IC-32 **Optimal Control** **Spring 2009**

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Problem Set #8

1. The objective of this problem is to solve, via the **direct simultaneous approach**, the following scalar optimal control problem:

$$\text{minimize:} \quad \int_0^1 \tfrac{1}{2}[u(t)]^2 \, \mathrm{d}t \tag{1}$$

$$\text{subject to:} \quad \dot{x}(t) = -x(t) + u(t); \qquad x_1(0) = 1 \tag{2}$$

$$x(1) = 0. \tag{3}$$

For simplicity, a **piecewise-constant control parameterization** over n_s stages is considered for approximating the control profile,

$$u(t) = \omega^k, \quad t_{k-1} \leq t \leq t_k, \quad k = 1, \ldots, n_s,$$

with the time stages being equally spaced, $t_k = \frac{k}{n_s}$.

On the other hand, the response is to be approximated, on each stage, by **Lagrange polynomials of degree** $N = 3$,

$$x(t) = \sum_{i=0}^{3} \xi_i^k \phi_i^{(3)} \left(\tfrac{t - t_{k-1}}{t_k - t_{k-1}} \right), \quad t_{k-1} \leq t \leq t_k, \quad k = 1, \ldots, n_s,$$

with

$$\phi_i^{(3)}(\tau) = \prod_{\substack{q=0 \\ q \neq i}}^{3} \frac{\tau - \tau_q}{\tau_i - \tau_q},$$

and the collocation points $\tau_0, \ldots, \tau_3$ as given in the following table:

τ_0	τ_1	τ_2	τ_3
0	0.25	0.5	0.75

Questions:

(a) Reformulate the optimal control problem into the Mayer form, then discretize this problem as per the direct simultaneous approach; give the expressions of the cost and all the constraints in the resulting NLP problem.

(b) In MatLab®, write two m-files that calculate the cost value and the constraint values in the foregoing NLP problem, respectively, for a variable number n_s of stages and corresponding values of the control coefficients $\omega^1, \ldots, \omega^{n_s}$ and the state coefficients $\xi^1, \ldots, \xi^{n_s}, \zeta^1, \ldots, \zeta^{n_s}$.

  - Use the m-files `lagrange.m` and `dlagrange.m` to calculate the Lagrange polynomials and their time derivatives; these functions can be retrieved from the class website

(c) Solve the fully-discretized NLP problem using the `fmincon` function in MatLab's Optimization Toolbox:

  - Set all the control and state coefficients equal to zero as the initial guess

- For simplicity, let `fmincon` calculate a finite-difference approximation of the cost and constraint derivatives; in particular, set the minimum change in variable for finite differencing to 10^{-10}
- Make sure to select the medium-scale SQP algorithm with quasi-Newton update and line-search, and set the solution point tolerance, the function tolerance and the constraint tolerance all to 10^{-6}

Plot the results for $n_s = 2$, 5, 10 and 20 stages.

(d) Compare these results with the analytical solution, as given in Example 4.23 (pp.133-134) of the class textbook.

(e) Modify the m-files so that the cost and constraint derivatives with respect to the control/state coefficients are also calculated analytically.

- Needless to say, always cross-check the analytical derivatives by comparison with forward finite differences

Resolve the fully-discretized NLP problem for $n_s = 2$, 5, 10 and 20 stages.

2. Consider the functional:

$$\mathcal{F}(\mathbf{p}) = x(1)$$

$$\text{with:} \quad \dot{x}(t) = -x(t) + p_1; \quad x(0) = p_2$$

(a) Determine the expression of $\mathcal{F}(\mathbf{p})$ by first solving the ODE analytically; then, determine analytical expression for the derivatives $\mathcal{F}_{p_1}(\mathbf{p})$ and $\mathcal{F}_{p_2}(\mathbf{p})$.

(b) Derive the sensitivity equations and their respective initial conditions, then find an analytical solution to these equations; apply the forward sensitivity formula to calculate $\mathcal{F}_{p_1}(\mathbf{p})$ and $\mathcal{F}_{p_2}(\mathbf{p})$, then compare with the expressions obtained ealier.

(c) Derive the adjoint equation and its terminal condition, then find an analytical solution to this equation; apply the adjoint sensitivity formula to calculate $\mathcal{F}_{p_1}(\mathbf{p})$ and $\mathcal{F}_{p_2}(\mathbf{p})$, then compare with the expressions obtained ealier.