Optimal Control
Lectures 25-27: Maximum Principles

Benoît Chachuat <benoit@mcmaster.ca>

McMaster University

Department of Chemical Engineering

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Variational Approach: Summary

$$\min_{(\mathbf{u},t_f)\in\hat{\mathcal{C}}[t_0,T]^{n_u}\times\mathbb{R}} \int_{t_0}^{t_f} \ell(t,\mathbf{x}(t),\mathbf{u}(t))\,dt + \phi(t_f,\mathbf{x}(t_f))$$
$$\text{s.t. } \dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)); \quad \mathbf{x}(t_0)=\mathbf{x}_0$$
$$\psi_k(t_f,\mathbf{x}(t_f)) = 0, \quad k=1,\ldots,n_\psi$$

Necessary Conditions for $(\mathbf{u}^*, t_f^*, \mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ to be Optimal

- Euler-Lagrange Equations ($\mathcal{H} \stackrel{\Delta}{=} \ell + \boldsymbol{\lambda}^\mathsf{T}\mathbf{f}$):
$$\dot{\mathbf{x}} = \mathcal{H}_{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} = -\mathcal{H}_{\mathbf{x}}, \qquad \mathbf{0} = \mathcal{H}_{\mathbf{u}}, \qquad t_0 \le t \le t_f$$
- Legendre-Clebsch Condition:
$$\mathcal{H}_{\mathbf{uu}} \text{ semi-definite positive}, \quad t_0 \le t \le t_f$$
- Transversal Conditions:
$$[\mathbf{x}-\mathbf{x}_0]_{t_0} = \mathbf{0}, \qquad [\boldsymbol{\lambda} - \phi_{\mathbf{x}} + \boldsymbol{\nu}^\mathsf{T}\boldsymbol{\psi}_{\mathbf{x}}]_{t_f} = \mathbf{0}$$
$$[\mathcal{H} + \phi_t + \boldsymbol{\nu}^\mathsf{T}\boldsymbol{\psi}_t]_{t_f} = 0, \text{ if } t_f \text{ is free}$$
$$[\boldsymbol{\psi}]_{t_f} = \mathbf{0}, \text{ and } \boldsymbol{\psi} \text{ satisfy a regularity condition}$$

Pontryagin Maximum Principle

Motivations:

1. Encompass optimal control problems with path constraints in the control and/or state variables
2. Tighten the necessary conditions for optimality obtained with the variational approach

Base Problem Formulation:

$$\text{minimize: } \int_{t_0}^{t_f} \ell(\mathbf{x}(t),\mathbf{u}(t))\,dt$$
$$\text{subject to: } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)); \quad \mathbf{x}(t_0)=\mathbf{x}_0, \quad \mathbf{x}(t_f)=\mathbf{x}_f$$
$$\mathbf{u} \in \mathcal{U}[t_0,T] \stackrel{\Delta}{=} \{\mathbf{u}\in\hat{\mathcal{C}}[t_0,T]^{n_u} : \mathbf{u}(t)\in U, t_0 \le t \le t_f\}$$

- Prescribed final state $\mathbf{x}_f$ and free final time t_f
- Control region U same at all times
- Autonomous problem: no explicit dependence of ℓ and $\mathbf{f}$ in t

Pontryagin Maximum Principle: Statement

Theorem. Suppose that $(\mathbf{u}^*, t_f^*) \in \hat{\mathcal{C}}[t_0,T]^{n_u} \times [t_0,T)$ is optimal, with corresponding response $\mathbf{x}^* \in \hat{\mathcal{C}}^1[t_0,T]^{n_x}$. Then, there exist $(\lambda_0^*, \boldsymbol{\lambda}^*) \in \hat{\mathcal{C}}^1[t_0,T]^{n_x+1}$ such that:

1. $[\lambda_0^*(t), \boldsymbol{\lambda}^*(t)] \neq [0,\mathbf{0}], \; t_0 \le t \le t_f^*$
2. $\dot{\lambda}_0(t) = 0, \quad \dot{\boldsymbol{\lambda}}(t) = -\mathcal{H}_{\mathbf{x}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda_0^*(t),\boldsymbol{\lambda}^*(t))$, a.e. in $[t_0,t_f^*]$, with: $\mathcal{H}(\mathbf{x},\mathbf{u},\lambda_0,\boldsymbol{\lambda}) \stackrel{\Delta}{=} \lambda_0\ell(\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}^\mathsf{T}\mathbf{f}(\mathbf{x},\mathbf{u})$
3. $\mathcal{H}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda_0^*(t),\boldsymbol{\lambda}^*(t)) \le \mathcal{H}(\mathbf{x}^*(t),\mathbf{v},\lambda_0^*(t),\boldsymbol{\lambda}^*(t)), \; \forall \mathbf{v}\in U$, a.e. in $[t_0,t_f^*]$
4. $\lambda_0^*(t) = \text{constant} \ge 0, \quad \mathcal{H}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda_0^*(t),\boldsymbol{\lambda}^*(t)) = \text{constant}$ ($= 0$ if t_f is free)

Pontryagin Maximum Principle: Remarks

Conditions 1-4. Complete set of conditions to determine $(\mathbf{u}^*, \mathbf{x}^*, \lambda_0^*, \boldsymbol{\lambda}^*)$, along with t_f^* (if free)

Condition 4. Either one of 2 situations:

- Normal case: $\lambda_0(t) > 0$
  - $\lambda_0, \lambda_1, \ldots, \lambda_{n_x}$ defined up to a constant only
  - Need to fix λ_0, e.g., $\lambda_0(t) = 1, \forall t$
- Abnormal case: $\lambda_0(t) = 0$
  - $\lambda_0, \lambda_1, \ldots, \lambda_{n_x}$ uniquely defined, but NCO become independent of ℓ!
  - Abnormal problems are those for which the terminal conditions $\mathbf{x}(t_f) = \mathbf{x}_f$ fail to satisfy a regularity condition
- Case of a maximize problem:
  - Replace by $\lambda_0(t) \leq 0$
  - Do not change inequality sign in condition 3!

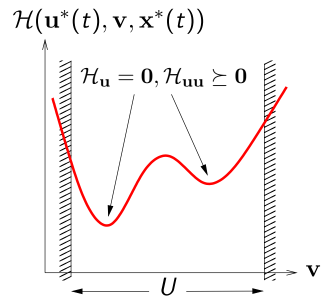
Pontryagin Maximum Principle: Remarks (cont'd)

Conditions 3. Extremely powerful result!

- Can be rewritten in the form:

$$\mathbf{u}^*(t) \in \arg\min_{\mathbf{v}} \{\mathcal{H}(\mathbf{x}^*(t), \mathbf{v}, \lambda_0^*(t), \boldsymbol{\lambda}^*(t)) : \mathbf{v} \in U\}$$

  - Yields a minimum condition – Originally, formulated as a maximum condition (Pontryagin)
  - Handles Control bounds in a very natural way: Solve an NLP problem at each time along $[t_0, t_f^*]$!



- On interior arcs, $\mathbf{u}^*(t) \in \text{int}(U)$,
  - $\mathcal{H}_\mathbf{u}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t)) = \mathbf{0}$
  - $\mathcal{H}_\mathbf{uu}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t)) \succeq \mathbf{0}$

  PMP implies the Euler-Lagrange and Legendre-Clebsch conditions!

Case Study: Linear Time-Optimal Control

$$\text{minimize:} \quad \mathcal{J}(\mathbf{u}, t_f) \triangleq \int_{t_0}^{t_f} dt = t_f - t_0$$

$$\text{subject to:} \quad \dot{\mathbf{x}}(t) = \mathbf{F}(t)\,\mathbf{x}(t) + \mathbf{G}(t)\,\mathbf{u}(t); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{0}$$

$$\mathbf{u}^L \leq \mathbf{u}(t) \leq \mathbf{u}^U, \quad t_0 \leq t \leq t_f$$

- If $(\mathbf{u}^*, t_f^*, \mathbf{x}^*, \boldsymbol{\lambda}^*, \lambda_0^* \equiv 1)$ is an optimal solution, then

$$\mathbf{u}^*(t) \in \arg\min_{\mathbf{v}} \left\{ 1 + \boldsymbol{\lambda}^*(t)^\mathsf{T} \left(\mathbf{F}(t)\,\mathbf{x}^*(t) + \mathbf{G}(t)\,\mathbf{v}\right) : \mathbf{u}^L \leq \mathbf{v} \leq \mathbf{u}^U \right\}$$

- If $\boldsymbol{\lambda}^*(t)^\mathsf{T}\mathbf{G}(t)$ vanishes only at isolated times,

$$u_i^*(t) = \begin{cases} u_i^U & \text{if } \boldsymbol{\lambda}^*(t)^\mathsf{T}\mathbf{G}_i(t) < 0 \\ u_i^L & \text{if } \boldsymbol{\lambda}^*(t)^\mathsf{T}\mathbf{G}_i(t) > 0 \end{cases}$$

- **The optimal control is said to be of bang-bang type:**
  - $u_i^*(t)$ switches instantaneously as $\boldsymbol{\lambda}^*(t)^\mathsf{T}\mathbf{G}_i(t)$ changes sign
  - $\boldsymbol{\lambda}^*(t)^\mathsf{T}\mathbf{G}_i(t)$ is called the switching function

Solving Linear Time-Optimal Control Problems

Class Exercise: Characterize the optimal solutions to the problem:

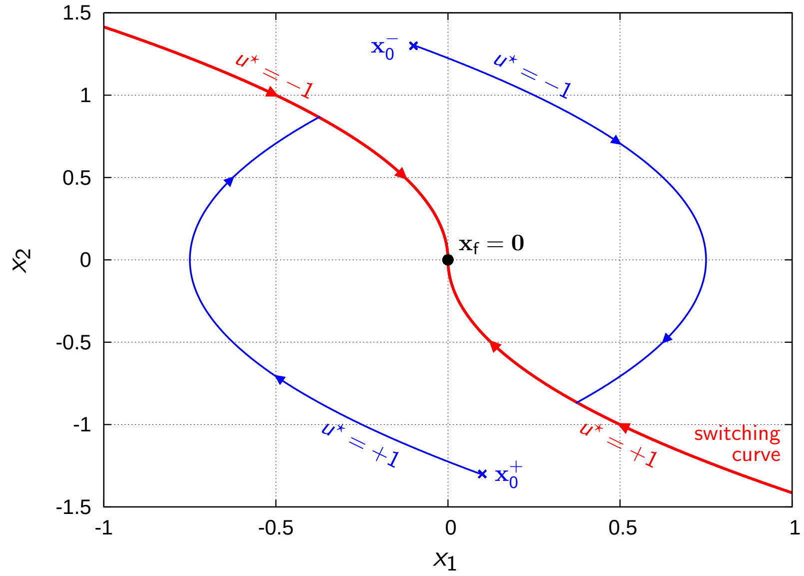
$$\min_{u, t_f} \quad \mathcal{J}(u, t_f) \triangleq \int_0^{t_f} dt = t_f$$

$$\text{s.t.} \quad \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{0}$$

$$-1 \leq u(t) \leq 1, \quad \forall t.$$

Solving Linear Time-Optimal Control Problems (cont'd)



Pontryagin Maximum Principle: Extensions

Non-Autonomous Control Problems:

$$\text{minimize:} \quad \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$$

$$\text{subject to:} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u} \in \mathcal{U}[t_0, T] \triangleq \{\mathbf{u} \in \hat{\mathcal{C}}[t_0, T]^{n_u} : \mathbf{u}(t) \in U, t_0 \leq t \leq t_f\}$$

Trick:

1. Define the extra state variable x_{n_x+1} as follows:
$$\dot{x}_{n_x+1}(t) = 1; \qquad x_{n_x+1}(t_0) = t_0$$
2. Replace:
$$\ell(t, \mathbf{x}(t), \mathbf{u}(t)) \to \ell(x_{n_x+1}(t), \mathbf{x}(t), \mathbf{u}(t)),$$
$$\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) \to \mathbf{f}(x_{n_x+1}(t), \mathbf{x}(t), \mathbf{u}(t))$$
3. Apply the PMP for autonomous systems!

Pontryagin Maximum Principle: Extensions (cont'd)

General Terminal Constraints: $\mathbf{x}(t_f) \in X_f \subset \mathbb{R}^{n_x}$

- Suppose that X_f has dimension n_f locally at $\mathbf{x}^*(t_f^*)$
  - n_f transversal conditions are given by: $\mathbf{x}(t_f) \in X_f$
  - the complementary $n_x - n_f$ conditions read:

$$\boxed{\boldsymbol{\lambda}^*(t_f^*)^{\mathsf{T}} \mathbf{d} = 0, \quad \forall \mathbf{d} \in \mathcal{T}_{X_f}(\mathbf{x}^*(t_f^*))}$$

Application: $X_f \triangleq \{\mathbf{x} \in \mathbb{R}^{n_x} : \Psi_1(\mathbf{x}) = \cdots = \Psi_{n_\Psi}(\mathbf{x}) = 0\}$

- If the n_Ψ terminal constraints are regular, $\text{rank}\,(\boldsymbol{\Psi}_{\mathbf{x}}(\mathbf{x}^*(t_f^*))) = n_\Psi$,

$$\mathcal{T}_{X_f}(\mathbf{x}^*(t_f^*)) = \{\mathbf{d} \in \mathbb{R}^{n_x} : \boldsymbol{\Psi}_{\mathbf{x}}(\mathbf{x}^*(t_f^*))\,\mathbf{d} = \mathbf{0}\}$$

- There exist (unique) Lagrange multipliers $\boldsymbol{\nu}^* \in \mathbb{R}^{n_\Psi}$ such that:

$$\boxed{\boldsymbol{\lambda}^*(t_f^*) = \boldsymbol{\nu}^{*\mathsf{T}} \boldsymbol{\Psi}_{\mathbf{x}}(\mathbf{x}^*(t_f^*))}$$

Case Study: Affine-Control Problems

Class Exercise: Discuss the possible values that can be taken by an optimal solution $\mathbf{u}^*(t)$ to to the affine-control problem:

$$\min_{u, t_f} \quad \mathcal{J}(u, t_f) \triangleq \int_{t_0}^{t_f} \ell^0(t, \mathbf{x}(t)) + u(t)\,\ell^1(t, \mathbf{x}(t))\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{f}^0(t, \mathbf{x}(t)) + u(t)\mathbf{f}^1(t, \mathbf{x}(t)); \quad \mathbf{x}(0) = \mathbf{x}_0; \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$u^L \leq u(t) \leq u^U, \quad \forall t.$$

Singular Optimal Control: Definition

minimize: $\int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$

subject to: $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$

$\mathbf{u} \in \mathcal{U}[t_0, T] \triangleq \{\mathbf{u} \in \hat{\mathcal{C}}[t_0, T]^{n_u} : \mathbf{u}(t) \in U, t_0 \leq t \leq t_f\}$

$\Downarrow$ Optimal Solution: $\mathbf{u}^*, t_f^* \mathbf{x}^*, \boldsymbol{\lambda}^*, \lambda_0^*$

Singular control problems are obtained when the Hessian matrix $\mathcal{H}_{\mathbf{uu}}(t, \mathbf{x}^*(t), \cdot, \boldsymbol{\lambda}^*(t), \lambda_0^*(t))$ is singular on the control region U

- In singular scalar problems, the stationarity condition:

$$\mathcal{H}_u(t, \mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t)) = 0,$$

is trivially satisfied for any value of $u \in [u^L, u^U]$, over some finite time interval $(\theta_1, \theta_2) \in [t_0, t_f]$

Singular Optimal Control: Optimality Conditions

Consider a Singular Arc (θ_1, θ_2),

$$\mathcal{H}_u(t, \mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t), \lambda_0^*(t)) = 0, \quad \forall u \in U, \quad \forall t \in (\theta_1, \theta_2)$$

- Successive Time Differentiation:

$$\frac{d^q}{dt^q}\mathcal{H}_u(t, \mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t), \lambda_0^*(t)) = 0, \quad \forall t \in (\theta_1, \theta_2), \quad \forall q \geq 0$$

- The smallest positive integer σ (if any) such that:

$$\frac{\partial}{\partial u}\left[\frac{d^\sigma}{dt^\sigma}\mathcal{H}_u(t, \mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t), \lambda_0^*(t))\right] \neq 0,$$

  is called the order of singularity
- If σ exists, it is even
- Generalized Legendre-Clebsch Condition:

$$(-1)^{\frac{\sigma}{2}}\frac{\partial}{\partial u}\left[\frac{d^\sigma}{dt^\sigma}\mathcal{H}_u(t, \mathbf{x}^*(t), u, \boldsymbol{\lambda}^*(t), \lambda_0^*(t))\right] \geq 0$$

Solving Singular Control Problems

Class Exercise: Characterize the optimal solutions to the problem:

minimize: $\mathcal{J}(u) \triangleq \int_0^2 \frac{1}{2}[x(t)]^2\, dt$

subject to: $\dot{x}_1(t) = x_2(t) + u(t); \quad x_1(0) = 1; \quad x_1(2) = 0$

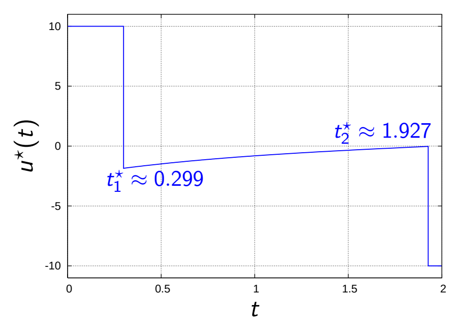
$\dot{x}_2(t) = -u(t); \quad x_2(0) = 1; \quad x_2(2) = 0$

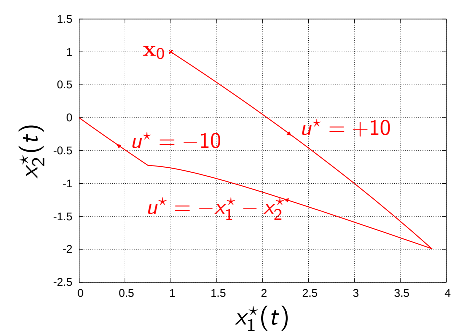
$-10 \leq u(t) \leq 10, \quad \forall t \in [0, 2].$

Solving Singular Control Problems (cont'd)

Optimal Solution Structure:

- Set of active terminal constraints
- Sequence of interior/boundary arcs in the optimal control

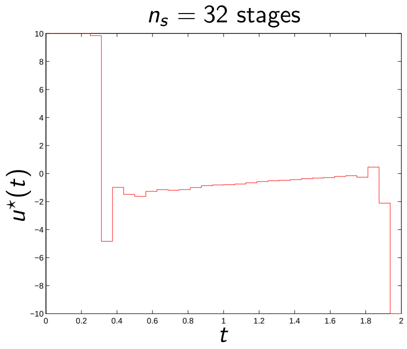


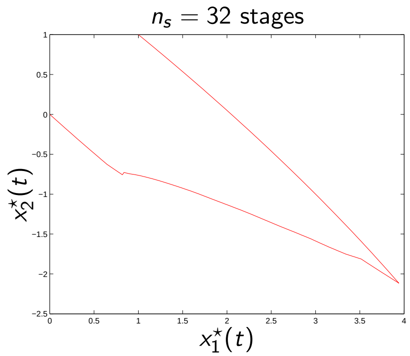


Solving Singular Control Problems (cont'd)

Characterizing the optimal solution structure is a very difficult task:

- Even simple problems can have an infinite number of arcs!
- Use direct numerical methods to guess the structure





Optimal Control with State Path Constraints

- State inequality constraints arise frequently in practical applications
  - Notoriously hard to solve
  - Various forms of the NCOs – Sometimes ambiguous theory!
- Types of inequality path constraints:

Mixed State-Control Constraints:

$$g(t, \mathbf{x}(t), \mathbf{u}(t)) \leq 0, \quad \forall t$$

- **Explicit** dependence in $\mathbf{u}$:

$$g_{\mathbf{u}}(t, \mathbf{x}, \mathbf{u}) \neq \mathbf{0}$$

Pure State Constraints:

$$h(t, \mathbf{x}(t)) \leq 0, \quad \forall t$$

- **Implicit** dependence in $\mathbf{u}$ via the differential system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

Mixed Control-State Constrained Problems

Base Problem Formulation:

$$\text{minimize:} \quad \int_{t_0}^{t_f} \ell(\mathbf{x}(t), \mathbf{u}(t))\, dt$$

$$\text{subject to:} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$g_k(t, \mathbf{x}(t), \mathbf{u}(t)) \leq 0, \quad k = 1, \ldots, n_g$$

- Encompasses the PMP formulation:

$$\mathbf{u} \in \mathcal{U}[t_0, T] \triangleq \{\mathbf{u} \in \hat{\mathcal{C}}[t_0, T]^{n_u} : \mathbf{u}(t) \in U, t_0 \leq t \leq t_f\}$$

- **Idea:** Form a Lagrangian function,

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu}) \triangleq \mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}) + \boldsymbol{\mu}^\mathsf{T} \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

with:
  - $\mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}) \triangleq \lambda_0 \ell(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^\mathsf{T} \mathbf{f}(t, \mathbf{x}, \mathbf{u})$
  - $\boldsymbol{\mu} \in \hat{\mathcal{C}}[t_0, t_f]^{n_g}$ Lagrange multiplier vector function

Mixed Control-State Constrained Problems (cont'd)

- Let $(\mathbf{u}^*, t_f^*) \in \hat{\mathcal{C}}[t_0, T]^{n_u} \times [t_0, T)$ be an optimal solution
- Let $\mathbf{x}^* \in \hat{\mathcal{C}}^1[t_0, T]^{n_x}$ be the optimal response
- Suppose the constraint qualification: rank $\left[\ \mathbf{g_u} \quad \text{diag}\,(\mathbf{g}) \ \right] = n_g$, holds along $(t, \mathbf{x}^*, \mathbf{u}^*)$, $t_0 \leq t \leq t_f^*$

Necessary Conditions for Optimality:

There exist $(\lambda_0^*, \boldsymbol{\lambda}^*) \in \hat{\mathcal{C}}^1[t_0, T]^{n_x+1}$ and $\boldsymbol{\mu}^* \in \hat{\mathcal{C}}[t_0, T]^{n_g}$ such that:

1. $(\lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)) \neq \mathbf{0}, \quad \lambda_0^*(t) = \text{constant} \geq 0$
2. $\mathbf{u}^*(t) \in \arg\min_{\mathbf{v} \in \mathbb{R}^{n_u}} \{\mathcal{H}(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*(t), \boldsymbol{\lambda}^*(t)) : \mathbf{g}(t, \mathbf{x}^*(t), \mathbf{v}) \leq \mathbf{0}\}$
3. $\begin{cases} \dot{\mathbf{x}}^*(t) &= \mathcal{L}_{\boldsymbol{\lambda}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)) \\ \dot{\boldsymbol{\lambda}}^*(t) &= -\mathcal{L}_{\mathbf{x}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)) \\ \mathbf{0} &= \mathcal{L}_{\mathbf{u}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\mu}^*(t)) \end{cases}$
4. $\boldsymbol{\mu}^*(t)^\mathsf{T} \mathbf{g}(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) = 0; \quad \boldsymbol{\mu}^*(t) \geq \mathbf{0}$
5. $\mathcal{H}(t_f^*, \mathbf{x}^*(t_f^*), \mathbf{u}^*(t_f^*), \lambda_0^*(t_f^*), \boldsymbol{\lambda}^*(t_f^*)) = 0$, if t_f is free

Solving Mixed Control-State Constrained Problems

Class Exercise: Consider the optimal control problem:

$$\begin{aligned}\text{minimize:}\quad & \mathcal{J}(u) \triangleq \int_0^1 u(t)\, dt\\ \text{subject to:}\quad & \dot{x}(t) = -u(t);\quad x(0) = -1\\ & u(t) \leq 0,\quad x(t) - u(t) \leq 0,\quad 0 \leq t \leq 1\end{aligned}$$

1. Is $u(t) = x(t)$, $0 \leq t \leq 1$, a candidate optimal control?

Pure State Constrained Problems

$$\begin{aligned}\text{minimize:}\quad & \int_{t_0}^{t_f} \ell(\mathbf{x}(t), \mathbf{u}(t))\, dt\\ \text{subject to:}\quad & \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t));\quad \mathbf{x}(t_0) = \mathbf{x}_0,\quad \mathbf{x}(t_f) = \mathbf{x}_f\\ & h_k(t, \mathbf{x}(t)) \leq 0,\quad k = 1, \ldots, n_h\end{aligned}$$

- No explicit dependence of h_k in the control $\mathbf{u}$
- **Idea:** Apply similar concepts as with singular control problems

Differentiate each h_k w.r.t. t as many times as needed:

$$\begin{aligned} h_k^0(t, \mathbf{x}) &\triangleq h_j(t, \mathbf{x})\\ &\vdots\\ h_k^{i+1}(t, \mathbf{x}) &\triangleq \frac{d}{dt} h_k^i(t, \mathbf{x}) = (h_k^i)_{\mathbf{x}}(t, \mathbf{x})\, \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + (h_k^i)_t(t, \mathbf{x})\\ &\vdots\end{aligned}$$

Pure State Constrained Problems

$$\begin{aligned}\text{minimize:}\quad & \int_{t_0}^{t_f} \ell(\mathbf{x}(t), \mathbf{u}(t))\, dt\\ \text{subject to:}\quad & \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t));\quad \mathbf{x}(t_0) = \mathbf{x}_0,\quad \mathbf{x}(t_f) = \mathbf{x}_f\\ & h_k(t, \mathbf{x}(t)) \leq 0,\quad k = 1, \ldots, n_h\end{aligned}$$

- No explicit dependence of h_k in the control $\mathbf{u}$
- **Idea:** Apply similar concepts as with singular control problems

until the controls $\mathbf{u}$ appear explicitly:

$$(h_k^i)_{\mathbf{u}} = \mathbf{0},\quad 0 \leq i \leq \sigma_k - 1;\qquad (h_k^{\sigma_k})_{\mathbf{u}} \neq \mathbf{0},$$

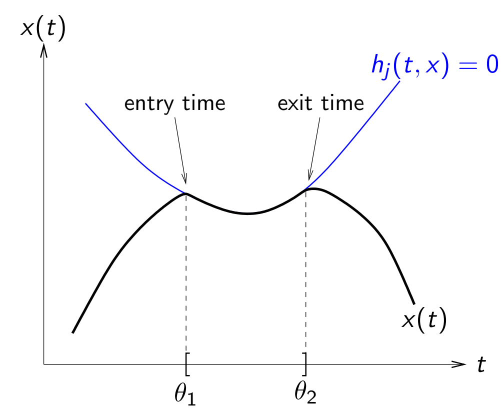
σ_k is said to be the **degree** (or **order**) of the state constraint h_k

Class Exercise: What is the degree σ of the state constraint $x_2(t) \leq x_2^{\max}$, subject to the differential system $\dot{x}_1(t) = u(t)$, $\dot{x}_2(t) = x_1(t)$?

1st-Order State Constraints: Indirect Adjoining Approach

- Let $[\theta_1, \theta_2]$ be a boundary arc for h_k in $(\mathbf{u}^*, \mathbf{x}^*)$,

$$h_k(t, \mathbf{x}^*(t)) = 0 \quad\Rightarrow\quad \begin{cases} h_k^1(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \leq 0\\ h_k(\theta_1^-, \mathbf{x}^*(\theta_1^-)) < 0\\ h_k(\theta_1^+, \mathbf{x}^*(\theta_1^+)) = 0\end{cases}$$



1st-Order State Constraints: Indirect Adjoining Approach

- Let $[\theta_1, \theta_2]$ be a boundary arc for h_k in $(\mathbf{u}^*, \mathbf{x}^*)$,

$$h_k(t, \mathbf{x}^*(t)) = 0 \quad \Rightarrow \quad \begin{cases} h_k^1(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \leq 0 \\ h_k(\theta_1^-, \mathbf{x}^*(\theta_1^-)) < 0 \\ h_k(\theta_1^+, \mathbf{x}^*(\theta_1^+)) = 0 \end{cases}$$

- Activation represented by means of a multiplier function $\eta_k(t)$,

$$\eta_k(t) h_k(t, \mathbf{x}^*(t)) = 0; \quad \eta_k(t) \geq 0; \quad \eta_k(t) h_k^1(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) \leq 0$$

- Interior-point constraint induce jumps in λ^* and $\mathcal{H}$ at θ_1,

$$\boldsymbol{\lambda}^*(\theta_1^-) = \boldsymbol{\lambda}^*(\theta_1^+) + \pi_k (h_k)_{\mathbf{x}}(\theta_1, \mathbf{x}^*(\theta_1))$$
$$\mathcal{H}[\theta_1^-] = \mathcal{H}[\theta_1^+] - \pi_k (h_k)_t(\theta_1, \mathbf{x}^*(\theta_1))$$

with Lagrange multiplier π_k satisfying

$$\pi_k h_k(\theta_1, \mathbf{x}^*(\theta_1)) = 0; \quad \pi_k \geq \eta_k(\theta_1^+)$$

Apply the NCOs for mixed control-state constrained problems, with:

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\eta}) \triangleq \mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}) + \boldsymbol{\eta}^{\mathsf{T}} \mathbf{h}^1(t, \mathbf{x}, \mathbf{u})$$

First-Order Pure State Constrained Problems (cont'd)

- Let $(\mathbf{u}^*, t_f^*) \in \hat{\mathcal{C}}[t_0, T]^{n_u} \times [t_0, T)$ be an optimal solution
- Let $\mathbf{x}^* \in \hat{\mathcal{C}}^1[t_0, T]^{n_x}$ be the optimal response
- Suppose the constraint qualification: $\operatorname{rank} \begin{bmatrix} \mathbf{h}_{\mathbf{u}}^1 & \operatorname{diag}(\mathbf{h}) \end{bmatrix} = n_h$, holds along $(t, \mathbf{x}^*, \mathbf{u}^*)$, $t_0 \leq t \leq t_f^*$

Necessary Conditions for Optimality:

There exist $(\lambda_0^*, \boldsymbol{\lambda}^*) \in \hat{\mathcal{C}}^1[t_0, T]^{n_x+1}$, $\boldsymbol{\eta}^* \in \hat{\mathcal{C}}^1[t_0, T]^{n_h}$, $\boldsymbol{\pi}^* \in \mathbb{R}^{n_h}$ s.t.

1. $(\lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)) \neq \mathbf{0}, \quad \lambda_0^*(t) = \text{constant} \geq 0$
2. $\mathbf{u}^*(t) \in \arg\min_{\mathbf{v} \in \mathbb{R}^{n_u}} \{\mathcal{H}(t, \mathbf{x}^*(t), \mathbf{v}, \lambda_0^*(t), \boldsymbol{\lambda}^*(t)) : \boldsymbol{\eta}^*(t)^{\mathsf{T}} \mathbf{h}^1(t, \mathbf{x}^*(t), \mathbf{v}) \leq \mathbf{0}\}$
3. $\begin{cases} \dot{\mathbf{x}}^*(t) &= \mathcal{L}_{\boldsymbol{\lambda}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)) \\ \dot{\boldsymbol{\lambda}}^*(t) &= -\mathcal{L}_{\mathbf{x}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)) \\ \mathbf{0} &= \mathcal{L}_{\mathbf{u}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \lambda_0^*(t), \boldsymbol{\lambda}^*(t), \boldsymbol{\eta}^*(t)) \end{cases}$
4. $\boldsymbol{\eta}^*(t)^{\mathsf{T}} \mathbf{h}(t, \mathbf{x}^*(t), \mathbf{u}^*(t)) = 0; \quad \boldsymbol{\eta}^*(t) \geq \mathbf{0}; \quad \dot{\boldsymbol{\eta}}^*(t) \leq \mathbf{0}$

First-Order Pure State Constrained Problems (cont'd)

5. At each entry time θ_1,

$$\boldsymbol{\lambda}^*(\theta_1^-) = \boldsymbol{\lambda}^*(\theta_1^+) + \boldsymbol{\pi}_k^{*\mathsf{T}} \mathbf{h}_{\mathbf{x}}(\theta_1, \mathbf{x}^*(\theta_1))$$
$$\mathcal{H}[\theta_1^-] = \mathcal{H}[\theta_1^+] - \boldsymbol{\pi}_k^{*\mathsf{T}} \mathbf{h}_t(\theta_1, \mathbf{x}^*(\theta_1))$$
$$\pi_k^* h_k(\theta_1, \mathbf{x}^*(\theta_1)) = 0; \quad \pi_k^* \geq \eta_k^*(\theta_1^+)$$

6. $\mathcal{H}(t_f^*, \mathbf{x}^*(t_f^*), \mathbf{u}^*(t_f^*), \lambda_0^*(t_f^*), \boldsymbol{\lambda}^*(t_f^*)) = 0$, if t_f is free

Combined (1st-Order) Pure and Mixed State Inequality Constraints:

- Mix the previous sets of optimality conditions, with

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}) \triangleq \mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda_0, \boldsymbol{\lambda}) + \boldsymbol{\mu}^{\mathsf{T}} \mathbf{g}(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\eta}^{\mathsf{T}} \mathbf{h}^1(t, \mathbf{x}, \mathbf{u})$$

- Strengthened C.Q.: $\operatorname{rank} \begin{bmatrix} \mathbf{g}_{\mathbf{u}} & \operatorname{diag}(\mathbf{g}) & \mathbf{0} \\ \mathbf{h}_{\mathbf{u}}^1 & \mathbf{0} & \operatorname{diag}(\mathbf{h}) \end{bmatrix} = n_g + n_h$

Caution: A general theorem proving the foregoing NCOs is still lacking!

Solving Pure/Mixed Path Constrained Problems

Class Exercise: Consider the optimal control problem:

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(u) \triangleq \int_0^3 e^{-\varrho t}\, u(t)\, dt \\ \text{subject to:} \quad & \dot{x}(t) = u(t); \quad x(0) = 0 \\ & 1 - x(t) - (t-2)^2 \leq 0, \quad 0 \leq t \leq 3 \\ & 0 \leq u(t) \leq 3, \quad 0 \leq t \leq 3 \end{aligned}$$

1. Is $u(t) = \begin{cases} 0, & 0 \leq t \leq 1^- \\ -2(t-2), & 1^+ \leq t \leq 2^- \\ 0, & 2^+ \leq t \leq 3^- \end{cases}$ a candidate optimal control?

Solving Pure/Mixed Path Constrained Problems

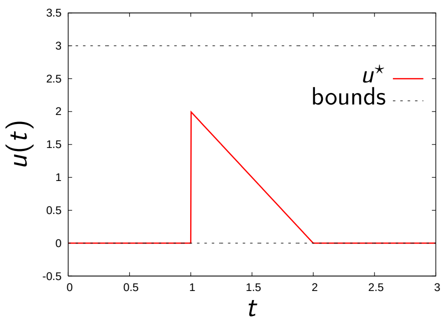
Class Exercise: Consider the optimal control problem:

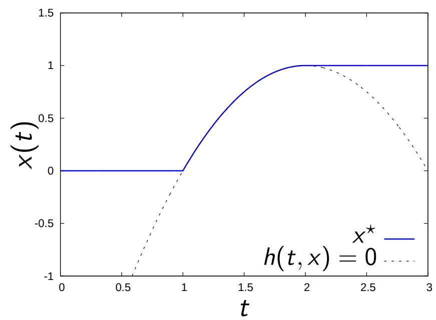
$$\text{minimize:} \quad \mathcal{J}(u) \triangleq \int_0^3 \mathrm{e}^{-\varrho t}\, u(t)\, \mathrm{d}t$$

$$\text{subject to:} \quad \dot{x}(t) = u(t); \quad x(0) = 0$$

$$1 - x(t) - (t-2)^2 \leq 0, \quad 0 \leq t \leq 3$$

$$0 \leq u(t) \leq 3, \quad 0 \leq t \leq 3$$





General Constrained Optimal Control Problems

Extended Problem Formulation:

- Cost functional: $\int_{t_0}^{t_f} \ell(\mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t + \phi(t_f, \mathbf{x}(t_f))$
- Terminal constraints: $\psi_k(t_f, \mathbf{x}(t_f)) \leq 0, \quad k = 1, \ldots, n_\psi$ (instead of $\mathbf{x}(t_f) = \mathbf{x}_f$)
- Suppose the constraint qualification: rank $\left[\ \boldsymbol{\psi}_\mathbf{x} \quad \mathrm{diag}\,(\boldsymbol{\psi})\ \right] = n_\psi$, holds at $(t_f^\star, \mathbf{x}(t_f^\star))$

Modified/Extra Necessary Conditions for Optimality:

There exist Lagrange multipliers $\boldsymbol{\nu}^\star \in \mathbb{R}^{n_\psi}$ such that:

1. $\psi(t_f^\star, \mathbf{x}^\star(t_f^\star)) \leq 0; \quad \boldsymbol{\nu}^\star(t)^\mathsf{T} \psi(t_f^\star, \mathbf{x}^\star(t_f^\star)) = 0; \quad \boldsymbol{\nu}^\star(t) \geq \mathbf{0}$
2. $\left[\boldsymbol{\lambda}^\star - \lambda_0^\star \phi_\mathbf{x} - \boldsymbol{\nu}^{\star\mathsf{T}} \boldsymbol{\psi}_\mathbf{x}\right]_{t_f^\star} = \mathbf{0}$
3. $\left[\mathcal{H} + \lambda_0^\star \phi_t + \boldsymbol{\nu}^{\star\mathsf{T}} \boldsymbol{\psi}_t\right]_{t_f^\star} = 0$, if t_f is free

General Constrained Optimal Control Problems (cont'd)

Optimal Solution Structure:

- Sequence of active/inactive path constraints and junction times
- Set of active terminal constraints
- Use direct numerical methods to guess the optimal solution structure

Identifying Candidate Optimal Solutions:

1. Postulate an optimal solution structure
2. Check whether a pair $(\mathbf{u}^\star, \mathbf{x}^\star)$, final time $t_f^\star$, functions $(\lambda_0^\star, \boldsymbol{\lambda}^\star)$ and multipliers $(\boldsymbol{\mu}^\star, \boldsymbol{\eta}^\star, \boldsymbol{\nu}^\star$ can be found that satisfy all the NCOs
3. Go back to step 1

Conditions at a Junction Time, $\theta \in (t_0, t_f^\star)$:

$$\mathbf{x}^\star(\theta^-) = \mathbf{x}^\star(\theta^+)$$

$$\boldsymbol{\lambda}^\star(\theta^-) = \boldsymbol{\lambda}^\star(\theta^+)$$

$$\mathcal{H}(\theta, \mathbf{x}^\star(\theta), \mathbf{u}^\star(\theta^-), \lambda_0^\star(\theta), \boldsymbol{\lambda}^\star(\theta)) = \mathcal{H}(\theta, \mathbf{x}^\star(\theta), \mathbf{u}^\star(\theta^+), \lambda_0^\star(\theta), \boldsymbol{\lambda}^\star(\theta))$$

General Constrained Optimal Control Problems (cont'd)

Consider the scalar optimal control problem to

$$\textbf{minimize:} \quad \mathcal{J}(u) := \int_0^1 \left([x_1(t)]^2 + [x_2(t)]^2\right)\, \mathrm{d}t$$

$$\textbf{subject to:} \quad \dot{x}_1(t) = x_2(t); \quad x_1(0) = 0$$

$$\dot{x}_2(t) = -x_2(t) + u(t); \quad x_2(0) = -1$$

$$x_2(t) + 0.5 - 8[t - 0.5]^2 \leq 0, \forall t$$

$$-20 \leq u(t) \leq 20, \forall t$$