Robust Loop Shaping Controller Design for Spectral Models by Quadratic Programming

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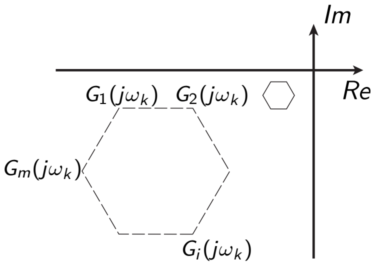
Introduction

Class of models

$$\mathcal{P} = \left\{ \sum_{i=1}^{m} \lambda_i G_i(j\omega_k) : \sum_{i=1}^{m} \lambda_i = 1; k = 1, N \right\}$$



- Non parametric models.
- Multimodel uncertainty as well as unstructured frequency-domain uncertainty.

Introduction (cont.)

<u>Class of controllers</u>

$$K(s) = \rho^T \phi(s)$$
$$\rho^T = [\rho_1 \ \rho_2 \ \dots \ \rho_n]$$
$$\phi^T(s) = [\phi_1(s) \ \phi_2(s) \ \dots \ \phi_n(s)]$$

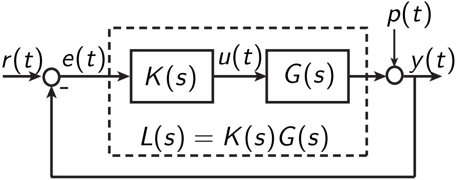
Property:

$$\begin{aligned} K(j\omega_k)G_i(j\omega_k) &= \rho^T \phi(j\omega_k) G_i(j\omega_k) \\ &= \rho^T (\mathcal{R}_i(j\omega_k) + j\mathcal{I}_i(j\omega_k)) \end{aligned}$$

Linear in the controller parameters.

Introduction (cont.)

Design specifications



- Open-loop shaping

Example: $|L(s)|$

- Large at low frequencies.
- -20dB/decade near the crossover frequency.
- Small at high frequencies.

- Closed-loop shaping

$\mathcal{S}(s) = [1 + L(s)]^{-1}$
$\mathcal{U}(s) = K(s)\mathcal{S}(s)$
$\mathcal{T}(s) = L(s)\mathcal{S}(s)$

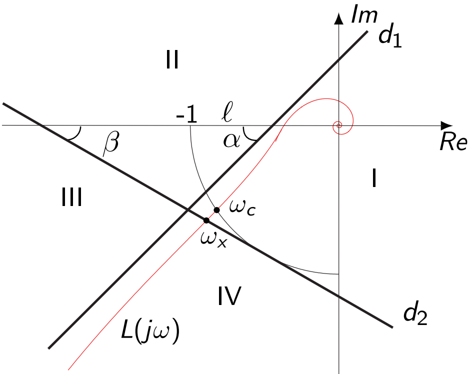
Example: $|\mathcal{S}(s)|$

- Small at low frequencies.
- Moderate $||\mathcal{S}||_\infty$: robustness and performance.

Open-loop shaping method

Linear programming approach

- Classical robustness margins (gain, phase and modulus margins) are **nonlinear** with respect to the parameters of linearly parameterized controllers.
- Introduce a **linear** robustness margin ℓ in the controller parameters.



Two optimization problems are proposed and solved by linear programming:

- **Performance**: maximization of K_i for PID controller.
- **Robustness**: maximization of ℓ.

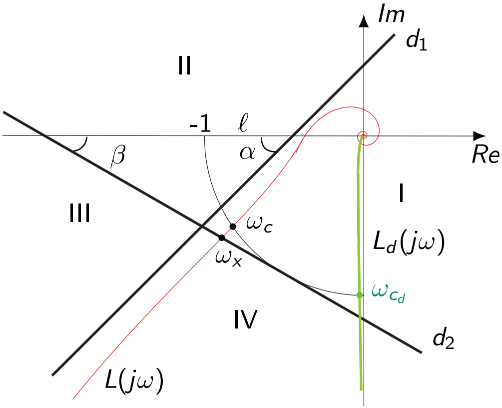
Typical values:

- $\ell \in [0.5, 0.8]$.
- $\alpha \in [0^o, 90^o]$.

Open-loop shaping method (cont.)

Quadratic programming approach

- A desired open-loop transfer function $L_d(s)$ corresponding to an appropriate closed-loop behavior is chosen (for example an integrator: $L_d(s) = \omega_{c_d}/s$).



- The 2-norm of the difference between the open-loop transfer function $L(j\omega)$ and $L_d(j\omega)$ is minimized subject to the linear constraints assuring the robustness margins.
- If $L(s)$ contains only one integrator, d_2 is not always needed.

Open-loop shaping method (cont.)

- Minimizing the 2-norm leads to a quadratic cost function which can be solved by standard quadratic programming algorithms:
  - Design method for an $L(s)$ containing one integrator:

Convex optimization problem

$$\text{Minimize} \quad \|\rho^T \phi(j\omega_k) G(j\omega_k) - L_d(j\omega_k)\|_2^2$$

Subject to:

$$\rho^T(\cot\alpha\,\mathcal{I}(\omega_k) - \mathcal{R}(\omega_k)) + \ell \leq 1 \quad \text{for all } \omega_k$$

  - The problem can be extended to polytopic models like $\mathcal{P}$:

Convex optimization problem

$$\text{Minimize} \quad \sum_{i=1}^{m} \|\rho^T \phi(j\omega_k) G_i(j\omega_k) - L_{d_i}(j\omega_k)\|_2^2$$

Subject to:

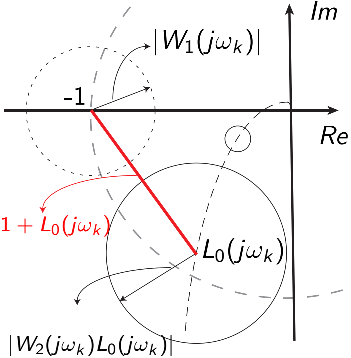
$$\rho^T\left(\cot\alpha\,\mathcal{I}_i(\omega_k) - \mathcal{R}_i(\omega_k)\right) + \ell \leq 1 \quad \text{for all } \omega_k$$

Closed-loop shaping method

Robust performance condition

Necessary and sufficient condition for **robust performance**:

$$\| |W_1 \mathcal{S}| + |W_2 \mathcal{T}| \|_\infty < 1$$



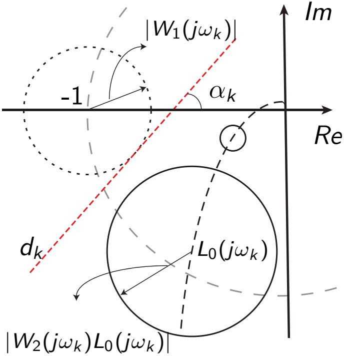
$$\frac{|W_1|}{|1+L_0|} + \frac{|W_2 L_0|}{|1+L_0|} < 1 \quad \forall \omega$$

$$|W_1| + |W_2 L_0| < |1 + L_0| \quad \forall \omega$$

- The uncertainty circle of radius $|W_2(j\omega_k)L_0(j\omega_k)|$ centered at $L_0(j\omega_k)$ should not intersect the performance circle of radius $|W_1(j\omega_k)|$ centered at -1: **Non-convex** constraint on the controller parameters.

Closed-loop shaping method (cont.)

New robust performance condition



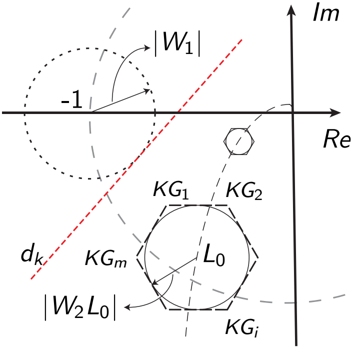
- Performance circle replaced by a tangential d_k line with angle α_k:

$$\rho^T \left(\cos\alpha_k \mathcal{I}(\omega_k) - \sin\alpha_k \mathcal{R}(\omega_k) \right) + |W_1(j\omega_k)| \leq \sin\alpha_k$$

- How to choose α_k?
  - Low frequencies, $\omega_k < \omega_{c_d} : \alpha_k = 0^\circ$
  - High frequencies, $\omega_k > \omega_{c_d} : \alpha_k = 90^\circ$
  - ω_{c_d}: desired crossover frequency $\simeq$ desired closed-loop bandwidth.

Closed-loop shaping method (cont.)

Approximation of model uncertainty



- At each ω_k, a polygon circumscribing the uncertainty circle is defined.
- Each vertex of this polygon defines a model in the set $\mathcal{P}$:

$$\left\{ \sum_{i=1}^{m} \lambda_i G_i(j\omega_k) : \sum_{i=1}^{m} \lambda_i = 1; k = 1, N \right\}$$

- If all points $KG_i(j\omega_k), i = 1 \dots m$, $k = 1 \dots N$, are on the right hand side of its corresponding line d_k, then the **robust performance** condition is satisfied.

Closed-loop shaping method (cont.)

Convex optimization problem

$$\text{Minimize} \quad \sum_{i=1}^{m} \left\| \rho^T \phi(j\omega_k) G_i(j\omega_k) - L_{d_i}(j\omega_k) \right\|_2^2$$

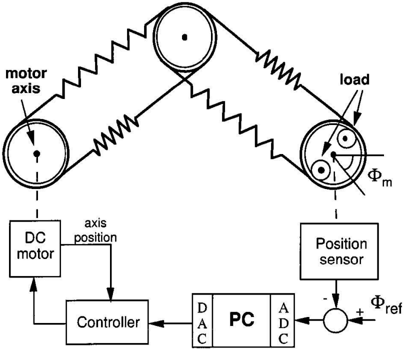
Subject to:

$$\rho^T \mathcal{I}_i(\omega_k) \leq -|W_{1_i}(j\omega_k)| \qquad \text{for } \omega_k \leq \omega_c, \quad i = 1, \ldots, m$$

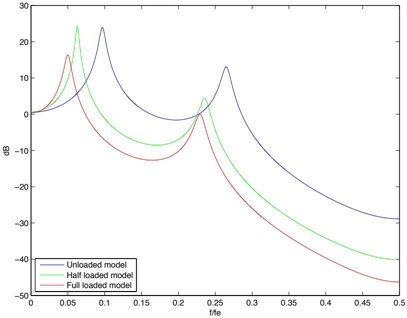
$$\rho^T \mathcal{R}_i(\omega_k) \geq |W_{1_i}(j\omega_k)| - 1 \qquad \text{for } \omega_k > \omega_c, \quad i = 1, \ldots, m$$

Application

- Flexible transmission benchmark



- Plant magnitude diagram



Application (cont.)

- Three different operating points:

$$G_i(q^{-1}) = \frac{q^{-d}B_i(q^{-1})}{A_i(q^{-1})} \qquad i = 1, 2, 3 \qquad h = 1/20s$$

  - Unloaded model:

$$A_1(q^{-1}) = 1 - 1.1483q^{-1} + 1.5894q^{-2} - 1.31608q^{-3} + 0.88642q^{-4}$$
$$B_1(q^{-1}) = 0.28261q^{-1} + 0.50666q^{-2}$$

  - Half loaded model:

$$A_2(q^{-1}) = 1 - 1.9918q^{-1} + 2.2026q^{-2} - 1.84083q^{-3} + 0.89413q^{-4}$$
$$B_2(q^{-1}) = 0.1027q^{-1} + 0.18123q^{-2}$$

  - Fully loaded model:

$$A_3(q^{-1}) = 1 - 2.0968q^{-1} + 2.3196q^{-2} - 1.93353q^{-3} + 0.87129q^{-4}$$
$$B_3(q^{-1}) = 0.06408q^{-1} + 0.10407q^{-2}$$

Application (cont.)

- Required specifications:
  - Rise time: 90% of the final value in less than 1 s.
  - Overshoot: less than 10%.
  - Rejection of 90% of the output disturbance filtered by $1/A_i$ in less than 1.2 s.
  - Perfect rejection of a constant disturbance.
  - Disturbance attenuation at low frequencies.
  - Maximum output sensitivity function $\mathcal{S}$ less than 6 dB (modulus margin greater than 0.5).
  - Delay margin of at least 40 ms.
  - Maximum value of less than 10 dB for the input sensitivity function $\mathcal{U}$ at high frequencies (between 8 to 10 Hz).

Application (cont.)

- Discrete-time two-degree of freedom polynomial RST controller:

$$S(q^{-1})u(t) = T(q^{-1})r(t) - R(q^{-1})y(t)$$

$$\begin{aligned} R(q^{-1}) &= (\rho_0 + \rho_1 q^{-1} + \ldots + \rho_6 q^{-6})(1 + q^{-1}) \\ S(q^{-1}) &= (1 - q^{-1}) \\ T(q^{-1}) &= t_0 \end{aligned}$$

  - Fixed term $(1 - q^{-1})$ in S for integral action and disturbance attenuation at low frequencies.
  - Fixed term $(1 + q^{-1})$ in R for reducing $\mathcal{U}$ at high frequencies.
  - $\alpha = 80^\circ$ and $\ell = 0.5/sin\alpha$ to assure modulus margin of 0.5.
  - Time domain performances are tuned using $L_{d_i}(s) = \omega_{c_i}/s$.
  - For the filtered disturbance rejection, a constraint can be added:

$$\left\| \frac{\mathcal{S}_i}{A_i} \right\|_\infty < \gamma$$

    In conventional notation, $W_{1_i} = 1/(\gamma A_i)$, $\gamma = 10^{27/20}$ (equal to 27 dB).
  - $N = 8000$ equally spaced frequency points between 0 and ω_N.

Application (cont.)

Convex optimization problem

$$\text{Minimize} \quad \sum_{i=1}^{3} \left\| \rho^T \phi(j\omega_k) G_i(j\omega_k) - \omega_{c_i}/(j\omega_k) \right\|_2^2$$

Subject to:

$$\rho^T \mathcal{I}_i \leq -|\gamma A_i^{-1}(j\omega_k)| \qquad \text{for } \omega_k \leq \omega_{c_i}, \quad i = 1,2,3$$

$$\rho^T \mathcal{R}_i \geq |\gamma A_i^{-1}(j\omega_k)| - 1 \qquad \text{for } \omega_k > \omega_{c_i}, \quad i = 1,2,3$$

$$\rho^T (\cot \alpha \, \mathcal{I}_i - \mathcal{R}_i) + \ell \leq 1 \qquad \text{for all } \omega_k, \quad i = 1,2,3$$

where

$$\begin{aligned} \mathcal{R}_i &= Re[\phi(j\omega_k) G_i(j\omega_k)] \\ \mathcal{I}_i &= Im[\phi(j\omega_k) G_i(j\omega_k)] \end{aligned}$$

$$\omega_{c_1} = 2.6 \text{ rad/s} \quad \omega_{c_2} = 1.2 \text{ rad/s} \quad \omega_{c_3} = 1.2 \text{ rad/s}$$

and

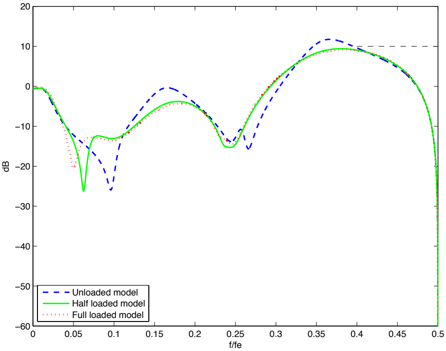
$$\phi(j\omega_k) = \frac{1 + e^{-j\omega_k h}}{1 - e^{-j\omega_k h}} [1 \; e^{-j\omega_k h} \; \ldots \; e^{-6j\omega_k h}]$$

- Solution

$$R(q^{-1}) = (0.4485 - 1.716q^{-1} + 2.916q^{-2} - 3.238q^{-3} + 2.675q^{-4} - 1.474q^{-5} + 0.4126q^{-6})(1 + q^{-1})$$
$$S(q^{-1}) = 1 - q^{-1}$$
$$T(q^{-1}) = 0.0483$$

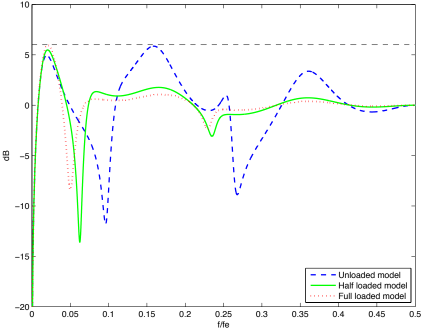
Specification	No load	Half load	Full load
Rise Time [s]	0.923	0.895	0.802
Overshoot[%]	6.1	7.8	6.7
Disturbance reject. [s]	1.15	1.17	1.00
Maximum $\mathcal{S}$ [dB]	5.86	5.48	5.96
Delay Margin [ms]	76	159	338
Maximum $\mathcal{U}$ [dB]	9.59	9.04	8.95

- Input sensitivity function $\mathcal{U}$

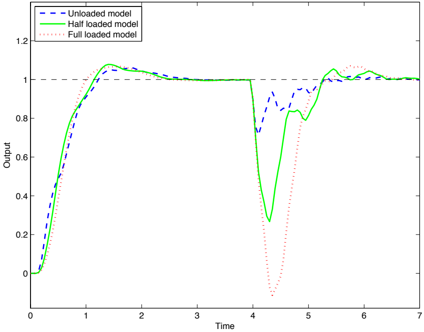


Application (cont.)

- Output sensitivity function $\mathcal{S}$



- Step response and perturbation rejection



Concluding remarks

- Convex optimization solved by standard quadratic programming algorithms.
- Spectral models, identification of parametric models is not needed.
- Few design variables directly related to robust and performance.
- Simple to understand and easy to implement.
- Design of continuous-time and discrete-time linearly parametrized controllers.
- Multimodel uncertainty can be taken into account.
- Application to a challenging benchmark problem illustrates effectiveness of the proposed method (smallest order controller meeting all specifications).