APPENDIX A

A.1 NOTATIONS

The following notations are used throughout the textbook. Scalars are denoted by lowercase Roman or Greek letters, e.g., k, α, and μ. $\mathbb{R}^n$ denotes the n-dimensional *real Euclidean space*, composed of all real vectors of dimension n; such vectors are denoted by boldface lowercase Roman or Greek letters, e.g., $\mathbf{x}$, $\mathbf{y}$, and $\boldsymbol{\nu}$. All vectors are column vectors unless stated otherwise. Row vectors are the transpose of column vectors, e.g., $\mathbf{x}^\mathsf{T}$ denotes the row vector $(x_1, \ldots, x_n)$. Matrices are denoted by san serif capital Roman or boldface capital Greek letters, e.g., A, B, and $\boldsymbol{\Psi}$.

$\mathcal{C}([a,b])$ [resp. $\mathcal{C}^k([a,b])$] stands for the set of real-valued, continuous [resp. k times continuously differentiable] functions on the interval $[a,b]$; such functions are denoted by lowercase Roman or Greek letters, e.g., f, g, and φ. $\mathcal{C}([a,b])^n$ [resp. $\mathcal{C}^k([a,b])^n$] stands for the set of vector-valued, continuous [resp. k times continuously differentiable] functions on the interval $[a,b]$; vector-valued functions are denoted by boldface lowercase Roman letters, e.g., $\mathbf{h}$ and $\boldsymbol{\psi}$.

Finally, the following abbreviations are used in this textbook:

$:=$	*"defined as..."*
$\in$	*"is an element of..."* or *"is in..."*
$\notin$	*"is not an element of..."*
$\exists$	*"there exists..."*
$\forall$	*"for each..."* or *"for every..."*
$\Box$	*"end of proof."*

A.2 ELEMENTARY CONCEPTS FROM REAL ANALYSIS

We recall some elementary concepts from real analysis [50].

Definition A.1 (Open Ball). *The* open ball *of radius ε centered at $\bar{\mathbf{x}}$ is defined to be the set*

$$\mathcal{B}_{\varepsilon}\left(\bar{\mathbf{x}}\right) := \{\mathbf{x} \in \mathrm{I\!R}^n : \|\bar{\mathbf{x}} - \mathbf{x}\| < \varepsilon\},$$

in any norm $\|\cdot\|$. The corresponding deleted open ball *is defined by*

$$\dot{\mathcal{B}}_{\varepsilon}\left(\bar{\mathbf{x}}\right) := \mathcal{B}_{\varepsilon}\left(\bar{\mathbf{x}}\right) \setminus \{\bar{\mathbf{x}}\}.$$

Definition A.2 (Interior Point, Openness, Limit Point, Closedness, Boundedness, Compactness, Boundary Point, Closure). *Let D be a set in $\mathrm{I\!R}^n$, $n \geq 1$.*

Interior Point *A point $\mathbf{x} \in \mathrm{I\!R}^n$ is said to be an* interior point *of D if there is an open ball $\mathcal{B}_{\varepsilon}(\mathbf{x})$ such that $\mathcal{B}_{\varepsilon}(\mathbf{x}) \subset D$. The* interior *of a set D, denoted* $\text{int}\,(D)$*, is the set of interior points of D. A point $\mathbf{x} \in \mathrm{I\!R}^n$ is said to be an* exterior point *of D if it is an interior point of $\mathrm{I\!R}^n \setminus D$.*

Openness *D is said to be* open *if every point of D is an interior point of D. Obviously, if D is open then* $\text{int}\,(D) = D$.

Limit Point *A point $\bar{\mathbf{x}} \in \mathrm{I\!R}^n$ is said to be a* limit point *of the set D if every open ball $\mathcal{B}_{\varepsilon}(\bar{\mathbf{x}})$ contains a point $\mathbf{x} \neq \bar{\mathbf{x}}$ such that $\mathbf{x} \in D$. Note in particular that $\bar{\mathbf{x}}$ does not necessarily have to be an element of D to be a limit point of D.*

Closedness *D is said to be* closed *if every limit point of D is an element of D. Note that there do exist sets that are both open and closed, as well as sets that are neither closed nor open.*

Boundedness *D is said to be* bounded *if there is a real number M such that*

$$\|\mathbf{x}\| \leq M \quad \forall \mathbf{x} \in D$$

in any norm.

Compactness *In $\mathrm{I\!R}^n$, D is said to be* compact *if it is both closed and bounded.*

Boundary Point *A point $\bar{\mathbf{x}} \in \mathrm{I\!R}^n$ is said to be a* boundary point *of the set D if every neighborhood of $\bar{\mathbf{x}}$ contains points both inside and outside of D. The set of boundary point of D is denoted by ∂D.*

Closure *The* closure *of D is the set* $\text{cl}\,(D) := D \cup L$ *where L denotes the set of all limit points of D.*

A.3 CONVEX ANALYSIS

This subsection summarizes a number of important definitions and results related to convex sets (§ A.3.1) and convex functions (§ A.3.2 and A.3.3). Indeed, the notions of convex sets and convex functions play a crucial role in the theory of optimization. In particular, strong theoretical results are available for convex optimization problems (see, e.g., § 1.3).

A.3.1 Convex Sets

Definition A.3 (Convex Set). *A set $C \subset \mathbb{R}^n$ is said to be* convex *if for every points $\mathbf{x}, \mathbf{y} \in C$, the points*

$$\mathbf{z} := \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \quad \forall \lambda \in [0,1],$$

are also in the set C.

It is important to note that for a set C to be convex, (A.3) must hold for **all** pairs of points in the set C. Geometrically, for C to be convex every point on the line segment connecting any two points in C must also be in C. Fig. A.1. show an example of a convex set. Note that the line segment joining the points $\mathbf{x}$ and $\mathbf{y}$ lies completely inside C, and this is true for all pairs of points in C. On the other hand, Fig. A.2. shows an example of a *nonconvex* set (i.e., a set that is not convex). Observe that not all points on the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ lie in the set D, immediately indicating that D in nonconvex.

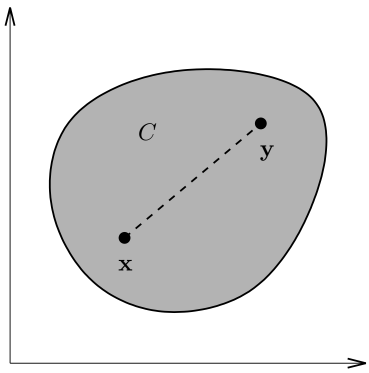

Figure A.1. A convex set.

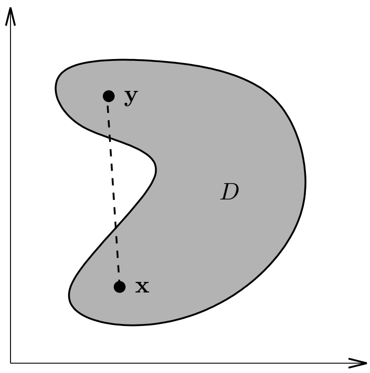

Figure A.2. A nonconvex set.

Example A.4. The set $C := \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ is convex.

Lemma A.5 (Intersection of Convex Sets). *Let C_1 and C_2 be convex sets in $\mathbb{R}^n$. Then, $C_1 \cap C_2$ is convex, i.e., the intersection of two convex sets in a convex set.*

Proof. The proof is left to the reader as an exercise. □

Remark A.6. Note that by Definition A.3, an empty set is convex (this is because no counterexample can be found to show that it is nonconvex). That is, Lemma A.5 holds even if C_1 and C_2 do not share any common elements. Note also that Lemma A.5 can be readily extended, by induction, to the intersection of any family of convex sets.

Definition A.7 (Hyperplane, Halfspace). *Let $\mathbf{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then, $H := \{\mathbf{x} : \mathbf{a}^\mathsf{T}\mathbf{x} = c\}$ is said to be a* hyperplane *and $H^+ := \{\mathbf{x} : \mathbf{a}^\mathsf{T}\mathbf{x} \geq c\}$ is said to be a* halfspace.

Theorem A.8 (Separation of a Convex Set and a Point). *Let C be a nonempty, convex set in $\mathbb{R}^n$ and let $\mathbf{y} \notin C$. Then, there exists a nonzero vector $\mathbf{a} \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$ such that:*

$$\mathbf{a}^\mathsf{T}\mathbf{y} > c \quad \textit{and} \quad \mathbf{a}^\mathsf{T}\mathbf{x} \leq c \quad \forall \mathbf{x} \in C.$$

Proof. See, e.g., [6, Theorem 2.4.4] for a proof. ☐

In fact, $\mathbf{a}^\mathsf{T}\mathbf{y} = c$ defines a separating hyperplane, as illustrated in Fig. A.3. below.

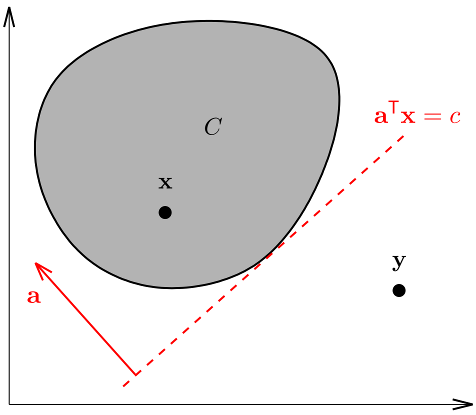

Figure A.3. Illustration of the Separation Theorem.

Theorem A.9 (Separation of Two Convex Sets). *Let C_1 and C_2 be two nonempty, convex set in $\mathbb{R}^n$ and suppose that $C_1 \cap C_2 = \emptyset$. Then, there exists a hyperplane that separates C_1 and C_2; that is, there exists a nonzero vector $\mathbf{p} \in \mathbb{R}^n$ such that*

$$\mathbf{p}^\mathsf{T}\mathbf{x_1} \geq \mathbf{p}^\mathsf{T}\mathbf{x_2} \quad \forall \mathbf{x_1} \in \mathrm{cl}\,(C_1)\,,\ \forall \mathbf{x_2} \in \mathrm{cl}\,(C_2)\,.$$

Proof. See, e.g., [6, Theorem 2.4.8] for a proof. ☐

Definition A.10 (Cone, Convex Cone). *A nonempty set $C \subset \mathbb{R}^n$ is said to be a* cone *if for every point $\mathbf{x} \in C$,*

$$\alpha\mathbf{x} \in C \quad \forall \alpha \geq 0.$$

If, in addition, C is convex then it is said to be a convex cone*.*

A.3.2 Convex and Concave Functions

Definition A.11 (Convex Function, Strictly Convex Function). *A function $f : C \to \mathbb{R}$ defined on a convex set $C \in \mathbb{R}^n$ is said to be* convex *if*

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}), \tag{A.1}$$

for each $\mathbf{x}, \mathbf{y} \in C$ and each $\lambda \in (0,1)$; that is, the value of the function on the line segment connecting any two points in the convex set C lies below the line segment in $C \times \mathbb{R}$ connecting the value of the function at the same two points in C. Moreover, f is said to be strictly convex *if*

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}), \tag{A.2}$$

for each $\mathbf{x}, \mathbf{y} \in C$ and each $\lambda \in (0,1)$.

The left plot in Fig. A.4. illustrates the definition: the line segment connecting the values of the function at any two points $\mathbf{x}$ and $\mathbf{y}$ in C lies above the function between $\mathbf{x}$ and $\mathbf{y}$. It should be noted that this alone does not establish that the function is convex on C; the set C itself should be a convex set. A function that is not convex is said to be *nonconvex*. the right plot in Fig. A.4. shows an example of a nonconvex function on the set C. Note that the dotted portion of the line segment connecting the values of the function at $\mathbf{x}$ and $\mathbf{y}$ lies below the function. Yet, this function is convex on the set C'.

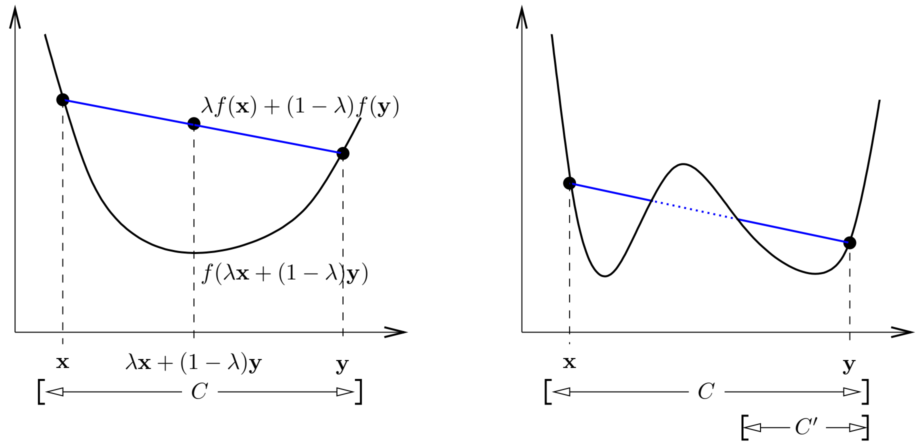

Figure A.4. Illustration of a convex function on C (left plot) and a nonconvex function on C (right plot).

Example A.12. The function $f(x) = |x|$ is convex on $\mathbb{R}$.

Definition A.13 (Concave Function, Strictly Concave Function). *A function $g : C \to \mathbb{R}$ defined on a convex set $C \in \mathbb{R}^n$ is said to be* concave *if the function $f := -g$ is convex on C. The function g is said to be* strictly concave *on C if $-g$ is strictly convex on C.*

Often, it is required that only those $\mathbf{x} \in \mathbb{R}^n$ with $g_i(\mathbf{x}) \leq 0$, $i = 1, \ldots, m$, are feasible points of an optimization problem (i.e., a finite number of inequality constraints are imposed – see, e.g., Chapter 1).

Theorem A.14. *Let C be a convex set in $\mathbb{R}^n$ and let $f : C \to \mathbb{R}$ be a convex function. Then, the level set $C_\alpha := \{\mathbf{x} \in C : f(\mathbf{x}) \leq \alpha\}$, where α is a real number, is a convex set.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in C_\alpha$. Clearly, $\mathbf{x}_1, \mathbf{x}_2 \in C$, and $f(\mathbf{x}_1) \leq \alpha$ and $f(\mathbf{x}_2) \leq \alpha$. Let $\lambda \in (0, 1)$ and $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$. By convexity of C, $\mathbf{x} \in C$. Moreover, by convexity of f on C,

$$f(\mathbf{x}) \leq \lambda f(\mathbf{x}_1) + (1 - \lambda) f(\mathbf{x}_2) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha,$$

i.e., $\mathbf{x} \in C_\alpha$. □

Corollary A.15. *Let C be a convex set in $\mathbb{R}^n$ and let $g_i : C \to \mathbb{R}$, $i = 1, \ldots, m$, be convex functions on C. Then, the set defined by*

$$F := \{\mathbf{x} \in C : g_i(\mathbf{x}) \leq 0, \forall i = 1, \ldots, n\}$$

is convex.

Proof. The result is immediately evident from Theorem A.14 and Lemma A.5. □

It is not uncommon that the feasible set in an optimization problems be also defined in terms of equality constraints. Imposing an equality constraint such as $h(\mathbf{x}) = 0$ is obviously equivalent to imposing the pair of inequality constraints $h(\mathbf{x}) \leq 0$ and $-h(\mathbf{x}) \leq 0$. In particular, an affine equality constraint, $\mathbf{a}^\mathsf{T}\mathbf{x} = b$, defines a convex feasible set, for it is both convex and concave for the pair of inequality constraints. With a few trivial exceptions, **most nonlinear equality constraints define a nonconvex feasible set**.

A.3.3 How to Detect Convexity?

In an optimization problem, convexity of the objective function and constraints is crucial, because convex programs posses nicer theoretical properties and can be more efficiently solved numerically than general nonconvex programs. Henceforth, it is important to know whether a given function is convex or not.

Proposition A.16 (Operations that Preserve Convexity of Functions).

- Stability under Nonnegative Weighted Sums. *Let C be a convex set in $\mathbb{R}^n$. If $\mathbf{f} : C \to \mathbb{R}^m$ and $\mathbf{g} : C \to \mathbb{R}^m$ are convex on C, then their linear combination $\lambda\mathbf{f} + \mu\mathbf{g}$, with* nonnegative *coefficients λ and μ, is also convex on C.*

- Stability under Composition with an Affine Mapping. *Let C_1 and C_2 be convex sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $g : C_1 \to \mathbb{R}$ is a convex function on C_1, and $\mathbf{h} : C_2 \to \mathbb{R}^m$ is an affine mapping (i.e., $\mathbf{h}(\mathbf{x}) := \mathbf{A}(\mathbf{x}) + \mathbf{b}$) with* $\text{range}(\mathbf{h}) \subset C_1$, *then the composite function $f : C_2 \to \mathbb{R}$ defined as $f(\mathbf{x}) := g[\mathbf{h}(\mathbf{x})]$ is convex on C_2.*

- Stability under (Scalar) Composition with a Nondecreasing Convex Function. *Let C_1 and C_2 be convex sets in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. If $g : C_1 \to \mathbb{R}$ is a nondecreasing, convex function on C_1, and $h : C_2 \to \mathbb{R}$ is a convex function with range$(h) \subset C_1$, then the composite function $f : C_2 \to \mathbb{R}$ defined as $f(\mathbf{x}) := g[h(\mathbf{x})]$ is convex on C_2.*

- Stability under Pointwise Supremum. *Let C be a convex set in $\mathbb{R}^n$. If $g_\alpha : C \to \mathbb{R}^m$, $\alpha = 1, 2, \ldots$, are convex functions on C, then the function $\mathbf{x} \mapsto \sup_\alpha g_\alpha(\mathbf{x})$ is convex on C.*

Proof. The proofs are left to the reader as an exercise. □

We shall now have a look to which standard functions these operations can be applied to. The usual way of checking convexity of a "simple" function is based on *differential criteria* of convexity.

Theorem A.17 (First-Order Condition of Convexity). *Let C be a convex set in $\mathbb{R}^n$ with a nonempty interior, and let $f : C \to \mathbb{R}$ be a function. Suppose f is continuous on C and differentiable on* $\text{int}(C)$. *Then f is convex on* $\text{int}(C)$ *if and only if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \boldsymbol{\nabla} f(\mathbf{x})^\mathsf{T}[\mathbf{y} - \mathbf{x}]$$

holds for any two points $\mathbf{x}, \mathbf{y} \in C$.

Theorem A.18 (Second-Order Condition of Convexity). *Let C be a convex set in $\mathbb{R}^n$ with a nonempty interior, and let $f : C \to \mathbb{R}$ be a function. Suppose f is continuous on C and twice differentiable on* int (C). *Then f is convex on* int (C) *if and only if its Hessian matrix $\mathbf{H}(\mathbf{x})$ is positive semidefinite at each $\mathbf{x} \in$* int (C).

With the foregoing result, it is straightforward to verify that a great variety of functions is convex. However, a difficulty arises, e.g., if the set C is closed, since convexity can be established on the interior of C only. The following result can be used to overcome this difficulty.

Lemma A.19. *Let C be a convex set in $\mathbb{R}^n$ with a nonempty interior, and let $f : C \to \mathbb{R}$ be a function. If f is continuous on C and convex on* int (C), *then it is also convex on C.*

With the foregoing rules, convexity can be established for a great variety of complicated functions. This is illustrated in the following example.

Example A.20. Consider the exponential posynomial function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(\mathbf{x}) = \sum_{i=1}^{N} c_i \exp(\mathbf{a}_i^{\mathsf{T}}\mathbf{x}),$$

with positive coefficients c_i. The function $\mathbf{x} \mapsto \exp(\mathbf{x})$ is convex on $\mathbb{R}^n$, for its Hessian matrix is positive definite at each $\mathbf{x} \in \mathbb{R}^n$. All functions $\mathbf{x} \mapsto \exp(\mathbf{a}_i^{\mathsf{T}}\mathbf{x})$ are therefore convex on $\mathbb{R}^n$ (stability of convexity under composition with an affine mapping). Finally, f is convex on $\mathbb{R}^n$ (stability of convexity under taking linear combinations with nonnegative coefficients).

A.4 LINEAR SPACES

The problems considered in the Chapters 2 and 3 of this textbook consist of optimizing a real valued function $\mathcal{J}$ defined on a subset $\mathcal{D}$ of a linear space $\mathcal{X}$. This section gives a summary of standard results for linear spaces, presupposing some familiarity with vector space operations in $\mathbb{R}^d$.

The principal requirement for a *(real) linear space*, also called *(real) vector space*, is that it contain the sums and (real) scalar multiples of its elements. In other words, a linear space must be closed under the operations of addition and scalar multiplication.

Definition A.21 (Linear Space). *A* real linear space *is a nonempty set $\mathcal{X}$ for which two operations called addition (denoted $+$) and (real) scalar multiplication (denoted $\cdot$) are defined. Addition is commutative and associative, making $\mathcal{X}$ an Abelian group under addition. Multiplication by scalars from the real number field is associative, and distributive with respect to $+$ as well as addition of scalars.*

We remark without proof that the set of real-valued functions f, g, on a (nonempty) set S forms a real linear space (or vector space) with respect to the operations of pointwise addition:

$$(f + g)(x) \; = \; f(x) + g(x) \quad \forall x \in S,$$

and scalar multiplication:

$$(\alpha f)(x) \;=\; \alpha f(x) \quad \forall x \in S,\ \alpha \in \mathbb{R}.$$

Likewise, for each $d = 1, 2, \ldots$ the set of all d-dimensional real vector valued functions on this set S forms a linear space with respect to the operations of component-wise addition and scalar multiplication.

If continuity is definable on S, then $\mathcal{C}(S)$ $(:= \mathcal{C}^0(S))$, the set of continuous real-valued functions on S, will be a real linear space since the sum of continuous functions, or the multiple of a continuous function by a real constant, is again a continuous function. Similarly, for each *open* subset D of a Euclidean space and each $k = 1, 2, \ldots$, $\mathcal{C}^k(D)$, the set of functions on D having continuous partial derivatives of order lower than or equal to k, is a real linear space, since the laws of differentiation guarantee that the sum or scalar multiple of such functions will be another. In addition, if D is bounded with boundary ∂D, and $\bar{D} := D \cup \partial D$, then $\mathcal{C}^k(\bar{D})$, the subset of $\mathcal{C}^k(\bar{D}) \cup \mathcal{C}(\bar{D})$ consisting of those functions whose partial derivatives of order lower than or equal to k each admit continuous extension to $\bar{D}$, is a real linear space. For example, a function x, which is continuous on $[a, b]$, is in $\mathcal{C}^1([a, b])$ if it is continuously differentiable in (a, b) and its derivative $\dot{x}$ has finite limiting values from the right at a (denoted $\dot{x}(a^+)$) and from the left at b (denoted $\dot{x}(b^-)$).

Example A.22. The function $x \mapsto x^{\frac{3}{2}}$ defines a function in $\mathcal{C}^1([0, 1])$, but $x \mapsto x^{\frac{1}{2}}$ does not.

For $d = 1, 2, \ldots$, $[\mathcal{C}(S)]^d$, $[\mathcal{C}^k(D)]^d$, and $[\mathcal{C}^k(\bar{D})]^d$, the sets of d-dimensional vector valued functions whose components are in $\mathcal{C}(S)$, $\mathcal{C}^k(D)$, and $\mathcal{C}^k(\bar{D})$, respectively, also form real linear spaces.

Definition A.23 (Linear Subspace). *A* linear subspace*, or simply a* subspace*, of the linear space $\mathcal{X}$ is a subset which is itself a linear space under the same operations.*

We note that subsets $\mathcal{D}$ of these spaces provide natural domain for optimization of real-valued functions in Chapters 2 and 3. However, these subsets do *not* in general constitute linear spaces themselves.

Example A.24. The subset

$$\mathcal{D} \;:=\; \{x \in \mathcal{C}([a, b]) : x(a) = 0, x(b) = 1\},$$

is *not* a linear space since if $x \in \mathcal{D}$ then $2x \notin \mathcal{D}$. $(2x(b) = 2 \neq 1.)$ On the other hand,

$$\mathcal{D} \;:=\; \{x \in \mathcal{C}([a, b]) : x(a) = 0, x(b) = 0\},$$

is a linear space.

Definition A.25 (Functional). *A function defined on a linear space $\mathcal{X}$ with range in $\mathbb{R}$ is called a* functional*.*

Definition A.26 (Continuous Functional). *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. A functional $\mathcal{F} : \mathcal{X} \to \mathbb{R}$ is said to be* continuous at $\mathbf{x} \in \mathcal{X}$*, if $\{\mathcal{F}(\mathbf{x}_k)\} \to \mathcal{F}(\mathbf{x})$, for any* *convergent sequence* $\{\mathbf{x}_k\} \to \mathbf{x}$ *in* $\mathcal{X}$*. A functional is said to be* continuous *on* $\mathcal{X}$*, if it is continuous at any* $\mathbf{x} \in \mathcal{X}$.

Analysis in $\mathbb{R}^d$ is described most easily through inequalities between the lengths of its vectors. Similarly, in the real linear space $\mathcal{X}$, we shall assume that we can assign to each $\mathbf{x} \in \mathcal{X}$ a nonnegative number, denoted $\|\mathbf{x}\|$:

Definition A.27 (Norm). *A* norm $\|\cdot\|$ *on a linear space* $\mathcal{X}$ *is a nonnegative functional such that*

$$\begin{aligned} \|\mathbf{x}\| &= 0 \quad \text{if and only if} \quad \mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathcal{X} \text{ (positive definite)} \\ \|\alpha \mathbf{x}\| &= |\alpha|\, \|\mathbf{x}\| \quad \forall \mathbf{x} \in \mathcal{X},\ \alpha \in \mathbb{R} \text{ (positive homogeneous)} \\ \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X} \text{ (triangle inequality).} \end{aligned}$$

There may be more than one norm for a linear space, although in a specific example, one may be more natural or more useful than another. Every norm also satisfies the so-called *reverse triangle inequality*:

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X} \text{ (reverse triangle inequality).}$$

Definition A.28 (Normed Linear Space). *A* normed linear space *is a linear space with the topology induced by the norm defined on it: neighborhoods of any point* $\bar{\mathbf{x}}$ *are the balls*

$$\mathcal{B}_\eta(\bar{\mathbf{x}}) := \{\mathbf{x} \in \mathcal{X} : \|\mathbf{x} - \bar{\mathbf{x}}\| < \eta\},$$

with $\eta > 0$.

Two possible norms on the linear space of continuous real-values functions on $[a, b]$ are:

$$\|x\|_\infty := \max_{a \leq t \leq b} |x(t)| \tag{A.3}$$

$$\|x\|_p := \left(\int_a^b |x(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}. \tag{A.4}$$

Further, since $\mathcal{C}^k[a, b] \subset \mathcal{C}[a, b]$, for each $k = 1, 2, \ldots$, it follows that (A.3) and (A.4) also define norms on $\mathcal{C}^k[a, b]$. However, these norms do not take cognizance of the differential properties of the functions and supply control only over their continuity. Alternative norms on $\mathcal{C}^k[a, b]$ supplying control over the k first derivatives are:

$$\|x\|_{k,\infty} := \|x\|_\infty + \|x^{(k)}\|_\infty = \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} |x^{(k)}(t)| \tag{A.5}$$

$$\|x\|_{k,p} := \|x\|_p + \|x^{(k)}\|_p = \left(\int_a^b |x(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} + \left(\int_a^b |x^{(k)}(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}. \tag{A.6}$$

Norms can be defined in a likewise fashion for the linear spaces $\mathcal{C}[a, b]^d$, and $\mathcal{C}^k[a, b]^d$, with $d = 1, 2, \ldots$. For example,

$$\|\mathbf{x}\|_\infty := \max_{a \leq t \leq b} \|\mathbf{x}(t)\|, \tag{A.7}$$

defines a norm on $\mathcal{C}[a, b]^d$, and

$$\|\mathbf{x}\|_{k,\infty} := \|\mathbf{x}\|_\infty + \|\mathbf{x}^{(k)}\|_\infty = \max_{a \leq t \leq b} \|\mathbf{x}(t)\| + \max_{a \leq t \leq b} \|\mathbf{x}^{(k)}(t)\|, \tag{A.8}$$

defines a norm on $\mathcal{C}^k[a,b]^d$, where $\|\mathbf{x}(t)\|$ stands for any norm in $\mathbb{R}^d$.

Definition A.29 (Equivalent Norms). *Let $\|\cdot\|$ and $\|\cdot\|'$ be two norm on a linear space $\mathcal{X}$. These norm are said to be* equivalent norms *if there exist positive real numbers α, β such that*

$$\alpha\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq \beta\|\mathbf{x}\|.$$

While all norms can be shown to be equivalent on a finite dimensional linear space, this result does *not* hold on infinite dimensional spaces. This is illustrated in the following:

Example A.30. Consider the linear space of continuously differentiable real-valued functions $\mathcal{C}^1[0,1]$, supplied with the maximum norms $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$, as defined in (A.3) and (A.5), respectively. Let sequence of functions $\{x_k\} \in \mathcal{C}^1[0,1]$ be defined as

$$x_k(t) \;:=\; 2^{2k}t^k(1-t)^k.$$

It is easily shown that

$$\|x_k\|_\infty = \left|x_k\left(\tfrac{1}{2}\right)\right| = 1 \quad \text{for each } k \geq 1.$$

On the other hand, the maximum value of the first derivative $|\dot{x}_k(t)|$ on $[0,1]$ is attained at $t^\pm = \frac{1}{2}\frac{\sqrt{2k-1}\pm 1}{\sqrt{2k-1}}$, yielding

$$\|\dot{x}_k\|_\infty = |\dot{x}_k(t^\pm)| = 2\sqrt{2k-1}\frac{k}{k-1}\left(\frac{\sqrt{2k-1}+1}{\sqrt{2k-1}}\right)^k\left(\frac{\sqrt{2k-1}-1}{\sqrt{2k-1}}\right)^k,$$

As k grows large, we thus have $\|\dot{x}_k\|_{1,\infty} = \|x_k\|_\infty + \|\dot{x}_k\|_\infty \sim \sqrt{k}$. Hence, for any $\beta > 0$, there is always a k such that

$$\|\dot{x}_k\|_{1,\infty} \;>\; \beta\|\dot{x}_k\|_\infty.$$

This proves that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$ are not equivalent on $\mathcal{C}^1[0,1]$.

Definition A.31 (Convergent Sequence). *A sequence $\{\mathbf{x}_k\}$ in a normed linear space $(\mathcal{X}, \|\cdot\|)$ is said to be* convergent *if there is an element $\bar{x} \in \mathcal{X}$ such that:*

$$\forall\varepsilon > 0, \exists N(\varepsilon) > 0 \textit{ such that } \|\mathbf{x}_k - \bar{\mathbf{x}}\| < \varepsilon, \forall k \geq N.$$

We say that $\{\mathbf{x}_k\}$ converges to $\bar{\mathbf{x}}$.

Hence, the convergence of a sequence in a linear space is reduced to the convergence of a sequence of real numbers via the use of a norm. A point $\bar{\mathbf{x}}$ in a normed linear space $\mathcal{X}$, is said to be a *limit point* of a set $\mathcal{D} \in \mathcal{X}$ if there exists a sequence $\{\mathbf{x}_k\}$ in $\mathcal{D}$ such that $\{\mathbf{x}_k\} \to \bar{\mathbf{x}}$.

Definition A.32 (Cauchy Sequence). *A sequence $\{\mathbf{x}_k\}$ in a normed linear space $(\mathcal{X}, \|\cdot\|)$ is said to be a* Cauchy sequence *if*

$$\forall\varepsilon > 0, \exists N > 0 \textit{ such that } \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon, \forall n, m \geq N.$$

While every convergent sequence is a Cauchy sequence, **the reverse does not necessarily hold true in a normed linear space**. This motivates the following:

Definition A.33 (Completeness, Banach Space). *A normed linear space* $(\mathcal{X}, \|\cdot\|)$ *in which every Cauchy sequence is a convergent sequence in* $\mathcal{X}$ *is said to be* complete. *A complete normed linear space is called a* Banach space.

Example A.34 (Complete Function Space). The linear space of continuous functions, $\mathcal{C}([a,b])$, equipped with the maximum norm $\|\cdot\|_\infty$, is a Banach space. The linear space of continuously differentiable functions, $\mathcal{C}^1([a,b])$, equipped with the maximum norm $\|\cdot\|_{1,\infty}$, is a Banach space too.

Example A.35 (Incomplete Function Space). Consider the function space $\mathcal{C}^1[0,1]$, supplied with the norm $\|\cdot\|_p$ as defined in (A.4), and let $\{x_k\} \in \mathcal{C}^1[0,1]$ be defined as earlier in Example A.30,

$$x_k(t) \;:=\; 2^{2k} t^k (1-t)^k.$$

It can be established that the limit $\bar{x}$ of $\{x_k\}$ as $k \to +\infty$ is a real-valued function given by

$$\bar{x}(t) = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{otherwise,} \end{cases}$$

which is not in $\mathcal{C}^1[0,1]$. In fact, $\bar{x}(t) \in L^p[0,1]$[1], and $\{x_k\}$ is convergent in the function space $L^p[0,1]$. Therefore, $\{x_k\}$ is a Cauchy sequence in $\mathcal{C}^1[0,1]$ relative to the norm $\|\cdot\|_p$, which does not have a limit in $\mathcal{C}^1[0,1]$. We have thus established that $(\mathcal{C}^1[0,1], \|\cdot\|_p$ is not a complete normed linear space.

Definition A.36 (Ball). *Let* $(\mathcal{X}, \|\cdot\|)$ *be a normed linear space. Given a point* $\bar{\mathbf{x}} \in \mathcal{X}$ *and a real number* $r > 0$*, a* ball centered at $\bar{\mathbf{x}}$ and of radius r *is the set*

$$\mathcal{B}_r(\bar{\mathbf{x}}) := \{\mathbf{x} \in \mathcal{X} : \|\mathbf{x} - \bar{\mathbf{x}}\| < r\}$$

Definition A.37 (Open Set, Closed Set). *A subset* $\mathcal{D}$ *of a normed linear space* $(\mathcal{X}, \|\cdot\|)$ *is said to be* open *if it contains a ball arount each of its points. A subset* $\mathcal{K}$ *of* $\mathcal{X}$ *is said to be* closed *if its complement in* $\mathcal{X}$ *is open.*

Theorem A.38 (Closed Set). *Let* $\mathcal{K}$ *be a nonempty subset of a normed linear space* $(\mathcal{X}, \|\cdot\|)$*. Then,* $\mathcal{K}$ *is closed if and only if every convergent sequence* $\{\mathbf{x}_k\} \in \mathcal{K}$ *converges to an element* $\bar{\mathbf{x}} \in \mathcal{K}$.

Definition A.39 (Totally Bounded Set). *Let* $(\mathcal{X}, \|\cdot\|)$ *be a normed linear space. A set* $\mathcal{D} \subset \mathcal{X}$ *is said to be* totally bounded *if*

$$\forall \varepsilon > 0, \exists n \geq 1 \text{ (finite) and} (d_1, \ldots, d_n) \in \mathcal{D} \text{ such that } \mathcal{D} \subseteq \bigcup_{k=1}^{n} \mathcal{B}_\varepsilon(d_k).$$

[1] $L^p(\Omega)$, $p \geq 1$, stands for the linear space of p-integrable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ with $\int_\Omega f(x)^p \mathrm{d}x < \infty$.

Definition A.40 ((Sequentially) Compact Normed Linear Space). *A normed linear space* $(\mathcal{X}, \|\cdot\|)$ *is said to be* sequentially compact *if every sequence in* $\mathcal{X}$ *has a convergent subsequence in* $\mathcal{X}$.

Theorem A.41 (Characterization of Compact Normed Linear Spaces). *A normed linear space* $(\mathcal{X}, \|\cdot\|)$ *is (sequentially) compact if and only if* $\mathcal{X}$ *is* totally bounded *and* complete.

Definition A.42 ((Sequentially) Compact Set). *Let* $(\mathcal{X}, \|\cdot\|)$ *be a normed linear space. A set* $\mathcal{K} \subset \mathcal{X}$ *is said to be* sequentially compact*, or simply* compact*, if every subsequence in* $\mathcal{K}$ *has a subsequence that converges to a point in* $\mathcal{K}$.

Theorem A.43 (Characterization of Compact Sets). *A subset* $\mathcal{K}$ *of a compact normed linear space* $(\mathcal{X}, \|\cdot\|)$ *is compact if and only if it is closed.*

In particular, it should be noted that a compact subset of a normed linear space is both closed and bounded. However, the converse is true only for finite dimensional spaces.

We close this subsection with the extension of Weierstrass' Theorem 1.14 (p. 7) to general normed linear spaces:

Theorem A.44 (Weierstrass' Theorem for Normed Linear Spaces). *A continuous functional* $\mathcal{J}$ *on a compact subset* $\mathcal{K}$ *of a compact normed linear space* $(\mathcal{X}, \|\cdot\|)$ *assumes both its maximum and minimum values at points in* $\mathcal{K}$. *In particular, these values are finite.*

A.5 FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

This section states some fundamental properties of the solutions of *ordinary differential equations (ODEs)*, such as existence, uniqueness, continuous dependence on initial conditions and parameters, and differentiabililty. These properties are *essential* for the state equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ to be a useful mathematical model of a physical system.

A.5.1 Existence and Uniqueness

For a mathematical model of a given system to predict the future state of that system from its current state at t_0, the *initial value problem (IVP)*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{A.9}$$

with $\mathbf{x} \in \mathbb{R}^{n_x}$ and $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, must have a unique solution. By a solution of (A.9) over an interval $[t_0, t_f]$, we mean a continuous vector-valued function $\mathbf{x} : [t_0, t_f] \to \mathbb{R}^{n_x}$, such that $\dot{\mathbf{x}}(t)$ is defined and $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$, for all $t \in [t_0, t_f]$. If $\mathbf{f}(t, \mathbf{x})$ is continuous both in t and $\mathbf{x}$, then the solution $\mathbf{x}(t)$ will be continuously differentiable. We shall assume herein that $\mathbf{f}(t, \mathbf{x})$ is continuous in $\mathbf{x}$, but only piecewise continuous in t, in which case, a solution $\mathbf{x}(t)$ could only be piecewise continuously differentiable, i.e., $\mathbf{x} \in \hat{\mathcal{C}}^1[t_0, t_f]$. The assumption that $\mathbf{f}(t, \mathbf{x})$ be piecewise continuous in t allows us to include the case wherein $\mathbf{f}(t, \mathbf{x}(t)) := \mathbf{g}(t, \mathbf{u}(t), \mathbf{x}(t))$ depends on a time-varying input $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ that may experience step changes with time, e.g., $\mathbf{u} \in \hat{\mathcal{C}}[t_0, t_f]$.

Prior to giving local existence and uniqueness conditions for the solution to an IVP in ODEs, we need the following:

Definition A.45 (Local Lipschitzness). *The function* $\mathbf{f}(\mathbf{x})$ *is said to be* Lipschitz *at* $\mathbf{x}_0 \in \mathbb{R}^{n_x}$ *if there exist constants* $K \geq 0$ *and* $\eta > 0$ *such that*[2]

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|, \tag{A.10}$$

for every $\mathbf{x}, \mathbf{y} \in \mathcal{B}_\eta(\mathbf{x}_0)$*. Moreover,* $\mathbf{f}(\mathbf{x})$ *is said to be* locally Lipschitz *on* X*, an open connected subset of* $\mathbb{R}^{n_x}$*, if it is Lipschitz at each* $\mathbf{x}_0 \in X$*.*

Likewise, the function $\mathbf{f}(t, \mathbf{x})$*,* $t \in [t_0, t_\mathrm{f}]$*, is said to be Lipschitz at* $\mathbf{x}_0 \in \mathbb{R}^{n_x}$ *provided the Lipschitz condition* (A.10) *holds* uniformly *in* $t \in [t_0, t_\mathrm{f}]$*;* $\mathbf{f}(t, \mathbf{x})$ *is said to be locally Lipschitz on* X *for* $t \in [t_0, t_\mathrm{f}]$ *provided that it is Lipschitz at any point* $\mathbf{x}_0 \in X$*.*

Note, in particular, that the Lipschitz property is stronger than continuity, but weaker than continuous differentiability. We are now ready to state the following:

Theorem A.46 (Local Existence and Uniqueness). *Let* $\mathbf{f}(t, \mathbf{x})$ *be piecewise continuous in t, and Lipschitz at* $\mathbf{x}_0 \in \mathbb{R}^{n_x}$ *for* $t \in [t_0, t_\mathrm{f}]$*. Then, there exists some* $\delta > 0$ *such that the state equation* $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *has a unique solution over* $[t_0, t_0 + \delta]$*.*

The key assumption in Theorem A.46 is the Lipschitz condition at $\mathbf{x}_0$. Strictly, only continuity of $\mathbf{f}(t, \mathbf{x})$ with respect to $\mathbf{x}$ is needed to ensure existence of a solution. However, continuity is not sufficient to ensure uniqueness of that solution as illustrated subsequently:

Example A.47. The scalar differential equation

$$\dot{x}(t) = [x(t)]^{\frac{1}{3}},$$

with initial condition $x(0) = 0$ has a solution $x(t) = [\frac{2}{3}t]^{\frac{3}{2}}$ for each $t > 0$. This solution is not unique, however, since $x(t) = 0, \forall t > 0$ is another solution.

The foregoing Theorem A.46 gives conditions under which a solution to an IVP in ODEs of the form (A.9) exists and is unique over an interval $[t_0, t_0+\delta]$, where δ may be very small. In other words, we have no control on δ, and cannot guarantee existence and uniqueness over a given time interval $[t_0, t_\mathrm{f}]$. Starting at time t_0, with an initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, Theorem A.46 shows that there is a positive constant δ (dependent on $\mathbf{x}_0$) such that the state equation (A.9) has a unique solution over the time interval $[t_0, t_0 + \delta]$. Then, taking $t_0 + \delta$ as a new initial time and $\mathbf{x}(t_0 + \delta)$ as a new initial state, one may try to apply Theorem A.46 to establish existence and uniqueness of the solution beyond $t_0 + \delta$. If the conditions of the theorem are satisfied at $(t_0 + \delta, \mathbf{x}(t_0 + \delta))$, then there exists $\delta_2 > 0$ such that the equation has a unique solution over $[t_0+\delta, t_0+\delta+\delta_2]$, that passes through the point $(t_0+\delta, \mathbf{x}(t_0+\delta))$. The solutions over $[t_0, t_0+\delta]$ and $[t_0+\delta, t_0+\delta+\delta_2]$ can now be pieced together to establish the existence of a unique solution over $[t_0, t_0 + \delta + \delta_2]$. This idea can be repeated to keep extending the solution. However, in general, the interval of existence of the solution cannot be extended indefinitely because the conditions of Theorem A.46 may cease to hold. In other words, there is a *maximum interval* $[t_0, T)$ where the unique solution starting at (t_0, x_0) exists. Clearly, T may be less than t_f, in which case the solution leaves any compact set over which $\mathbf{f}$ is locally Lipschitz in $\mathbf{x}$ as $t \to T$. The term *finite escape time* is used to describe this phenomenon.

[2] Here, $\|\cdot\|$ stands for any norm in $\mathbb{R}^{n_x}$.

Example A.48. Consider the scalar differential equation

$$\dot{x}(t) = [x(t)]^2,$$

with $x(0) = 1$. The function $f(x) = x^2$ is locally Lipschitz on $\mathbb{R}$. Hence, it is Lipschitz on any compact subset of $\mathbb{R}$. However, the unique solution

$$x(t) = \frac{1}{1-t},$$

passing through the point $(0, 1)$, exists over $[0, 1)$. As $t \to 1$, $x(t)$ leaves any compact set.

In view of the preceding discussion, one may ask the question whether additional conditions could be imposed, if any, so that a solution can be extended indefinitely. Prior to giving one such condition, we need the following:

Definition A.49 (Global Lipschitzness). *The function* $\mathbf{f}(\mathbf{x})$ *is said to be* globally Lipschitz *if there exists a constant* $K \geq 0$ *such that*

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|, \tag{A.11}$$

for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n_x}$. *(Here,* K *must be the same for every pair of points* x, y *in* $\mathbb{R}^{n_x}$.*) Likewise, the function* $\mathbf{f}(t, \mathbf{x})$ *is said to be globally Lipschitz for* $t \in [t_0, t_\mathrm{f}]$, *provided that* (A.11) *holds for every* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n_x}$, uniformly *in* $t \in [t_0, t_\mathrm{f}]$.

The global Lipschitzness property is sufficient for a solution to be extended indefinitely:

Theorem A.50 (Global Existence and Uniqueness I). *Let* $\mathbf{f}(t, \mathbf{x})$ *be piecewise continuous in t, and globally Lipschitz in* $\mathbf{x}$, *for* $t \in [t_0, t_\mathrm{f}]$. *Then, the state equation* $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *has a unique solution over* $[t_0, t_\mathrm{f}]$.

Example A.51. Consider the linear system

$$\dot{\mathbf{x}}(t) = \mathsf{A}(t)\mathbf{x}(t) + \mathbf{b}(t), \tag{A.12}$$

where the elements of $\mathsf{A} \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{b} \in \mathbb{R}^{n_x}$ are piecewise continuous functions of t. Over any finite time interval $[t_0, t_\mathrm{f}]$, the elements of $\mathsf{A}(t)$ are bounded. Hence, $\|\mathsf{A}(t)\| \leq a$, where $\|\cdot\|$ stands for any any matrix norm, and we have

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| = \|\mathsf{A}(t)\,(\mathbf{x} - \mathbf{y})\,\| = \|\mathsf{A}(t)\|\;\|\mathbf{x} - \mathbf{y}\| \leq a\|\mathbf{x} - \mathbf{y}\|,$$

for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n_x}$. Therefore, Theorem A.50 applies and the linear system (A.12) has a unique solution over $[t_0, t_\mathrm{f}]$. Since t_1 can be arbitrarily large, we can also conclude that a linear system has a unique solution, provided that $\mathsf{A}(t)$ and $\mathbf{b}(t)$ are piecewise continuous for all $t \geq t_0$. Hence, a linear system *cannot* have a finite escape time.

In view of the conservative nature of the global Lipschitz condition, it would be useful to have a global existence and uniqueness theorem requiring the function $\mathbf{f}$ to be only locally Lipschitz. The next theorem achieves that at the expense of having to know more about the solution of the system:

Theorem A.52 (Global Existence and Uniqueness II). *Let* $\mathbf{f}(t,\mathbf{x})$ *be piecewise continuous in* t*, and locally Lipschitz in* $\mathbf{x}$*, for all* $t \geq t_0$ *and all* $\mathbf{x} \in D \subset \mathbb{R}^{n_x}$*. Let also* X *be a compact subset of* D*,* $\mathbf{x}_0 \in X$*, and suppose it is known that every solution of*

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}); \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

lies entirely in X*. Then, there is a unique solution that is defined for all* $t \geq t_0$*.*

Example A.53. Consider the scalar differential equation

$$\dot{x}(t) = f(x) := -[x(t)]^3,$$

with $x(0) = a$. Observe that the function f is locally Lipschitz on $\mathbb{R}$, but does *not* satisfy a global Lipschitz condition since the Jacobian $f_x(x) = -3x^2$ is not globally bounded. That is, Theorem A.50 does not apply. Now, remarking that, at any time instant $t \geq 0$, $\dot{x}(t) \leq 0$ when $x(t) \geq 0$ and $\dot{x}(t) \geq 0$ when $x(t) \leq 0$, a solution cannot leave the compact set $X := \{x \in \mathbb{R} : |x| \leq |a|\}$. Thus, without calculating the solution, we conclude by Theorem A.52 that the differential equation has a unique solution for all $t \geq 0$.

A.5.2 Continuous Dependence on Initial Conditions and Parameters

We now turn to the problem of continuous dependence on initial conditions and parameters for the IVP in ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x},\mathbf{p}); \quad \mathbf{x}(t_0;\mathbf{p}) = \mathbf{x}_0, \tag{A.13}$$

with $\mathbf{p} \in \mathbb{R}^{n_p}$ being constant parameters, e.g., representing physical parameters of the system. Continuous dependence is an important property that any model of interest should possess. It is defined as follows:

Definition A.54 (Continuous Dependence on Initial Conditions and Parameters). *Let* $\mathbf{x}(t;\mathbf{p}^0)$ *be a solution of* (A.13)*, with* $\mathbf{x}(t_0;\mathbf{p}^0) = \mathbf{x}_0^0$*, defined on* $[t_0,t_\mathrm{f}]$*. Then,* $\mathbf{x}(t;\mathbf{p}^0)$ *is said to depend continuously on* $\mathbf{x}_0$ *and* $\mathbf{p}$ *if, for any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that for all* $\mathbf{x}_0^1 \in \mathcal{B}_\delta\left(\mathbf{x}_0^0\right)$ *and* $\mathbf{p}^1 \in \mathcal{B}_\delta\left(\mathbf{p}^0\right)$*,* (A.13) *has a unique solution* $\mathbf{x}(t;\mathbf{p}^1)$*, defined on* $[t_0,t_\mathrm{f}]$*, with* $\mathbf{x}(t_0;\mathbf{p}^1) = \mathbf{x}_0^1$*, and* $\mathbf{x}(t;\mathbf{p}^1)$ *satisfies*

$$\|\mathbf{x}(t;\mathbf{p}^1) - \mathbf{x}(t;\mathbf{p}^0)\| < \varepsilon, \quad \forall t \in [t_0,t_\mathrm{f}].$$

We can now state the main theorem on the continuity of solutions in terms of initial states and parameters:

Theorem A.55 (Continuous Dependence on Initial Conditions and Parameters). *Let* $X \subset \mathbb{R}^{n_x}$ *be an open connected set, and* $\mathbf{p}_0 \in \mathbb{R}^{n_p}$*. Suppose that* $\mathbf{f}(t,\mathbf{x},\mathbf{p})$ *is piecewise continuous in* $(t,\mathbf{x},\mathbf{p})$ *and locally Lipschitz (uniformly in t and* $\mathbf{p}$*) on* $[t_0,t_\mathrm{f}] \times X \times \mathcal{B}_\eta\left(\mathbf{p}^0\right)$*, for some* $\eta > 0$*. Let* $\mathbf{x}(t;\mathbf{p}^0)$ *be a solution of* $\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x},\mathbf{p})$ *with* $\mathbf{x}(t_0;\mathbf{p}^0) := \mathbf{x}_0^0 \in X$*, and suppose that* $\mathbf{x}(t;\mathbf{p}^0)$ *exists and belongs to* X *for all* $t \in [t_0,t_\mathrm{f}]$*. Then,* $\mathbf{x}(t;\mathbf{p}^0)$ *depends continuously on* $\mathbf{x}_0$ *and* $\mathbf{p}$*.*

A.5.3 Differentiability of Solutions

Suppose that $\mathbf{f}(t, \mathbf{x}, \mathbf{p})$ is continuous in $(t, \mathbf{x}, \mathbf{p})$ and has continuous first partial derivatives with respect to $\mathbf{x}$ and $\mathbf{p}$ for all $(t, \mathbf{x}, \mathbf{p}) \in [t_0, t_\mathrm{f}] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_p}$. Suppose also that $\mathbf{h}(\mathbf{p})$ is continuous and has continuous first partial derivatives with respect to $\mathbf{p}$ in $\mathbb{R}^{n_p}$. Let $\mathbf{p}^0$ be a nominal value of $\mathbf{p}$, and suppose that the nominal state equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}^0); \quad \mathbf{x}(t_0; \mathbf{p}^0) = \mathbf{h}(\mathbf{p}^0), \tag{A.14}$$

has a unique solution $\mathbf{x}(t; \mathbf{p}^0)$ over $[t_0, t_\mathrm{f}]$. From Theorem A.55, we know that for all $\mathbf{p}$ sufficiently close to $\mathbf{p}^0$, the state equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}); \quad \mathbf{x}(t_0; \mathbf{p}) = \mathbf{h}(\mathbf{p}),$$

has a unique solution $\mathbf{x}(t; \mathbf{p})$ over $[t_0, t_\mathrm{f}]$ that is close to the nominal solution $\mathbf{x}(t; \mathbf{p}^0)$. The continuous differentiability of $\mathbf{f}$ with respect to $\mathbf{x}$ and $\mathbf{p}$, together with the continuous differentiability of $\mathbf{h}$ with respect to $\mathbf{p}$, implies the additional property that $\mathbf{x}(t; \mathbf{p})$ is differentiable with respect to $\mathbf{p}$ near $\mathbf{p}^0$, at each $t \in [t_0, t_\mathrm{f}]$.[3] This is easily seen upon writing

$$\mathbf{x}(t; \mathbf{p}) = \mathbf{h}(\mathbf{p}) + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{x}(\tau; \mathbf{p}), \mathbf{p}) \, \mathrm{d}\tau,$$

and then taking partial derivatives with respect to $\mathbf{p}$,

$$\mathbf{x}_\mathbf{p}(t; \mathbf{p}) = \mathbf{h}_\mathbf{p}(\mathbf{p}) + \int_{t_0}^{t} \left[\mathbf{f}_\mathbf{x}(\tau, \mathbf{x}(\tau; \mathbf{p}), \mathbf{p})\mathbf{x}_\mathbf{p}(\tau; \mathbf{p}) + \mathbf{f}_\mathbf{p}(\tau, \mathbf{x}(\tau; \mathbf{p}), \mathbf{p})\right] \, \mathrm{d}\tau. \tag{A.15}$$

(See Theorem 2.A.59 on p. 102 for differentiation under the integral sign.)

A.6 NOTES AND REFERENCES

There are many excellent textbooks on real analysis. We just mention here the textbooks by Rudin [46] and Sohrab [50].

The material presented in Section A.3 is mostly a summary of the material in Chapters 2 and 3 of the book by Bazaraa, Sherali and Shetty [6], where most of the omitted proofs can be found. See also Chapters 2 and 3 of the book by Boyd and Vandenberghe [10].[4] The classical, comprehensive text on convex analysis is Rockafellar's book [44].

Similarly, there are many excellent textbooks on functional analysis. A particularly accessible introductory textbook is that by Kreysig [33], whose only real prerequisites are a solid understanding of calculus and some familiarity with Linear Algebra. The classical textbooks in functional analysis are Rudin's book [47] as well as Lax's book [34].

Finally, the summary on nonlinear differential equations presented in Section A.3 is mostly taken from the excellent textbook by Khalil [30, Chapter 3].

[3] This result can be readily extended to the solution $\mathbf{x}(t; \mathbf{p})$ being $\mathcal{C}^k$ with respect to $\mathbf{p}$ near $\mathbf{p}^0$, at each $t \in [t_0, t_\mathrm{f}]$, when $\mathbf{f}$ and $\mathbf{h}$ are themselves $\mathcal{C}^k$ with respect to $(\mathbf{x}, \mathbf{p})$ and $\mathbf{p}$, respectively.

[4] An electronic copy of this book can be obtained at `http://www.stanford.edu/ boyd/cvxbook/`.

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