Pontryagin Maximum Principle

Motivations:

- Encompass optimal control problems with path constraints in the control and/or state variables
- Tighten the necessary conditions for optimality obtained with the variational approach

Base Problem Formulation:

\[
\begin{align*}
\text{minimize:} & \int_{t_0}^{t_f} \ell(t, x(t), u(t)) \, dt \\
\text{subject to:} & \quad \dot{x}(t) = f(x(t), u(t)); \quad x(t_0) = x_0, \quad x(t_f) = x_f \\
& \quad u \in U = \{ u \in \mathcal{U}[t_0, T] : u(t) \in U, t_0 \leq t \leq t_f \}
\end{align*}
\]

- Prescribed final state \(x_f\) and free final time \(t_f\)
- Control region \(U\) same at all times
- Autonomous problem: no explicit dependence of \(\ell\) and \(f\) in \(t\)

Pontryagin Maximum Principle: Statement

Theorem. Suppose that \((u^*, t_f^*) \in \mathcal{C}[t_0, T]^n_u \times \{ t_0, T \}\) is optimal, with corresponding response \(x^* \in \mathcal{C}^1[0, T]^{n_x}\). Then, there exist \((\lambda^*_0, \lambda^*) \in \mathcal{C}^1[0, T]^{n_x+1}\) such that:

\[
\begin{align*}
& [\lambda^*_0(t), \lambda^*(t)] \neq [0, 0], \quad t_0 \leq t \leq t_f^* \\
& \dot{\lambda}_0(t) = 0, \quad \dot{\lambda}(t) = -\mathcal{H}_x(x^*(t), u^*(t), \lambda^*_0(t), \lambda^*(t)), \quad \text{a.e. in } [t_0, t_f^*], \\
& \mathcal{H}(x^*(t), u^*(t), \lambda^*_0(t), \lambda^*(t)) \leq \mathcal{H}(x^*(t), v, \lambda^*_0(t), \lambda^*(t)), \quad \forall v \in U, \quad \text{a.e. in } [t_0, t_f^*] \\
& \lambda^*_0(t) = \text{constant} \geq 0, \quad \mathcal{H}(x^*(t), u^*(t), \lambda^*_0(t), \lambda^*(t)) = \text{constant} \quad (= 0 \text{ if } t_f^* \text{ is free})
\end{align*}
\]
Pontryagin Maximum Principle: Remarks

**Conditions 1-4.** Complete set of conditions to determine \((u^*, x^*, \lambda^*_0, \lambda^*_\ast)\), along with \(t^*_f\) (if free)

**Condition 4.** Either one of 2 situations:

- **Normal case:** \(\lambda_0(t) > 0\)
  - \(\lambda_0, \lambda_1, \ldots, \lambda_n\) defined up to a constant only
  - Need to fix \(\lambda_0\), e.g., \(\lambda_0(t) = 1, \forall t\)

- **Abnormal case:** \(\lambda_0(t) = 0\)
  - \(\lambda_0, \lambda_1, \ldots, \lambda_n\) uniquely defined, but NCO become independent of \(\ell\)!
  - Abnormal problems are those for which the terminal conditions \(x(t_f) = x_f\) fail to satisfy a regularity condition

- Case of a maximize problem:
  - Replace by \(\lambda_0(t) \leq 0\)
  - Do **not** change inequality sign in condition 3!

---

Case Study: Linear Time-Optimal Control

minimize: \(J(u, t_f) \triangleq \int_{t_0}^{t_f} dt = t_f - t_0\)

subject to: \[
\begin{align*}
\dot{x}(t) &= F(t)x(t) + G(t)u(t); \quad x(t_0) = x_0, \quad x(t_f) = 0 \\
u_L &\leq u(t) \leq u_U, \quad t_0 \leq t \leq t_f
\end{align*}
\]

If \((u^*, t^*_f, x^*, \lambda^*_0, \lambda^*_\ast) \equiv 1\) is an optimal solution, then

\[
u^*(t) \in \arg \min_v \left\{ 1 + \lambda^*(t)^T (F(t)x^*(t) + G(t)v) : u_L \leq v \leq u_U \right\}
\]

If \(\lambda^*(t)^T G(t)\) vanishes only at isolated times,

\[
u^*_i(t) = \begin{cases} 
u^*_U & \text{if } \lambda^*(t)^T G_i(t) < 0 \\
u^*_L & \text{if } \lambda^*(t)^T G_i(t) > 0 \end{cases}
\]

**The optimal control is said to be of bang-bang type:**

- \(u^*_i(t)\) switches instantaneously as \(\lambda^*(t)^T G_i(t)\) changes sign
- \(\lambda^*(t)^T G_i(t)\) is called the **switching function**

---

Solving Linear Time-Optimal Control Problems

**Class Exercise:** Characterize the optimal solutions to the problem:

\[
\begin{align*}
\min_{u, t_f} & \quad J(u, t_f) \triangleq \int_{0}^{t_f} dt = t_f \\
\text{s.t.} & \quad \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t) \\
& \quad x(0) = x_0, \quad x(t) = 0 \\
& \quad -1 \leq u(t) \leq 1, \quad \forall t.
\end{align*}
\]
Pontryagin Maximum Principle: Extensions (cont’d)

General Terminal Constraints: $x(t_f) \in X_f \subset \mathbb{R}^{n_x}$

- Suppose that $X_f$ has dimension $n_f$ locally at $x^*(t_f^*)$
  - $n_f$ transversal conditions are given by: $x(t_f) \in X_f$
  - the complementary $n_x - n_f$ conditions read:
    $$\lambda^*(t_f^*)^T d = 0, \quad \forall d \in \mathcal{T}_X(x^*(t_f^*))$$

Application: $X_f \triangleq \{ x \in \mathbb{R}^{n_x} : \psi_1(x) = \cdots = \psi_{n_{\psi}}(x) = 0 \}$

- If the $n_{\psi}$ terminal constraints are regular, $\text{rank}(\Psi_x(x^*(t_f^*))) = n_{\psi}$,
  $$\mathcal{T}_X(x^*(t_f^*)) = \{ d \in \mathbb{R}^{n_x} : \Psi_x(x^*(t_f^*)) d = 0 \}$$

- There exist (unique) Lagrange multipliers $\nu^* \in \mathbb{R}^{n_{\nu}}$ such that:
  $$\lambda^*(t_f^*) = \nu^T \Psi_x(x^*(t_f^*))$$

Case Study: Affine-Control Problems

Class Exercise: Discuss the possible values that can be taken by an optimal solution $u^*(t)$ to the affine-control problem:

$$\min_{u(t)} J(u(t)) = \int_{t_0}^{t_f} \ell(t, x(t), u(t)) \, dt$$

s.t. $x(t) = f(t, x(t), u(t))$; $x(0) = x_0$, $x(t_f) = x_f$

$$u(t) \leq u_U, \quad \forall t.$$
Singular Optimal Control: Definition

\[
\begin{align*}
\text{minimize:} & \quad \int_{t_0}^{t_f} \ell(t, x(t), u(t)) \, dt \\
\text{subject to:} & \quad \dot{x}(t) = f(t, x(t), u(t)); \quad x(t_0) = x_0, \quad x(t_f) = x_f \\
& \quad u \in U[t_0, T] \Delta \{ u \in \mathcal{C}[t_0, T]^{2u} : u(t) \in U, t_0 \leq t \leq t_f \}
\end{align*}
\]

Optimal Solution:
\[
u^*, t^*_x, \lambda^*, \lambda^*_0
\]

Singular control problems are obtained when the Hessian matrix \( \mathcal{H}_{uu}(t, x^*(t), \cdot, \lambda^*(t), \lambda^*_0(t)) \) is singular on the control region \( U \).

- In singular scalar problems, the stationarity condition:

\[
\mathcal{H}_u(t, x^*(t), u, \lambda^*(t)) = 0,
\]

is trivially satisfied for any value of \( u \in [u_L, u_U] \), over some finite time interval \( (\theta_1, \theta_2) \in [t_0, t_f] \).

Singular Optimal Control: Optimality Conditions

Consider a Singular Arc \((\theta_1, \theta_2)\),

\[
\mathcal{H}_u(t, x^*(t), u, \lambda^*(t), \lambda^*_0(t)) = 0, \quad \forall u \in U, \quad \forall t \in (\theta_1, \theta_2)
\]

- Successive Time Differentiation:

\[
\frac{d^q}{dt^q} \mathcal{H}_u(t, x^*(t), u, \lambda^*(t), \lambda^*_0(t)) = 0, \quad \forall t \in (\theta_1, \theta_2), \quad \forall q \geq 0
\]

- The smallest positive integer \( \sigma \) (if any) such that:

\[
\left. \frac{\partial}{\partial u} \left[ \frac{d^\sigma}{dt^\sigma} \mathcal{H}_u(t, x^*(t), u, \lambda^*(t), \lambda^*_0(t)) \right] \right| \neq 0,
\]

is called the order of singularity.

- If \( \sigma \) exists, it is even

- Generalized Legendre-Clebsch Condition:

\[
(-1)^{\frac{\sigma}{2}} \frac{\partial}{\partial u} \left[ \frac{d^\sigma}{dt^\sigma} \mathcal{H}_u(t, x^*(t), u, \lambda^*(t), \lambda^*_0(t)) \right] \geq 0
\]

Solving Singular Control Problems

Class Exercise: Characterize the optimal solutions to the problem:

\[
\begin{align*}
\text{minimize:} & \quad \mathcal{J}(u) \triangleq \int_0^2 \frac{1}{2} x(t)^2 \, dt \\
\text{subject to:} & \quad x_1(t) = x_2(t) + u(t); \quad x_1(0) = 1, \quad x_1(2) = 0 \\
& \quad x_2(t) = -u(t); \quad x_2(0) = 1, \quad x_2(2) = 0 \\
& \quad -10 \leq u(t) \leq 10, \quad \forall t \in [0, 2].
\end{align*}
\]

Optimal Solution Structure:

- Set of active terminal constraints

- Sequence of interior/boundary arcs in the optimal control

\[
\begin{align*}
\text{Optimal Solution:} & \quad \mathcal{J}^* = 1.927 \\
& \quad u(t) \approx 1.299 \\
& \quad x_1(t) \approx 0.27 \
\end{align*}
\]
Solving Singular Control Problems (cont’d)

Characterizing the optimal solution structure is a very difficult task:
- Even simple problems can have an infinite number of arcs!
- Use direct numerical methods to guess the structure

Mixed Control-State Constrained Problems

Base Problem Formulation:

minimize: \( \int_{t_0}^{t_f} \ell(x(t), u(t)) \, dt \)
subject to: \( \dot{x}(t) = f(t, x(t), u(t)); \ x(t_0) = x_0, \ x(t_f) = x_f \)
\( g_k(t, x(t), u(t)) \leq 0, \quad k = 1, \ldots, n_g \)

- Encompasses the PMP formulation:
  \[ u \in U; t \rightleftarrows \tilde{C}[t_0, T] \ni \{ u \in \tilde{C}[t_0, T]^{n_u}: u(t) \in U, t_0 \leq t \leq t_f \} \]

- Idea: Form a Lagrangian function,
  \[ \mathcal{L}(t, x, u, \lambda_0, \lambda, \mu) = \mathcal{H}(t, x, u, \lambda_0, \lambda) + \mu^T g(t, x, u) \]
  with:
  - \( \mathcal{H}(t, x, u, \lambda_0, \lambda) = \lambda_0 \ell(t, x, u) + \lambda^T f(t, x, u) \)
  - \( \mu \in \tilde{C}[t_0, t_f]^{n_u} \) Lagrange multiplier vector function

Optimal Control with State Path Constraints

- State inequality constraints arise frequently in practical applications
  - Notoriously hard to solve
  - Various forms of the NCOs – Sometimes ambiguous theory!

- Types of inequality path constraints:

Mixed State-Control Constraints:

- \( g(t, x(t), u(t)) \leq 0, \quad \forall t \)
  - \textbf{Explicit} dependence in \( u \):
    \[ g_u(t, x, u) \neq 0 \]

Pure State Constraints:

- \( h(t, x(t)) \leq 0, \quad \forall t \)
  - \textbf{Implicit} dependence in \( u \) via the differential system:
    \[ \dot{x}(t) = f(t, x(t), u(t)); \ x(t_0) = x_0 \]

Mixed Control-State Constrained Problems (cont’d)

- Let \((u^*, t^*_f) \in \tilde{C}[t_0, T]^{n_u} \times [t_0, T] \) be an optimal solution
- Let \( x^* \in \tilde{C}^{1}[t_0, T]^{n_s} \) be the optimal response
- Suppose the constraint qualification: \( \text{rank} \left[ g_u(t, x^*, u^*); \ \text{diag}(g) \right] = n_g \), holds along \((t, x^*, u^*); t_0 \leq t \leq t_f^* \)

Necessary Conditions for Optimality:

There exist \((\lambda_0^*, \lambda^*), \mu^* \in \tilde{C}[t_0, T]^{n_u+1} \) such that:
- \( (\lambda_0^*(t), \lambda^*(t), \mu^*(t)) \neq 0 \)
- \( \mu^*(t) = \text{constant} \geq 0 \)
- \( u^*(t) \in \arg \min_{v \in R^{n_u}} \{ g(t, x^*(t), v, \lambda_0^*(t), \lambda^*(t)) : g(t, x^*(t), v) \leq 0 \} \)
- \( u^*(t) = \arg \min_{v \in R^{n_u}} \{ \mathcal{H}(t, x^*(t), v, \lambda_0^*(t), \lambda^*(t)) : g(t, x^*(t), v) \leq 0 \} \)
- \( \dot{x}^*(t) = g^{\perp}(t, x^*(t), u^*(t), \lambda_0^*(t), \lambda^*(t), \mu^*(t)) \)
- \( \mu^*(t) \geq 0 \)
- \( \mathcal{H}(t^*_f, x^*(t^*_f), u^*(t^*_f), \lambda_0^*(t^*_f), \lambda^*(t^*_f)) = 0, \quad \text{if } t_f \text{ is free} \)

Mixed State-Control Constrained Problems

- Let \((u^*, t^*_f) \in \tilde{C}[t_0, T]^{n_u} \times [t_0, T] \) be an optimal solution
- Let \( x^* \in \tilde{C}^{1}[t_0, T]^{n_s} \) be the optimal response
- Suppose the constraint qualification: \( \text{rank} \left[ g_u(t, x^*, u^*); \ \text{diag}(g) \right] = n_g \), holds along \((t, x^*, u^*); t_0 \leq t \leq t_f^* \)

Necessary Conditions for Optimality:

There exist \((\lambda_0^*, \lambda^*), \mu^* \in \tilde{C}[t_0, T]^{n_u+1} \) such that:
- \( (\lambda_0^*(t), \lambda^*(t), \mu^*(t)) \neq 0 \)
- \( \mu^*(t) = \text{constant} \geq 0 \)
- \( u^*(t) \in \arg \min_{v \in R^{n_u}} \{ g(t, x^*(t), v, \lambda_0^*(t), \lambda^*(t)) : g(t, x^*(t), v) \leq 0 \} \)
- \( u^*(t) = \arg \min_{v \in R^{n_u}} \{ \mathcal{H}(t, x^*(t), v, \lambda_0^*(t), \lambda^*(t)) : g(t, x^*(t), v) \leq 0 \} \)
- \( \dot{x}^*(t) = g^{\perp}(t, x^*(t), u^*(t), \lambda_0^*(t), \lambda^*(t), \mu^*(t)) \)
- \( \mu^*(t) \geq 0 \)
- \( \mathcal{H}(t^*_f, x^*(t^*_f), u^*(t^*_f), \lambda_0^*(t^*_f), \lambda^*(t^*_f)) = 0, \quad \text{if } t_f \text{ is free} \)
Solving Mixed Control-State Constrained Problems

**Class Exercise:** Consider the optimal control problem:

\[
\text{minimize: } J(u) \triangleq \int_0^1 u(t) \, dt \\
\text{subject to: } \dot{x}(t) = f(t, x(t), u(t)); \quad x(0) = x_0, \quad x(t_f) = x_f \\
h_k(t, x(t)) \leq 0, \quad k = 1, \ldots, n_h
\]

Is \( u(t) = x(t), \; 0 \leq t \leq 1 \) a candidate optimal control?

---

**Pure State Constrained Problems**

\[
\text{minimize: } \int_0^b \ell(x(t), u(t)) \, dt \\
\text{subject to: } \dot{x}(t) = f(t, x(t), u(t)); \quad x(t_0) = x_0, \quad x(t_f) = x_f \\
h_k(t, x(t)) \leq 0, \quad k = 1, \ldots, n_h
\]

- No explicit dependence of \( h_k \) in the control \( u \)
- **Idea:** Apply similar concepts as with singular control problems

Differentiate each \( h_k \) w.r.t. \( t \) as many times as needed:

\[
h_k^0(t, x) \triangleq h_j(t, x) \\
\vdots \\
h_k^{i+1}(t, x) \triangleq \frac{d}{dt} h_k^i(t, x) = (h_k^i)_x(t, x) f(t, x, u) + (h_k^i)_t(t, x) \\
\vdots
\]

---

**1st-Order State Constraints: Indirect Adjoining Approach**

Let \([\theta_1, \theta_2]\) be a boundary arc for \( h_k \) in \((u^*, x^*)\),

\[
h_k(t, x^*(t)) = 0 \quad \Rightarrow \quad \begin{cases} 
  h_k^1(t, x^*(t), u^*(t)) \leq 0 \\
  h_k(\theta_1^+, x^*(\theta_1^+)) < 0 \\
  h_k(\theta_1^-, x^*(\theta_1^-)) = 0 
\end{cases}
\]

**Class Exercise:** What is the degree \( \sigma \) of the state constraint \( x_2(t) \leq x_2^{\text{max}} \), subject to the differential system \( \dot{x}_1(t) = u(t) \), \( \dot{x}_2(t) = x_1(t) \)?
1st-Order State Constraints: Indirect Adjoining Approach

- Let \([\theta_1, \theta_2]\) be a boundary arc for \(h_k\) in \((u^*; x^*)\).

\[
h_k(t, x^*(t)) = 0 \quad \Rightarrow \quad \begin{cases} h^k_1(t, x^*(t), u^*(t)) \leq 0 \\
 h^k_2(\theta^*_1; x^*(\theta^*_1)) < 0 \\
 h^k_3(\theta^*_0; x^*(\theta^*_0)) = 0 
\end{cases}
\]

- Activation represented by means of a multiplier function \(\eta_k(t)\).

\[
\eta_k(t)h_k(t, x^*(t)) = 0; \quad \eta_k(t) \geq 0; \quad \eta_k(t)h_k(t, x^*(t), u^*(t)) \leq 0
\]

- Interior-point constraint induce jumps in \(\lambda^*\) and \(H\) at \(\theta_1\),

\[
\lambda^*(\theta^*_1) = \lambda^*(\theta^*_0) + \pi_k(h_k(x_1, x^*(\theta_1)))
\]

\[
H[\theta^*_1] = H[\theta^*_0] - \pi_k(h_k(x_1, x^*(\theta_1)))
\]

with Lagrange multiplier \(\pi_k\) satisfying

\[
\pi_k h_k(\theta_1, x^*(\theta_1)) = 0; \quad \pi_k \geq \eta_k(\theta^*_1)
\]

Apply the NCOs for mixed control-state constrained problems, with:

\[
L(t, x, u, \lambda, \eta) = H(t, x, u, \lambda, \eta) + \eta^T H^1(t, x, u)
\]

First-Order Pure State Constrained Problems (cont’d)

- At each entry time \(\theta_1\),

\[
\lambda^*(\theta^*_1) = \lambda^*(\theta^*_0) + \pi_k^T h_x(\theta_1, x^*(\theta_1))
\]

\[
H[\theta^*_1] = H[\theta^*_0] - \pi_k^T h_x(\theta_1, x^*(\theta_1))
\]

\[
\pi_k^T h_k(\theta_1, x^*(\theta_1)) = 0; \quad \pi_k \geq \eta_k(\theta^*_1)
\]

\[
H(t^*_f, x^*(t^*_f), u^*(t^*_f), \lambda_0^*(t^*_f), \lambda^*(t^*_f)) = 0, \text{ if } t_f \text{ is free}
\]

Combined (1st-Order) Pure and Mixed State Inequality Constraints:

- Mix the previous sets of optimality conditions, with

\[
L(t, x, u, \lambda, \mu, \eta) = H(t, x, u, \lambda) + \mu^T g(t, x, u) + \eta^T H^1(t, x, u)
\]

- Strengthened C.Q.: \(\text{rank} \begin{bmatrix} g_u & \text{diag}(g) & 0 \\ h_u^1 & \text{diag}(h) & 0 \end{bmatrix} = n_g + n_h\)

Caution: A general theorem proving the foregoing NCOs is still lacking!
Solving Pure/Mixed Path Constrained Problems

**Class Exercise:** Consider the optimal control problem:

\[
\begin{align*}
\text{minimize:} & \quad J(u) = \int_0^3 e^{-at} u(t) \, dt \\
\text{subject to:} & \quad \dot{x}(t) = u(t); \quad x(0) = 0 \\
& \quad 1 - x(t) - (t - 2)^2 \leq 0, \quad 0 \leq t \leq 3 \\
& \quad 0 \leq u(t) \leq 3, \quad 0 \leq t \leq 3
\end{align*}
\]

**Conditions at a Junction Time, \( \theta \in (t_0, t_f^*) \):**

\[
\begin{align*}
x^*(\theta^-) &= x^*(\theta^+) \\
\lambda^*(\theta^-) &= \lambda^*(\theta^+) \\
\mathcal{H}(\theta, x^*(\theta), u^*(\theta^-), \lambda^*(\theta), \lambda^*(\theta)) &= \mathcal{H}(\theta, x^*(\theta), u^*(\theta^+), \lambda^*(\theta), \lambda^*(\theta))
\end{align*}
\]

**Optimal Solution Structure:**
- Sequence of active/inactive path constraints and junction times
- Set of active terminal constraints
- Use direct numerical methods to guess the optimal solution structure

**Identifying Candidate Optimal Solutions:**
- Postulate an optimal solution structure
- Check whether a pair \((u^*, x^*)\), final time \(t_f^*\), functions \((\lambda^*_0, \lambda^*_1)\) and multipliers \((\mu^*, \eta^*, \nu^*)\) can be found that satisfy all the NCOs
- Go back to step 1

---

**General Constrained Optimal Control Problems (cont’d)**

**Extended Problem Formulation:**
- Cost functional: \( J(u) = \int_{t_0}^{T_f} \ell(x(t), u(t)) \, dt + \phi(t_f, x(t_f)) \)
- Terminal constraints: \( \psi_k(t_f, x(t_f)) \leq 0, \quad k = 1, \ldots, n_\psi \)
- Suppose the constraint qualification: rank \([ \psi \kappa \quad \text{diag}(\psi) \] = \( n_\psi \), holds at \((t_f^*, x(t_f^*))\)

**Modified/Extra Necessary Conditions for Optimality:**

There exist Lagrange multipliers \( \nu^* \in \mathbb{R}^{n_\psi} \) such that:
- \( \psi(t_f^*, x^*(t_f^*)) \leq 0; \quad \nu^*(t)^T \psi(t^*(t_f^*)) = 0; \quad \nu^*(t) \geq 0 \)
- \( \left[ \lambda^*-\lambda^*_0 \phi_x-\nu^* \psi_x \right] \psi(t_f^*) = 0 \)
- \( [\mathcal{H} + \lambda^*_0 \phi_t + \nu^* \psi_t] \psi(t_f^*) = 0, \) if \( t_f \) is free

---

**General Constrained Optimal Control Problems (cont’d)**

Consider the scalar optimal control problem to

\[
\begin{align*}
\text{minimize:} & \quad J(u) = \int_0^L \left( [x_1(t)]^2 + [x_2(t)]^2 \right) \, dt \\
\text{subject to:} & \quad \dot{x}_1(t) = x_2(t); \quad x_1(0) = 0 \\
& \quad \dot{x}_2(t) = -x_2(t) + u(t); \quad x_2(0) = -1 \\
& \quad x_2(t) + 0.5 - 8[t - 0.5]^2 \leq 0, \quad \forall t \leq 20 \leq u(t) \leq 20, \forall t
\end{align*}
\]

---

**Optimal Solution (CVP 100 Stages)**

\[
\begin{align*}
u(t) & \quad (u(t) \quad x(t)) \\
\end{align*}
\]

---

**Optimal Solution (CVP 100 Stages)**

\[
\begin{align*}
u(t) & \quad (u(t) \quad x(t)) \\
\end{align*}
\]