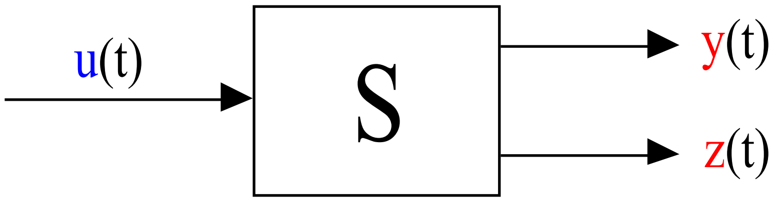
Représentation par variable d'état

L'état d'un système dynamique déterministe à un instant donné est l'information minimale qui permet, à partir des entrées futures, de déterminer de façon univoque le comportement futur du système

$$\left.\begin{array}{l}\text{état à } t_0 \\ u_{[t_0,\, \infty)}\end{array}\right\} \longrightarrow \text{comportement pour } t \geq t_0$$

En d'autres termes, l'état résume le passé d'un système nécessaire à la détermination de son futur

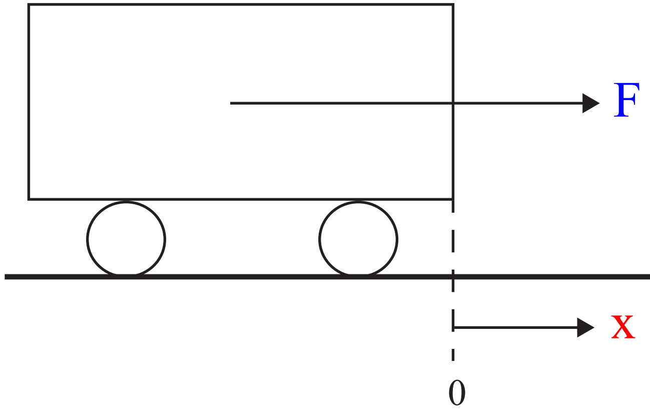
Conditions initiales d'un système d'équations dif férentielles : état du système à l'état initial



$$\begin{aligned} \dot{y}(t) &= -3y(t) + 2u(t)z(t) \qquad y(0) \\ \dot{z}(t) &= 2y(t) - z(t) \qquad z(0) \end{aligned}$$

Exemple de variables d'état

a)

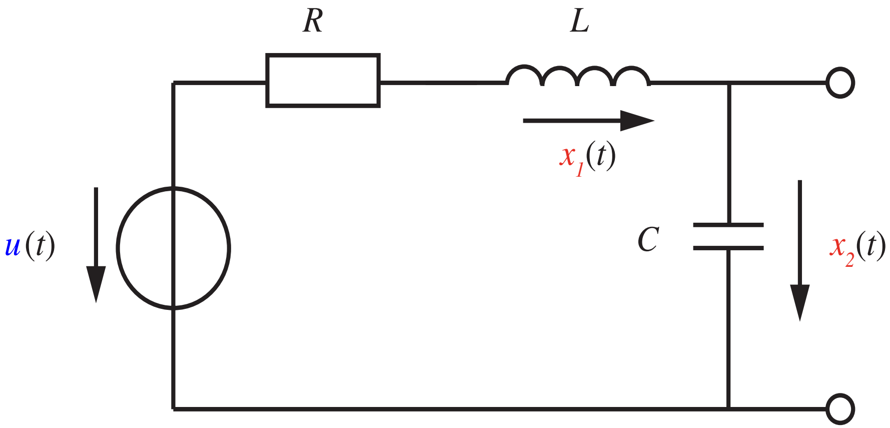


Pour déterminer de façon univoque le mouvement du chariot pour $t \geq t_0$, il faut connaître

– Force $F(t)$, $t \geq t_0$

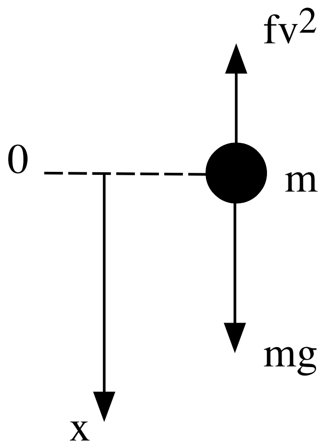
$$\left.\begin{array}{l} -\ x(t_0) \\ -\ \dot{x}(t_0) \end{array}\right\} \text{état à } t_0$$

b)



Etat : - courant à travers la bobine
- tension aux bornes du condensateur

Modèle d'état



Equation de mouvement

$$m\ddot{x} = mg - f\dot{x}^2 \qquad \begin{array}{l} x(0) \\ \dot{x}(0) \end{array}$$

2 variables d'état : $x_1 := x$

$x_2 := \dot{x}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g - \dfrac{f}{m}x_2^2 \end{cases} \qquad \begin{array}{l} x_1(0) = x(0) \\ x_2(0) = \dot{x}(0) \end{array}$$

entrée ?
sortie ?

Modèle d’état

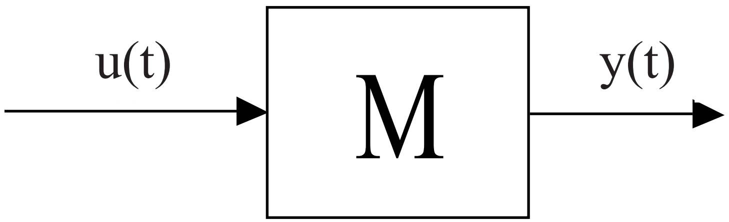
Ordre d’un système dynamique

nombre de variables d’état = nombre de conditions initiales

Etat n’est pas unique

$$\begin{matrix} x(t) \\ \dot{x}(t) \end{matrix} \quad \text{mais aussi} \quad \begin{matrix} 2x(t) \\ 5\,\dot{x}(t) \end{matrix} \quad \text{ou} \quad \begin{matrix} x(t) \\ x(t) - 2\,\dot{x}(t) \end{matrix}$$

Etat de dimension infinie



$y(t) \;=\; u(t-1)$ système d’ordre infini

$$\left.\begin{matrix} \text{état à } t_0 : u_{[t_0 - 1,\, t_0]} \\ u_{[t_0,\, \infty)} \end{matrix}\right\} \rightarrow y_{[t_0,\, \infty)}$$

Sélection des variables d'état

$$a\ddot{w} + \sin \dot{z} = u_1^2$$

$$\sqrt{\dot{v}} + \cos \dot{z} = u_2$$

$$\dot{z} + z = \alpha t$$

i	grandeur γ_i	ordre ρ_i	variables d'état
1	v	1	$x_1 = v$
2	w	2	$x_2 = w, x_3 = \dot{w}$
3	z	1	$x_4 = z$

Ordre du système: $n = \rho_1 + \rho_2 + \rho_3 = 4$

$$\begin{aligned}
\dot{x}_1 &= \dot{v} = (u_2 - \cos \dot{z})^2 = [u_2 - \cos(-x_4 + \alpha t)]^2 \\
\dot{x}_2 &= \dot{w} = x_3 \\
\dot{x}_3 &= \ddot{w} = \frac{1}{a}\left[u_1^2 - \sin(-x_4 + \alpha t)\right] \\
\dot{x}_4 &= \dot{z} = -x_4 + \alpha t
\end{aligned}$$

Modèle d'état

$$
\left|\begin{array}{lll}
\dot{x}_1(t) & = f_1[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t] & \qquad x_1(0) = x_{1,\,0} \\
\dot{x}_2(t) & = f_2[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t] & \qquad x_2(0) = x_{2,\,0} \\
 & \vdots & \\
\dot{x}_n(t) & = f_n[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t] & \qquad x_n(0) = x_{n,\,0}
\end{array}\right.
$$

$$
\left|\begin{array}{ll}
y_1(t) & = g_1[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t] \\
y_2(t) & = g_2[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t] \\
 & \vdots \\
y_q(t) & = g_q[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_p(t), t]
\end{array}\right.
$$

Modèle d'état (suite)

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_q(t) \end{bmatrix}$$

Equation d'état

$$\dot{x}(t) = f[x(t), u(t), t] \qquad x(0) = x_0$$

Equation de sortie

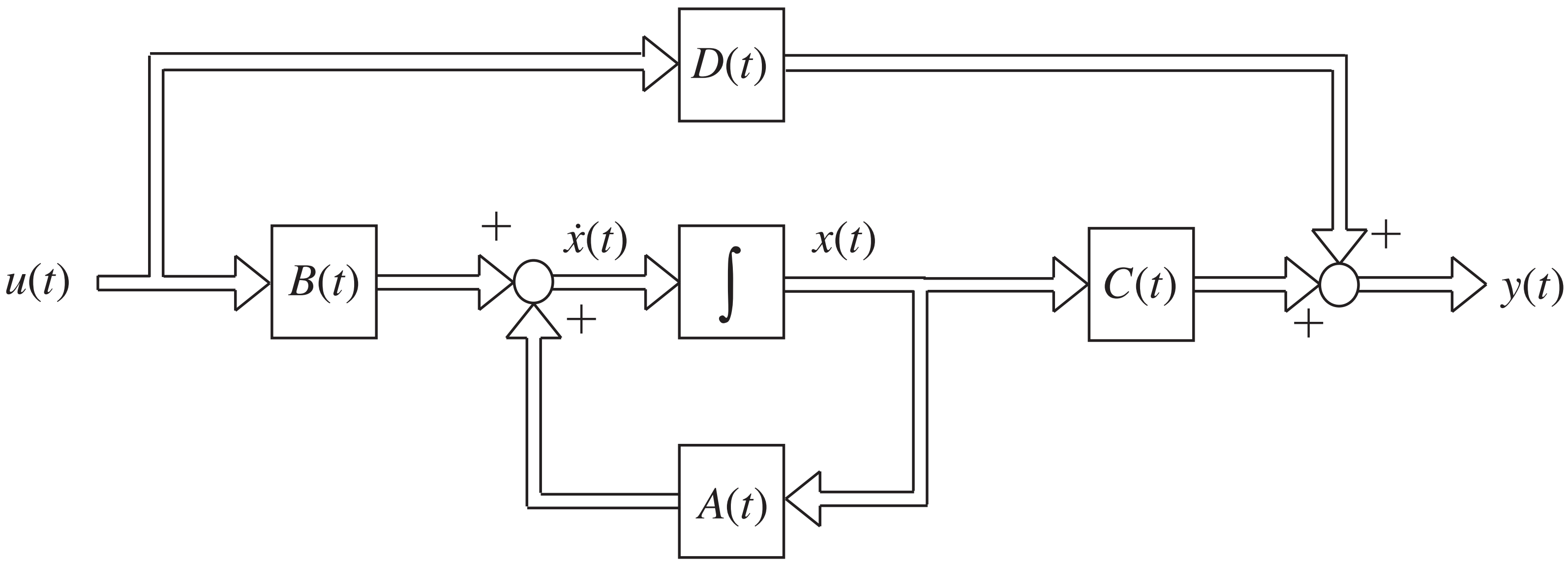
$$y(t) = g[x(t), u(t), t]$$

Modèle d'état linéaire

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x(0) = x_0$$

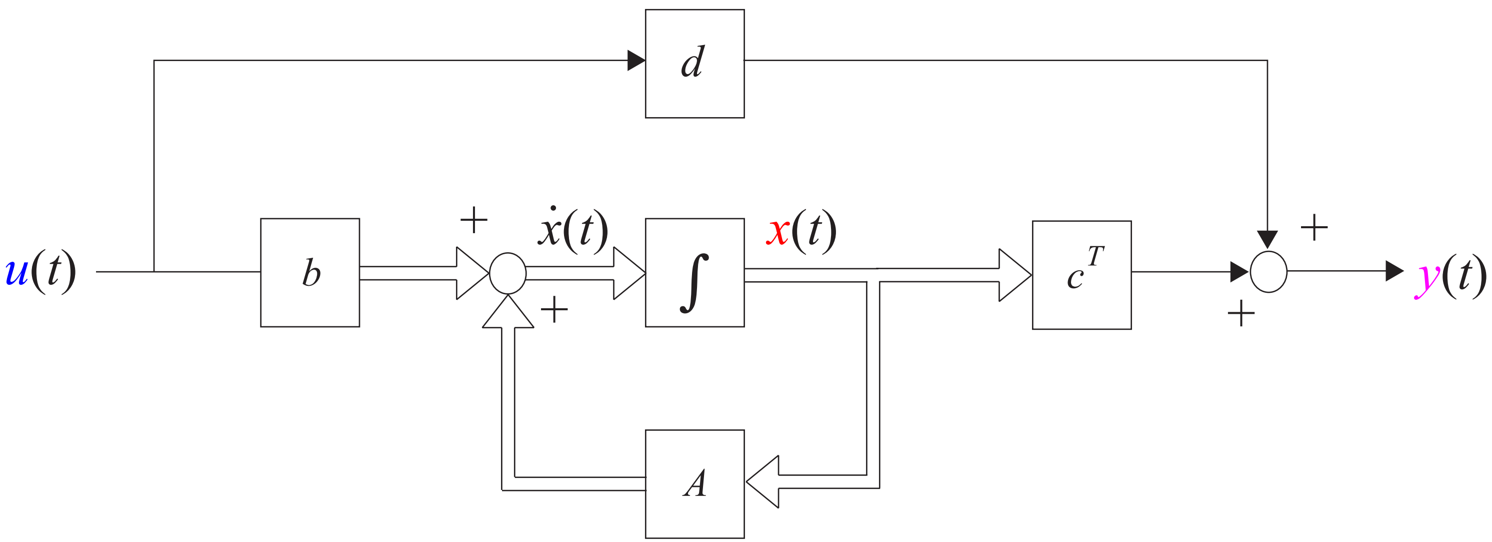
$$y(t) = C(t)x(t) + D(t)u(t)$$

Modèle multivariable non stationnaire

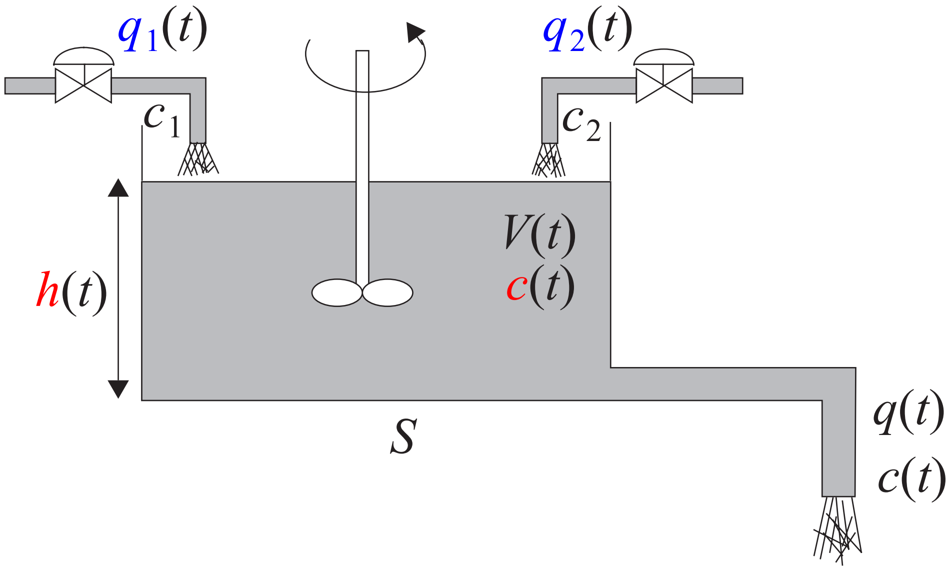


Modèle d’état linéaire

Modèle monovariable stationnaire



Cuve de mélange



$$q(t) = k\sqrt{h(t)}$$

$$\rho S\frac{d}{dt}h(t) = \rho q_1(t) + \rho q_2(t) - \rho q(t) \qquad \left[\frac{kg}{s}\right]$$

$$\rho S\frac{d}{dt}[c(t)h(t)] = \rho c_1 q_1(t) + \rho c_2 q_2(t) - \rho c(t)q(t) \qquad \left[\frac{kgP}{s}\right]$$

Cuve de mélange (suite)

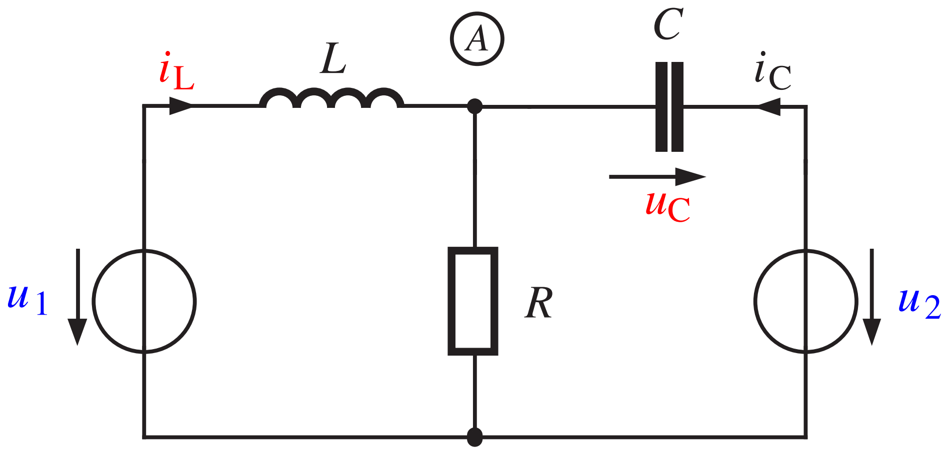
$$x_1(t) = h(t) \qquad u_1(t) = q_1(t) \qquad y_1(t) = h(t)$$

$$x_2(t) = c(t) \qquad u_2(t) = q_2(t) \qquad y_2(t) = c(t)/c_1$$

$$\Rightarrow \quad \dot{x}_1(t) = \frac{1}{S}[u_1(t) + u_2(t) - k\sqrt{x_1(t)}] \qquad x_1(0) = h(0)$$

$$\dot{x}_2(t) = \frac{1}{Sx_1(t)}[[c_1 - x_2(t)]u_1(t) + [c_2 - x_2(t)]u_2(t)] \qquad x_2(0) = c(0)$$

Circuit RLC



$$\frac{di_L}{dt} = \frac{1}{L}(u_1 - u_C - u_2) \qquad i_L(0) = i_{L,0}$$

$$\frac{du_C}{dt} = \frac{1}{C}i_L - \frac{1}{RC}(u_C + u_2) \qquad u_C(0) = u_{C,0}$$

$$\begin{aligned} x_1 &= i_L \\ x_2 &= u_C \end{aligned} \longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{aligned} x_1(0) &= i_L(0) \\ x_2(0) &= u_C(0) \end{aligned}$$

$$\begin{aligned} y_1 &= i_L \\ y_2 &= i_C \end{aligned} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Equation différentielle d'ordre n

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = bu(t)$$

$$\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= y^{(1)}(t) \\
&\vdots \\
x_{n-1}(t) &= y^{(n-2)}(t) \\
x_n(t) &= y^{(n-1)}(t)
\end{aligned}$$

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
&\vdots \\
\dot{x}_{n-1}(t) &= x_n(t) \\
\dot{x}_n(t) &= -a_{n-1}x_n(t) - a_{n-2}x_{n-1}(t) - \ldots - a_0 x_1(t) + bu(t)
\end{aligned}$$

Equation d'état

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t)$$

Equation de sortie

$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

Simulation

La simulation d'un système dynamique consiste à déterminer l'état $x(t)$ et la sortie $y(t)$ pour $t \geq t_0$ une fois l'état initial $x(t_0)$ et l'entrée du système $u(t)$ pour $t \geq t_0$ spécifiés

Pour t = kh

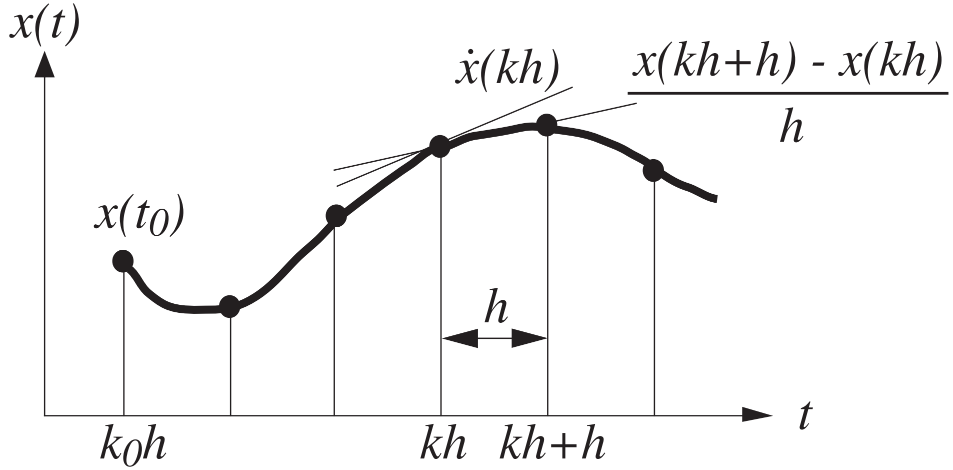
$$\dot{x}(kh) = f[x(kh), u(kh)] \qquad x(k_0 h) = x_0$$

$$y(kh) = g[x(kh), u(kh)]$$

Euler

$$\dot{x}(kh) \approx \frac{x(kh+h) - x(kh)}{h} \qquad x(t)$$

$$x(kh+h) \approx x(kh) + hf[x(kh), u(kh)]$$

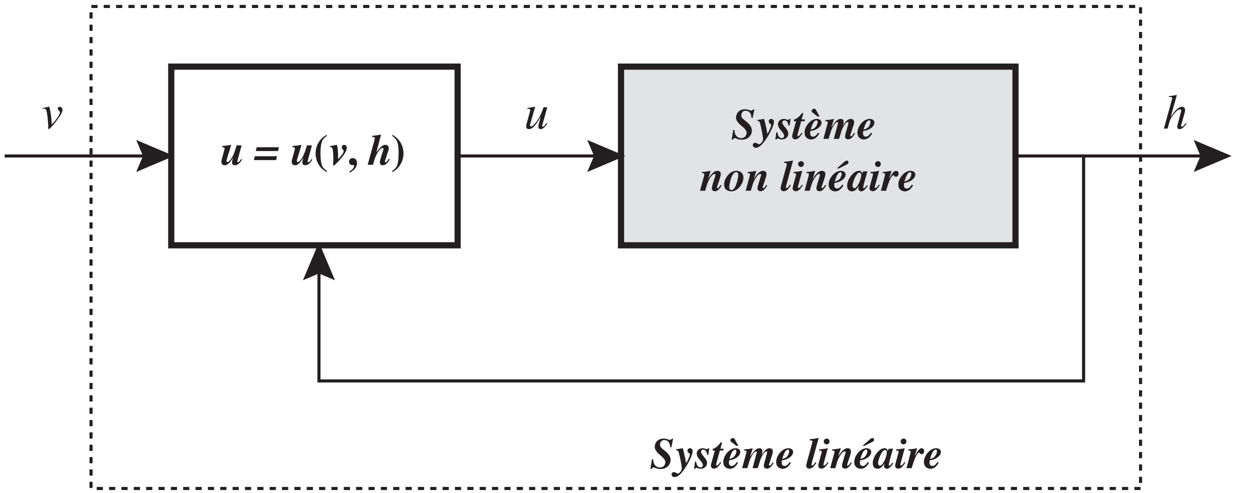


Linéarisation exacte par rétroaction

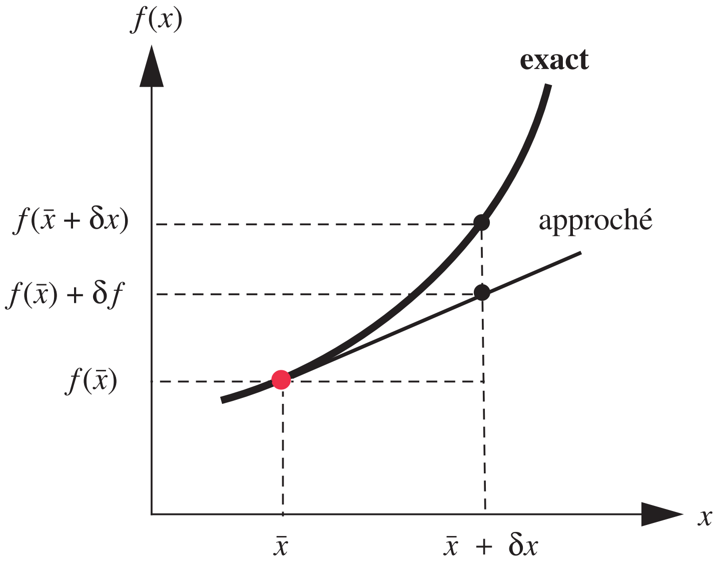
$$A\dot{h}(t) = u(t) - k\sqrt{h(t)}$$

Définir $v(t)$ tel que $$u(t) - k\sqrt{h(t)} = Av(t)$$

$$\rightarrow \quad \dot{h}(t) = v(t)$$



Linéarisation par approximation



$$f(\bar{x} + \delta x) \cong f(\bar{x}) + \frac{df}{dx}(\bar{x})\delta x = f(\bar{x}) + \delta f$$

Trajectoires nominales

$$\dot{\bar{x}} = f[\bar{x}(t), \bar{u}(t), t] \qquad \bar{x}(0) = x_0$$

$$\bar{y}(t) = g[\bar{x}(t), \bar{u}(t), t]$$

Point de fonctionnement stationnaire

$$0 = f[\bar{x}, \bar{u}]$$

$$\bar{y} = g[\bar{x}, \bar{u}]$$

Procédure de linéarisation

$$\dot{x}(t) = f[x(t), u(t)] \qquad x(0) = x_0$$
$$y(t) = g[x(t), u(t)]$$

Point de fonctionnement stationnaire

$$\frac{d}{dt}\bar{x} = 0 = f[\bar{x}, \bar{u}]$$

$$\bar{y} = g[\bar{x}, \bar{u}]$$

Développement en série de Taylor

$$\dot{x} = f[\bar{x}, \bar{u}] + \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\, \bar{x}} (x - \bar{x}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\, \bar{x}} (u - \bar{u}) \ + \text{tos}$$

$$y = g[\bar{x}, \bar{u}] + \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\, \bar{x}} (x - \bar{x}) + \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\, \bar{x}} (u - \bar{u}) \ + \text{tos}$$

Procédure de linéarisation (suite)

Variables écart

$$\delta x(t) := x(t) - \bar{x}$$

$$\delta u(t) := u(t) - \bar{u}$$

$$\delta y(t) := y(t) - \bar{y}$$

Approximation linéaire

$$\delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\,\bar{x}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\,\bar{x}} \delta u \qquad \delta x(0) = x_0 - \bar{x}$$

$$\delta y = \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\,\bar{x}} \delta x + \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\,\bar{x}} \delta u$$

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

$$y = Cx + Du$$

Matrices du modèle d'état

$$A := \left.\frac{\partial f}{\partial x}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(n \times n)$

$$B := \left.\frac{\partial f}{\partial u}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(n \times p)$

$$C := \left.\frac{\partial g}{\partial x}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_q}{\partial x_1} & \frac{\partial g_q}{\partial x_2} & \cdots & \frac{\partial g_q}{\partial x_n} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(q \times n)$

$$D := \left.\frac{\partial g}{\partial u}\right|_{\bar{u},\,\bar{x}} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_p} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_q}{\partial u_1} & \frac{\partial g_q}{\partial u_2} & \cdots & \frac{\partial g_q}{\partial u_p} \end{bmatrix}\right|_{\bar{u},\,\bar{x}}$$

$(q \times p)$

Exemple – Méthode 1

$$\dot{x} = -2x + 0{,}5(x+1)u \qquad x(0) = 1 \tag{1}$$

Point de fonctionnement

$$0 = -2\bar{x} + 0{,}5(\bar{x}+1)\bar{u} \tag{2}$$

Pour $\bar{u} = 2$, on obtient $\bar{x} = 1$

Approximation linéaire

$$xu \approx \bar{x}\bar{u} + \bar{u}(x-\bar{x}) + \bar{x}(u-\bar{u}) \tag{3}$$

Variables écart

$$\delta x = x - \bar{x} \tag{4}$$

$$\delta u = u - \bar{u} \tag{5}$$

Approximation linéaire de (1) - (2)

$$\begin{aligned}\dot{x} &= [-2x + 0{,}5(\bar{x}\bar{u} + \bar{u}\delta x + \bar{x}\delta u) + 0{,}5u] - [-2\bar{x} + 0{,}5\bar{x}\bar{u} + 0{,}5\bar{u}] \\ &= -2\delta x + 0{,}5(\bar{u}\delta x + \bar{x}\delta u) + 0{,}5\delta u\end{aligned} \tag{6}$$

$$\delta\dot{x} = \dot{x} \tag{7}$$

$$\delta\dot{x} = -\delta x + \delta u \qquad \delta x(0) = 0 \tag{8}$$

Méthode 2

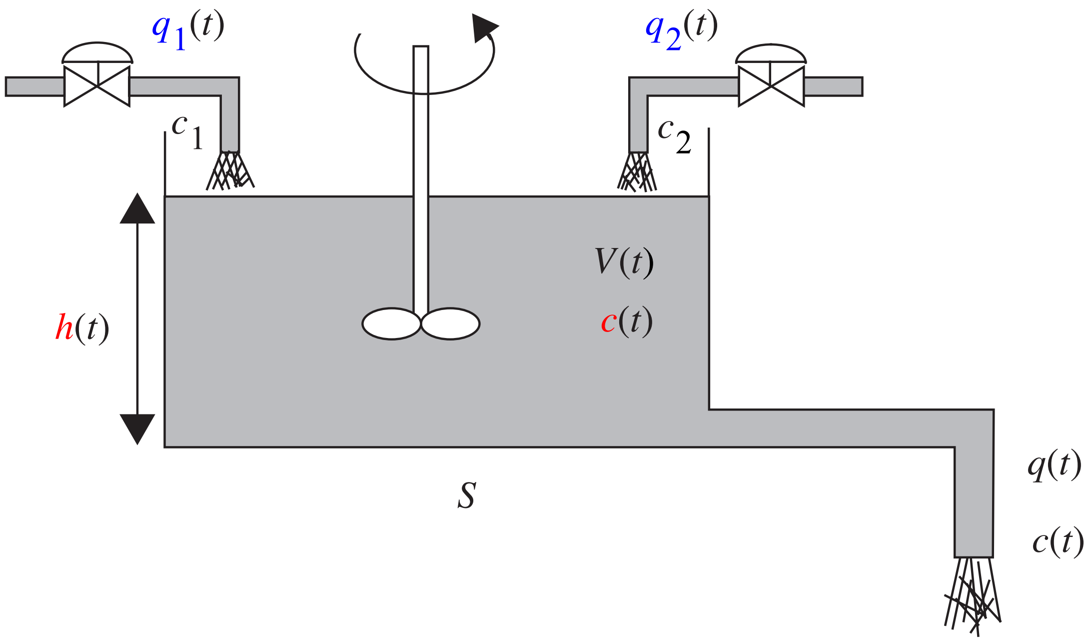
Système monovariable: $n = p = 1$

$$A = \left. \frac{\partial\left(-2x + 0,5xu + 0,5u\right)}{\partial x} \right|_{\bar{u},\ \bar{x}} = -2 + 0,5\bar{u} = -1$$

$$B = \left. \frac{\partial\left(-2x + 0,5xu + 0,5u\right)}{\partial u} \right|_{\bar{u},\ \bar{x}} = 0,5\bar{x} + 0,5 = 1$$

$$\delta\dot{x} = -\delta x + \delta u \qquad \delta x(0) = x(0) - \bar{x} = 0$$

Cuve de mélange



$$S\dot{h} = q_1 + q_2 - k\sqrt{h} \qquad h(0) = h_0$$

$$\dot{c} = \frac{q_1}{Sh}(c_1 - c) + \frac{q_2}{Sh}(c_2 - c) \qquad c(0) = c_0$$

$$y_1 = h$$

$$y_2 = \frac{c}{c_1}$$

Cuve de mélange (suite)

Point de fonctionnement $\left(\bar{q}_1,\, \bar{q}_2,\, \bar{h},\, \bar{c}\right)$

Système multivariable $n = p = q = 2$

$$a_{11} = \left.\frac{\partial f_1}{\partial h}\right|_{\text{éq}} = -\frac{k}{2S\sqrt{\bar{h}}} \qquad a_{12} = \left.\frac{\partial f_1}{\partial c}\right|_{\text{éq}} = 0$$

$$a_{21} = \left.\frac{\partial f_2}{\partial h}\right|_{\text{éq}} = -\frac{\bar{q}_1}{S\bar{h}^2}\left(c_1 - \bar{c}\right) - \frac{\bar{q}_2}{S\bar{h}^2}\left(c_2 - \bar{c}\right) = \frac{1}{S\bar{h}^2}\left[-\bar{q}_1 c_1 + \bar{q}_1 \bar{c} - \bar{q}_2 c_2 + \bar{q}_2 \bar{c}\right]$$

$$a_{22} = \left.\frac{\partial f_2}{\partial c}\right|_{\text{éq}} = -\frac{\bar{q}_1}{S\bar{h}} - \frac{\bar{q}_2}{S\bar{h}}$$

Cuve de mélange (suite)

$$b_{11} = \left.\frac{\partial f_1}{\partial q_1}\right|_{\text{éq}} = \frac{1}{S} \qquad b_{12} = \left.\frac{\partial f_1}{\partial q_2}\right|_{\text{éq}} = \frac{1}{S}$$

$$b_{21} = \left.\frac{\partial f_2}{\partial q_1}\right|_{\text{éq}} = \frac{c_1 - \bar{c}}{S\bar{h}} \qquad b_{22} = \left.\frac{\partial f_2}{\partial q_2}\right|_{\text{éq}} = \frac{c_2 - \bar{c}}{S\bar{h}}$$

$$c_{11} = 1 \qquad c_{12} = 0$$

$$c_{21} = 0 \qquad c_{22} = \frac{1}{c_1}$$

Temps de résidence

$$\theta := \frac{S\bar{h}}{\bar{q}_1 + \bar{q}_2} = \frac{S\sqrt{\bar{h}}}{k} \quad [\mathrm{s}]$$

Cuve de mélange (suite)

Modèle linéaire

$$\begin{bmatrix} \delta\dot{h} \\ \delta\dot{c} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{bmatrix} \begin{bmatrix} \delta h \\ \delta c \end{bmatrix} + \begin{bmatrix} \frac{1}{S} & \frac{1}{S} \\ \frac{c_1 - \bar{c}}{S\bar{h}} & \frac{c_2 - \bar{c}}{S\bar{h}} \end{bmatrix} \begin{bmatrix} \delta q_1 \\ \delta q_2 \end{bmatrix} \qquad \begin{bmatrix} \delta h(0) \\ \delta c(0) \end{bmatrix} = \begin{bmatrix} h_0 - \bar{h} \\ c_0 - \bar{c} \end{bmatrix}$$

$$\begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c_1} \end{bmatrix} \begin{bmatrix} \delta h \\ \delta c \end{bmatrix}$$