IC-32 # Optimal Control **Spring 2009**

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Problem Set #11

1. The objective of this problem is to analyze Fuller's famous problem and calculate an approximate solution using the sequential approach,

$$\begin{aligned}
\text{minimize:} \quad & \mathcal{J}(u) := \int_0^4 [x_1(t)]^2 \, dt && (1)\\
\text{subject to:} \quad & \dot{x}_1(t) = x_2(t); \quad x_1(0) = 2 && (2)\\
& \dot{x}_2(t) = u(t); \quad x_2(0) = -2 && (3)\\
& -1 \le u(t) \le 1, \forall t \in [0,4]. && (4)
\end{aligned}$$

(a) Using the Pontryagin Maximum Principle, find out the possible types of arc that may be contained in an optimal solution. In particular, show that an optimal solution for the Fuller's problem may have singular arcs of order $p = 2$, and derive the expression u_{sing} of the optimal control along such arcs.

(b) Although the various types of arcs composing an optimal solution are known, it is hard to find out *a priori* what the optimal arc sequence should be. As a first guess, we shall suppose here that the optimal control sequence consists of 2 arcs,

$$u^{(1)}(t) = \begin{cases} +1 & \text{for } 0 \le t \le t_1 \\ u_{\text{sing}} & \text{for } t_1 \le t \le 4 \end{cases}$$

In MatLab®, write a m-file calculating the optimal values of the junction time t_1 in the above expression of $u^{(1)}(\cdot)$, so that $\mathcal{J}(u^{(1)})$ is minimized. For example, this can be done via the sequential approach by determining the lengths Δt_1 and Δt_2 of each arc,

$$\begin{aligned}
\text{minimize:} \quad & x_3(\Delta t_1 + \Delta t_2) && (5)\\
\text{subject to:} \quad & \dot{x}_1(t) = x_2(t); \quad x_1(0) = 2 && (6)\\
& \dot{x}_2(t) = \begin{cases} u^U & \text{for } 0 \le t \le \Delta t_1 \\ u_{\text{sing}} & \text{for } \Delta t_1 \le t \le \Delta t_1 + \Delta t_2 \end{cases} \quad ; \quad x_2(0) = -2 && (7)\\
& \dot{x}_3(t) = [x_1(t)]^2; \quad x_3(0) = 0 && (8)\\
& \Delta t_1 + \Delta t_2 = 4 && (9)\\
& 0 \le \Delta t_1, \Delta t_2 \le 4. && (10)
\end{aligned}$$

Indications:

- Use the function fmincon to solve the parametric optimization problem (set the solution point tolerance, function tolerance and constraint tolerance to 10^{-10} in fmincon);
- Use the function ode15s to integrate the differential equations (6–8) on each arc, with state continuity between the arcs (set both the absolute and relative integration tolerances to 10^{-8});

- Consider arcs of equal durations as the initial guess for the optimization;
- For simplicity, let `fmincon` calculate a **finite-difference approximation** of the gradients of the objective function; in particular, set the minimum change in variables for finite differencing to 10^{-4}.

Plot the optimal response vs. time, as well as the optimal response in the phase space (i.e., x_2 vs. x_1). Also get the optimal values of the cost and calculate the function time t_1.

(c) Repeat the foregoing optimization approach for the following arc sequences:

$$u^{(3)}(t) = \begin{cases} -1 & \text{for } 0 \leq t \leq t_1 \\ +1 & \text{for } t_1 \leq t \leq t_2 \\ -1 & \text{for } t_2 \leq t \leq t_3 \\ u_{\text{sing}} & \text{for } t_3 \leq t \leq 4, \end{cases} \quad \text{and} \quad u^{(4)}(t) = \begin{cases} -1 & \text{for } 0 \leq t \leq t_1 \\ +1 & \text{for } t_1 \leq t \leq t_2 \\ -1 & \text{for } t_2 \leq t \leq t_3 \\ +1 & \text{for } t_3 \leq t \leq t_4 \\ u_{\text{sing}} & \text{for } t_4 \leq t \leq 4, \end{cases}$$

For each arc sequence, calculate the optimal cost value and junction times, and plot the optimal response vs. time, as well as the optimal response in the phase space. Comment the results.