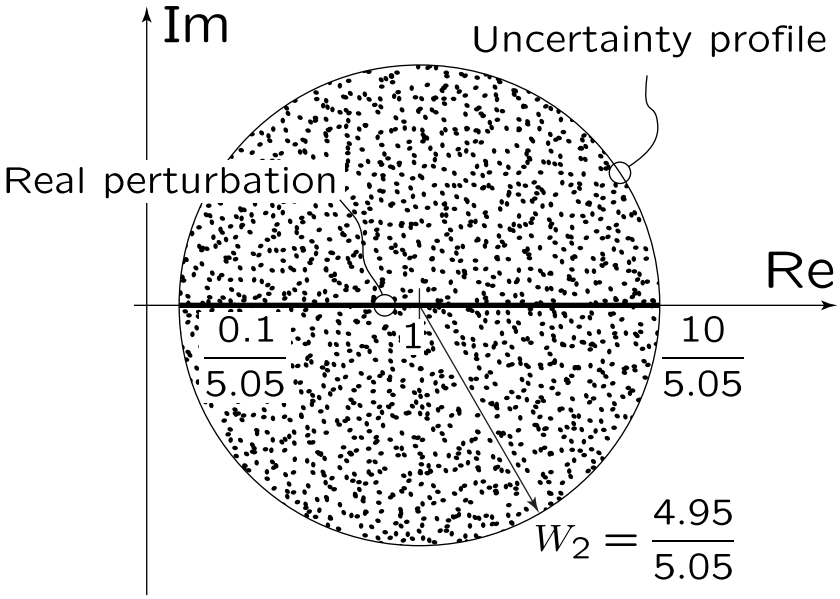
$$\left\{ \frac{\tilde{G}(j\omega)}{G(j\omega)} : 0 \leq T \leq 0.1 \right\} \subset \{ s \in \mathbb{C} : |s - 1| \leq |W_2(j\omega)| \} :$$

Conservatism

Example: Uncertain Gain

Uncertain gain (parametric uncertainty)

$$\tilde{G}(s) = G(s)\,\gamma \qquad \gamma \in [0.1, 10]$$



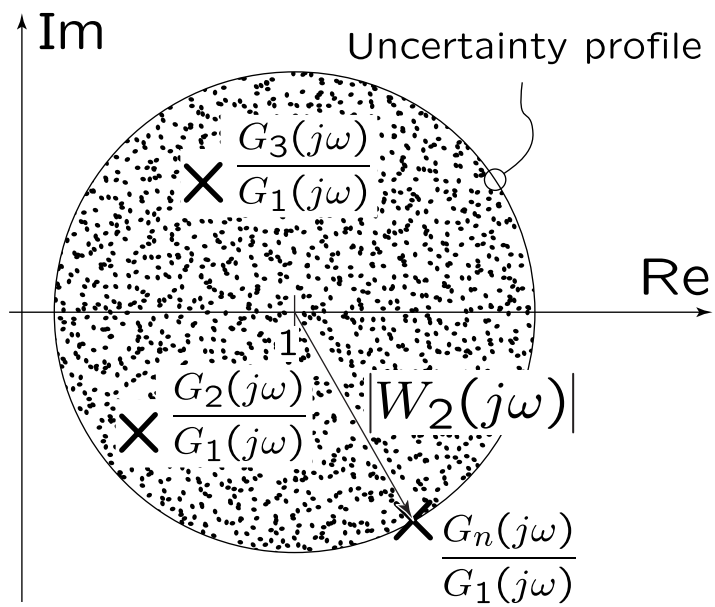
$$\left\{\frac{\tilde{G}(j\omega)}{G(j\omega)} : 0.1 \le \gamma \le 10\right\} \subset \left\{s \in \mathbb{C} : |s-1| \le \frac{4.95}{5.05}\right\} :$$

Conservatism

Example: Multiple Models

Finite set of models

$G_1(s)$: Nominal plant transfer function



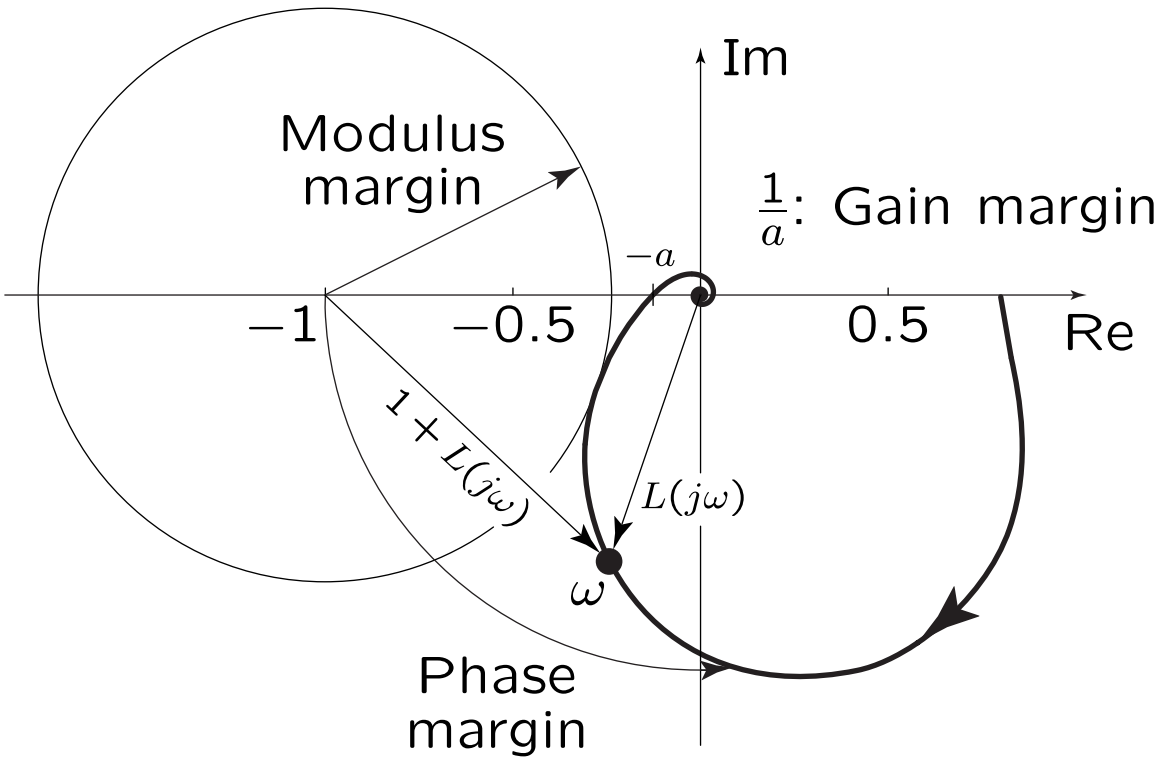
$$|W_2(j\omega)| = \max\left\{\left|\frac{G_i(j\omega)}{G_1(j\omega)} - 1\right| : 2 \le i \le n\right\}$$

$$\left\{\frac{G_i(j\omega)}{G_1(j\omega)} : 2 \le i \le n\right\} \subset \{s \in \mathbb{C} : |s-1| \le |W_2(j\omega)|\} :$$

Conservatism

Robust Stability

Classical measures of stability margin



Modulus margin

$$\begin{aligned}\inf_{\omega} |1 + L(j\omega)| &= \frac{1}{\sup_{\omega} \frac{1}{|1 + L(j\omega)|}} \\ &= \frac{1}{\|S\|_\infty}\end{aligned}$$

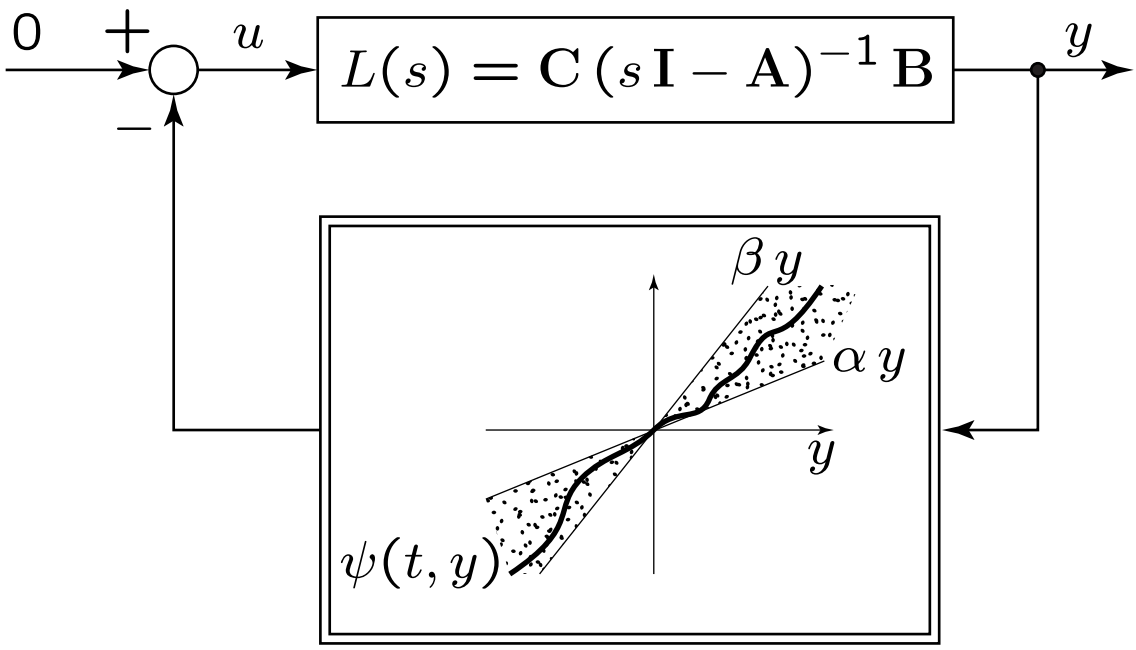
Absolute Stability

$$\left.\begin{array}{rcl}\dot{\mathbf{x}} & = & \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u \\ y & = & \mathbf{C}\,\mathbf{x}\end{array}\right\}$$: Controllable and observable

$u = -\psi(t, y)$

$\psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$: Piecewise continuous in t and Lipschitz in y

$\alpha\, y^2 \leq y\, \psi(t, y) \leq \beta\, y^2$, $\alpha < \beta$: Sector condition



Absolute stability: The origin is globally uniformly asymptotically stable for any nonlinearity in the sector.

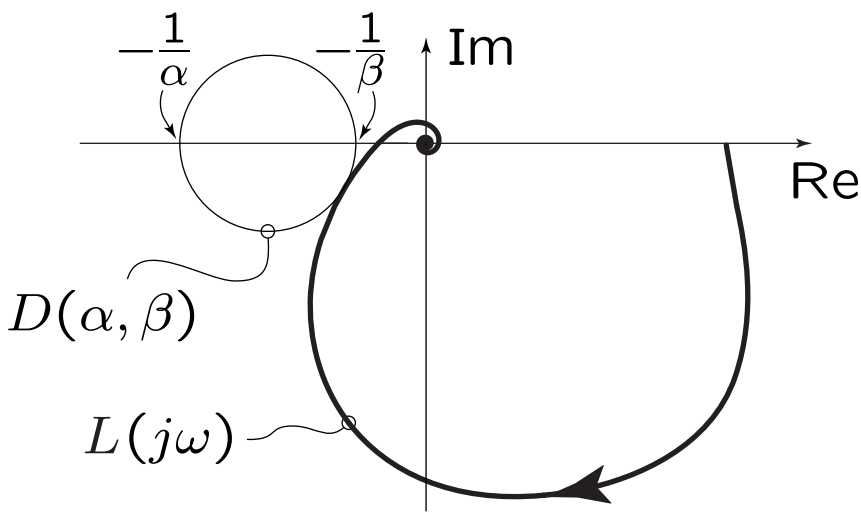
Absolute Stability (cont.)

Theorem (circle criterion): The system is absolutely stable if one of the following conditions is satisfied:

(i) If $0 < \alpha < \beta$, the Nyquist plot of $L(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it P times in the counterclockwise direction, where P is the number of poles of $L(s)$ with positive real parts.

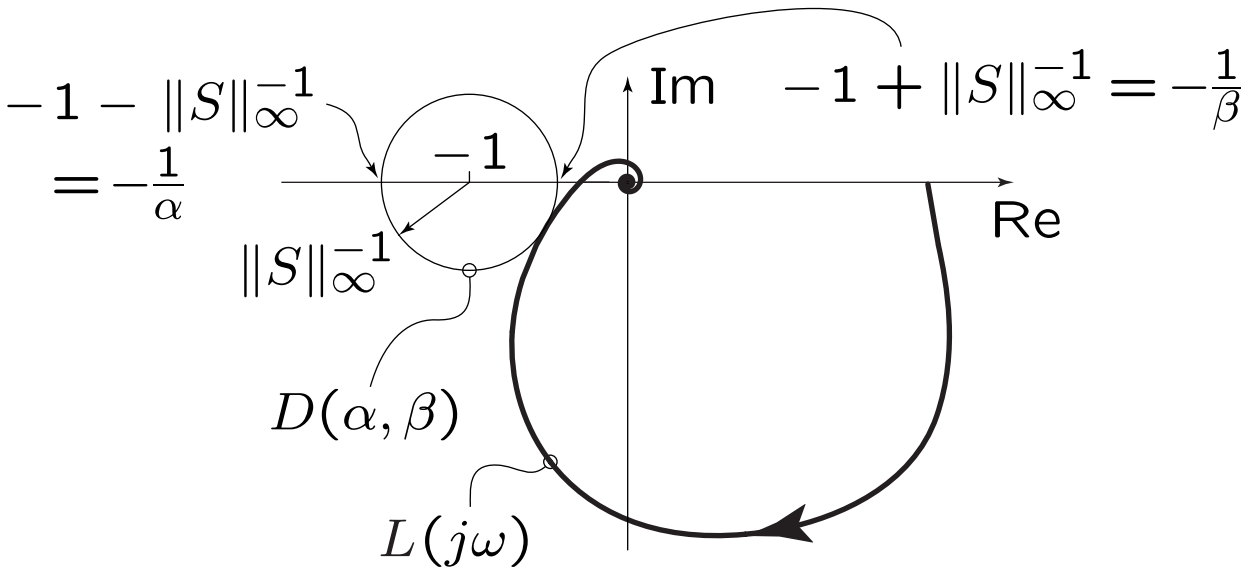
(ii) If $0 = \alpha < \beta$, $L(s)$ is Hurwitz and the Nyquist plot of $L(j\omega)$ lies to the right of the line $s = -\frac{1}{\beta}$.

(iii) If $\alpha < 0 < \beta$, $L(s)$ is Hurwitz and the Nyquist plot of $L(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$.

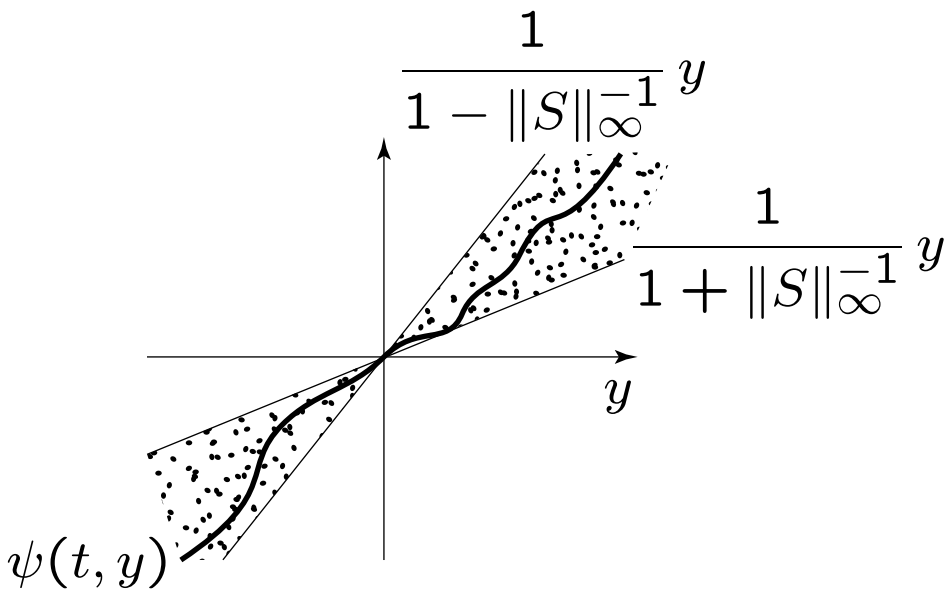


Absolute Stability (cont.)

Modulus margin: $\|S\|_\infty^{-1}$



$$\alpha = \frac{1}{1 + \|S\|_\infty^{-1}} \qquad \beta = \frac{1}{1 - \|S\|_\infty^{-1}}$$

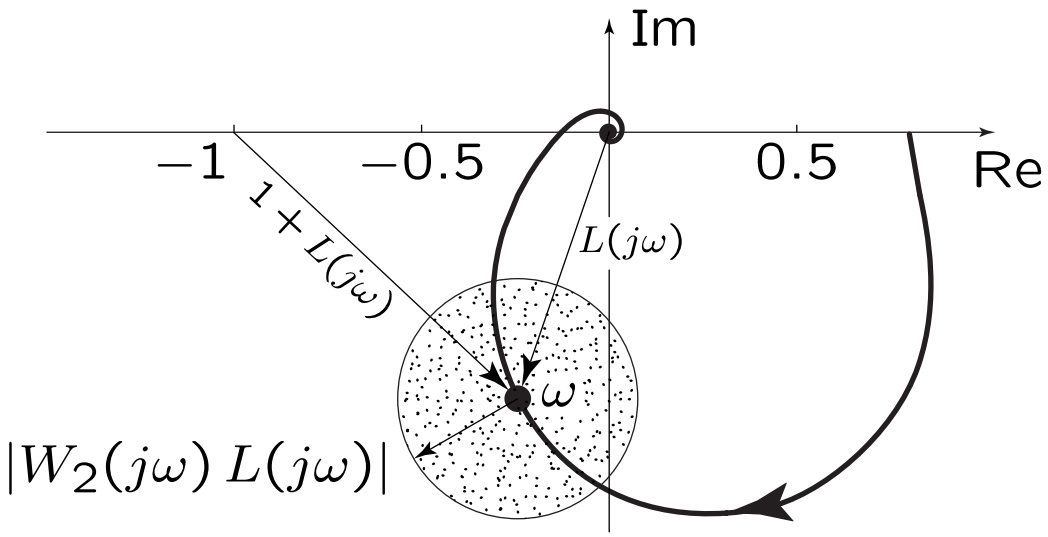


Robust Stability

Robust stability: Internal stability for all $\tilde{G}(s) \in \mathcal{G}$.

Theorem: The controller $K(s)$ provides robust stability iff $\|W_2 T\|_\infty < 1$.

$$
\begin{aligned}
& \|W_2 T\|_\infty < 1 \\
\Leftrightarrow \quad & \left| W_2(j\omega) \frac{L(j\omega)}{1 + L(j\omega)} \right| < 1 \qquad \forall \omega \\
\Leftrightarrow \quad & |W_2(j\omega)\, L(j\omega)| < |1 + L(j\omega)| \quad \forall \omega
\end{aligned}
$$



At each frequency, the critical point lies outside the disk of center $L(j\omega)$, radius $|W_2(j\omega)\, L(j\omega)|$.

Robust Performance

Robust performance: Internal stability and performance for all $\tilde{G}(s) \in \mathcal{G}$.

$$s \mapsto |W_1(s)\,S(s)| + |W_2(s)\,T(s)| = |W_1\,S| + |W_2\,T|$$

Theorem: The controller $K(s)$ provides robust performance iff $\| \, |W_1\,S| + |W_2\,T| \, \|_\infty < 1$.

$$\| \, |W_1\,S| + |W_2\,T| \, \|_\infty < 1$$

$$\Updownarrow$$

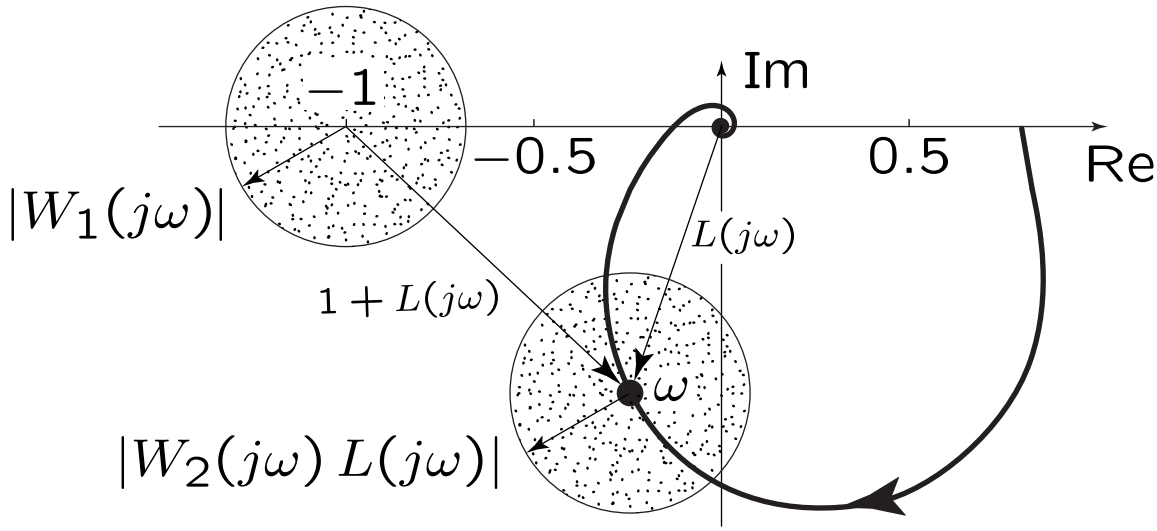
$$\left| W_1(j\omega)\,\frac{1}{1+L(j\omega)} \right| + \left| W_2(j\omega)\,\frac{L(j\omega)}{1+L(j\omega)} \right| < 1 \quad \forall\omega$$

$$\Updownarrow$$

$$\frac{|W_1(j\omega)| + |W_2(j\omega)\,L(j\omega)|}{|1+L(j\omega)|} < 1 \qquad \forall\omega$$

$$\Updownarrow$$

$$|W_1(j\omega)| + |W_2(j\omega)\,L(j\omega)| < |1+L(j\omega)| \qquad \forall\omega$$



At each frequency, the disks of center $L(j\omega)$, radius $|W_2(j\omega)\,L(j\omega)|$, and of center -1, radius $|W_1(j\omega)|$, are disjoint.

Design Constraints

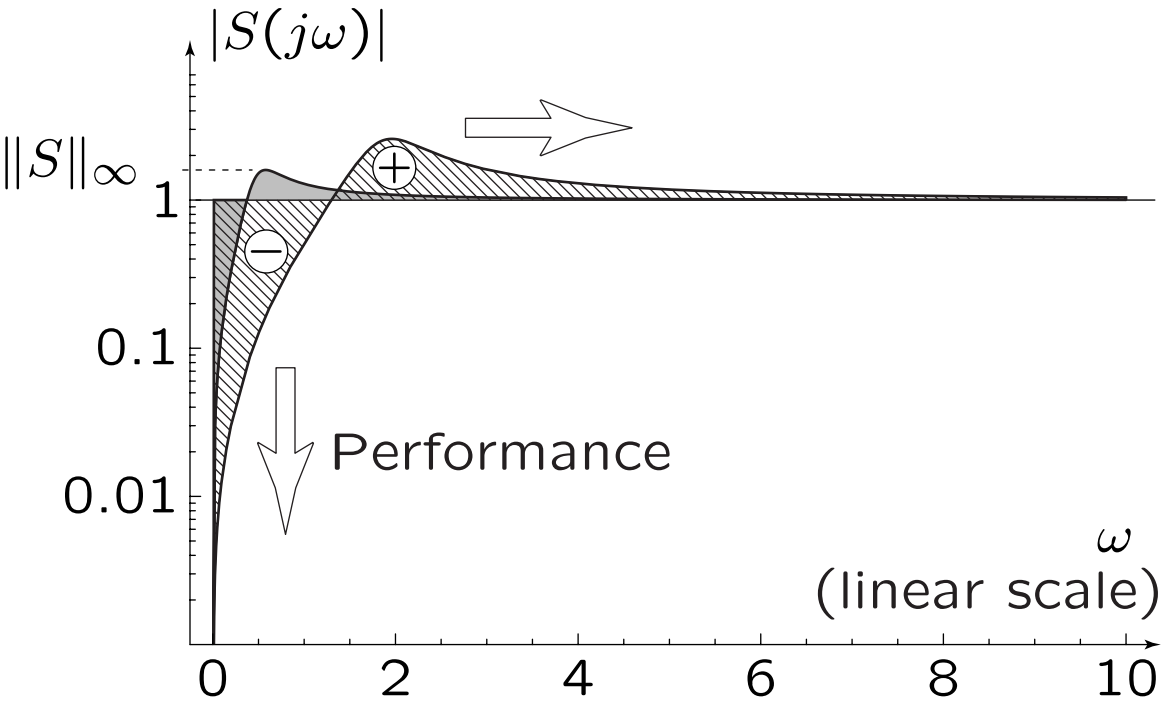
$$S(s) + T(s) = \frac{1}{1 + L(s)} + \frac{L(s)}{1 + L(s)} = 1$$

Robust performance

$$\Rightarrow \ \min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1 \quad \forall\omega$$

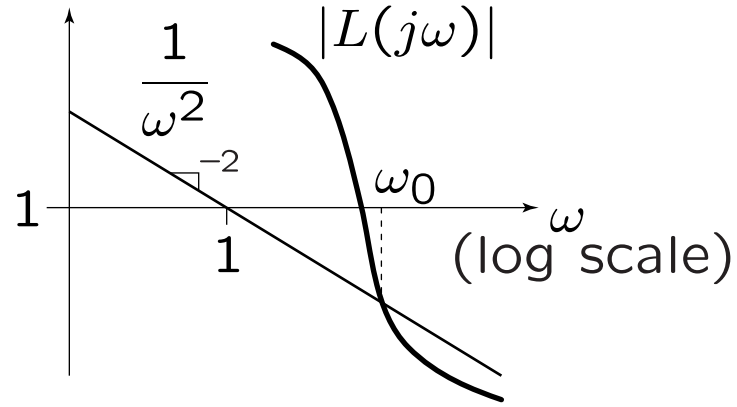
Theorem (area formula): Assume that the relative degree of $L(s)$ is at least 2 and let $\{p_i : i = 1, 2, \ldots, n\}$ denote the set of poles of $L(s)$ in $\operatorname{Re} s > 0$. Then

$$\int_0^\infty \log|S(j\omega)|\,\mathrm{d}\omega = \pi\,(\log e)\,\sum_{i=1}^{n} \operatorname{Re} p_i$$



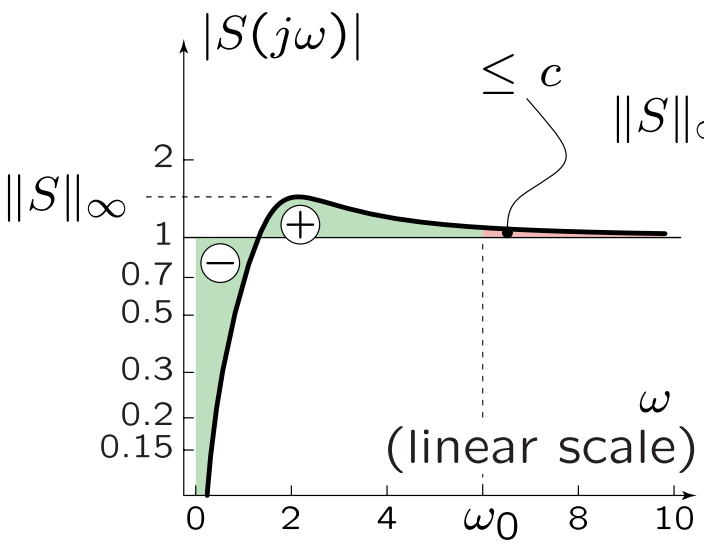
Waterbed Effect

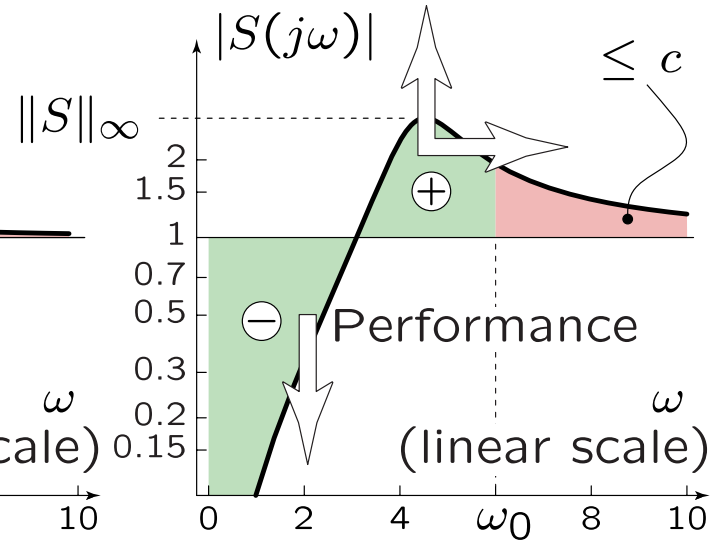
$$|L(j\omega)| < \frac{1}{\omega^2} \qquad \forall \omega \geq \omega_0 > 1$$



$$|S(j\omega)| = \frac{1}{|1 + L(j\omega)|} \leq \frac{1}{1 - |L(j\omega)|}$$
$$< \frac{1}{1 - \frac{1}{\omega^2}} = \frac{\omega^2}{\omega^2 - 1}$$

$$\int_{\omega_0}^{\infty} \log |S(j\omega)| \, \mathrm{d}\omega \leq \int_{\omega_0}^{\infty} \log \frac{\omega^2}{\omega^2 - 1} \, \mathrm{d}\omega = c < \infty$$



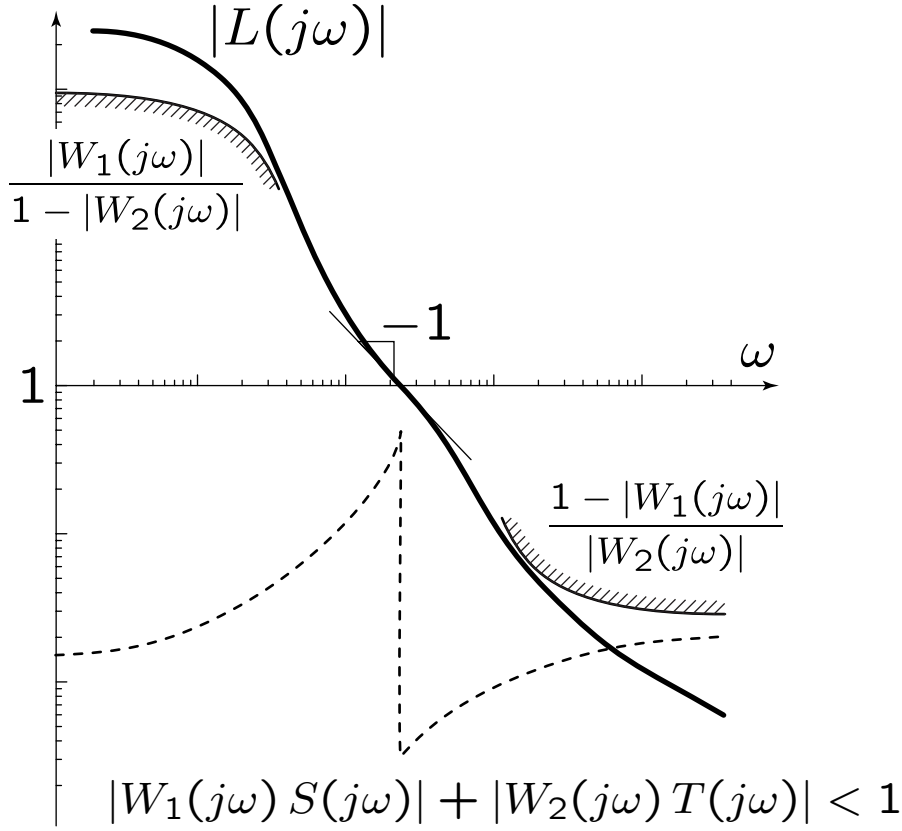


Standard Loopshaping

$G(s)$: Stable and minimum phase

$$|W_1(j\omega)| \gg 1 > |W_2(j\omega)| \;\; : \;\; |L(j\omega)| > \frac{|W_1(j\omega)|}{1 - |W_2(j\omega)|}$$

$$|W_1(j\omega)| < 1 \ll |W_2(j\omega)| \;\; : \;\; |L(j\omega)| < \frac{1 - |W_1(j\omega)|}{|W_2(j\omega)|}$$



Concluding Remarks

Robust stability, robust performance, design constraints, loopshaping, *etc.*, nicely extend classical control theory.

Extension to MIMO systems:

$$\|G\|_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$$

$\bar{\sigma}(G(j\omega))$: Largest singular value of $G(j\omega)$

Conservatism
High order controllers
Small set of performance specifications
Provides what can be achieved
etc.

Numerous other approaches to robust control

Kharitonov polynomials
Multimodel approach
Singular perturbation
Sliding mode control
Lyapunov methods
etc.