IC-32 Optimal Control Spring 2009

Benoît Chachuat
ME C2 426, Ph: 35148, benoit@mcmaster.ca

Problem Set #10 (With Solutions)

1. The objective of this problem is to calculate, by three different methods, the gradient of a functional

$$\mathcal{F}(\mathbf{p}) := \phi(\mathbf{x}(t_\mathsf{f}), \mathbf{p}), \tag{1}$$

where $\mathbf{p} \in \mathbb{R}^{n_p}$ is a set of time-invariant parameters, and the state variables $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ are the solutions of an initial value problem (IVP)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{p}); \qquad \mathbf{x}(t_0) = \mathbf{h}(\mathbf{p}). \tag{2}$$

The functions ϕ, $\mathbf{f}$ and $\mathbf{h}$ are assumed to be continuously differentiable with respect to all their argument throughout.

For each $i = 1, \ldots, n_p$, estimates of the gradient can be calculated as follows:

- Finite differences approach:

$$\boldsymbol{\nabla}_{p_i}\mathcal{F}(\mathbf{p}) \approx \frac{\phi(\mathbf{x}(t_\mathsf{f}), p_1, \ldots, p_i + \delta p_i, \ldots, p_{n_p}) - \phi(\mathbf{x}(t_\mathsf{f}), \mathbf{p})}{\delta p_i}; \tag{3}$$

- Forward sensitivity approach:

$$\boldsymbol{\nabla}_{p_i}\mathcal{F}(\mathbf{p}) = \phi_\mathbf{x}(\mathbf{x}(t_\mathsf{f}), \mathbf{p})\ \mathbf{x}_{p_i}(t_\mathsf{f}) + \phi_{p_i}(\mathbf{x}(t_\mathsf{f}), \mathbf{p}) \tag{4}$$

$$\text{with:}\quad \dot{\mathbf{x}}_{p_i}(t) = \mathbf{f}_\mathbf{x}(t, \mathbf{x}(t), \mathbf{p})\ \mathbf{x}_{p_i}(t) + \mathbf{f}_{p_i}(t, \mathbf{x}(t), \mathbf{p}); \qquad \mathbf{x}_{p_i}(t_0) = \mathbf{h}_{p_i}(\mathbf{p}); \tag{5}$$

- Adjoint (reverse) sensitivity approach:

$$\boldsymbol{\nabla}_{p_i}\mathcal{F}(\mathbf{p}) = \phi_{p_i}(\mathbf{x}(t_\mathsf{f}), \mathbf{p}) + z_i(t_0) + \boldsymbol{\lambda}(t_0)^\mathsf{T}\mathbf{h}_{p_i}(\mathbf{p}) \tag{6}$$

$$\text{with:}\quad \dot{\boldsymbol{\lambda}}(t) = -\mathbf{f}_\mathbf{x}(t, \mathbf{x}(t), \mathbf{p})^\mathsf{T}\boldsymbol{\lambda}(t); \qquad \boldsymbol{\lambda}(t_\mathsf{f}) = \phi_\mathbf{x}(\mathbf{x}(t_\mathsf{f}), \mathbf{p}) \tag{7}$$

$$\dot{z}_i(t) = -\mathbf{f}_{p_i}(t, \mathbf{x}(t), \mathbf{p})^\mathsf{T}\boldsymbol{\lambda}(t); \qquad z_i(t_\mathsf{f}) = 0. \tag{8}$$

As an application, we shall consider the problem of a batch reactor doing the degradation of a substrate S by a biomass X, according to the auto-catalyzed reaction $S \rightarrow X$. The differential equations for the system are simply:

$$\dot{S}(t) = -\mu(S)X \tag{9}$$

$$\dot{X}(t) = \frac{1}{Y}\mu(S)X, \tag{10}$$

where the rate of reaction $\mu(S)$ is given by

$$\mu(S) = \mu_M \frac{S}{K + S}.$$

The parameter and initial concentration values are $Y = 0.75$, $K = 10$, $\mu_M = 0.1$, $S(0) = 5$, and $X(0) = 10$.

(a) In MATLAB®, calculate the concentration of biomass $X(t_f)$ at $t_f = 10$, by solving the ODEs (9,10).

Solution. A possible implementation is as follows:

main.m

```
clear all;
clf;
format long;

% Options for ODE solver
opt = odeset( 'RelTol', 1e-6, 'AbsTol', 1e-6 );

% Parameters
t0  = 0;
tf  = 10;
muM = 0.1;
K   = 10;
R   = 0.75;
S0  = 5;
X0  = 10;

% Forward state integration: t0 -> tf
[tx,yx] = ode23s(@(t,y)syst(t,y,muM,K,R), [t0,tf], [S0,X0], opt);
Xf=yx(end,2);
disp(sprintf('Biomass concentration at final time: %d', Xf));
```

syst.m

```
function [dy] = syst( t, y, muM, K, R )
  % State equations
  mu = muM * y(1) / ( K + y(1) );
  dy = [ - mu * y(2);
           mu / R * y(2) ];
end
```

With this implementation, the concentration of biomass at final time is found to be

$$X(10) \approx 13.9876.$$

(b) Using the finite difference approach, estimate the gradient of $X(t_f)$ with respect to the parameter μ_M and the initial concentration of biomass $X(0)$. In particular, try out different values for the perturbation parameter δp_i in (3).

Solution. A possible implementation is as follows (with the same function `syst.m` as before):

main.m

```
clear all;
clf;
format long;

% Options for ODE solver
opt = odeset( 'RelTol', 1e-6, 'AbsTol', 1e-6 );

% Parameters
t0  = 0;
tf  = 10;
```

```
muM = 0.1;
K   = 10;
R   = 0.75;
S0  = 5;
X0  = 10;

% Forward state integration: t0 -> tf, with default parameters
[tx,yx] = ode23s(@(t,y)syst(t,y,muM,K,R), [t0,tf], [S0,X0], opt);
Xf=yx(end,2);
disp(sprintf('Biomass concentration at final time: %d', Xf));

%Forward Finite Differences Approach (for various perturbations)
for iexp = -3:-2:-9
  tol = 10^iexp;
  disp(sprintf('\nFinite Differences Approach (Perturbation: %d):', tol));

  % Forward state integration: t0 -> tf, with perturbed muM
  muMp = muM*(1+tol);
  [tx,yx] = ode23s(@(t,y)syst(t,y,muMp,K,R), [t0,tf], [S0,X0], opt);
  Xfper=yx(end,2);
  dXf_FD(1) = (Xfper-Xf)/(muMp-muM);
  disp(sprintf('Sens. of biomass concentration at final time w.r.t. muM: %d',...
               dXf_FD(1)));

  % Forward state integration: t0 -> tf, with perturbed x0
  X0p = X0*(1+tol);
  [tx,yx] = ode23s(@(t,y)syst(t,y,muM,K,R), [t0,tf], [S0,X0p], opt);
  Xfper=yx(end,2);
  dXf_FD(2) = (Xfper-Xf)/(X0p-X0);
  disp(sprintf('Sens. of biomass concentration at final time w.r.t. X0:  %d',...
               dXf_FD(2)));
end

```

With the foregoing implementation, one gets the following results:

Perturbation	$X_{\mu_M}(10)$	$X_{X_0}(10)$
10^{-3}	31.19419	1.260091
10^{-5}	31.20564	1.260203
10^{-7}	31.20575	1.260205
10^{-9}	31.20670	1.260228

For the specified integration tolerances (10^{-6}), a suitable perturbation magnitude appears to be in the order of 10^{-5}–10^{-7}. That is, approximate sensitivity values for the biomass concentration at final time with respect to μ_M and X_0 are as follows:

$$X_{\mu_M}(10) \approx 31.2057, \qquad X_{X_0}(10) \approx 1.26020.$$

Next, we shall confirm these values based on forward and reverse sensitivity analyses.

(c) Write down the sensitivity equations (5) corresponding to (9,10), for the sensitivity parameters taken as μ_M and $X(0)$. In MATLAB®, solve the resulting equations, along with (9,10), and then calculate the gradient of $X(t_f)$ with respect to μ_M and $X(0)$ based on (4).

Solution. The sensitivity equations for the model (9,10) with respect to μ_M and $X(0)$ are given by:

$$
\begin{aligned}
\dot{S}_{\mu_M}(t) &= -\mu_S(S(t))\; X(t)\; S_{\mu_M}(t) - \mu(S(t))\; X_{\mu_M}(t) - \mu_{\mu_M}(S(t))\; X(t); && S_{\mu_M}(0) = 0\\
\dot{X}_{\mu_M}(t) &= \frac{1}{Y}\left[\mu_S(S(t))\; X(t)\; S_{\mu_M}(t) + \mu(S(t))\; X_{\mu_M}(t) + \mu_{\mu_M}(S(t))\; X(t)\right]; && X_{\mu_M}(0) = 0
\end{aligned}
$$

$$
\begin{aligned}
\dot{S}_{X_0}(t) &= -\mu_S(S(t))\; X(t)\; S_{X_0}(t) - \mu(S(t))\; X_{X_0}(t); && S_{X_0}(0) = 0\\
\dot{X}_{X_0}(t) &= \frac{1}{Y}\left[\mu_S(S(t))\; X(t)\; S_{X_0}(t) + \mu(S(t))\; X_{X_0}(t)\right]; && X_{X_0}(0) = 1
\end{aligned}
$$

with:

$$
\begin{aligned}
\mu_S(S) &= \mu_M \frac{K}{(K+S)^2}\\
\mu_{\mu_M}(S) &= \frac{S}{K+S}.
\end{aligned}
$$

An implementation of the forward sensitivity approach is given subsequently. Observe that the sensitivity equations are integrated, forward in time, along with the state equations.

main.m

```
clear all;
clf;
format long;

% Options for ODE solver
opt = odeset( 'RelTol', 1e-6, 'AbsTol', 1e-6 );

% Parameters
t0  = 0;
tf  = 10;
muM = 0.1;
K   = 10;
R   = 0.75;
S0  = 5;
X0  = 10;

% Forward state/sensitivity integration: t0 -> tf
[ts,ys] = ode23s(@(t,y)sens(t,y,muM,K,R), [t0,tf], [S0,X0,0,0,0,1], opt);

% Results
disp(sprintf('\nForward Sensitivity Approach:'));
dXf_FS(1) = ys(end,4);
disp(sprintf('Sens. of biomass concentration at final time w.r.t. muM: %d',...
             dXf_FS(1)));
dXf_FS(2) = ys(end,6);
disp(sprintf('Sens. of biomass concentration at final time w.r.t. X0:  %d',...
             dXf_FS(2)));
```

sens.m

```
function [dy] = sens( t, y, muM, K, R )
  % State & sensitivity equations
  mu = muM * y(1) / ( K + y(1) );
  dmudmuM = y(1) / ( K + y(1) );
  dmudS   = muM * K / ( K + y(1) )^2;
  dy = [ - mu * y(2);
           mu / R * y(2);
```

```
            - dmudS * y(2) * y(3) - mu * y(4) - dmudmuM * y(2);
              dmudS * y(2) / R * y(3) + mu / R * y(4) + dmudmuM / R * y(2);
            - dmudS * y(2) * y(5) - mu * y(6);
              dmudS * y(2) / R * y(5) + mu / R * y(6) ];
end
```

With the foregoing implementation, one gets the following results:

$$X_{\mu_M}(10) \approx 31.20519, \qquad X_{X_0}(10) \approx 1.260201,$$

which are rather close to those obtained via finite differences.

(d) Write down the adjoint equations (7,8) corresponding to (9,10). In MATLAB®, solve the resulting equations backward in time, by interpolating the state variables, and then calculate the gradient of $X(t_f)$ with respect to μ_M and $X(0)$ based on (6).

Solution. In the reverse mode of sensitivity analysis, the sensitivity values for the biomass concentration at final time with respect to μ_M and X_0 are obtained as:

$$\begin{aligned} X_{\mu_M}(10) &= z_{\mu_M}(0) \\ X_{X_0}(10) &= \lambda_X(0), \end{aligned}$$

where the adjoint equations for the model (9,10) are given by:

$$\begin{aligned}
\dot{\lambda}_S(t) &= \mu_S(S(t))\ X(t) \left(\lambda_S(t) - \frac{1}{Y}\lambda_X(t)\right); && \lambda_S(10) = 0 \\
\dot{\lambda}_X(t) &= \mu(S(t)) \left(\lambda_S(t) - \frac{1}{Y}\lambda_X(t)\right); && \lambda_X(10) = 1 \\
\dot{z}_{\mu_M}(t) &= \mu_{\mu_M}(S(t))\ X(t) \left(\lambda_S(t) - \frac{1}{Y}\lambda_X(t)\right); && z_{\mu_M}(10) = 0.
\end{aligned}$$

with the functions $\mu_S(\cdot)$ and $\mu_{\mu_M}(\cdot)$ defined earlier. Note also that $z_{X_0}(t) = 0, \forall t$, so the corresponding differential equation has been omitted.

An implementation of the adjoint sensitivity approach is given subsequently. Observe that the state equations are integrated first, forward in time; then, the adjoint equation are integrated, backward in time, using interpolated values for the state variables.

main.m

```
clear all;
clf;
format long;

% Options for ODE solver
opt = odeset( 'RelTol', 1e-6, 'AbsTol', 1e-6 );

% Parameters
t0  = 0;
tf  = 10;
muM = 0.1;
K   = 10;
R   = 0.75;
S0  = 5;
X0  = 10;

% Forward state integration: t0 -> tf
```

```
[tx,yx] = ode23s(@(t,y)syst(t,y,muM,K,R), [t0,tf], [S0,X0], opt);

% Backward adjoint integration: tf -> t0
[ta,ya] = ode23s(@(t,y)adj(t,y,muM,K,R,tx,yx), [tf,t0], [0,1,0], opt);

% Results
disp(sprintf('\n Reverse (Adjoint) Sensitivity Approach'));
dXf_AS(1) = ya(end,3);
disp(sprintf('Sens. of biomass concentration at final time w.r.t. muM: %d',...
             dXf_AS(1)));
dXf_AS(2) = ya(end,2);
disp(sprintf('Sens. of biomass concentration at final time w.r.t. X0:  %d',...
             dXf_AS(2)));
```

adj.m

```
function [dy] = adj( t, y, muM, K, R, tspan, xspan )
  % Calculate the states [S,X] at current time t by interpolation
  x = interp1( tspan, xspan(:,1:2), t, 'spline' );
  % Adjoint equations
  mu = muM * x(1) / ( K + x(1) );
  dmudmuM = x(1) / ( K + x(1) );
  dmudS   = muM * K / ( K + x(1) )^2;
  dy = [ dmudS * x(2) * ( y(1) - 1 / R * y(2) );
         mu * ( y(1) - 1 / R * y(2) );
         dmudmuM * x(2) * ( y(1) - 1 / R * y(2) ) ];
end
```

With the foregoing implementation, the following sensitivity values are obtained:

$$X_{\mu_M}(10) \approx 31.20510, \qquad X_{X_0}(10) \approx 1.260200,$$

which are again close to those obtained via forward sensitivity analysis as well as finite differences.

Indications:

- Use the function `ode15s` to integrate the differential equations (set both the absolute and relative integration tolerances to 10^{-6});
- In the adjoint approach, use the function `interp1` to interpolate the values of the state variables (use the option '`spline`' of `interp1`).

---

2. In this exercise, we consider the problem to find a minimizing control for the Bolza-type functional

$$\int_{t_0}^{t_\mathrm{f}} \ell(t, \mathbf{x}, \mathbf{u})\ \mathrm{d}t + \phi(\mathbf{x}(t_\mathrm{f})), \tag{11}$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}, \mathbf{u}); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{12}$$

where $\mathbf{u} \in \mathcal{C}[t_0, t_\mathrm{f}]$, and both the initial time t_0 and the terminal time t_f are fixed. It shall be supposed throughout that ℓ and $\mathbf{f}$ are continuous in $(t, \mathbf{x}, \mathbf{u})$ and have continuous first partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$, for all $(t, \mathbf{x}, \mathbf{u}) \in [t_0, t_\mathrm{f}] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$; ϕ is continuous and has continuous first partial derivatives with respect to $\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^{n_x}$.

(a) Prove that, if $\mathbf{u}^\star \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ is a (local) minimizer for (11,12), with response $\mathbf{x}^\star \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$, then there is a function $\boldsymbol{\lambda}^\star \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$ such that the triple $(\mathbf{u}^\star,\mathbf{x}^\star,\boldsymbol{\lambda}^\star)$ satisfies the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)); \qquad \mathbf{x}(t_0)=\mathbf{x}_0 \tag{13}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\ell_{\mathbf{x}}(t,\mathbf{x}(t),\mathbf{u}(t)) - \mathbf{f}_{\mathbf{x}}(t,\mathbf{x}(t),\mathbf{u}(t))^\mathsf{T}\boldsymbol{\lambda}(t); \qquad \boldsymbol{\lambda}(t_\mathrm{f}) = \boldsymbol{\phi}_{\mathbf{x}}(\mathbf{x}(t_\mathrm{f}) \tag{14}$$

$$\mathbf{0} = \ell_{\mathbf{u}}(t,\mathbf{x}(t),\mathbf{u}(t)) + \mathbf{f}_{\mathbf{u}}(t,\mathbf{x}(t),\mathbf{u}(t))^\mathsf{T}\boldsymbol{\lambda}(t). \tag{15}$$

for $t_0 \le t \le t_\mathrm{f}$.
[Hint: Use similar arguments to the proof of Theorem 3.9 in the class textbook.]

Solution. Consider a one-parameter family of comparison controls $\mathbf{v}(t;\eta) := \mathbf{u}^\star(t) + \eta\boldsymbol{\omega}(t)$, where $\boldsymbol{\omega}(t) \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ is some fixed function, and η is a (scalar) parameter. Based on the continuity and differentiability properties of $\mathbf{f}$, we know that there exists $\bar{\eta} > 0$ such that the response $\mathbf{y}(t;\eta) \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$ associated to $\mathbf{v}(t;\eta)$ through (12) exists, is unique, and is differentiable with respect to η, for all $\eta \in \mathcal{B}_{\bar{\eta}}(0)$ and for all $t \in [t_0,t_\mathrm{f}]$. Clearly, $\eta = 0$ provides the optimal response $\mathbf{y}(t;0) \equiv \mathbf{x}^\star(t)$, $t_0 \le t \le t_\mathrm{f}$.
Since the control $\mathbf{v}(t;\eta)$ is admissible and its associated response is $\mathbf{y}(t,\eta)$, we have

$$\begin{aligned}
\mathcal{J}(\mathbf{v}(t;\eta)) &= \int_{t_0}^{t_\mathrm{f}} \Big[\ell(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) + \boldsymbol{\lambda}(t)^\mathsf{T}\left[\mathbf{f}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) - \dot{\mathbf{y}}(t;\eta)\right]\Big]\ \mathrm{d}t + \phi(\mathbf{y}(t_\mathrm{f};\eta)) \\
&= \int_{t_0}^{t_\mathrm{f}} \Big[\ell(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) + \boldsymbol{\lambda}(t)^\mathsf{T}\mathbf{f}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) + \dot{\boldsymbol{\lambda}}(t)^\mathsf{T}\mathbf{y}(t;\eta)\Big]\ \mathrm{d}t \\
&\quad - \boldsymbol{\lambda}(t_\mathrm{f})^\mathsf{T}\mathbf{y}(t_\mathrm{f};\eta) + \boldsymbol{\lambda}(t_0)^\mathsf{T}\mathbf{y}(t_0;\eta) + \phi(\mathbf{y}(t_\mathrm{f};\eta)),
\end{aligned}$$

for any $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$ and for each $\eta \in \mathcal{B}_{\bar{\eta}}(0)$. Based on the differentiability properties of ℓ, ϕ, and $\mathbf{y}$, we have

$$\begin{aligned}
\frac{\partial}{\partial\eta}\mathcal{J}(\mathbf{v}(t;\eta)) &= \int_{t_0}^{t_\mathrm{f}} \Big[\ell_{\mathbf{u}}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) + \mathbf{f}_{\mathbf{u}}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta))^\mathsf{T}\boldsymbol{\lambda}(t)\Big]^\mathsf{T}\boldsymbol{\omega}(t)\ \mathrm{d}t \\
&\quad + \int_{t_0}^{t_\mathrm{f}} \Big[\ell_{\mathbf{x}}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta)) + \mathbf{f}_{\mathbf{x}}(t,\mathbf{y}(t;\eta),\mathbf{v}(t;\eta))^\mathsf{T}\boldsymbol{\lambda}(t) + \dot{\boldsymbol{\lambda}}(t)\Big]^\mathsf{T}\mathbf{y}_\eta(t;\eta)\ \mathrm{d}t \\
&\quad + \left[\phi_{\mathbf{x}}(\mathbf{y}(t_\mathrm{f};\eta)) - \boldsymbol{\lambda}(t_\mathrm{f})\right]^\mathsf{T}\mathbf{y}_\eta(t_\mathrm{f};\eta) + \boldsymbol{\lambda}(t_0)^\mathsf{T}\mathbf{y}_\eta(t_0;\eta),
\end{aligned}$$

for any $\boldsymbol{\omega} \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ and any $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$. Taking the limit as $\eta \to 0$, and since $\mathbf{y}_\eta(t_0;\eta) = 0$, we get

$$\begin{aligned}
\delta\mathcal{J}(\mathbf{u}^\star;\boldsymbol{\omega}) &= \int_{t_0}^{t_\mathrm{f}} \Big[\ell_{\mathbf{u}}(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t)) + \mathbf{f}_{\mathbf{u}}(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t))^\mathsf{T}\boldsymbol{\lambda}(t)\Big]^\mathsf{T}\boldsymbol{\omega}(t)\ \mathrm{d}t \\
&\quad + \int_{t_0}^{t_\mathrm{f}} \Big[\ell_{\mathbf{x}}(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t)) + \mathbf{f}_{\mathbf{x}}(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t))^\mathsf{T}\boldsymbol{\lambda}(t) + \dot{\boldsymbol{\lambda}}(t)\Big]^\mathsf{T}\mathbf{y}_\eta(t;0)\ \mathrm{d}t \\
&\quad + \left[\phi_{\mathbf{x}}(\mathbf{x}^\star(t_\mathrm{f})) - \boldsymbol{\lambda}(t_\mathrm{f})\right]^\mathsf{T}\mathbf{y}_\eta(t_\mathrm{f};0),
\end{aligned}$$

which is finite for each $\boldsymbol{\omega} \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ and each $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$, since the integrand is continuous on $[t_0,t_\mathrm{f}]$. That is, $\delta\mathcal{J}(\mathbf{u}^\star;\boldsymbol{\omega})$ exists for each $\boldsymbol{\omega} \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ and each $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$.
Since $\mathbf{u}^\star$ is a local minimizer, we get

$$0 = \int_{t_0}^{t_\mathrm{f}} \Big[\ell_{\mathbf{x}}^\star + {\mathbf{f}_{\mathbf{x}}^\star}^\mathsf{T}\boldsymbol{\lambda}(t) + \dot{\boldsymbol{\lambda}}(t)\Big]^\mathsf{T}\mathbf{y}_\eta(t;0) + \Big[\ell_{\mathbf{u}}^\star + {\mathbf{f}_{\mathbf{u}}^\star}^\mathsf{T}\boldsymbol{\lambda}(t)\Big]^\mathsf{T}\boldsymbol{\omega}(t)\ \mathrm{d}t + \left[\phi_{\mathbf{x}}^\star - \boldsymbol{\lambda}\right]_{t=t_\mathrm{f}}^\mathsf{T}\mathbf{y}_\eta(t_\mathrm{f};0),$$

for each $\boldsymbol{\omega} \in \mathcal{C}[t_0,t_\mathrm{f}]^{n_u}$ and each $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0,t_\mathrm{f}]^{n_x}$; where the compressed notations $\ell_z^\star := \ell_z(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t))$, $\mathbf{f}_z^\star := \mathbf{f}_z(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t))$, and $\phi_z^\star = \phi_z(\mathbf{x}^\star(t_\mathrm{f}))$ are used.
Because the effect of a variation of the control on the course of the response is hard to determine (i.e., $\mathbf{y}_\eta(t;0)$), we choose $\boldsymbol{\lambda}^\star(t)$, $t_0 \le t \le t_\mathrm{f}$, so as to obey the differential equation

$$\dot{\boldsymbol{\lambda}}(t) = -{\mathbf{f}_{\mathbf{x}}^\star}^\mathsf{T}\boldsymbol{\lambda}(t) - \ell_{\mathbf{x}}^\star, \tag{16}$$

with the terminal condition

$$\boldsymbol{\lambda}(t_{\rm f}) = \phi^{\star}_{\mathbf{x}}.$$

Note that (16) being a linear system of ODEs, and from the regularity assumptions on ℓ and $\mathbf{f}$, the solution $\boldsymbol{\lambda}^\star$ exists and is unique over $[t_0, t_{\rm f}]$, i.e., $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0, t_{\rm f}]^{n_x}$. That is, the condition

$$0 = \int_{t_0}^{t_{\rm f}} \left[\ell^{\star}_{\mathbf{u}} + {\mathbf{f}^{\star}_{\mathbf{u}}}^{\mathsf{T}} \boldsymbol{\lambda}^\star(t)\right]^{\mathsf{T}} \boldsymbol{\omega}(t)\ \mathrm{d}t,$$

must hold for any $\boldsymbol{\omega} \in \mathcal{C}[t_0, t_{\rm f}]^{n_u}$. In particular, for $\boldsymbol{\omega}(t) := \ell^{\star}_{\mathbf{u}} + {\mathbf{f}^{\star}_{\mathbf{u}}}^{\mathsf{T}} \boldsymbol{\lambda}^\star(t)$, we get

$$0 = \int_{t_0}^{t_{\rm f}} \left[\ell^{\star}_{\mathbf{u}} + {\mathbf{f}^{\star}_{\mathbf{u}}}^{\mathsf{T}} \boldsymbol{\lambda}^\star(t)\right]^2 \mathrm{d}t,$$

which, in turn, implies the necessary condition that

$$0 = \ell_{\mathbf{u}}(t, \mathbf{x}^\star(t), \mathbf{u}^\star(t)) + \mathbf{f}_{\mathbf{u}}(t, \mathbf{x}^\star(t), \mathbf{u}^\star(t))^{\mathsf{T}} \boldsymbol{\lambda}^\star(t),$$

for each $t \in [t_0, t_{\rm f}]$.

(b) Justify the adjoint terminal condition $\boldsymbol{\lambda}^\star(t_{\rm f}) = \boldsymbol{\phi}_{\mathbf{x}}(\mathbf{x}(t_{\rm f}))$ in light of the discussion help in §3.4.4 of the class textbook.

Solution. We saw in §3.4.4 of the class textbook that $\boldsymbol{\lambda}^\star(t)$, $t_0 \leq t \leq t_{\rm f}$, can be interpreted as the rate of change of the optimal cost value if the state of the system were slightly perturbed at time t. Here, it is easy to see that the integral part of the optimal cost value would *not* change if the state at $t_{\rm f}$ were slightly perturbed. On the other hand, the terminal term would be modified as

$$\phi(\mathbf{x}^\star(t_{\rm f}) + \boldsymbol{\delta}\mathbf{x}(t_{\rm f})) = \phi(\mathbf{x}^\star(t_{\rm f})) + \phi_{\mathbf{x}}(t_{\rm f}, \mathbf{x}(t_{\rm f}))^{\mathsf{T}} \boldsymbol{\delta}\mathbf{x}(t_{\rm f}) + o(\boldsymbol{\delta}\mathbf{x}(t_{\rm f})).$$

That is, the first variation $\delta\mathcal{J}$ in the optimal cost value would be

$$\delta\mathcal{J} = \phi_{\mathbf{x}}(\mathbf{x}^\star(t_{\rm f}))^{\mathsf{T}} \boldsymbol{\delta}\mathbf{x}(t_{\rm f}),$$

This interpretation thus confirms the terminal adjoint condition

$$\boldsymbol{\lambda}^\star(t_{\rm f}) = \phi_{\mathbf{x}}(\mathbf{x}^\star(t_{\rm f})).$$

(c) **Application:** Consider the optimal control problem

$$\text{minimize:} \quad \int_0^1 [x(t) - u(t)]\ \mathrm{d}t + x(1) \qquad (17)$$

$$\text{subject to:} \quad \dot{x}(t) \ = \ 1 + [u(t)]^2; \quad x(0) \ = \ 1. \qquad (18)$$

i. Find candidate solutions to the problem (17,18) based on the first-order necessary conditions developed above.

Solution. We start by forming the Hamiltonian function

$$\mathcal{H}(x, u, \lambda) \ = \ x - u + \lambda(1 + u^2).$$

Necessary conditions of optimality are given by (18) and

$$\begin{aligned} \dot{x}(t) &= \mathcal{H}_\lambda \ = \ 1 + [u(t)]^2; \quad x(0) \ = \ 1 \\ \dot{\lambda}^\star(t) &= -\mathcal{H}_x \ = \ -1; \quad \lambda^\star(1) \ = \ \phi_{\mathbf{x}} \ = \ 1 \\ 0 &= \mathcal{H}_u \ = \ -1 + 2\lambda^\star u^\star. \end{aligned}$$

The adjoint equation is trivially integrated as

$$\lambda^\star(t) \;=\; 2-t,$$

and from the stationarity condition $\mathcal{H}_u = 0$, we get

$$u^\star(t) \;=\; \frac{1}{2(2-t)}.$$

(Note that $u^\star$ is indeed a candidate *minimum* solution for the problem since $\mathcal{H}_{uu} = 2(2-t) > 0$ for each $0 \leq t \leq 1$.) Finally, substituting the optimal control candidate back into (18) yields

$$\dot{x}^\star(t) \;=\; 1 + \frac{1}{4(2-t)^2}; \quad x(0) \;=\; 1.$$

Integrating this last equation, and drawing the results together, we obtain

$$u^\star(t) \;= \frac{1}{2(2-t)} \tag{19}$$

$$x^\star(t) \;= t + \frac{1}{4(2-t)} + \frac{7}{8} \tag{20}$$

$$\lambda^\star(t) \;= 2-t. \tag{21}$$

ii. Check that the Hamiltonian function $\mathcal{H} = \ell + \boldsymbol{\lambda}^\mathsf{T}\mathbf{f}$ is constant along the candidate optimal solution(s).

Solution. Since the problem is autonomous (neither ℓ nor $\mathbf{f}$ depend explicitely on t), the Hamiltonian function should be constant along $(u^\star, x^\star, \lambda^\star)$. Here, it is easily established that

$$\mathcal{H}(x^\star(t), u^\star(t), \lambda^\star(t)) \;=\; \frac{23}{8},$$

for each $t \in [0,1]$.

(d) Let $(\mathbf{u}^\star, \mathbf{x}^\star, \boldsymbol{\lambda}^\star)$ be a solution to the Euler-Lagrange equations (13–15). Prove that for $\mathbf{u}^\star$ to be a global minimizer to (11,12), it is sufficient that:

(i) ℓ and $\mathbf{f}$ be jointly convex in $(\mathbf{u}, \mathbf{x})$, for each $(t, \mathbf{u}, \mathbf{x}) \in [t_0, t_\mathrm{f}] \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x}$;

(ii) ϕ be convex in $\mathbf{x}$, for each $\mathbf{x} \in \mathbb{R}^{n_x}$; and

(iii) $\boldsymbol{\lambda}^\star(t) \geq \mathbf{0}$ (component-wise), for each $t \in [t_0, t_\mathrm{f}]$.

[Hint: Use similar arguments to the proof of Theorem 3.11 in the class textbook.]

Solution. Provided that conditions (i) and (ii) hold, for any feasible control $\mathbf{u}(t)$ and its associated response $\mathbf{x}(t)$, we have

$$\begin{aligned}
\mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}^\star) \;&=\; \int_{t_0}^{t_\mathrm{f}} \left[\ell(t, \mathbf{x}(t), \mathbf{u}(t)) - \ell(t, \mathbf{x}^\star(t), \mathbf{u}^\star(t))\right]\, \mathrm{d}t + \left[\phi(\mathbf{x}(t_\mathrm{f})) - \phi(\mathbf{x}^\star(t_\mathrm{f}))\right] \\
&\geq\; \int_{t_0}^{t_\mathrm{f}} \left({\ell_\mathbf{x}^\star}^\mathsf{T} \left[\mathbf{x}(t) - \mathbf{x}^\star(t)\right] + {\ell_\mathbf{u}^\star}^\mathsf{T} \left[\mathbf{u}(t) - \mathbf{u}^\star(t)\right]\right) \mathrm{d}t + {\phi_\mathbf{x}^\star}^\mathsf{T} \left[\mathbf{x}(t_\mathrm{f}) - \mathbf{x}^\star(t_\mathrm{f})\right],
\end{aligned}$$

with the usual compressed notation. Since the triple $(\mathbf{u}^\star, \mathbf{x}^\star, \boldsymbol{\lambda}^\star)$ satisfies the Euler-Lagrange equations (13–15), we obtain

$$\begin{aligned}
\mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}^\star) \;\geq\; &\int_{t_0}^{t_\mathrm{f}} \left(-\left[{\mathbf{f}_\mathbf{x}^\star}^\mathsf{T} \boldsymbol{\lambda}^\star(t) + \dot{\boldsymbol{\lambda}}^\star(t)\right]^\mathsf{T} \left[\mathbf{x}(t) - \mathbf{x}^\star(t)\right] - \left[{\mathbf{f}_\mathbf{u}^\star}^\mathsf{T} \boldsymbol{\lambda}^\star(t)\right]^\mathsf{T} \left[\mathbf{u}(t) - \mathbf{u}^\star(t)\right]\right) \mathrm{d}t \\
&+ {\phi_\mathbf{x}^\star}^\mathsf{T} \left[\mathbf{x}(t_\mathrm{f}) - \mathbf{x}^\star(t_\mathrm{f})\right],
\end{aligned}$$

Integrating by part the term in $\dot{\boldsymbol{\lambda}}^\star(t)$, and rearranging the terms, we get

$$\begin{aligned}\mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}^\star) \;\geq\; & \int_{t_0}^{t_f} \boldsymbol{\lambda}^\star(t)^\mathsf{T} \left(\mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)) - \mathbf{f}(t,\mathbf{x}^\star(t),\mathbf{u}^\star(t)) - \mathbf{f}_\mathbf{x}^\star\left[\mathbf{x}(t)-\mathbf{x}^\star(t)\right] - \mathbf{f}_\mathbf{u}^\star\left[\mathbf{u}(t)-\mathbf{u}^\star(t)\right]\right)\,\mathrm{d}t \\ & + \left[\phi_\mathbf{x}^\star - \boldsymbol{\lambda}^\star(t_f)\right]^\mathsf{T}\left[\mathbf{x}(t_f)-\mathbf{x}^\star(t_f)\right] + \boldsymbol{\lambda}^\star(t_0)^\mathsf{T}\left[\mathbf{x}(t_0)-\mathbf{x}^\star(t_0)\right].\end{aligned}$$

Note that the integrand is positive provided that conditions (i) and (iii) hold; the remaining two terms are equal to zero due to the optimal adjoint boundary conditions and the prescribed state initial conditions, respectively, in the Euler-Lagrange equations. That is,

$$\mathcal{J}(\mathbf{u}) \;\geq\; \mathcal{J}(\mathbf{u}^\star),$$

for each feasible control if conditions (i), (ii), and (iii) hold.

(e) **Application:** What can be said about the candidate optimal control(s) found previously for the problem (17,18)?

Solution. The integrand term is linear (and thus convex) in (u, x), the salvage term is also linear with respect to x, and the right-hand side of the differential equation (18) is (strictly) convex in u (and independent of x). Moreover, the (unique) solution (19,21) to the Euler-Lagrange equations (13–15) is such that

$$\lambda^\star(t) \;:=\; 2-t \;\geq 0,$$

for each $t \in [0,1]$. We can therefore conclude that $u^\star(t) := \frac{1}{2(2-t)}$, $0 \leq t \leq 1$, is a (global) optimal control for the problem (17,18).